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THE EFFROS-RUAN CONJECTURE FOR BILINEAR FORMS ON C*-ALGEBRAS

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Abstract. In 1991 Effros and Ruan conjectured that a certain Grothendieck-type inequality for a bilinear form on C*-algebras holds if (and only if) the bilinear form is jointly completely bounded. In 2002 Pisier and Shlyakhtenko proved that this inequality holds in the more general setting of operator spaces, provided that the operator spaces in question are exact. Moreover, they proved that the conjecture of Effros and Ruan holds for pairs of C*-algebras, of which at least one is exact. In this paper we prove that the Effros-Ruan conjecture holds for general C*-algebras, with constant one. More precisely, we show that for every jointly completely bounded (for short, j.c.b.) bilinear form on a pair of C*-algebras A and B, there exist states f_1, f_2 on A and g_1, g_2 on B such that for all $a \in A$ and $b \in B$,

$$|u(a,b)| \leq \|u\|_{\mathrm{jcb}}(f_1(aa^*)^{1/2}g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}) .$$

While the approach by Pisier and Shlyakhtenko relies on free probability techniques, our proof uses more classical operator algebra theory, namely, Tomita-Takesaki theory and special properties of the Powers factors of type III_λ, $0 < \lambda < 1$.

1. Introduction

In 1956 Grothendieck published the celebrated "Résumé de la théorie métrique des produits tensoriels topologiques", containing a general theory of tensor norms on tensor products of Banach spaces, describing several operations to generate new norms from known ones, and studying the duality theory between these norms. Since 1968 it has had considerable influence on the development of Banach space theory (see e.g., [12]). The highlight of the paper [8], now referred to as the "Résumé" is a result that Grothendieck called "The fundamental theorem on the metric theory of tensor products". Grothendieck's theorem asserts that given compact spaces K_1 and K_2 and a bounded bilinear form $u : C(K_1) \times C(K_2) \to \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), then there exist probability measures μ_1 and μ_2 on K_1 and K_2, respectively, such that

$$|u(f,g)| \leq K_G^{\mathbb{K}}\|u\| \left(\int_{K_1} |f(t)|^2 \, d\mu_1(t)\right)^{1/2} \left(\int_{K_2} |g(t)|^2 \, d\mu_2(t)\right)^{1/2} ,$$

for all $f \in C(K_1)$ and $g \in C(K_2)$, where $K_G^{\mathbb{K}}$ is a universal constant.

The non-commutative version of Grothendieck's inequality (conjectured in the "Résumé") was first proved by Pisier under some approximability assumption (cf. [13]), and obtained in full generality in [9]. The theorem asserts that given C*-algebras A and B and a bounded bilinear form $u : A \times B \to \mathbb{C}$, then there exist states f_1, f_2 on A and states g_1, g_2 on B such that for all $a \in A$ and $b \in B$,

$$|u(a,b)| \leq \|u\|(f_1(a^*a) + f_2(aa^*))^{1/2}(g_1(b^*b) + g_2(bb^*))^{1/2} .$$

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As a corollary, it was shown in [9] that given C*-algebras A and B, then any bounded linear operator $T : A \to B^*$ admits a factorization $T = SR$ through a Hilbert space H, where $A \xrightarrow{R} H \xrightarrow{S} B^*$, and

$$\|R\|\|S\| \leq 2\|T\|\,.$$

Let $E \subseteq A$ and $F \subseteq B$ be operator spaces sitting in C*-algebras A and B, and let $u : E \times F \to \mathbb{C}$ be a bounded bilinear form. Then, there exists a unique bounded linear operator $\widetilde{u} : E \to F^*$ such that

$$u(a\,,b) := \langle \widetilde{u}(a)\,, b\rangle\,, \quad a \in E\,, b \in F\,, \tag{1.1}$$

where $\langle\,\cdot\,,\cdot\,\rangle$ denotes the duality bracket between F and F^*. The map u is called *jointly completely bounded* (for short, j.c.b.) if the associated map $\widetilde{u} : E \to F^*$ is completely bounded, in which case we set

$$\|u\|_{\mathrm{jcb}} := \|\widetilde{u}\|_{\mathrm{cb}}\,. \tag{1.2}$$

(Otherwise, we set $\|u\|_{\mathrm{jcb}} = \infty$.) It is easily checked that

$$\|u\|_{\mathrm{jcb}} = \sup_{n\in\mathbb{N}} \|u_n\|\,, \tag{1.3}$$

where for every $n \geq 1$, the map $u_n : M_n(E) \otimes M_n(F) \to M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is given by

$$u_n\left(\sum_{i=1}^{k} a_i \otimes c_i\,, \sum_{j=1}^{l} b_j \otimes d_j\right) = \sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i, b_j) c_i \otimes d_j\,,$$

for all finite sequences $\{a_i\}_{1\leq i\leq k}$ in E, $\{b_j\}_{1\leq j\leq l}$ in F, $\{c_i\}_{1\leq i\leq k}$ and $\{d_j\}_{1\leq j\leq l}$ in $M_n(\mathbb{C})$, $k\,,l \in \mathbb{N}$. Moreover, $\|u\|_{\mathrm{jcb}}$ is the smallest constant κ_1 for which, given arbitrary C*-algebras C and D and finite sequences $\{a_i\}_{1\leq i\leq k}$ in E, $\{b_j\}_{1\leq j\leq l}$ in F, $\{c_i\}_{1\leq i\leq k}$ in C and $\{d_j\}_{1\leq j\leq l}$ in D, where $k\,,l \in \mathbb{N}$, the following inequality holds

$$\left\|\sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i\,, b_j) c_i \otimes d_j\right\|_{C\otimes_{\min} D} \leq \kappa_1 \left\|\sum_{i=1}^{k} a_i \otimes c_i\right\|_{E\otimes_{\min} C} \left\|\sum_{j=1}^{l} b_j \otimes d_j\right\|_{F\otimes_{\min} D}. \tag{1.4}$$

For a reference, see the discussion following Definition 1.1 in [16]

It was conjectured by Effros and Ruan in 1991 (cf. [5] and [16], Conjecture 0.1) that if A and B are C*-algebras and $u : A \times B \to \mathbb{C}$ is a jointly completely bounded bilinear form, then there exist states f_1, f_2 on A and states g_1, g_2 on B such that for all $a \in A$ and $b \in B$,

$$|u(a, b)| \leq K\|u\|_{\mathrm{jcb}}(f_1(aa^*)^{1/2} g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2} g_2(bb^*)^{1/2})\,, \tag{1.5}$$

where K is a universal constant.

In [16] Pisier and Shlyakhtenko proved an operator space version of (1.5), namely, if $E \subseteq A$ and $F \subseteq B$ are exact operator spaces with exactness constants $\mathrm{ex}(E)$ and $\mathrm{ex}(F)$, respectively, and $u : E \times F \to \mathbb{C}$ is a j.c.b. bilinear form, then there exist states f_1, f_2 on A and states g_1, g_2 on B such that for all $a \in E$ and $b \in F$,

$$|u(a, b)| \leq 2^{3/2}\mathrm{ex}(E)\mathrm{ex}(F)\|u\|_{\mathrm{jcb}}(f_1(aa^*)^{1/2} g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2} g_2(bb^*)^{1/2})\,.$$

Moreover, by the same methods they were able to prove the Effros-Ruan conjecture for C*-algebras with constant $K = 2^{3/2}$, provided that at least one of the C*-algebras A, B is exact (cf. [16], Theorem 0.5).

The main result of this paper is that the Effros-Ruan conjecture is true. Moreover, it holds with constant $K = 1$, that is,

Theorem 1.1. *Let A and B be C*-algebras and $u : A \times B \to \mathbb{C}$ a jointly completely bounded bilinear form. Then there exist states f_1, f_2 on A and states g_1, g_2 on B such that for all $a \in A$ and $b \in B$,*

$$|u(a,b)| \leq \|u\|_{\mathrm{jcb}}(f_1(aa^*)^{1/2}g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}) .$$

It follows from Theorem 1.1 that every completely bounded linear map $T : A \to B^*$ from a C*-algebra A to the dual B^* of a C*-algebra B has a factorization $T = vw$ through $H_r \oplus K_c$ (the direct sum of a row Hilbert space and a column Hilbert space), such that

$$\|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \leq 2\|T\|_{\mathrm{cb}} .$$

(See Proposition 3.5 of this paper.) Theorem 1.1 also settles in the affirmative a related conjecture by Blecher (cf. [1]; see also [16], Conjecture 0.2). For details, see Remark 3.2 of this paper.

Furthermore, thanks to Theorem 1.1 we can strengthen a number of results from [16], cf. Corollaries 3.7 through 3.10 in this paper. For instance, it follows that if an operator space E and its dual E^* both embed in noncommutative L_1-spaces, then E is completely isomorphic to a quotient of a subspace of $H_r \oplus K_c$, for some Hilbert spaces H and K.

It also follows from Theorem 1.1 that if $u : A \times B \to \mathbb{C}$ is a j.c.b. bilinear form on C*-algebras A and B, then the inequality (1.4) holds, as well, when the $C \otimes_{\min} D$-norm on the left-hand side is replaced by the $C \otimes_{\max} D$-norm (with constant $2\|u\|_{\mathrm{jcb}}$ instead of $\|u\|_{\mathrm{jcb}}$), cf. Proposition 3.11. Moreover, we show that for bilinear forms u on operator spaces $E \subseteq A$ and $F \subseteq B$ sitting in C*-algebras A and B, the above mentioned variant of (1.4), namely the inequality

$$\left\| \sum_{i=1}^{m}\sum_{j=1}^{n} u(a_i, b_j)c_i \otimes d_j \right\|_{C\otimes_{\max}D} \leq \kappa_4 \left\| \sum_{i=1}^{m} a_i \otimes c_i \right\|_{E\otimes_{\min}C} \left\| \sum_{j=1}^{n} b_j \otimes d_j \right\|_{F\otimes_{\min}D}$$

(where C and D are arbitrary C*-algebras) characterizes those j.c.b. bilinear forms that satisfy an Effros-Ruan type inequality. That is, there exists a constant $\kappa_2 \geq 0$ and states f_1, f_2 on A and states g_1, g_2 on B such that, for all $a \in E$ and $b \in F$,

$$|u(a,b)| \leq \kappa_2(f_1(aa^*)^{1/2}g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}) .$$

For details on operator spaces and completely bounded maps we refer to the monographs [7] and [15].

2. Proof of the Effros-Ruan Conjecture

Let $0 < \lambda < 1$ be fixed, and let $(\mathcal{M}, \phi)$ be the Powers factor of type III_λ with product state ϕ, that is,

$$(\mathcal{M}, \phi) = \bigotimes_{n=1}^{\infty}(M_2(\mathbb{C}), \omega_\lambda) ,$$

where $\phi = \bigotimes_{n=1}^{\infty} \omega_\lambda$, $\omega_\lambda(\,\cdot\,) = \mathrm{Tr}(h_\lambda\,\cdot\,)$ and $h_\lambda = \begin{pmatrix} \frac{\lambda}{1+\lambda} & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix}$ (cf. [4]). The modular automorphism group $(\sigma_t^\phi)_{t\in\mathbb{R}}$ of ϕ is given by

$$\sigma_t^\phi = \bigotimes_{n=1}^{\infty} \sigma_t^{\omega_\lambda} ,$$

where for any matrix $x = [x_{ij}]_{1\leq i,j\leq 2} \in M_2(\mathbb{C})$,

$$\sigma_t^{\omega_\lambda}(x) = h_\lambda^{it} x h_\lambda^{-it} = \begin{pmatrix} x_{11} & \lambda^{it}x_{12} \\ \lambda^{-it}x_{21} & x_{22} \end{pmatrix} , \quad t \in \mathbb{R} .$$

Therefore $\sigma_t^{\omega_\lambda}$ and σ_t^ϕ are periodic in $t \in \mathbb{R}$ with minimal period

$$t_0 := -\frac{2\pi}{\log \lambda}\,.$$

Let $\mathcal{M}_\phi$ denote the centralizer of ϕ , that is,

$$\mathcal{M}_\phi := \{x \in \mathcal{M} : \sigma_t^\phi(x) = x\,, \forall t \in \mathbb{R}\}\,.$$

It was proved by Connes (cf. [3] , Theorem 4.26) that the relative commutant of $\mathcal{M}_\phi$ in $\mathcal{M}$ is trivial, i.e.,

$$\mathcal{M}_\phi' \cap \mathcal{M} = \mathbb{C}1_{\mathcal{M}}\,, \tag{2.1}$$

where $1_{\mathcal{M}}$ denotes the identity of $\mathcal{M}$. In particular, ϕ is homogeneous in the sense of Takesaki (cf. [17]).

Furthermore, it is shown in [10] , (see Theorem 3.1 therein) that the following strong Dixmier property holds for the Powers factor $\mathcal{M}$. Namely, for all $x \in \mathcal{M}$,

$$\phi(x) \cdot 1_{\mathcal{M}} \in \overline{\mathrm{conv}\{vxv^* : v \in \mathcal{U}(\mathcal{M}_\phi)\}}^{\|\cdot\|}\,, \tag{2.2}$$

where the closure is taken in norm topology and $\mathcal{U}(\mathcal{M}_\phi)$ denotes the unitary group on $\mathcal{M}_\phi$. Moreover, by Corollary 3.4 in [10] , this can be extended to finite sets in $\mathcal{M}$, i.e., for every finite set $\{x_1\,, \ldots\,, x_n\} \in \mathcal{M}$ and every $\varepsilon > 0$, there exists a convex combination α of elements from $\{\mathrm{ad}(v) : v \in \mathcal{U}(\mathcal{M}_\phi)\}$ such that

$$\|\alpha(x_i) - \phi(x_i) \cdot 1_{\mathcal{M}}\| < \varepsilon\,, \quad 1 \le i \le n\,.$$

By standard arguments, it follows that there exists a net $\{\alpha_i\}_{i\in I} \subseteq \mathrm{conv}\{\mathrm{ad}(v) : v \in \mathcal{U}(\mathcal{M}_\phi)\}$ such that

$$\lim_{i\in I} \|\alpha_i(x) - \phi(x) \cdot 1_{\mathcal{M}}\| = 0\,, \quad x \in \mathcal{M}\,. \tag{2.3}$$

In the following, we will identify $\mathcal{M}$ with $\pi_\phi(\mathcal{M})$, where $(\pi_\phi\,, H_\phi\,, \xi_\phi)$ is the GNS representation of $\mathcal{M}$ associated to the state ϕ . Then

$$H_\phi := \overline{\mathcal{M}\xi_\phi} = L^2(\mathcal{M}\,, \phi)\,.$$

By Tomita-Takesaki theory (cf. [18]), the operator S_0 defined by

$$S_0(x\xi_\phi) = x^*\xi_\phi\,, \quad x \in \mathcal{M}$$

is closable. Its closure $S := \overline{S_0}$ has a unique polar decomposition

$$S = J\Delta^{1/2}\,, \tag{2.4}$$

where Δ is a positive self-adjoint unbounded operator on $L^2(\mathcal{M}\,, \phi)$ and J is a conjugate-linear involution. Moreover, for all $t \in \mathbb{R}$,

$$\sigma_t^\phi(x) = \Delta^{it} x \Delta^{-it}\,, \quad x \in \mathcal{M}$$

and

$$J\mathcal{M}J = \mathcal{M}'\,,$$

where $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$.

Following Takesaki's construction from [17] , define for all $n \in \mathbb{Z}$

$$\mathcal{M}_n := \{x \in \mathcal{M} : \sigma_t^\phi(x) = \lambda^{int} x\,, \ \forall t \in \mathbb{R}\}\,.$$

Then, by Lemma 1.16 in [17] ,

$$\mathcal{M}_n = \{x \in \mathcal{M} : \phi(xy) = \lambda^n \phi(yx)\,, \ \forall y \in \mathcal{M}\}\,.$$

In particular, $\mathcal{M}_\phi = \mathcal{M}_0$. It was proved in [17] (cf. Lemma 1.10) that $\mathcal{M}_n \neq \{0\}$, for all $n \in \mathbb{Z}$. Furthermore, by a combination of Lemma 1.4 and Corollary 1.16 in [17], it follows that for all $n \in \mathbb{Z}$,

$$\Delta(\eta) = \lambda^n \eta\,, \quad \eta \in \overline{\mathcal{M}_n \xi_\phi}$$

and that

$$L^2(\mathcal{M}\,,\phi) = \bigoplus_{n=-\infty}^{\infty} \overline{\mathcal{M}_n \xi_\phi}\,.$$

As a consequence, one has the following

Lemma 2.1. *For every $n \in \mathbb{Z}$, there exists $c_n \in \mathcal{M}$ such that*

$$\phi(c_n^* c_n) = \lambda^{-n/2}\,, \quad \phi(c_n c_n^*) = \lambda^{n/2} \tag{2.5}$$

and, moreover,

$$\langle c_n J c_n J \xi_\phi\,, \xi_\phi \rangle_{H_\phi} = 1\,. \tag{2.6}$$

Proof. Let $n \in \mathbb{Z}$. Take $z \in \mathcal{M}_n \setminus \{0\}$. Then $\phi(zz^*) = \lambda^n \phi(z^* z)$. Moreover,

$$JzJ\xi_\phi = S\Delta^{-1/2} z \xi_\phi = S(\lambda^{-n/2} z)\xi_\phi = \lambda^{-n/2} z^* \xi_\phi\,.$$

Therefore, $\langle zJzJ\xi_\phi\,, \xi_\phi\rangle_{H_\phi} = \lambda^{-n/2}\langle zz^*\xi_\phi\,, \xi_\phi\rangle_{H_\phi} = \lambda^{-n/2}\phi(zz^*) = \lambda^{n/2}\phi(z^*z)$. Hence

$$c_n := (\lambda^{n/2}\phi(z^* z))^{-1/2} z$$

satisfies relations (2.5) and (2.6). □

Remark 2.2. Lemma 2.1 holds with the same proof for every type III_λ factor equipped with a normal, faithful state ϕ for which σ_t^ϕ has minimal period $-2\pi/\log\lambda$. However, for the special case we are interested in, namely the Powers factor $\mathcal{M}$ of type III_λ equipped with the product state $\phi = \otimes_{k=1}^{\infty} \omega_\lambda$, a more direct proof can be given. Namely, set $c_0 = 1$ and define for every positive integer n

$$\begin{aligned} c_n &:= \left(\lambda^{1/2} + \lambda^{-1/2}\right)^n (\otimes_{k=1}^n e_{12}) \otimes I_2 \otimes \dots \\ c_{-n} &:= \left(\lambda^{1/2} + \lambda^{-1/2}\right)^n (\otimes_{k=1}^n e_{21}) \otimes I_2 \otimes \dots\,, \end{aligned}$$

where $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $I_2 \in M_2(\mathbb{C})$ is the identity matrix. It is then elementary to check that the operators $\{c_n\}_{n\in\mathbb{Z}}$ above defined satisfy the conditions in Lemma 2.1. □

Since the Powers factor $\mathcal{M}$ is injective, it is known (cf. [4]) that for all finite sequences $x_1, \dots, x_n \in \mathcal{M}$ and $y_1, \dots, y_n \in \mathcal{M}'$, where n is a positive integer, the following holds

$$\left\| \sum_{i=1}^n c_i d_i \right\|_{\mathcal{B}(L^2(\mathcal{M}\,,\phi))} = \left\| \sum_{i=1}^n c_i \otimes d_i \right\|_{\mathcal{M}\otimes_{\min}\mathcal{M}'}\,. \tag{2.7}$$

That is, the map defined by $c \otimes d \mapsto cd$, where $c \in \mathcal{M}$ and $d \in \mathcal{M}'$ extends uniquely to a C*-algebra isomorphism of $\mathcal{M} \otimes_{\min} \mathcal{M}'$ onto $C^*(\mathcal{M}\,, \mathcal{M}')$.

Now let A and B be C*-algebras and let $u : A \times B \to \mathbb{C}$ be a jointly completely bounded bilinear form.

Proposition 2.3. *There exists a bounded bilinear form* $\widehat{u} : (A \otimes_{\min} \mathcal{M}) \times (B \otimes_{\min} \mathcal{M}') \to \mathbb{C}$ *such that*

$$\widehat{u}(a \otimes c\,, b \otimes d) = u(a\,, b)\langle cd\xi_\phi\,, \xi_\phi\rangle_{H_\phi}\,, \quad a \in A\,, b \in B\,, c \in \mathcal{M}\,, d \in \mathcal{M}'\,, \tag{2.8}$$

and, moreover,

$$\|\widehat{u}\| \le \|u\|_{\mathrm{jcb}}\,. \tag{2.9}$$

Proof. Let $a_1\,, \ldots\,, a_m \in A\,,\ b_1\,, \ldots\,, b_n \in B\,,\ c_1\,, \ldots\,, c_m \in \mathcal{M}\,,\ d_1\,, \ldots\,, d_n \in \mathcal{M}'$, where m and n are positive integers. Then, by (2.7) ,

$$\begin{aligned}
\left|\sum_{i=1}^{m}\sum_{j=1}^{n} u(a_i\,, b_j)\langle c_i d_j \xi_\phi\,, \xi_\phi\rangle_{H_\phi}\right| &\le \left\|\sum_{i=1}^{m}\sum_{j=1}^{n} u(a_i\,, b_j) c_i d_j\right\|_{\mathcal{B}(L^2(\mathcal{M}\,,\phi))} \\
&= \left\|\sum_{i=1}^{m}\sum_{j=1}^{n} u(a_i\,, b_j) c_i \otimes d_j\right\|_{\mathcal{M}\otimes_{\min}\mathcal{M}'} \\
&\le \|u\|_{\mathrm{jcb}} \left\|\sum_{i=1}^{m} a_i \otimes c_i\right\|_{A\otimes_{\min}\mathcal{M}} \left\|\sum_{j=1}^{n} b_j \otimes d_j\right\|_{B\otimes_{\min}\mathcal{M}'}\,,
\end{aligned}$$

which yields the conclusion. $\square$

Lemma 2.4. *Let* $v \in \mathcal{U}(\mathcal{M}_\phi)$ *and set* $v' = JvJ \in \mathcal{M}'$. *Then, for all* $x \in A \otimes_{\min} \mathcal{M}$ *and* $y \in B \otimes_{\min} \mathcal{M}'$,

$$\widehat{u}((Id_A \otimes ad(v))(x)\,, (Id_B \otimes ad(v'))(y)) = \widehat{u}(x\,, y)\,. \tag{2.10}$$

Proof. It suffices to prove that formula (2.10) holds for elementary tensors $x = a \otimes c$ and $y = b \otimes d$, where $a \in A$, $b \in B$, $c \in \mathcal{M}$ and $d \in \mathcal{M}'$. By (2.8), it is enough to show that for all $c \in \mathcal{M}$ and $d \in \mathcal{M}'$,

$$\langle vcv^* v'd(v')^*)\xi_\phi\,, \xi_\phi\rangle_{H_\phi} = \langle cd\xi_\phi\,, \xi_\phi\rangle_{H_\phi}\,. \tag{2.11}$$

Since $\{v\,, c\,, v^*\}$ commutes with $\{v'\,, d\,, (v')^*\}$, we have

$$\begin{aligned}
\langle vcv^* v'd(v')^*\xi_\phi\,, \xi_\phi\rangle_{H_\phi} &= \langle v'vcdv^*(v')^*\xi_\phi\,, \xi_\phi\rangle_{H_\phi} \\
&= \langle cdv^*(v')^*\xi_\phi\,, v^*(v')^*\xi_\phi\rangle_{H_\phi}\,.
\end{aligned} \tag{2.12}$$

But since $J\xi_\phi = \xi_\phi$, we deduce that

$$v^*(v')^*\xi_\phi = v^*(JvJ)^*\xi_\phi = v^*Jv^*J\xi_\phi = v^*Jv^*\xi_\phi\,.$$

Furthermore, since $v^* \in \mathcal{M}_\phi$ and $\Delta^{it}\xi_\phi = \xi_\phi$, for all $t \in \mathbb{R}$, we have

$$\Delta^{it}(v^*\xi_\phi) = \sigma_t^\phi(v^*)\Delta^{it}\xi_\phi = v^*\xi_\phi\,, \quad t \in \mathbb{R}\,.$$

Hence $v^*\xi_\phi$ is an eigenvector for Δ with corresponding eigenvalue equal to 1. Using the polar decomposition (2.4) of S, we infer that

$$v^*Jv^*\xi_\phi = v^*S\Delta^{-1/2}u^*\xi_\phi = v^*Sv^*\xi_\phi = v^*v\xi_\phi = \xi_\phi\,,$$

i.e., $v^*(v')^*\xi_\phi = \xi_\phi$. Therefore

$$\langle cdv^*(v')^*\xi_\phi\,, v^*(v')^*\xi_\phi\rangle_{H_\phi} = \langle cd\xi_\phi\,, \xi_\phi\rangle_{H_\phi}\,.$$

This gives (2.12) , which completes the proof of the lemma. $\square$

Lemma 2.5. *Let* $\{\alpha_i\}_{i\in I} \subseteq \mathrm{conv}\{ad(v) : v \in \mathcal{U}(\mathcal{M}_\phi)\}$ *be a net satisfying (2.3). For every* $i \in I$ *, consider the corresponding map* α_i' *on* $\mathcal{M}' = J\mathcal{M}J$ *given by*

$$\alpha_i'(JxJ) = J\alpha_i(x)J\,, \quad x \in \mathcal{M}\,.$$

Moreover, let ϕ' *be the state on* $\mathcal{M}'$ *defined by*

$$\phi'(JxJ) := \overline{\phi(x)}\,, \quad x \in \mathcal{M}\,. \tag{2.13}$$

Furthermore, let $\hat{f}$ *be a state on* $A \otimes_{\min} \mathcal{M}$ *and* $\hat{g}$ *be a state on* $B \otimes_{\min} \mathcal{M}'$ *, arbitrarily chosen, and define states* f *on* A *, respectively,* g *on* B *by*

$$f(a) = \hat{f}(a \otimes 1_{\mathcal{M}})\,, \quad a \in A \tag{2.14}$$

$$g(b) = \hat{g}(b \otimes 1_{\mathcal{M}'})\,, \quad b \in B\,, \tag{2.15}$$

where $1_{\mathcal{M}'}$ *denotes the identity of* $\mathcal{M}'$ *. Then,*

$$\lim_{i\in I} \hat{f}((Id_A \otimes \alpha_i)(z)) = (f \otimes \phi)(z)\,, \quad z \in A \otimes_{\min} \mathcal{M}\,, \tag{2.16}$$

and, respectively,

$$\lim_{i\in I} \hat{g}((Id_B \otimes \alpha_i')(w)) = (g \otimes \phi')(w)\,, \quad w \in B \otimes_{\min} \mathcal{M}'\,. \tag{2.17}$$

Proof. Note that for $i \in I$, $\|\alpha_i\|_{\mathrm{cb}} \le 1$ and $\|\alpha_i'\|_{\mathrm{cb}} \le 1$. Therefore, $\mathrm{Id}_A \otimes \alpha_i$ and $\mathrm{Id}_B \otimes \alpha_i'$ are well-defined contractions on $A \otimes_{\min} \mathcal{M}$ and $B \otimes_{\min} \mathcal{M}'$, respectively. Hence, in order to prove (2.16) and (2.17), it suffices to consider elementary tensors $z = a \otimes c$ and $w = b \otimes d$, where $a \in A$, $b \in B$, $c \in \mathcal{M}$ and $d \in \mathcal{M}'$.

Let $a \in A$ and $c \in \mathcal{M}$. By (2.3) we deduce that the following holds in norm topology

$$\lim_{i\in I}(\mathrm{Id}_A \otimes \alpha_i)(a \otimes c) = \lim_{i\in I} a \otimes \alpha_i(c) = \phi(c)(a \otimes 1_{\mathcal{M}})\,.$$

It follows that

$$\lim_{i\in I} \hat{f}((\mathrm{Id}_A \otimes \alpha_i)(a \otimes c)) = \phi(c)\hat{f}(a \otimes 1_{\mathcal{M}}) = \phi(c)f(a) = (f \otimes \phi)(a \otimes c)\,,$$

which proves (2.16). Further, for all $x \in \mathcal{M}$,

$$\begin{aligned}\lim_{i\in I} \alpha_i'(JxJ) = \lim_{i\in I} J\alpha_i(x)J &= J(\phi(x) \cdot 1_{\mathcal{M}})J \\ &= \overline{\phi(x)}J \cdot J = \overline{\phi(x)} \cdot 1_{\mathcal{M}} = \phi'(JxJ) \cdot 1_{\mathcal{M}}\,,\end{aligned}$$

where the limit is taken in norm topology. Then (2.17) can be proved in the same way as (2.16). ☐

Proposition 2.6. *Let* u *,* $\widehat{u}$ *and* ϕ' *be as above. Then there exist states* f_1 *,* f_2 *on* A *and states* g_1 *,* g_2 *on* B *such that for all* $x \in A \otimes_{\min} \mathcal{M}$ *and* $y \in B \otimes_{\min} \mathcal{M}'$ *,*

$$|\widehat{u}(x,y)| \le \|u\|_{\mathrm{jcb}} \left((f_1 \otimes \phi)(xx^*) + (f_2 \otimes \phi)(x^*x)\right)^{1/2} \left((g_1 \otimes \phi')(y^*y) + (g_2 \otimes \phi')(yy^*)\right)^{1/2}\,. \tag{2.18}$$

Proof. By the Grothendieck inequality for C*-algebras (cf. [9]) applied to the bilinear form $\widehat{u}$, there exist states $\hat{f}_1$, $\hat{f}_2$ on $A \otimes_{\min} \mathcal{M}$ and states $\hat{g}_1$, $\hat{g}_2$ on $B \otimes_{\min} \mathcal{M}'$ such that for all $x \in A \otimes_{\min} \mathcal{M}$ and $y \in B \otimes_{\min} \mathcal{M}'$,

$$\begin{aligned}|\widehat{u}(x,y)| &\le \|\widehat{u}\|(\hat{f}_1(xx^*) + \hat{f}_2(x^*x))^{1/2}(\hat{g}_1(y^*y) + \hat{g}_2(yy^*))^{1/2} \\ &\le \|u\|_{\mathrm{jcb}}(\hat{f}_1(xx^*) + \hat{f}_2(x^*x))^{1/2}(\hat{g}_1(y^*y) + \hat{g}_2(yy^*))^{1/2}\,,\end{aligned} \tag{2.19}$$

wherein we have used inequality (2.9).

Since $\sqrt{\alpha\beta} \leq (\alpha+\beta)/2$ for all $\alpha, \beta \geq 0$, it follows that

$$|\widehat{u}(x,y)| \quad \leq \quad \frac{1}{2}\|u\|_{\mathrm{jcb}}\left(\hat{f}_1(xx^*) + \hat{f}_2(x^*x) + \hat{g_1}(y^*y) + \hat{g_2}(yy^*)\right).$$

For $i = 1, 2$, let f_i be the state on A constructed from $\hat{f}_i$ by formula (2.14), and, respectively, let g_i be the state on B constructed from $\hat{g}_i$ by formula (2.15). We show in the following that these are the states we are looking for.

By Lemma 2.4, we deduce that for all $v \in \mathcal{U}(\mathcal{M}_\phi)$ (and $v' := JvJ$, as defined therein),

$$\begin{aligned}(2.20)\qquad |\widehat{u}(x,y)| \quad \leq \quad & \frac{1}{2}\|u\|_{\mathrm{jcb}}\Big[\hat{f}_1((\mathrm{Id}_A \otimes \mathrm{ad}(v))(xx^*)) + \hat{f}_2((\mathrm{Id}_A \otimes \mathrm{ad}(v))(x^*x)) + \\ & +\hat{g_1}((\mathrm{Id}_B \otimes \mathrm{ad}(v'))(y^*y)) + \hat{g_2}((\mathrm{Id}_B \otimes \mathrm{ad}(v'))(yy^*))\Big].\end{aligned}$$

Next choose nets $\{\alpha_i\}_{i\in I}$ and $\{\alpha_i'\}_{i\in I}$ as in Lemma 2.5. For all $i \in I$, it follows that

$$\begin{aligned}(2.21)\qquad |\widehat{u}(x,y)| \quad \leq \quad & \frac{1}{2}\|u\|_{\mathrm{jcb}}\Big[\hat{f}_1((\mathrm{Id}_A \otimes \alpha_i)(xx^*)) + \hat{f}_2((\mathrm{Id}_A \otimes \alpha_i)(x^*x)) + \\ & +\hat{g_1}((\mathrm{Id}_B \otimes \alpha_i')(y^*y)) + \hat{g_2}((\mathrm{Id}_B \otimes \alpha_i')(yy^*))\Big],\end{aligned}$$

since the right-hand side of (2.21) is a convex combination of the possible right-hand sides of (2.20). Then, by Lemma 2.5 we obtain in the limit that

$$(2.22)\qquad |\widehat{u}(x,y)| \leq \frac{1}{2}\|u\|_{\mathrm{jcb}}((f_1\otimes\phi)(xx^*) + (f_2\otimes\phi)(x^*x) + (g_1\otimes\phi')(y^*y) + (g_2\otimes\phi')(yy^*)).$$

Recall that x and y were arbitrarily chosen in $A \otimes_{\min} \mathcal{M}$ and $B \otimes_{\min} \mathcal{M}'$, respectively. Hence, replacing x by $t^{1/2}x$ and y by $t^{-1/2}y$, where $t > 0$, we deduce that the following inequality holds for all $x \in A \otimes_{\min} \mathcal{M}$, $y \in B \otimes_{\min} \mathcal{M}'$ and $t > 0$:

$$(2.23)\qquad |\widehat{u}(x,y)| \leq \frac{1}{2}\|u\|_{\mathrm{jcb}}\left(t(f_1\otimes\phi)(xx^*) + t(f_2\otimes\phi)(x^*x) + \frac{1}{t}(g_1\otimes\phi')(y^*y) + \frac{1}{t}(g_2\otimes\phi')(yy^*)\right).$$

Since for all $\alpha, \beta \geq 0$, we have

$$(2.24)\qquad \inf_{t>0}(t\alpha + t^{-1}\beta) = 2\sqrt{\alpha\beta},$$

the assertion then follows by taking infimum over all $t > 0$ in (2.23). $\square$

Lemma 2.7. *Let* $\alpha, \beta \geq 0$. *Then*

$$(2.25)\qquad \inf_{n\in\mathbb{Z}}(\lambda^n\alpha + \lambda^{-n}\beta) \leq (\lambda^{1/2} + \lambda^{-1/2})\sqrt{\alpha\beta}.$$

Proof. The statement is obvious if $\alpha = 0$ or $\beta = 0$. Assume that $\alpha, \beta > 0$. Since $0 < \lambda < 1$, it follows that $(0,\infty) = \bigcup_{n\in\mathbb{Z}}[\lambda^{2n+1}, \lambda^{2n-1}]$. Hence, we can choose $n \in \mathbb{Z}$ such that

$$\lambda^{2n+1} \leq \beta/\alpha \leq \lambda^{2n-1}.$$

Set $\alpha_1 := \lambda^n\alpha$ and $\beta_1 := \lambda^{-n}\beta$. Then $\lambda \leq \beta_1/\alpha_1 \leq 1/\lambda$. Since the function $t \mapsto t^{1/2} + t^{-1/2}$ is decreasing on $[\lambda, 1]$ and increasing on $[1, 1/\lambda]$, it follows that

$$\max\{t^{1/2} + t^{-1/2} : t \in [\lambda, 1/\lambda]\} = \lambda^{1/2} + \lambda^{-1/2}.$$

Hence, we deduce that

$$\begin{aligned}\lambda^n\alpha+\lambda^{-n}\beta \;=\; \alpha_1+\beta_1 &= \left(\sqrt{\alpha_1/\beta_1}+\sqrt{\beta_1/\alpha_1}\right)\sqrt{\alpha_1\beta_1}\\ &\leq (\lambda^{1/2}+\lambda^{-1/2})\sqrt{\alpha_1\beta_1} \;=\; (\lambda^{1/2}+\lambda^{-1/2})\sqrt{\alpha\beta}\,,\end{aligned}$$

which proves (2.25). □

Proposition 2.8. *Set*

$$C(\lambda):=\sqrt{\left(\lambda^{1/2}+\lambda^{-1/2}\right)/2}\,.$$

Let u be as above and let $f_1\,,f_2$ be states on A, respectively, $g_1\,,g_2$ be states on B as in Proposition 2.6. Then, for all $a\in A$ and $b\in B$,

$$|u(a,b)|\leq C(\lambda)\|u\|_{\mathrm{jcb}}\left(f_1(aa^*)^{1/2}g_1(b^*b)^{1/2}+f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}\right)\,. \tag{2.26}$$

that is, the Effros-Ruan conjecture holds with constant $C(\lambda)$.

Proof. Let $n\in\mathbb{Z}$ and choose $c_n\in\mathcal{M}$ as in Lemma 2.1. Then, for all $a\in A$ and all $b\in B$, it follows by (2.8) and (2.6) that

$$\widehat{u}(a\otimes c_n\,,b\otimes Jc_nJ)=u(a,b)\langle c_nJc_nJ\xi_\phi\,,\xi_\phi\rangle_{H_\phi}=u(a,b)\,.$$

By Proposition 2.6, together with (2.13) and (2.5), it follows that

$$\begin{aligned}|u(a,b)|^2 &= |\widehat{u}(a\otimes c_n\,,b\otimes Jc_nJ)|^2\\ &\leq \|u\|_{\mathrm{jcb}}^2\left(f_1(aa^*)\phi(c_nc_n^*)+f_2(a^*a)\phi(c_n^*c_n)\right)\left(g_1(b^*b)\phi({c_n}^*c_n)+g_2(bb^*)\phi(c_n{c_n}^*)\right)\\ &= \|u\|_{\mathrm{jcb}}^2\left(\lambda^{n/2}f_1(aa^*)+\lambda^{-n/2}f_2(a^*a)\right)\left(\lambda^{-n/2}g_1(b^*b)+\lambda^{n/2}g_2(bb^*)\right)\\ &= \|u\|_{\mathrm{jcb}}^2\left(f_1(aa^*)g_1(b^*b)+f_2(a^*a)g_2(bb^*)+\lambda^nf_1(aa^*)g_2(bb^*)+\lambda^{-n}f_2(a^*a)g_1(b^*b)\right)\,.\end{aligned} \tag{2.27}$$

Note that $\lambda^{1/2}+\lambda^{-1/2}=2C(\lambda)^2$. By taking infimum in (2.27) over all $n\in\mathbb{Z}$, we deduce from Lemma 2.7 that

$$\begin{aligned}|u(a,b)|^2 &\leq \|u\|_{\mathrm{jcb}}^2\left(f_1(aa^*)g_1(b^*b)+f_2(a^*a)g_2(bb^*)+2C(\lambda)^2f_1(a^*a)^{\frac12}g_1(b^*b)^{\frac12}f_2(aa^*)^{\frac12}g_2(bb^*)^{\frac12}\right)\\ &\leq C(\lambda)^2\|u\|_{\mathrm{jcb}}^2\left(f_1(aa^*)^{\frac12}g_1(b^*b)^{\frac12}+f_2(a^*a)^{\frac12}g_2(bb^*)^{\frac12}\right)^2\,,\end{aligned}$$

wherein we have used the fact that $C(\lambda)>1$. The assertion follows now by taking square roots. □

Proof of Theorem 1.1: Thus far we have proved that given C*-algebras A and B and a j.c.b. bilinear form $u:A\times B\to\mathbb{C}$, then the Effros-Ruan conjecture holds with constant $C(\lambda)=\sqrt{\left(\lambda^{1/2}+\lambda^{-1/2}\right)/2}$, for every $0<\lambda<1$. Now recall that the sets

$$Q(A):=\{f\in A_+^*:\|f\|\leq 1\}\,,\quad Q(B):=\{g\in B_+^*:\|g\|\leq 1\}$$

are compact in the weak*-topology, where A_+^* and B_+^* denote the sets of positive functionals on A and B, respectively. Since $C(\lambda)\to 1$ as $\lambda\to 1$, by using a simple compactness argument it follows from Proposition 2.8 that there exist $f_1^0\,,f_2^0\in Q(A)$ and $g_1^0\,,g_2^0\in Q(B)$ such that for all $a\in A$ and $b\in B$,

$$|u(a,b)|\leq\|u\|_{\mathrm{jcb}}\left(f_1^0(aa^*)^{1/2}g_1^0(b^*b)^{1/2}+f_2^0(a^*a)^{1/2}g_2^0(bb^*)^{1/2}\right)\,.$$

But $f_i^0 \leq f_i$, respectively, $g_i^0 \leq g_i$, $i = 1, 2$, where f_1 , f_2 are states on A and g_1 , g_2 are states on B . Therefore the Effros-Ruan conjecture holds with constant one. $\square$

Remark 2.9. In this remark we will outline a slightly different proof of the Effros-Ruan conjecture, which was kindly suggested to us by the referee. The purpose is to show that one can avoid using the strong Dixmier property (2.2) for type III_λ- factors, and instead apply only the classical Dixmier property (cf. [11], Section 8.3), which implies that for every finite factor $\mathcal{N}$ with (unique) trace state τ, there exists a net $(\beta_i)_{i\in I} \subseteq \mathrm{conv}\{\mathrm{ad}_{\mathcal{N}}(v) : v \in \mathcal{N}\}$ such that

$$\lim_{i\in I} \|\beta_i(x) - \tau(x)1_{\mathcal{N}}\| = 0 , \quad x \in \mathcal{N} . \tag{2.28}$$

Using this property instead of (2.3), it is then possible to prove weaker versions of Lemma 2.5 and of Proposition 2.6, which are, nonetheless, sufficiently strong in order to prove Proposition 2.8, and hence to conclude the proof of the Effros-Ruan conjecture.

To simplify the argument, we will assume that A and B are unital C^*-algebras, which implies that the state spaces $S(A \otimes_{\min} \mathcal{M})$ and $S(B \otimes_{\min} \mathcal{M}')$ are w^*-compact. By (2.1), the centralizer $\mathcal{M}_\phi$ is a finite factor with trace state $\phi_{|\mathcal{M}_\phi}$. Choose now a net $(\beta_i)_{i\in I}$ in $\mathrm{conv}\{\mathrm{ad}_{\mathcal{M}_\phi}(v) : v \in \mathcal{M}_\phi\}$ such that (2.28) holds for $\mathcal{N} = \mathcal{M}_\phi$ and $\tau = \phi_{|\mathcal{M}_\phi}$. Clearly, each β_i can be extended to an operator $\alpha_i : \mathcal{M} \to \mathcal{M}$ in $\mathrm{conv}\{\mathrm{ad}_{\mathcal{M}}(v) : v \in \mathcal{M}_\phi\}$. Moreover, by (2.28) we have

$$\lim_{i\in I} \|\alpha_i(x) - \phi(x)1_{\mathcal{M}}\| = 0 , \quad x \in \mathcal{M}_\phi . \tag{2.29}$$

Using now (2.29) instead of (2.3) in the proof of Lemma 2.5, one obtains a weaker version of this lemma, namely, (2.16) and (2.17) therein must be replaced, respectively, by

$$\lim_{i\in I} \hat{f}((\mathrm{Id}_A \otimes \alpha_i)(z)) = (f \otimes \phi)(z) , \quad z \in A \otimes_{\min} \mathcal{M}_\phi , \tag{2.30}$$

$$\lim_{i\in I} \hat{g}((\mathrm{Id}_B \otimes \alpha_i')(w)) = (g \otimes \phi')(w) , \quad w \in B \otimes_{\min} (J\mathcal{M}_\phi J) , \tag{2.31}$$

As a consequence, Proposition 2.6 must be replaced by the following weaker statement:

There exists states $f_1 , f_2 \in S(A)$, $g_1 , g_2 \in S(B)$, $F_1 , F_2 \in S(A \otimes_{\min} \mathcal{M})$ and $G_1 , G_2 \in S(B \otimes_{\min} \mathcal{M}')$ such that for all $x \in A \otimes_{\min} \mathcal{M}$ and all $y \in B \otimes_{\min} \mathcal{M}'$,

$$|\widehat{u}(x, y)| \leq \|u\|_{\mathrm{jcb}}(F_1(xx^*) + F_2(x^*x))^{1/2}(G_1(y^*y) + G_2(yy^*))^{1/2} , \tag{2.32}$$

and, moreover, for $k = 1 , 2$,

$$F_k(z) = (f_k \otimes \phi)(z) , \quad z \in A \otimes_{\min} \mathcal{M}_\phi , \tag{2.33}$$

$$G_k(w) = (g_k \otimes \phi')(w) , \quad w \in B \otimes_{\min} (J\mathcal{M}_\phi J) . \tag{2.34}$$

In order to prove from (2.30) and (2.31) that such states exist, we start by choosing $\hat{f}_1 , \hat{f}_2 \in S(A \otimes_{\min} \mathcal{M})$ and $\hat{g}_1 , \hat{g}_2 \in S(B \otimes_{\min} \mathcal{M}')$ satisfying (2.19). Next we let (F_1 , F_2 , G_1 , G_2) be a weak*-limit point of the net $(\hat{f}_1 \circ (\mathrm{Id}_A \otimes \alpha_i), \hat{f}_2 \circ (\mathrm{Id}_A \otimes \alpha_i), \hat{g}_1 \circ (\mathrm{Id}_B \otimes \alpha_i'), \hat{g}_2 \circ (\mathrm{Id}_B \otimes \alpha_i'))_{i\in I}$ in the w^*-compact set

$(S(A\otimes_{\min}\mathcal{M}))^2\times(S(B\otimes_{\min}\mathcal{M}'))^2$. The convexity argument from the proof of Proposition 2.6 can now be applied to show that (2.32) holds. Further, for $k=1\,,2$ set

$$\begin{aligned} f_k(a) &= \hat{f}_k(a\otimes 1_{\mathcal{M}})\,, \quad a\in A\,,\\ g_k(b) &= \hat{g_k}(b\otimes 1_{\mathcal{M}})\,, \quad b\in B\,. \end{aligned}$$

Then equalities (2.33) and (2.34) follow from (2.30) and (2.31). This completes the proof of the new (above) version of Proposition 2.6 .

Observe now that the operators $(c_n)_{n\in\mathbb{Z}}$ in Lemma 2.1 (or Remark 2.2) satisfy the condition

$$c_n\in\mathcal{M}_n:=\{x\in\mathcal{M}:\sigma_t^{\phi}(x)=\lambda^{int}x\,,\ \forall t\in\mathbb{R}\}\,.$$

Hence $c_n^*c_n$ and $c_nc_n^*$ belong to the centralizer $\mathcal{M}_\phi$, for all $n\in\mathbb{Z}$. Therefore, relations (2.32) , (2.33) and (2.34) are sufficient to prove (2.27), and hence to conclude the proofs of Proposition 2.8 and, respectively, of Theorem 1.1 .

3. Applications

Let $E\subseteq A$ and $F\subseteq B$ be operator spaces sitting in C*-algebras A and B. Let $u:E\times F\to\mathbb{C}$ be a bounded bilinear form. Define $\|u\|_{\mathrm{ER}}$ to be the smallest constant $0\le\kappa_2\le\infty$ for which there exist states f_1 , f_2 on A and states g_1 , g_2 on B such that for all $a\in E$ and $b\in F$,

$$|u(a,b)|\le\kappa_2(f_1(aa^*)^{1/2}g_1(b^*b)^{1/2}+f_2(a^*a)^{1/2}g_2(bb^*)^{1/2})\,. \tag{3.1}$$

In the case when $E=A$ and $F=B$, we have from Theorem 1.1 that $\|u\|_{\mathrm{ER}}\le\|u\|_{\mathrm{jcb}}$. Moreover, if E and F are exact operator spaces and $u:E\times F\to\mathbb{C}$ is a j.c.b. bilinear form, then by [16] (cf. Theorem 0.3 and 0.4) ,

$$\|u\|_{\mathrm{ER}}\le 2^{3/2}\mathrm{ex}(E)\mathrm{ex}(F)\|u\|_{\mathrm{jcb}}\,.$$

However, for bilinear forms on general operator spaces E and F it can happen that $\|u\|_{\mathrm{jcb}}<\infty$, while $\|u\|_{\mathrm{ER}}=\infty$ (see Example 3.6 below) . Therefore Theorem 1.1 cannot be generalized to arbitrary operator spaces.

Recall that a bilinear map $u:E\times F\to\mathbb{C}$ is called *completely bounded* (in the sense of Christensen and Sinclair) (see [2] , [16] and the references given therein) if the bilinear forms $u_n:M_n(E)\times M_n(F)\to M_n(\mathbb{C})$ defined by

$$u_n(a\otimes x\,,b\otimes y):=u(a\,,b)xy\,,\quad a\in E\,,b\in F\,,x\,,y\in M_n(\mathbb{C})$$

are uniformly bounded, in which case we set

$$\|u\|_{\mathrm{cb}}:=\sup_{n\in\mathbb{N}}\|u_n\|\,. \tag{3.2}$$

Moreover, u is completely bounded if and only if there exists a constant $\kappa_3\ge 0$ and states f on A and g on B such that for all $a\in E$ and $b\in F$,

$$|u(a,b)|\le\kappa_3 f(aa^*)^{1/2}g(b^*b)^{1/2} \tag{3.3}$$

and $\|u\|_{\mathrm{cb}}$ is the smallest constant κ_3 for which (3.3) holds (see also the Introduction to [16]).

It was shown by Effros and Ruan (cf. [6]) that if $u:E\times F\to\mathbb{C}$ is completely bounded, then the associated map $\widetilde{u}:E\to F^*$ defined by (1.1) admits a factorization of the form $\widetilde{u}=vw$ through a row Hilbert space H_r , where $E\xrightarrow{v}H_r\xrightarrow{w}F^*$ and $\|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}}=\|u\|_{\mathrm{cb}}$. In particular, it follows that

$$\|u\|_{\mathrm{jcb}}:=\|\widetilde{u}\|_{\mathrm{cb}}\le\|u\|_{\mathrm{cb}}\,. \tag{3.4}$$

Lemma 3.1. (*cf.* [16] *and* [19]) *Let $u : E \times F \to \mathbb{C}$ be a bounded bilinear form on operator spaces $E \subseteq A$ and $F \subseteq B$ sitting in C*-algebras A and B. Let f_1 , f_2 be states on A and g_1 , g_2 be states on B such that for all $a \in E$ and $b \in F$,*

$$|u(a,b)| \leq \|u\|_{\mathrm{ER}}(f_1(aa^*)^{1/2}g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}) .$$

Then u can be decomposed as $u = u_1 + u_2$, where u_1 and u_2 are bilinear forms satisfying the following inequalities, for all $a \in A$ and $b \in B$:

$$|u_1(a,b)| \leq \|u\|_{\mathrm{ER}} f_1(aa^*)^{1/2}g_1(b^*b)^{1/2} \tag{3.5}$$

$$|u_2(a,b)| \leq \|u\|_{\mathrm{ER}} f_2(a^*a)^{1/2}g_2(bb^*)^{1/2} . \tag{3.6}$$

In particular,

$$\|u_1\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{ER}} , \quad \|u_2^t\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{ER}} ,$$

where $u_2^t(b,a) := u_2(a,b)$, for all $a \in E$ and $b \in F$.

Proof. Such a decomposition was obtained in [16] (cf. last statement in Theorem 0.4 in [16]), except that the states f_1 , f_2 , g_1 , g_2 satisfying (3.5) and (3.6) were possibly different from the original ones. Later, following a suggestion of Pisier, Xu proved the above decomposition without change of states. (See [19], Proposition 5.1 and the Remark following the proof of this proposition.) □

Remark 3.2. Note that our main result combined with the above splitting lemma solves conjecture (0.2′) in [16] (with constant $K = 2$) , and hence it solves Blecher's conjecture (cf. [1] and Conjecture (0.2) in [16]).

Proposition 3.3. (*i*) *Let $u : A \times B \to \mathbb{C}$ be a bounded bilinear form on C*-algebras A and B . Then*

$$\|u\|_{\mathrm{ER}} \leq \|u\|_{\mathrm{jcb}} \leq 2\|u\|_{\mathrm{ER}} . \tag{3.7}$$

(*ii*) *Let c_1 , c_2 denote the best constants in the inequalities*

$$c_1\|u\|_{\mathrm{ER}} \leq \|u\|_{\mathrm{jcb}} \leq c_2\|u\|_{\mathrm{ER}} , \tag{3.8}$$

where $u : A \times B \to \mathbb{C}$ is any bounded bilinear form on arbitrary C-algebras A and B . Then $c_1 = 1$ and $c_2 = 2$.*

Proof. (*i*) . The left-hand side inequality follows from our main theorem, while the right-hand side inequality follows from the splitting lemma above. Indeed, we can assume that $\|u\|_{\mathrm{ER}} < \infty$. Then with $u_1 , u_2 : A \times B \to \mathbb{C}$ as in Lemma 3.1 ,

$$\|u\|_{\mathrm{jcb}} \leq \|u_1\|_{\mathrm{jcb}} + \|u_2\|_{\mathrm{jcb}} = \|u_1\|_{\mathrm{jcb}} + \|u_2^t\|_{\mathrm{jcb}} \leq \|u_1\|_{\mathrm{cb}} + \|u_2^t\|_{\mathrm{cb}} \leq 2\|u\|_{\mathrm{ER}} .$$

(*ii*) . By (*i*) we know that $c_1 \geq 1$ and $c_2 \leq 2$. We now prove that $c_2 = 2$. Let τ be a tracial state on a C*-algebra A and define a bilinear form $u : A \times A \to \mathbb{C}$ by $u(a,b) := \tau(ab)$, for all $a , b \in A$. Then $\|u\|_{\mathrm{jcb}} \geq \|u\| = 1$, and for all $a , b \in A$,

$$|u(a,b)| \leq \tau(aa^*)^{1/2}\tau(b^*b)^{1/2} = \frac{1}{2}\left(\tau(aa^*)^{1/2}\tau(b^*b)^{1/2} + \tau(a^*a)^{1/2}\tau(bb^*)^{1/2}\right) ,$$

which implies that $\|u\|_{\mathrm{ER}} \leq \frac{1}{2}$. By (3.7), $\|u\|_{\mathrm{ER}} \geq \frac{1}{2}\|u\|_{\mathrm{jcb}}$. Hence $\|u\|_{\mathrm{ER}} = \frac{1}{2}$ and $\|u\|_{\mathrm{jcb}} = 1$, and the assertion follows. To prove that $c_1 = 1$, let ϕ be any state on a unital, properly infinite C*-algebra A. Let $u : A \otimes A \to \mathbb{C}$ be defined by $u(a,b) := \phi(ab)$, for all $a, b \in A$. Note that

$$\|u\|_{\mathrm{ER}} \leq \|u\|_{\mathrm{jcb}} \leq \|u\|_{\mathrm{cb}} \leq 1,$$

where the last inequality follows immediately from $(0.5')$ in [16] (by taking $f_1 = g_1 = \phi$ therein). We claim that $\|u\|_{\mathrm{ER}} = 1$. For this, let f_1, f_2, g_1, g_2 be states on A and let $\{s_n\}_{n\geq 1}$ be a sequence of isometries in A with orthogonal ranges. Then $f_k(s_n s_n^*) \to 0$ as $n \to \infty$, respectively $g_k(s_n s_n^*) \to 0$ as $n \to \infty$, for $k = 1, 2$. Note that $u(s_n, s_n^*) = 1$, for all $n \geq 1$, while

$$\lim_{n\to\infty} f_1(s_n s_n^*)^{1/2} g_1(s_n s_n^*)^{1/2} + f_2(s_n^* s_n)^{1/2} g_2(s_n^* s_n)^{1/2} = 1.$$

This shows that $\|u\|_{\mathrm{ER}} \geq 1$ and the assertion is proved. □

Lemma 3.4. *Let $E \subseteq A$ and $F \subseteq B$ be operator spaces sitting in C*-algebras A and B, and let $u : E \times F \to \mathbb{C}$ be a bounded bilinear form. If $\|u\|_{\mathrm{ER}} < \infty$, then the associated map $\widetilde{u} : E \to F^*$ admits a cb-factorization $\widetilde{u} = vw$ through $H_r \oplus K_c$ for some Hilbert spaces H and K, where $E \xrightarrow{v} H_r \oplus K_c \xrightarrow{w} F^*$, satisfying*

$$\|v\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \leq 2\|\widetilde{u}\|_{\mathrm{ER}}.$$

Proof. Choose states f_1, f_2 on A and states g_1, g_2 on B such that (3.1) holds. Then, by Lemma 3.1, u can be decomposed as $u = u_1 + u_2$, where u_1 and u_2 are bounded bilinear forms satisfying (3.5) and (3.6). The rest of the proof follows from the proof of Corollary 0.7 on p. 206 in [16]. □

Proposition 3.5. *Let A and B be C*-algebras. Then every completely bounded linear map $T : A \to B^*$ admits a cb-factorization $T = vw$ through $H_r \oplus K_c$ for some Hilbert spaces H and K, such that*

$$\|u\|_{\mathrm{cb}}\|w\|_{\mathrm{cb}} \leq 2\|T\|_{\mathrm{cb}}.$$

Proof. Let $T : A \to B^*$ be a completely bounded linear map. Then T is of the form $T = \widetilde{u}$, for a j.c.b. bilinear form $u : A \times B \to \mathbb{C}$ with $\|u\|_{\mathrm{jcb}} = \|T\|_{\mathrm{cb}}$. The assertion follows now from Lemma 3.4, by using the fact that $\|u\|_{\mathrm{ER}} \leq \|u\|_{\mathrm{jcb}}$. □

The following example is implicit in the proof of Corollary 3.2 in [16]:

Example 3.6. Let E be an operator space which is not Banach space isomorphic to a Hilbert space, and let $E \subseteq A$ and $E^* \subseteq B$ be completely isometric embeddings of E and E^*, respectively, into C*-algebras A and B. Define $u : E \times E^* \to \mathbb{C}$ by

$$u(a,b) := b(a), \quad a \in E, b \in E^*.$$

Then $\widetilde{u} : E \to E^{**}$ is the standard inclusion of E into its second dual. Therefore $\|u\|_{\mathrm{jcb}} = \|\widetilde{u}\|_{\mathrm{cb}} = 1$. We will show that $\|u\|_{\mathrm{ER}} = \infty$. If $\|u\|_{\mathrm{ER}} < \infty$, then it follows from Lemma 3.4 that $\widetilde{u}$ admits a cb-factorization through $H_r \oplus K_c$, for some Hilbert spaces H and K. In particular, $\widetilde{u} : E \to E^{**}$ has a Banach space factorization through a Hilbert space. This contradicts the assumption on E. Hence $\|u\|_{\mathrm{ER}} = \infty$.

The following result was proved in [16] with constant $2^{9/4}$ instead of $\sqrt{2}$ (see the second part of Corollary 3.4 in [16]) .

Corollary 3.7. *Let T be a completely bounded linear map from a C^*-algebra A to the operator Hilbert space $OH(I)$, I being an arbitrary index set. Then there exist states f_1 and f_2 on A such that*

$$\|T(a)\| \leq \sqrt{2} f_1(aa^*)^{1/4} f_2(a^*a)^{1/4}\,, \quad a \in A\,.$$

Proof. Given a vector space E , we let $\bar{E}$ denote the conjugate vector space. Let $J : OH(I) \to \overline{OH(I)^*}$ be the canonical cb-isomorphism of $OH(I)$ with the conjugate of its dual space (cf. [14]), and set

$$V := \overline{T^*} JT\,,$$

where $T^* : OH(I)^* \to A^*$ is the adjoint of T . Then V is a completely bounded linear map from A to $\overline{A^*} = (\bar{A})^*$. Therefore $V = \widetilde{v}$ for a j.c.b. bilinear form $v : A \times \bar{A} \to \mathbb{C}$. Moreover,

$$\|v\|_{\mathrm{jcb}} = \|V\|_{\mathrm{cb}} \leq \|T\|_{\mathrm{cb}}^2\,.$$

Actually, equality holds above (cf. [16], proof of Corollary 3.4), but we shall not need this. By our main theorem, there exist states $f_1^0\,, f_2^0$ on A and states $g_1^0\,, g_2^0$ on $\bar{A}$ such that for all $a \in A$ and $b \in \bar{A}$,

$$|v(a,b)| \leq \|T\|_{\mathrm{cb}}^2 \left(f_1^0(aa^*)^{1/2} g_1^0(b^*b)^{1/2} + f_2^0(a^*a)^{1/2} g_2^0(bb^*)^{1/2} \right)\,.$$

The canonical isomorphism J of $OH(I)$ onto $\overline{OH(I)^*}$ satisfies

$$\overline{J(x)}(x) = \|x\|^2 = J(x)(\bar{x})\,, \quad x \in OH(I)\,.$$

For all $a \in A$ we then have $v(a\,,\bar{a}) = (Va)(\bar{a}) = (\overline{T^*} JTa)(\bar{a}) = (JTa)(\,\overline{Ta}\,) = \|Ta\|^2$, and therefore

$$\begin{aligned} \|Ta\|^2 = |v(a\,,\bar{a})| &\leq \|T\|_{\mathrm{cb}}^2 \left(f_1^0(aa^*)^{1/2} g_1^0(\,\overline{a^*a}\,)^{1/2} + f_2^0(a^*a)^{1/2} g_2^0(\,\overline{aa^*}\,)^{1/2} \right) \\ &\leq \|T\|_{\mathrm{cb}}^2 \left(f_1^0(aa^*) + g_2^0(\,\overline{aa^*}\,) \right)^{1/2} \left(f_2^0(a^*a) + g_1^0(\,\overline{a^*a}\,) \right)^{1/2} \\ &\leq 2\|T\|_{\mathrm{cb}}^2 f_1(aa^*)^{1/2} f_2(a^*a)^{1/2}\,, \end{aligned}$$

where f_1 and f_2 are states on A given by

$$f_1(a) := \frac{1}{2}\left(f_1^0(a) + \overline{g_2^0(\bar{a})} \right)\,, \quad f_2(a) := \frac{1}{2}\left(f_2^0(a) + \overline{g_1^0(\bar{a})} \right)\,, \quad a \in A\,.$$

This completes the proof. □

As a consequence of Proposition 3.3 we also obtain (by adjusting the corresponding proofs in [16]) the following strengthening of Corollaries 3.1 and 3.3 in [16]:

Corollary 3.8. *Let E be an operator space such that E and its dual E^* embed completely isomorphically into preduals M_* and N_* , respectively, of von Neumann algebras M and N . Then E is cb-isomorphic to a quotient of a subspace of $H_r \oplus K_c$, for some Hilbert spaces H and K .*

Corollary 3.9. *Let E be an operator space and let $E \subseteq A$ and $E^* \subseteq B$ be completely isometric embeddings into C^*-algebras A and B such that both subspaces are completely complemented. Then E is cb-isomorphic to $H_r \oplus K_c$ for some Hilbert spaces H and K .*

Note that as another consequence of our main theorem we obtain (with essentially the same proof as the corresponding Corollary 0.6 in [16]) the following result:

Corollary 3.10. *Let A_0 , A , B_0 and B be C*-algebras such that $A_0 \subseteq A$ and $B_0 \subseteq B$. Then any j.c.b. bilinear form $u_0 : A_0 \times B_0 \to \mathbb{C}$ extends to a bilinear form $u : A \times B \to \mathbb{C}$ such that*

$$\|u\|_{\mathrm{jcb}} \leq 2\|u_0\|_{\mathrm{jcb}} .$$

Let $u : A \times B \to \mathbb{C}$ be a j.c.b. bilinear form on C*-algebras A and B . Recall that $\|u\|_{\mathrm{jcb}}$ is the smallest constant κ_1 for which inequality (1.4) holds, for arbitrary C*-algebras C and D . The following result shows that if the inequality (1.4) holds for the given bilinear form u with constant κ_1 , then the same inequality (with κ_1 replaced by $2\kappa_1$) holds for u , when the $(C \otimes_{\min} D)$-norm on the left-hand side is replaced by the $(C \otimes_{\max} D)$-norm.

Proposition 3.11. *Let A and B be C*-algebras, and let $u : A \times B \to \mathbb{C}$ be a j.c.b. bilinear form. Then, for all C*-algebras C and D , all $m, n \in \mathbb{N}$ and all finite sequences $a_1 , \ldots , a_m \in A$, $b_1 , \ldots , b_n \in B$, $c_1 , \ldots , c_m \in C$, $d_1 , \ldots , d_n \in D$,*

$$\left\| \sum_{i=1}^{m} \sum_{j=1}^{n} u(a_i, b_j) c_i \otimes d_j \right\|_{C \otimes_{\max} D} \leq 2\|u\|_{\mathrm{jcb}} \left\| \sum_{i=1}^{m} a_i \otimes c_i \right\|_{A \otimes_{\min} C} \left\| \sum_{j=1}^{n} b_j \otimes d_j \right\|_{B \otimes_{\min} D} . \tag{3.9}$$

Proof. There exist states f_1 , f_2 on A and g_1 , g_2 on B such that inequality (3.1) holds. Then, as explained in the proof of Lemma 3.4 , u can be decomposed as $u = u_1 + u_2$, where u_1 and u_2 are bounded bilinear forms satisfying (3.5) and (3.6) .

By the definition of $\| \cdot \|_{\max}$, in order to prove (3.9) we have to show that for all pairs of commuting representations $\pi : A \to \mathcal{B}(H)$, $\rho : B \to \mathcal{B}(H)$, where H is an arbitrary Hilbert space, and all finite sequences $a_1 , \ldots , a_m \in A$, $b_1 , \ldots , b_n \in B$, $c_1 , \ldots , c_m \in C$, $d_1 , \ldots , d_n \in D$, where $m, n \in \mathbb{N}$, we have

$$\left\| \sum_{i=1}^{m} \sum_{j=1}^{n} u(a_i, b_j) \pi(c_i) \rho(d_j) \right\| \leq 2\|u\|_{\mathrm{jcb}} \left\| \sum_{i=1}^{m} a_i \otimes c_i \right\|_{A \otimes_{\min} C} \left\| \sum_{j=1}^{n} b_j \otimes d_j \right\|_{B \otimes_{\min} D} . \tag{3.10}$$

By our main theorem, $\|u\|_{\mathrm{ER}} \leq \|u\|_{\mathrm{jcb}} < \infty$. Let ξ , η be unit vectors in H . Let $u = u_1 + u_2$ be the decomposition of u satisfying (3.5) and (3.6) as above. Then

$$\begin{aligned} &\left| \langle \sum_{i=1}^{m} \sum_{j=1}^{n} u(a_i, b_j) \pi(c_i) \rho(d_j) \xi, \eta \rangle \right| \\ &\qquad \leq \left| \langle \sum_{i=1}^{m} \sum_{j=1}^{n} u_1(a_i, b_j) \pi(c_i) \rho(d_j) \xi, \eta \rangle \right| + \left| \sum_{i=1}^{m} \sum_{j=1}^{n} \langle u_2(a_i, b_j) \pi(c_i) \rho(d_j) \xi, \eta \rangle \right| , \end{aligned} \tag{3.11}$$

where $\langle \cdot , \cdot \rangle$ denotes the inner product on H . By using the GNS construction for the states f_1 on A and g_1 on B and inequality (3.5), we obtain for any $a \in A$ and $b \in B$ that

$$\begin{aligned} |u_1(a, b)| &\leq \|u\|_{\mathrm{ER}} f_1(aa^*)^{1/2} g_1(b^*b)^{1/2} \\ &= \|u\|_{\mathrm{ER}} \|\pi_{f_1}(a^*)\xi_{f_1}\| \cdot \|\pi_{g_1}(b)\xi_{g_1}\| , \end{aligned}$$

where $(H_{f_1} , \pi_{f_1} , \xi_{f_1})$ is the GNS triple associated to (A, f_1) , respectively, $(H_{g_1} , \pi_{g_1} , \xi_{g_1})$ is the GNS triple associated to (B, g_1) . Hence, there exists $V_1 \in \mathcal{B}(H_{g_1} , H_{f_1})$ such that $\|V_1\| \leq \|u\|_{\mathrm{ER}}$, satisfying

$$u_1(a, b) = \langle V_1 \pi_{g_1}(b) \xi_{g_1} , \pi_{f_1}(a^*) \xi_{f_1} \rangle , \quad a \in A , b \in B .$$

Therefore, for any $a \in A$ and $b \in B$,

$$\left|\langle\sum_{i=1}^{m}\sum_{j=1}^{n} u_1(a_i, b_j)\pi(c_i)\rho(d_j)\xi, \eta\rangle\right| = \left|\langle\sum_{i=1}^{m}\sum_{j=1}^{n}\langle V_1\pi_{g_1}(b)\xi_{g_1}\,, \pi_{f_1}(a^*)\xi_{f_1}\rangle\rho(d_j)\pi(c_i)\xi\,, \eta\rangle\right| \tag{3.12}$$

$$= \left|\langle(V_1 \otimes 1_H)(\pi_{g_1} \otimes \rho)(\sum_{j=1}^{n} b_j \otimes d_j)(\xi_{g_1} \otimes \xi)\,, (\pi_{f_1} \otimes \pi)(\sum_{i=1}^{m} a_i^* \otimes c_i^*)(\xi_{f_1} \otimes \eta)\rangle\right|$$

$$\leq \|u\|_{ER}\left\|\sum_{i=1}^{m} a_i \otimes c_i\right\|_{A\otimes_{\min}C}\left\|\sum_{j=1}^{n} b_j \otimes d_j\right\|_{B\otimes_{\min}D},$$

wherein we used the fact that the representations π and ρ do commute, and that $\sum_i a_i^*\otimes c_i^* = (\sum_i a_i \otimes c_i)^*$.

Similarly, by using the GNS construction for the states f_2 on A and g_2 on B and inequality (3.6), we obtain for any $a \in A$ and $b \in B$ that

$$\begin{aligned}|u_2(a,b)| &\leq \|u\|_{\mathrm{ER}} f_2(a^*a)^{1/2} g_2(bb^*)^{1/2} \\ &= \|u\|_{\mathrm{ER}}\|\pi_{f_2}(a)\xi_{f_2}\| \cdot \|\pi_{g_2}(b^*)\xi_{g_2}\|\,,\end{aligned}$$

where $(H_{f_2}\,, \pi_{f_2}\,, \xi_{f_2})$ is the GNS triple associated to (A, f_2), respectively, $(H_{g_2}\,, \pi_{g_2}\,, \xi_{g_2})$ is the GNS triple associated to (B, g_2). Hence, there exists $V_2 \in \mathcal{B}(H_{f_2}\,, H_{g_2})$ such that $\|V_2\| \leq \|u\|_{\mathrm{ER}}$, satisfying

$$u_2(a,b) = \langle V_2\pi_{f_2}(a)\xi_{f_2}\,, \pi_{g_2}(b^*)\xi_{g_2}\rangle\,, \quad a \in A\,, b \in B\,.$$

Therefore, for any $a \in A$ and $b \in B$,

$$\left|\langle\sum_{i=1}^{m}\sum_{j=1}^{n} u_2(a_i, b_j)\pi(c_i)\rho(d_j)\xi, \eta\rangle\right| = \left|\langle\sum_{i=1}^{m}\sum_{j=1}^{n}\langle V_2\pi_{f_2}(a)\xi_{f_2}\,, \pi_{g_2}(b^*)\xi_{g_2}\rangle\pi(c_i)\rho(d_j)\xi\,, \eta\rangle\right| \tag{3.13}$$

$$= \left|\langle(V_2 \otimes 1_H)(\pi_{f_2} \otimes \pi)(\sum_{i=1}^{m} a_i \otimes c_i)(\xi_{f_2} \otimes \xi)\,, (\pi_{g_2} \otimes \rho)(\sum_{j=1}^{n} b_j^* \otimes d_j^*)(\xi_{g_2} \otimes \eta)\rangle\right|$$

$$\leq \|u\|_{\mathrm{ER}}\left\|\sum_{i=1}^{m} a_i \otimes c_i\right\|_{A\otimes_{\min}C}\left\|\sum_{j=1}^{n} b_j \otimes d_j\right\|_{B\otimes_{\min}D}.$$

The inequality (3.10) follows now by (3.11), (3.12) and (3.13), since $\|u\|_{\mathrm{ER}} \leq \|u\|_{\mathrm{jcb}}$. The proof is complete. □

Our next proposition gives a complete characterization of those bilinear forms $u : E\times F \to \mathbb{C}$ on operator spaces $E \subseteq A$ and $F \subseteq B$ sitting in C*-algebras A and B, for which $\|u\|_{\mathrm{ER}} < \infty$.

Proposition 3.12. *Let $E \subseteq A$ and $F \subseteq B$ be operator spaces sitting in C*-algebras A and B, and let $u : E \times F \to \mathbb{C}$ be a bounded bilinear map. The following two conditions are equivalent:*

(*i*) $\|u\|_{\mathrm{ER}} < \infty$.

(*ii*) *There exists a constant $\kappa_4 \geq 0$ such that for all C^*-algebras C and D, all $m, n \in \mathbb{N}$ and all $a_1, \ldots, a_m \in E$, $b_1, \ldots, b_n \in F$, $c_1, \ldots, c_m \in C$, $d_1, \ldots, d_n \in D$, we have*

$$\left\| \sum_{i=1}^{m} \sum_{j=1}^{n} u(a_i, b_j) c_i \otimes d_j \right\|_{C \otimes_{\max} D} \leq \kappa_4 \left\| \sum_{i=1}^{m} a_i \otimes c_i \right\|_{E \otimes_{\min} C} \left\| \sum_{j=1}^{n} b_j \otimes d_j \right\|_{F \otimes_{\min} D} . \tag{3.14}$$

Moreover, if $\kappa_4(u)$ denotes the best constant in (ii), then

$$\frac{1}{2} \|u\|_{\mathrm{ER}} \leq \kappa_4(u) \leq 2 \|u\|_{\mathrm{ER}} .$$

Proof. The implication $(i) \Rightarrow (ii)$ can be obtained from the proof of Proposition 3.11 with minor modifications. In the case when $E = A$ and $F = B$ we have by (3.12) and (3.13) that

$$\left\| \sum_{i=1}^{m} \sum_{j=1}^{n} u(a_i, b_j) c_i \otimes d_j \right\|_{C \otimes_{\max} D} \leq 2 \|u\|_{\mathrm{ER}} \left\| \sum_{i=1}^{m} a_i \otimes c_i \right\|_{A \otimes_{\min} C} \left\| \sum_{j=1}^{n} b_j \otimes d_j \right\|_{B \otimes_{\min} D} . \tag{3.15}$$

To extend the proof of (3.15) to the general case of operator spaces $E \subseteq A$ and $F \subseteq B$, the operators $V_1 \in \mathcal{B}(H_{g_1}, H_{f_1})$ and $V_2 \in \mathcal{B}(H_{f_2}, H_{g_2})$ will instead be operators in $\mathcal{B}(H^0_{g_1}, H^0_{f_1})$ and $\mathcal{B}(H^0_{f_2}, H^0_{g_2})$, respectively, where

$$H^0_{f_1} := \overline{\pi_{f_1}(E)^* \xi_{f_1}} , \quad H^0_{g_1} := \overline{\pi_{g_1}(F) \xi_{g_1}} , \quad H^0_{f_2} := \overline{\pi_{f_2}(E) \xi_{f_2}} , \quad H^0_{g_2} := \overline{\pi_{g_2}(F)^* \xi_{g_2}} .$$

The rest of the proof of the implication $(i) \Rightarrow (ii)$ can then be completed as in the Proof of Proposition 3.11. It also follows that $\kappa_4(u) \leq 2 \|u\|_{\mathrm{ER}}$.

The converse implication $(ii) \Rightarrow (i)$ can be obtained from the proof of Theorem 0.3 in [16]. For convenience of the reader, and in order to obtain a better constant, we include below a slightly modified argument.

Let $u : E \times F \to \mathbb{C}$ be a bounded bilinear form satisfying (3.14). We will show that $\|u\|_{\mathrm{ER}} \leq 2\kappa_4$. By Lemma 2.4 in [16], given a positive integer n and $\lambda_1, \ldots, \lambda_n > 0$, we can find two sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of operators on a Hilbert space H with a unit vector Ω such that the following properties hold:

(*a*) For all $a_1, \ldots, a_n \in E$ and all $b_1, \ldots, b_n \in B$,

$$\begin{aligned} \left\| \sum_{i=1}^{n} a_i \otimes x_i \right\| &\leq \left\| \sum_{i=1}^{n} \lambda_i a_i a_i^* \right\|^{1/2} + \left\| \sum_{i=1}^{n} \lambda_i^{-1} a_i^* a_i \right\|^{1/2} \\ \left\| \sum_{i=1}^{n} b_i \otimes y_i \right\| &\leq \left\| \sum_{i=1}^{n} \lambda_i b_i b_i^* \right\|^{1/2} + \left\| \sum_{i=1}^{n} \lambda_i^{-1} b_i^* b_i \right\|^{1/2} \end{aligned}$$

(*b*) The von Neumann algebra $W^*(x_1, \ldots, x_n)$ generated by $x_1, \ldots, x_n$ commutes with the von Neumann algebra $W^*(y_1, \ldots, y_n)$ generated by $y_1, \ldots, y_n$.

(*c*) $\langle x_i y_j \Omega, \Omega \rangle_H = \delta_{ij}$, for all $1 \leq i, j \leq n$.

Let now $n \in \mathbb{N}$ and let $\lambda_1, \dots, \lambda_n > 0$ (arbitrarily chosen), be fixed. Then by (3.14) we have for all $a_1, \dots, a_n \in E$ and $b_1, \dots, b_n \in F$, that

$$\begin{aligned}
\left|\sum_{i=1}^{n} u(a_i, b_i)\right| &= \left|\sum_{i,j=1}^{n} u(a_i, b_j)\langle x_i y_j \Omega, \Omega\rangle_H\right| \\
&\leq \left\|\sum_{i,j=1}^{n} u(a_i, b_j) x_i y_j\right\|_{\mathcal{B}(H)} \\
&\leq \left\|\sum_{i,j=1}^{n} u(a_i, b_j) x_i \otimes y_j\right\|_{W^*(x_1, \dots, x_n)\otimes_{\max} W^*(y_1, \dots, y_n)} \\
&\leq \kappa_4 \left\|\sum_{i=1}^{n} a_i \otimes x_i\right\|_{E\otimes_{\min} W^*(x_1, \dots, x_n)} \left\|\sum_{i=1}^{n} b_i \otimes y_i\right\|_{F\otimes_{\min} W^*(y_1, \dots, y_n)} \\
&\leq \kappa_4 \left(\left\|\sum_{i=1}^{n} \lambda_i a_i a_i^*\right\|^{\frac{1}{2}} + \left\|\sum_{i=1}^{n} \lambda_i^{-1} a_i^* a_i\right\|^{\frac{1}{2}}\right)\left(\left\|\sum_{i=1}^{n} \lambda_i b_i b_i^*\right\|^{\frac{1}{2}} + \left\|\sum_{i=1}^{n} \lambda_i^{-1} b_i^* b_i\right\|^{\frac{1}{2}}\right) \\
&\leq 2\kappa_4 \left(\left\|\sum_{i=1}^{n} \lambda_i a_i a_i^*\right\| + \left\|\sum_{i=1}^{n} \lambda_i^{-1} a_i^* a_i\right\|\right)^{\frac{1}{2}}\left(\left\|\sum_{i=1}^{n} \lambda_i b_i b_i^*\right\| + \left\|\sum_{i=1}^{n} \lambda_i^{-1} b_i^* b_i\right\|\right)^{\frac{1}{2}},
\end{aligned}$$

where we have used the well-known inequality

$$\sqrt{\alpha} + \sqrt{\beta} \leq \sqrt{2}\sqrt{\alpha+\beta}, \quad \alpha, \beta \geq 0.$$

Since $2\sqrt{\alpha\beta} \leq \alpha + \beta$ for all $\alpha, \beta \geq 0$, it follows that

$$\left|\sum_{i=1}^{n} u(a_i, b_i)\right| \leq \kappa_4 \left(\left\|\sum_{i=1}^{n} \lambda_i a_i^* a_i\right\| + \left\|\sum_{i=1}^{n} \lambda_i^{-1} a_i a_i^*\right\| + \left\|\sum_{i=1}^{n} \lambda_i b_i b_i^*\right\| + \left\|\sum_{i=1}^{n} \lambda_i^{-1} b_i^* b_i\right\|\right). \tag{3.16}$$

Using a Pietsch separation argument similar to the one given in the proof of Lemma 3.4 in [9], we infer the existence of states f_1, f_2 on A and g_1, g_2 on B such that for all $a \in E$, $b \in F$ and $\lambda > 0$,

$$|u(a,b)| \leq \kappa_4 \left(\lambda f_1(aa^*) + \lambda^{-1} f_2(a^*a) + \lambda g_2(bb^*) + \lambda^{-1} g_1(b^*b)\right).$$

Replacing now a by $t^{1/2}a$ and b by $t^{-1/2}b$, where $t > 0$, it follows that for all $a \in E$, $b \in F$, $t > 0$ and $\lambda > 0$,

$$|u(a,b)| \leq \kappa_4 \left(t\lambda f_1(aa^*) + t\lambda^{-1} f_2(a^*a) + \frac{1}{t}\lambda g_2(bb^*) + \frac{1}{t}\lambda^{-1} g_1(b^*b)\right).$$

By taking the infimum over all $t > 0$, we deduce by (2.24) that for all $a \in E$, $b \in F$ and all $\lambda > 0$,

$$|u(a,b)| \leq 2\kappa_4(\lambda f_1(aa^*) + \lambda^{-1} f_2(a^*a))^{1/2}(\lambda g_2(bb^*) + \lambda^{-1} g_1(b^*b))^{1/2}.$$

Therefore, for all $a \in E$, $b \in F$ and $\lambda > 0$,

$$|u(a,b)|^2 \leq (2\kappa_4)^2(f_1(aa^*)g_1(b^*b) + f_2(a^*a)g_2(bb^*) + \lambda^2 f_1(aa^*)g_2(bb^*) + \lambda^{-2} f_2(a^*a)g_1(b^*b)).$$

By taking infimum over $\lambda > 0$, a further application of (2.24) shows that for all $a \in E$ and $b \in F$,

$$\begin{aligned}
|u(a,b)|^2 &\leq (2\kappa_4)^2 \left(f_1(aa^*)g_1(b^*b) + f_2(a^*a)g_2(bb^*) + 2f_1(aa^*)^{1/2}g_1(b^*b)^{1/2}f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}\right) \\
&= (2\kappa_4)^2 \left(f_1(aa^*)^{1/2}g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}\right)^2.
\end{aligned}$$

This implies that $\|u\|_{\mathrm{ER}} \leq 2\kappa_4$, which completes the proof of the implication $(ii) \Rightarrow (i)$ and it also proves the inequality $\kappa_4 \geq \frac{1}{2}\|u\|_{\mathrm{ER}}$. $\square$

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