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CLASSIFICATION OF HYPERFINITE FACTORS UP TO COMPLETELY BOUNDED ISOMORPHISMS OF THEIR PREDUALS

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Abstract. In this paper we consider the following problem: When are the preduals of two hyperfinite (=injective) factors $\mathcal{M}$ and $\mathcal{N}$ (on separable Hilbert spaces) cb-isomorphic (i.e., isomorphic as operator spaces)? We show that if $\mathcal{M}$ is semifinite and $\mathcal{N}$ is type III, then their preduals are not cb-isomorphic. Moreover, we construct a one-parameter family of hyperfinite type III_0-factors with mutually non cb-isomorphic preduals, and we give a characterization of those hyperfinite factors $\mathcal{M}$ whose preduals are cb-isomorphic to the predual of the unique hyperfinite type III_1-factor. In contrast, Christensen and Sinclair proved in 1989 that all infinite dimensional hyperfinite factors with separable preduals are cb-isomorphic. More recently Rosenthal, Sukochev and the first-named author proved that all hyperfinite type III_λ-factors, where $0 < \lambda \leq 1$, have cb-isomorphic preduals.

1. Introduction and formulation of the main results

In the paper [1], Christensen and Sinclair proved that if $\mathcal{M}$ and $\mathcal{N}$ are infinite dimensional factors with separable preduals, then $\mathcal{M}$ and $\mathcal{N}$ are cb-isomorphic ($\mathcal{M} \overset{\mathrm{cb}}{\simeq} \mathcal{N}$), i.e., there exists a linear bijection ϕ of $\mathcal{M}$ onto $\mathcal{N}$ such that both ϕ and ϕ^{-1} are completely bounded. In 1993 Kirchberg (cf. [23]) proved a similar result for C^*-algebras, namely if A and B are simple, separable, nuclear, non-type I C^*-algebras, then $A \overset{\mathrm{cb}}{\simeq} B$.

However, if one turns to preduals of von Neumann algebras (on separable Hilbert spaces), the situation is very different. Rosenthal, Sukochev and the first-named author proved in [16] that if $\mathcal{M}$ is a II_1-factor and $\mathcal{N}$ is a properly infinite von Neumann algebra, then their preduals $\mathcal{M}_*$ and $\mathcal{N}_*$ are not isomorphic as Banach spaces, so in particular they are not cb-isomorphic. Moreover, the Banach space isomorphism classes (respectively, cb-isomorphism classes) of separable preduals of hyperfinite and semifinite von Neumann algebras are completely determined by [16], Theorem 5.1.

By a combination of two recent results of Pisier and Junge, the predual $\mathcal{M}_*$ of a semifinite factor $\mathcal{M}$ cannot be cb-isomorphic to the predual of R_∞, the unique hyperfinite factor of type III_1, because Pisier's operator Hilbert space OH does not cb-embed in $\mathcal{M}_*$ by [34], while OH admits a cb-embedding into $(R_\infty)_*$, as proved in [20] (see also [15]).

A von Neumann algebra $\mathcal{M}$ with separable predual is hyperfinite (or approximative finite dimensional) if $\mathcal{M}$ is the strong closure of an increasing union $\cup_{n=1}^\infty \mathcal{M}_n$ of finite dimensional $*$-subalgebras $\mathcal{M}_n$. By Connes' celebrated work [4], a factor $\mathcal{M}$ (with separable predual) is hyperfinite if and only if it is injective. It is well-known that the same holds for non-factors (see Section 6 of [12] and the references given therein), so in the following we do not need to distinguish between "hyperfiniteness" and "injectivity" for von Neumann algebras with separable preduals.

The main results of this paper are the following:

Theorem 1.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be hyperfinite von Neumann algebras with separable preduals $\mathcal{M}_*$ and $\mathcal{N}_*$, respectively. If $\mathcal{M}$ is type III and $\mathcal{N}$ is semifinite, then the preduals $\mathcal{M}_*$ and $\mathcal{N}_*$ are not cb-isomorphic. More generally, $\mathcal{M}_*$ is not cb-isomorphic to a cb-complemented subspace of $\mathcal{N}_*$.*

Theorem 1.2. *The predual of a hyperfinite type III-factor $\mathcal{M}$ (on a separable Hilbert space) is cb-isomorphic to the predual of the hyperfinite type III_1-factor R_∞ if and only if there exists a normal invariant state on the flow of weights $(Z(\mathcal{N}), (\theta_s)_{s\in\mathbb{R}})$ for $\mathcal{M}$.*

Theorem 1.3. *There exist uncountably many cb-isomorphism classes of preduals of hyperfinite type III_0-factors (on separable Hilbert spaces).*

Both Theorems 1.2 and 1.3 rely on Connes' classification of type III-factors (cf. [3]), and the Connes-Takesaki flow of weights for type III-factors (cf. [10]) and on Connes' classification of injective factors (see [4]), which we will outline below.

In [3] Connes introduced the subclassification of type III- factors into type III_λ-factors, where $0 \leq \lambda \leq 1$. Later, in [10] Connes and Takesaki introduced the "smooth flow of weights" (now called "the flow of weights") of a type III-factor. Following Takesaki's exposition in [40], Vol. II, pp. 364-368, the flow of weights can be constructed as follows. Let $\mathcal{M}$ be a type III-factor, and let ϕ be a normal, faithful state on $\mathcal{M}$. Consider the crossed product $\mathcal{N} := \mathcal{M} \rtimes_{\sigma^\phi} \mathbb{R}$, where $(\sigma_t^\phi)_{t\in\mathbb{R}}$ is the modular automorphism group associated with ϕ. Then $\mathcal{N}$ is generated by an embedding $\pi(\mathcal{M})$ of $\mathcal{M}$ into $\mathcal{N}$ and by a one-parameter group $(\lambda(t))_{t\in\mathbb{R}}$ of unitaries in $\mathcal{N}$. Moreover, there is a s.o.t-continuous dual action $(\widetilde{\theta_s})_{s\in\mathbb{R}}$ of $\mathbb{R}$ on $\mathcal{N}$, characterized by the relations

$$\begin{aligned} \widetilde{\theta}_s(\pi(x)) &= \pi(x)\,, \quad x \in \mathcal{M} \\ \widetilde{\theta}_s(\lambda(t)) &= e^{ist}\lambda(t)\,, \quad t \in \mathbb{R}\,, \end{aligned}$$

for all $s \in \mathbb{R}$. Let θ_s be the restriction of $\widetilde{\theta}_s$ to the center $Z(\mathcal{N})$ of $\mathcal{N}$. Then $(Z(\mathcal{N}), (\theta_s)_{s\in\mathbb{R}})$ is called *the flow of weights for* $\mathcal{M}$. It is independent (up to isomorphism) of the choice of the state ϕ on $\mathcal{M}$. Since $\mathcal{M}$ is a factor, $\theta = (\theta_s)_{s\in\mathbb{R}}$ acts ergodically on $Z(\mathcal{N})$, i.e., the fixed point algebra $Z(\mathcal{N})^\theta$ for the action θ is equal to $\mathbb{C}1$. Since $Z(\mathcal{N}) \simeq L^\infty(\Omega, \mu)$ for some standard Borel measure space (Ω, μ) , the flow $\theta = (\theta_s)_{s\in\mathbb{R}}$ can be realized as the flow associated to a one-parameter family $(\sigma_s)_{s\in\mathbb{R}}$ on non-singular Borel transformations of (Ω, μ) , that is, for all $s \in \mathbb{R}$,

$$(\theta_s f)(x) = f(\sigma_s^{-1}(x))\,, \quad f \in L^\infty(\Omega, \mu)\,, x \in \Omega\,.$$

The connection between Connes' type III_λ-factors and the flow of weights is given by **1.4.**, **1.5.** and **1.6.** below (cf. [10] , [39] and [40], Vol. II, Chapter XII).

Let $\mathcal{M}$ be a type III-factor (with separable predual). Then

1.4. $\mathcal{M}$ is of type III_0 if and only if the flow $(\theta_s)_{s\in\mathbb{R}}$ is non-periodic, i.e., $\theta_s \neq \mathrm{Id}_{Z(\mathcal{N})}$, for all $s \in \mathbb{R} \setminus \{0\}$. In this case the flow is non-transitive (=properly ergodic), which means that the measure μ described above is not concentrated on a single σ-orbit in Ω .

1.5. For $0 < \lambda < 1$, $\mathcal{M}$ is of type III_λ if and only if the flow $(\theta_s)_{s\in\mathbb{R}}$ is periodic with minimal period equal to $-\log\lambda$. In this case,

$$Z(\mathcal{N}) \simeq L^\infty(\mathbb{R}/((-\log\lambda)\mathbb{Z}))\,.$$

and for all $s \in \mathbb{R}$, θ_s is induced by the translation $\sigma_s : x \mapsto x + s$ on $\Omega = \mathbb{R}/((-\log\lambda)\mathbb{Z})$.

1.6. $\mathcal{M}$ is of type III_1 if and only if $\theta_s = \mathrm{Id}_{Z(\mathcal{N})}$, for all $s \in \mathbb{R}$. In this case $Z(\mathcal{N}) = \mathbb{C}1$.

For hyperfinite type III-factors with separable predual much more is known, owing to Connes' classification of injective factors [4], and related work by Krieger [25], Connes [5] and by the first-named author [14], namely:

1.7. The map $\mathcal{M} \mapsto ((Z(\mathcal{N}), (\theta_s)_{s\in\mathbb{R}})$ gives a one-to one correspondence between the set of (isomorphism classes of) hyperfinite type III_0-factors onto the set of (isomorphism classes of) non-transitive ergodic flows $(A, (\theta_s)_{s\in\mathbb{R}})$ on abelian von Neumann algebras A with separable predual. In particular, there are uncountably many isomorphism classes of hyperfinite type III_0-factors (cf. [4], [25]).

1.8. For each $0 < \lambda < 1$, there is exactly one (up to isomorphism) hyperfinite factor of type III_λ , namely the Powers factor

$$R_\lambda := \otimes_{n=1}^{\infty} (M_2(\mathbb{C}), \phi_\lambda) ,$$

where $\phi_\lambda = \mathrm{Tr}(h_\lambda \,\cdot\,)$, Tr being the non-normalized trace on $M_2(\mathbb{C})$ and $h_\lambda = \frac{1}{1+\lambda}\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. (See [4].)

1.9. There is only one (up to isomorphism) hyperfinite type III_1-factor, namely the Araki-Woods factor R_∞ , which can be expressed as the (von Neumann algebra) tensor product $R_\infty = R_{\lambda_1} \bar{\otimes} R_{\lambda_2}$ of two Powers factors with $\frac{\log\lambda_1}{\log\lambda_2} \notin \mathbb{Q}$ (cf. [5] and [14]).

Proofs of **1.7.** , **1.8.** and **1.9.** can also be found in [40], Vol. III, Chap. XVIII.

The rest of the paper is organized in the following way. In Section 2, we obtain some Stinespring-type results for completely positive and completely bounded maps, which will allow us to show that two properly infinite hyperfinite von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ with separable preduals have cb-isomorphic preduals if and only if there exist von Neumann algebras embeddings $i : \mathcal{M} \to \mathcal{N}$, $j : \mathcal{N} \to \mathcal{M}$ and normal conditional expectations $E : \mathcal{N} \to i(\mathcal{M})$, $F : \mathcal{M} \to j(\mathcal{N})$. From this result, Theorem 1.1 follows easily by results of Sakai [35] and Tomiyama [42] on normal conditional expectations.

In Section 3 we prove Theorem 1.2. The most difficult part is to show that R_∞ embeds into $\mathcal{M}$ as the range of a normal conditional expectation, provided that there exists a normal, invariant state on the flow of weights for $\mathcal{M}$. This part of the proof relies heavily on the main result from [17] on the classification of normal states on a von Neumann algebra up to approximative unitary equivalence.

Finally, in Section 4 we prove Theorem 1.3 by giving an explicit construction of a one-parameter family $(A^{(t)}, \theta^{(t)})_{0\le t<2}$ of non-transitive, ergodic flows $\theta^{(t)} = (\theta_s^{(t)})_{s\in\mathbb{R}}$ on abelian von Neumann algebras $A^{(t)}$ with separable preduals $A_*^{(t)}$, satisfying the following property:

$$\lim_{n\to\infty} \|\omega \circ \theta_{2^n}^{(t)} - \omega\| = t , \quad \omega \in A_*^{(t)} . \tag{1.1}$$

Then by **1.7.** above, $(A^{(t)}, \theta^{(t)})_{0\le t<2}$ are the flow of weights associated with hyperfinite type III_0 factors $(\mathcal{M}^{(t)})_{0\le t<2}$, and by (1.1) combined with the results of Section 2, we obtain that $\mathcal{M}_*^{(t_1)}$ and $\mathcal{M}_*^{(t_2)}$ are not cb-isomorphic when $t_1 \ne t_2$. It is interesting to note that the factors $(\mathcal{M}^{(t)})_{0\le t<2}$ cannot be separated by Connes' S- and T-invariants. Being type III_0-factors, $S(\mathcal{M}^{(t)}) = \{0, 1\}$, for all $t \in [0, 2)$, and in Theorem 4.5 we prove that $T(\mathcal{M}^{(t)}) = \left\{\frac{2\pi k}{2^n}; k \in \mathbb{Z}, n \in \mathbb{N}\right\}$, for all $t \in [0, 2)$.

For details on operator spaces and completely bounded maps we refer to the monographs [11, 32]. We shall briefly recall some definitions that are relevant for our paper. An operator space X is a Banach space given together with an isometric embedding $X \subseteq \mathcal{B}(H)$, the algebra of bounded linear operators on a Hilbert space H. For all $n \geq 1$, this embedding determines a norm on $M_n(X)$ (the $n \times n$ matrices over X), induced by the space $M_n(\mathcal{B}(H)) \cong \mathcal{B}(H^n)$. The morphisms in the category of operator spaces are *completely bounded maps.* Given a linear map $\phi : X \to Y$ between two operator spaces X and Y and $n \geq 1$, define $\phi_n : M_n(X) \to M_n(Y)$ by $\phi_n([x_{ij}]) = [\phi(x_{ij})]$, for all $[x_{ij}]_{i,j=1}^n \in M_n(X)$. Let $\|\phi\|_{cb} := \sup\{\|\phi_n\| ; n \in \mathbb{N}\}$. The map ϕ is called *completely bounded* (for short, *cb*) if $\|\phi\|_{cb} < \infty$, and ϕ is called *completely isometric* if all ϕ_n are isometries. The space of all cb maps from X to Y, denoted by $\mathcal{CB}(X,Y)$, is an operator space with matrix norms defined by $M_n(\mathcal{CB}(X,Y)) = \mathcal{CB}(X, M_n(Y))$, $n \geq 1$. The dual of an operator space X is, again, an operator space $X^* = \mathcal{CB}(X, \mathbb{C})$. A von Neumann algebra $\mathcal{M}$ carries a natural operator space structure, and its predual $\mathcal{M}_*$ carries the operator space structure induced by the completely isometric embedding into the dual $\mathcal{M}^*$ of $\mathcal{M}$.

2. Stinespring-type theorems and applications

Lemma 2.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras with separable preduals $\mathcal{M}_*$ and $\mathcal{N}_*$, respectively. Let $\alpha : \mathcal{M} \to \mathcal{N}$ be a completely positive map. Then there exists a completely positive unital map $\widetilde{\alpha} : \mathcal{M} \to \mathcal{N}$ such that*

$$\alpha(a) = \alpha(1)^{1/2}\widetilde{\alpha}(a)\alpha(1)^{1/2}, \quad a \in \mathcal{M},$$

where 1 denotes the unit of $\mathcal{M}$. Moreover, if α is normal, then $\widetilde{\alpha}$ can be chosen to be normal.

Proof. Assume first that $\mathrm{supp}(\alpha(1)) = 1_{\mathcal{N}}$, where $1_{\mathcal{N}}$ denotes the unit of $\mathcal{N}$. Let H be a separable Hilbert space with $\mathcal{N} \subseteq \mathcal{B}(H)$. The operator $\alpha(1)^{1/2}$ is one-to-one and has dense range, denoted by H_0. Hence we get a (possibly unbounded) map $\alpha(1)^{-1/2} : H_0 \to H$.

Now fix $a \in \mathcal{M}$, $a \geq 0$, and define a positive sesquilinear form on $H_0 \times H_0$ by

$$s(x,y) := \langle \alpha(a)\alpha(1)^{-1/2}x, \alpha(1)^{-1/2}y\rangle, \quad x, y \in H_0.$$

Note that s is positive, since α is so. We now show that s is a bounded sesquilinear form. For all $x \in H_0$,

$$s(x,x) \leq \|a\|\langle \alpha(1)\alpha(1)^{-1/2}x, \alpha(1)^{-1/2}x\rangle = \|a\|\|x\|^2.$$

By Schwarz's inequality,

$$|s(x,y)| \leq s(x,x)^{1/2}s(y,y)^{1/2} \leq \|a\|\|x\|\|y\|, \quad x, y \in H_0.$$

Hence there exists a unique operator $T \in \mathcal{B}(H)$ such that

$$\langle Tx, y\rangle = \langle \alpha(a)\alpha(1)^{-1/2}x\,, \alpha(1)^{-1/2}y\rangle, \quad x, y \in H_0.$$

Note first that $T \in \mathcal{N}$. This follows from the fact that for all $x, y \in H_0$ and all unitaries U in the commutant $\mathcal{N}'$ of $\mathcal{N}$,

$$\langle TUx\,, Uy\rangle = \langle Tx\,, y\rangle,$$

wherein we use the fact that $\alpha(a) \in \mathcal{N}$, and $\alpha(1)^{-1/2}$ is affiliated with $\mathcal{N}$. Clearly T is positive.

Since $\mathcal{M}$ is the span of its positive part $\mathcal{M}^+$, we infer that for all $a \in \mathcal{M}$, there exists a unique element $\widetilde{\alpha}(a) \in \mathcal{N}$ such that

$$\langle \widetilde{\alpha}(a)x, y\rangle = \langle \alpha(a)\alpha(1)^{-1/2}x\,, \alpha(1)^{-1/2}y\rangle, \quad x, y \in H_0.$$

By uniqueness, the map $\widetilde{\alpha}$ is linear. Also, clearly $\widetilde{\alpha}(a) \geq 0$, whenever $a \geq 0$ and $\widetilde{\alpha}(1) = 1_{\mathcal{N}}$. Looking at $n \times n$ matrices over $\mathcal{M}$ we infer that $\widetilde{\alpha}$ is completely positive. Moreover, for all $a \in \mathcal{M}$,

$$\alpha(a) = \alpha(1)^{1/2}\widetilde{\alpha}(a)\alpha(1)^{1/2},$$

since $\langle \alpha(a)x, y\rangle = \langle \widetilde{\alpha}(a)\alpha(1)^{1/2}x, \alpha(1)^{1/2}y\rangle$, for all $x, y \in H$. Note also that $\widetilde{\alpha}$ is normal if α is so.

It remains to consider the case when $p := \mathrm{supp}(\alpha(1)) \lneqq 1_{\mathcal{N}}$. Apply the previous argument to the mapping $\alpha : \mathcal{M} \to p\mathcal{N}p \subseteq \mathcal{N}$. We then obtain a completely positive map $\widetilde{\alpha} : \mathcal{M} \to p\mathcal{N}p$ such that $\widetilde{\alpha}(1) = p$ and $\alpha(a) = \alpha(1)^{1/2}\widetilde{\alpha}(a)\alpha(1)^{1/2}$, for all $a \in \mathcal{M}$. Choose a normal state ϕ on $\mathcal{M}$, and set

$$\widehat{\alpha}(a) := \widetilde{\alpha}(a) + \phi(a)(1_{\mathcal{N}} - p)\,, \quad a \in \mathcal{M}\,.$$

Then $\widehat{\alpha}$ is completely positive and $\widehat{\alpha}(1) = p + (1_{\mathcal{N}} - p) = 1_{\mathcal{N}}$. Furthermore, since $\mathrm{supp}(\alpha(1)^{1/2}) = p$,

$$\alpha(1)^{1/2}\widehat{\alpha}(a)\alpha(1)^{1/2} = \alpha(a) + \phi(a)\alpha(1)^{1/2}(1_{\mathcal{N}} - p)\alpha(1)^{1/2} = \alpha(a)\,.$$

Moreover, if α is normal, then both $\widetilde{\alpha}$ and $\widehat{\alpha}$ are normal. The proof is complete. □

Lemma 2.2. *Let $\mathcal{N}$ be a von Neumann algebra with separable predual. If $p \in \mathcal{N}$ is a properly infinite projection with central support equal to the identity 1 of $\mathcal{N}$, then $p \sim 1$.*

Proof. This result is well-known and it follows by standard comparison theory of projections. For convenience of the reader, we include a proof. Assume that $p \neq 1$. Choose a maximal family $(p_i)_{i\in I}$ of pairwise orthogonal non-zero projections such that $p_i \prec p$, for all $i \in I$. Note that I must be countable. We first show that

$$\sum_{i\in I} p_i = 1\,. \tag{2.1}$$

Suppose by contradiction that $\sum_{i\in I} p_i < 1$. Set $q := 1 - \sum_{i\in I} p_i$. If $c(p)$ and $c(q)$ denote the central support of p and q, respectively, then $c(q)c(p) = c(q) \neq 0$. This implies that there exist nonzero projections q_0 and p_0 such that $q_0 \leq q$, $p_0 \leq p$ and $q_0 \sim p_0$. Hence $q_0 \prec p$. Since $q_0 \leq q$, this contradicts the maximality assumption of the family $(p_i)_{i\in I}$, and (2.1) is proved. Since p is properly infinite, we can write

$$p = \sum_{i\in I} r_i\,, \tag{2.2}$$

where $(r_i)_{i\in I}$ are pairwise orthogonal projections so that $p \sim r_i$, for all $i \in I$. Hence $p_i \prec p \sim r_i$, for all $i \in I$. Together with (2.1) and (2.2), this implies that $1 \preceq p$. Clearly $p \preceq 1$, and therefore $p \sim 1$. □

We now prove the following Stinespring-Kasparov-type theorem (see [22], Theorem 3(1)):

Theorem 2.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras with separable preduals $\mathcal{M}_*$ and $\mathcal{N}_*$, respectively. Assume, moreover, that $\mathcal{N}$ is properly infinite. Let $\alpha : \mathcal{M} \to \mathcal{N}$ be a normal, completely positive map. Then there exists a normal $*$-representation $\pi : \mathcal{M} \to \mathcal{N}$ and an operator $V \in \mathcal{N}$ such that*

$$\alpha(a) = V^*\pi(a)V\,, \quad a \in \mathcal{M}\,.$$

Proof. By Lemma 2.1 we can assume without loss of generality that $\alpha(1) = 1_{\mathcal{N}}$, where 1 and $1_{\mathcal{N}}$ are the identities of $\mathcal{M}$ and $\mathcal{N}$, respectively. Following Stinespring's construction (see [29], Theorem 4.1), we define a positive sesquilinear form s on the algebraic tensor product $\mathcal{M} \odot H$, where H is a separable Hilbert space with $\mathcal{N} \subseteq \mathcal{B}(H)$, by

$$s(a \otimes x, b \otimes y) := \langle \alpha(b^*a)x, y\rangle\,, \quad a, b \in \mathcal{M}\,, x, y \in H\,.$$

Let $L := \{z \in \mathcal{M} \odot H; s(z,z) = 0\}$, and note that $(\mathcal{M} \odot H)/L$ is a prehilbert space whose completion we denote by K. For all $a \in \mathcal{M}$ and $x \in H$ let $[a \otimes x]$ denote the corresponding element in the quotient space $\mathcal{M} \odot H/L$. For all $a \in \mathcal{M}$ define $\pi_0(a)$ by

$$\pi_0(a)[b \otimes x] := [ab \otimes x]\,, \quad b \in \mathcal{M}\,, x \in H\,.$$

Then $\pi_0(a)$ is a densely-defined, bounded operator on the dense subspace $(\mathcal{M} \odot H)/L$ of K. Hence $\pi_0(a)$ extends to a bounded linear operator on the whole K, and this yields a map $\pi_0 : \mathcal{M} \to \mathcal{B}(K)$. It is easily checked that π_0 is a unital $*$-representation. Moreover, π_0 is normal. This follows immediately from the fact that α is normal and that

$$\langle \pi_0(a)[b \otimes x]\,, [c \otimes y] \rangle = \langle \alpha(c^*ab)x\,, y \rangle\,, \tag{2.3}$$

for all $a, b, c \in \mathcal{M}$, and all $x, y \in H$. Define now $W : H \to K$ by

$$Wx := [1 \otimes x]\,, \quad x \in H\,.$$

By (2.3), it follows that

$$\langle \pi_0(a)Wx, Wy \rangle = \langle \alpha(a)x, y \rangle\,, \quad a \in \mathcal{M}, x, y \in H\,.$$

Since $\alpha(1) = 1_{\mathcal{N}}$, W is an isometry of H into K, $W^*W = 1_{\mathcal{N}}$ and

$$\alpha(a) = W^*\pi_0(a)W\,, \quad a \in \mathcal{M}\,. \tag{2.4}$$

Let $\mathcal{N}'$ denote the commutant of $\mathcal{N}$ in $\mathcal{B}(H)$. Define a normal $*$-representation of $\mathcal{N}'$ on $(\mathcal{M} \odot H)/L$ by

$$\sigma_0(n')[a \otimes x] := [a \otimes n'x]\,, \quad n' \in \mathcal{N}'\,.$$

It is easily checked that σ_0 is a well-defined linear map on $(\mathcal{M} \odot H)/L$. Now let $n' \in \mathcal{N}'$. Then

$$\begin{aligned} \|[a \otimes n'x]\|^2 = \langle \alpha(a^*a)n'x, n'x \rangle = \langle (n')^*\alpha(a^*a)n'x, x \rangle &= \langle (n')^*n'\alpha(a^*a)x, x \rangle \\ &\leq \|n'\|^2 \|[a \otimes x]\|^2\,, \end{aligned}$$

wherein we have used the fact that $\alpha(\mathcal{M}) \subseteq \mathcal{N}$. We deduce that $\sigma_0(n')$ has a unique extension to an operator $\sigma(n') \in \mathcal{B}(K)$ such that

$$\sigma(n')[a \otimes x] = [a \otimes n'x]\,, \quad a \in \mathcal{M}\,, x \in H\,. \tag{2.5}$$

Note also that $\sigma(1_{\mathcal{N}}) = 1_K$, and it is easily checked that the map $\sigma : \mathcal{N}' \to \mathcal{B}(K)$ thus defined is a normal $*$-representation. Set

$$\widetilde{\mathcal{N}} := (\sigma(\mathcal{N}'))' \subseteq \mathcal{B}(K)\,.$$

Next, we check that $WW^* \in \widetilde{\mathcal{N}}$. Given any $a \in \mathcal{M}$ and $x \in H$, we have by the definition of W that

$$\langle W^*[a \otimes x]\,, y \rangle = \langle [a \otimes x]\,, Wy \rangle = \langle [a \otimes x]\,, [1 \otimes y] \rangle = \langle \alpha(a)x\,, y \rangle\,, \quad y \in H\,.$$

Hence $W^*[a \otimes x] = \alpha(a)x$, so $WW^*[a \otimes x] = [1 \otimes \alpha(a)x]$, which implies that

$$\sigma(n')WW^*[a \otimes x] = [1 \otimes n'\alpha(a)x]\,.$$

Moreover, by (2.5), $WW^*\sigma(n')[a \otimes x] = [1 \otimes \alpha(a)n'x] = [1 \otimes n'\alpha(a)x]$. We conclude that $WW^* \in \widetilde{\mathcal{N}}$. Note that

$$Wn' = \sigma(n')W\,, \quad n' \in \mathcal{N}'\,, \tag{2.6}$$

since for all $x \in H$, $Wn'x = [1 \otimes n'x]$, while $\sigma(n')Wx = \sigma(n')[1 \otimes x] = [1 \otimes n'x]$. Taking adjoints we get

$$n'W^* = W^*\sigma(n')\,, \quad n' \in \mathcal{N}'\,. \tag{2.7}$$

We next prove that WW^* is a properly infinite projection in $\widetilde{\mathcal{N}}$. Define

$$\rho(x) = WxW^*\,, \quad x \in \mathcal{N}\,.$$

Since $W^*W = 1_{\mathcal{N}}$, ρ is a (non-unital) $*$-homomorphism of $\mathcal{N}$ into $\mathcal{B}(K)$. Moreover,

$$WxW^*\sigma(n') = Wxn' = Wn'xW^* = \sigma(n')WxW^*\,, \quad x \in \mathcal{N}\,, n' \in \mathcal{N}'\,.$$

Hence $\rho(x) \in (\sigma(\mathcal{N}'))' = \widetilde{N}$, for all $x \in \mathcal{N}$. Since $\rho(1_{\mathcal{N}}) = WW^*$, we can consider ρ as a unital $*$-homomorphism of $\mathcal{N}$ into the corner algebra $(WW^*)\widetilde{N}(WW^*)$, and since $\mathcal{N}$ is properly infinite, it follows that $\rho(\mathcal{N})$ and $(WW^*)\widetilde{N}(WW^*)$ are also properly infinite von Neumann algebras. Hence WW^* is a properly infinite projection in $\widetilde{\mathcal{N}}$. Now let $c(WW^*)$ be the central support of WW^* in $\widetilde{\mathcal{N}}$ and put $q := 1 - c(WW^*)$. Then $q \in Z(\widetilde{N}) = Z(\sigma(\mathcal{N}'))$ and therefore $q = \sigma(q_0)$, for a projection $q_0 \in Z(\mathcal{N}')$. Since $\sigma(q_0)$ and WW^* are orthogonal projections, we have for all $x \in H$ that

$$Wq_0x = WW^*[1 \otimes q_0x] = WW^*\sigma(q_0)[1 \otimes x] = 0\,.$$

Therefore $q_0 = 0$, which implies that $c(WW^*) = 1_K$. By Lemma 2.2 , it follows that $WW^* \sim 1_K$ (in $\widetilde{\mathcal{N}}$) . Choose now $U \in \widetilde{\mathcal{N}}$ such that $U^*U = WW^*$ and $UU^* = 1_K$. Then $UW \in \mathcal{B}(H, K)$ and

$$(UW)^*UW = 1\,, \quad UW(UW)^* = 1_K\,,$$

i.e., UW is a unitary operator from H to K . Therefore

$$\pi(a) := (UW)^*\pi_0(a)UW\,, \quad a \in \mathcal{M}$$

defines a normal unital $*$-homomorphism of $\mathcal{M}$ into $\mathcal{B}(H)$. For all $a \in \mathcal{M}$,

$$U^*\pi_0(a)U \in \widetilde{\mathcal{N}} = (\sigma(\mathcal{N}'))'\,.$$

Hence, by (2.6) and (2.7) ,

$$\pi(a) = W^*(U^*\pi_0(a)U)W \in (\mathcal{N}')' = \mathcal{N}\,.$$

Next, set $V := W^*U^*W \in \mathcal{B}(H)$. Since $U^* \in \widetilde{\mathcal{N}} = (\sigma(\mathcal{N}'))'$, using again (2.6) and (2.7) we deduce that $V \in (\mathcal{N}')' = \mathcal{N}$. Moreover, for all $a \in \mathcal{M}$,

$$\alpha(a) = W^*\pi_0(a)W = W^*UW\pi(a)(UW)^*W = V^*\pi(a)V\,.$$

This completes the proof. □

Next we prove the following Stinespring-Paulsen-type theorem (cf. [28], Theorem 2.7; see also [29], Theorem 7.4):

Theorem 2.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras with separable preduals $\mathcal{M}_*$,$\mathcal{N}_*$. Assume, moreover, that $\mathcal{N}$ is properly infinite and injective. Let $\alpha : \mathcal{M} \to \mathcal{N}$ be a normal cb-map. Then there exists a normal $*$-representation $\pi : \mathcal{M} \to \mathcal{N}$ and operators $R, S \in \mathcal{N}$ such that*

$$\alpha(a) = R\pi(a)S\,, \quad a \in \mathcal{M} \tag{2.8}$$

and $\|R\|\|S\| = \|\alpha\|_{cb}$.

For the proof we need a few preliminary considerations. Recall that (see, e.g., [40] Vol.I, Theorem 2.14) if $\mathcal{M}$ is a von Neumann algebra, then any functional $\phi \in \mathcal{M}^*$ has a unique decomposition into its normal and singular part

$$\phi = \phi_n + \phi_s \,. \tag{2.9}$$

Moreover, if $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras (not necessarily with separable preduals), and $T \in \mathcal{B}(\mathcal{M}, \mathcal{N})$, then there exists a unique decomposition

$$T = T_n + T_s \,, \tag{2.10}$$

where $T_n \,, T_s \in \mathcal{B}(\mathcal{M}, \mathcal{N})$, and such that for any $\phi \in \mathcal{N}_*$, we have

$$\phi \circ T_n = (\phi \circ T)_n \,, \quad \phi \circ T_s = (\phi \circ T)_s \,, \tag{2.11}$$

(cf. [42], Theorem 1). Further, the following assertions hold:

(a) If T is positive, then both T_n and T_s in the decomposition (2.10) are positive.

(b) If T is completely positive, then both T_n and T_s in the decomposition (2.10) are completely positive.

Statement (a) follows by uniqueness, since for any positive functional $\phi \in \mathcal{M}^*$, both maps ϕ_n and ϕ_s in the decomposition (2.9) are positive. To justify (b), let k be a positive integer, and note that for any $\phi \in (M_k(\mathcal{M}))^*$, we have $\phi = (\phi_{ij})_{i,j=1}^k$ with $\phi_{ij} \in \mathcal{M}^*$, since, algebraically, $(M_k(\mathcal{M}))^* = M_k(\mathcal{M}^*)$. By uniqueness we deduce that $\phi_n = ((\phi_{ij})_n)_{i,j=1}^k$ and, respectively, $\phi_s = ((\phi_{ij})_s)_{i,j=1}^k$. Thus ($b$) follows.

Proof of Theorem 2.4. By Theorem 1.6 in [13], there exist completely positive maps $\beta, \gamma : \mathcal{M} \to \mathcal{N}$ such that the mapping σ defined by

$$\sigma(a) := \begin{pmatrix} \beta(a) & \alpha^*(a) \\ \alpha(a) & \gamma(a) \end{pmatrix} , \quad a \in \mathcal{M} \tag{2.12}$$

is a completely positive map from $\mathcal{M}$ into $M_2(\mathcal{N})$, where

$$\alpha^*(a) := \alpha(a^*)^* \,, \quad a \in \mathcal{M} \,.$$

Next, note that the maps β and γ can be chosen to be normal. For this, exchange (possibly) β and γ above with their normal parts β_n and γ_n , respectively. Then one can check that the map σ defined by (2.12) is still completely positive, and, moreover, normal. By Theorem 2.3, there exists a normal $*$-representation $\pi' : \mathcal{M} \to M_2(\mathcal{N})$ and an operator $V \in M_2(\mathcal{N})$ such that

$$\begin{pmatrix} \beta(a) & \alpha^*(a) \\ \alpha(a) & \gamma(a) \end{pmatrix} = V^* \pi'(a) V \,, \quad a \in \mathcal{M} \,.$$

Write now $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$. It then follows that

$$\alpha(a) = (V_{12}^* \,, V_{22}^*)\, \pi'(a) \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix} , \quad a \in \mathcal{M} \,.$$

Since $\mathcal{N}$ is properly infinite, $\mathcal{N} \cong M_2(\mathcal{N})$. Denote by 1 the identity of $\mathcal{N}$ and choose isometries $u_1, u_2 \in \mathcal{N}$ so that $u_1 u_1^*$ and $u_2 u_2^*$ are orthogonal projections with $u_1 u_1^* + u_2 u_2^* = 1$. Define $\pi : \mathcal{M} \to \mathcal{N}$ by

$$\pi(a) := (u_1 , u_2)\, \pi'(a) \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} \in \mathcal{N}, \quad a \in \mathcal{M}.$$

Then π is a $*$-representation because (u_1 , u_2) is a unitary from H to $H \oplus H$, as verified by the following computations: $(u_1 , u_2) \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = u_1 u_1^* + u_2 u_2^* = 1$, respectively, $\begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} (u_1 , u_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence

$$\pi'(a) = \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} \pi(a)(u_1 , u_2), \quad a \in \mathcal{M}.$$

Moreover, it is clear that π is normal, and that

$$\alpha(a) = (V_{12}^* , V_{22}^*) \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} \pi'(a)(u_1 , u_2) \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix}, \quad a \in \mathcal{M}. \tag{2.13}$$

Denote $(V_{12}^* , V_{22}^*) \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}$ by R, respectively, $(u_1 , u_2) \begin{pmatrix} V_{11} \\ V_{21} \end{pmatrix}$ by S. Then (2.13) yields (2.8), and the proof of theorem 2.4 is now complete. □

The following result is known as Pelczynski's trick (cf. [30]; see also [26], page 54):

Lemma 2.5. *Let X and Y be Banach spaces. Suppose that there exist Banach spaces V and W such that*

1) $X \cong Y \oplus W$
2) $Y \cong X \oplus V$
3) $X \oplus X \cong X$
4) $Y \oplus Y \cong Y$

Then X is isomorphic to Y.

Proof. For completeness, we include the short proof of this result. We have

$$X \cong X \oplus X \cong Y \oplus Y \oplus W \oplus W \cong Y \oplus W \oplus W \cong X \oplus W,$$

and therefore $X \cong Y \oplus W \cong (X \oplus V) \oplus W \cong (X \oplus W) \oplus V \cong X \oplus V \cong Y$, as wanted. □

Remark 2.6. A similar proof with isomorphisms being replaced by complete isomorphisms shows that Pelczynski's trick holds, more generally, in the category of operator spaces. Note also that if X and Y are properly infinite von Neumann algebras, or preduals of properly infinite von Neumann algebras, then conditions 3) and 4) above are automatically satisfied, in the operator space category, as justified in the proof of Theorem 6.2 in [16].

Proposition 2.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. The following statements are equivalent:*

1) *There exists a cb-embedding $i : \mathcal{M}_* \hookrightarrow \mathcal{N}_*$ such that $i(\mathcal{M}_*)$ is cb-complemented in $\mathcal{N}_*$.*
2) *There exist cb maps $\phi : \mathcal{M}_* \to \mathcal{N}_*$ and $\psi : \mathcal{N}_* \to \mathcal{M}_*$ such that $\psi \circ \phi = Id_{\mathcal{M}_*}$.*
3) *There exist normal cb-maps $\alpha : \mathcal{M} \to \mathcal{N}$ and $\beta : \mathcal{N} \to \mathcal{M}$ such that $\beta \circ \alpha = Id_{\mathcal{M}}$.*

Proof. 1) $\Rightarrow$ 2). By hypothesis, there exists a cb-projection $\rho : \mathcal{N}_* \to i(\mathcal{M}_*) \subseteq \mathcal{N}_*$. Set $\phi := i$, $\psi := i^{-1} \circ \rho$. Then ϕ, ψ are cb-maps and $\psi \circ \phi = \mathrm{Id}_{\mathcal{M}_*}$.
2) $\Rightarrow$ 1). Set $i := \phi, \rho := \phi \circ \psi$. Note that $i = \phi$ is a cb-embedding, since $\phi : \mathcal{M}_* \to \phi(\mathcal{M}_*)$ is a bijection with inverse $\phi^{-1} = \psi|_{\phi(\mathcal{M}_*)}$. Also, $\rho^2 = \phi \circ (\psi \circ \phi) \circ \psi = \phi \circ \psi = \rho$, and moreover, $\rho(i(\mathcal{M}_*)) = \phi(\mathcal{M}_*)$. Hence ρ is a cb projection of $\mathcal{N}_*$ onto $i(\mathcal{M}_*) = \phi(\mathcal{M}_*)$.
2) $\Rightarrow$ 3). Set $\alpha := \psi^* : \mathcal{M} \to \mathcal{N}$ and $\beta := \phi^* : \mathcal{N} \to \mathcal{M}$. Then α, β are cb normal maps with $\beta \circ \alpha = \mathrm{Id}_{\mathcal{M}}$. To prove 3) $\Rightarrow$ 2), take $\phi := \beta_*, \psi := \alpha_*$. Then ϕ, ψ are well-defined cb-maps with $\psi \circ \phi = \mathrm{Id}_{\mathcal{M}_*}$. □

Lemma 2.8. *Let $\mathcal{M}$ be an injective von Neumann algebra with separable predual. Let $\pi : \mathcal{M} \to \mathcal{B}(H)$ and $\rho : \mathcal{M} \to \mathcal{B}(K)$ be two normal, unital $*$-representations of $\mathcal{M}$ on separable Hilbert spaces H and K, respectively. Then, for all $T \in \mathcal{B}(H, K)$, there exists $T_0 \in \overline{conv\{\rho(u)T\pi(u)^*; u \in \mathcal{U}(\mathcal{M})\}}^{w^*}$, where $\mathcal{U}(\mathcal{M})$ is the set of unitaries in $\mathcal{M}$, such that*

$$T_0 = \rho(u)T_0\pi(u)^*, \quad u \in \mathcal{U}(\mathcal{M}).$$

Proof. Let us first recall the following definition due to J. Schwartz (cf. [37]; see also [36]). A von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ has property P if $\overline{\mathrm{conv}\{uTu^*; u \in \mathcal{U}(\mathcal{M})\}}^{w^*} \cap \mathcal{M}' \neq \emptyset$, for all $T \in \mathcal{B}(H)$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. It was proved by Sakai that if $\mathcal{M}$ is hyperfinite, then $\mathcal{M}$ has property P (see [36], Corollary 4.4.19). Now, let $T \in \mathcal{B}(H, K)$. Then, for any $u \in \mathcal{U}(\mathcal{M})$,

$$\begin{pmatrix} \pi(u) & 0 \\ 0 & \rho(u) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \begin{pmatrix} \pi(u)^* & 0 \\ 0 & \rho(u)^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \rho(u)T\pi(u)^* & 0 \end{pmatrix}. \tag{2.14}$$

Since π and ρ are normal, it follows that the von Neumann algebra $(\pi \oplus \rho)(\mathcal{M})$ is injective, and therefore, by the above discussion, it has property P. It follows that there exists an operator

$$T_0 \in \overline{\mathrm{conv}\{\rho(u)T\pi(u)^*; u \in \mathcal{U}(\mathcal{M})\}}^{w^*}$$

such that $\begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \in ((\pi \oplus \rho)(\mathcal{M}))'$. By applying (2.14) to the operator $T_0 \in \mathcal{B}(H, K)$, we deduce that

$$T_0 = \rho(u)T_0\pi(u)^*, \quad u \in \mathcal{U}(\mathcal{M}),$$

and the proof is complete. □

Proposition 2.9. *Let $\mathcal{M}$ and $\mathcal{N}$ be properly infinite von Neumann algebras with separable preduals $\mathcal{M}_*, \mathcal{N}_*$. If $\mathcal{N}$ is injective (=hyperfinite), then the following statements are equivalent:*

- *i) There exists a cb-embedding $i : \mathcal{M}_* \hookrightarrow \mathcal{N}_*$ such that $i(\mathcal{M}_*)$ is cb-complemented in $\mathcal{N}_*$.*
- *ii) There exists a von Neumann algebra embedding $\alpha : \mathcal{M} \hookrightarrow \mathcal{N}$ and a normal conditional expectation $\beta : \mathcal{N} \to \mathcal{M}$ such that $\beta \circ \alpha = Id_{\mathcal{M}}$.*

Moreover, if i) holds, then $\mathcal{M}$ is injective, as well.

Proof. We have to prove that $i) \Rightarrow ii)$, since by Proposition 2.7 we know already that $ii) \Rightarrow i)$. Suppose that $i)$ holds, then by Proposition 2.7 there exist normal completely bounded maps $\alpha : \mathcal{M} \to \mathcal{N}$ and $\beta : \mathcal{N} \to \mathcal{M}$ such that $\beta \circ \alpha = \mathrm{Id}_{\mathcal{M}}$. The first goal is to make α into a $*$-representation. Indeed, by Theorem 2.4 there exists a normal $*$-representation $\widehat{\pi} : \mathcal{M} \to \mathcal{N}$ and operators $R, S \in \mathcal{N}$ such that

$$\alpha(a) = R\widehat{\pi}(a)S, \quad a \in \mathcal{M}. \tag{2.15}$$

Note that α is one-to-one (since $\beta \circ \alpha = \mathrm{Id}_{\mathcal{M}}$), and by (2.15) this implies that $\widehat{\pi}$ is one-to-one, too. Hence $\widehat{\pi}$ is a $*$-isomorphism of $\mathcal{M}$ onto its image $\widehat{\pi}(\mathcal{M})$, the latter being a von Neumann subalgebra of $\mathcal{N}$. Set

$$\rho(b) := (\widehat{\pi} \circ \beta)(RyS)\,, \quad y \in \mathcal{N}\,.$$

Then, for all $b = \widehat{\pi}(a)$, where $a \in \mathcal{M}$, $\rho(b) = (\widehat{\pi} \circ \beta)(R\widehat{\pi}(a)S) = \widehat{\pi} \circ \beta \circ \alpha(a) = \widehat{\pi}(a) = b$. Thus $\rho(\widehat{\pi}(\mathcal{M})) = \widehat{\pi}(\mathcal{M})$. Since $\rho(\mathcal{N}) \subseteq \widehat{\pi}(\mathcal{M})$, we infer that $\rho(\mathcal{N}) = \widehat{\pi}(\mathcal{M})$, i.e., ρ is a projection of $\mathcal{N}$ onto $\widehat{\pi}(\mathcal{M})$. Hence we have proved that there exist a normal one-to-one $*$-representation $\widehat{\pi} : \mathcal{M} \to \mathcal{N}$ and a normal cb-projection $\rho : \mathcal{N} \to \widehat{\pi}(\mathcal{M})$. Let H be a separable Hilbert space with $\mathcal{N} \subseteq \mathcal{B}(H)$. Since $\mathcal{N}$ is injective, there exists a conditional expectation $E : \mathcal{B}(H) \to \mathcal{N}$. The composition $\hat{\pi}^{-1}\rho E : \mathcal{B}(H) \to \mathcal{M}$ is a cb-projection. By a result of Pisier (cf. Theorem 2.9 in [31]) and Christensen-Sinclair [2], it follows that $\mathcal{M}$ is injective, as well.

So far we have reduced the general case to the case when $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{N}$, $\alpha : \mathcal{M} \to \mathcal{N}$ is the inclusion map and $\beta : \mathcal{N} \to \mathcal{M}$ is a normal cb-projection of $\mathcal{N}$ onto $\mathcal{M}$. The next step is to change β into a normal conditional expectation. For this, apply now Theorem 2.4 to the map $\beta : \mathcal{N} \to \mathcal{M}$, and infer the existence of a normal $*$-representation $\pi : \mathcal{N} \to \mathcal{M}$ and operators $R, S \in \mathcal{M}$ such that $\beta(a) = R\pi(a)S$, for all $a \in \mathcal{N}$. Since $\beta|_{\mathcal{M}} = \mathrm{Id}_{\mathcal{M}}$ and $\mathcal{M} \subseteq \mathcal{N}$, it follows that

$$a = R\pi(a)S\,, \quad a \in \mathcal{M}\,. \tag{2.16}$$

Since $\mathcal{M}$ is injective, we get from Lemma 2.8 that there exists an operator

$$S_0 \in \overline{\mathrm{conv}\{\pi(u)Su^*; u \in \mathcal{U}(\mathcal{M})\}}^{w^*} \subseteq \mathcal{M}$$

such that

$$\pi(u)S_0u^* = S_0\,, \quad u \in \mathcal{U}(\mathcal{M})\,. \tag{2.17}$$

Note that by (2.16), $R\pi(u)Su^* = uu^* = 1$, for all $u \in \mathcal{U}(\mathcal{M})$. Hence $RS_0 = 1$. Therefore S_0 is bounded away from 0, i.e., $S_0| := (S_0^*S_0)^{\frac{1}{2}}$ is invertible. Let

$$S_0 := U_0|S_0|$$

be the polar decomposition of S_0. Since $S_0 \in \mathcal{M}$, it follows that $U_0 \in \mathcal{M}$ and $|S_0| \in \mathcal{M}$, as well. Moreover, $U_0 = S_0|S_0|^{-1}$ and $U_0^*U_0 = 1$. By (2.17), $\pi(u)S_0 = S_0u$, for all $u \in \mathcal{U}(\mathcal{M})$. Hence $\pi(a)S_0 = S_0a$, for all $a \in \mathcal{M} = \mathrm{Span}(\mathcal{U}(\mathcal{M}))$. By taking adjoints, it follows that $S_0^*\pi(a) = aS_0^*$, for all $a \in \mathcal{M}$. Hence

$$S_0^*S_0a = S_0^*\pi(a)S_0 = aS_0^*S_0\,, \quad a \in \mathcal{M}\,.$$

Therefore $|S_0| = (S_0^*S_0)^{\frac{1}{2}} \in Z(\mathcal{M})$ (the center of $\mathcal{M}$), which implies that

$$\pi(a)U_0 = \pi(a)S_0|S_0|^{-1} = S_0a|S_0|^{-1} = S_0|S_0|^{-1}a = U_0a\,, \quad a \in \mathcal{M}\,.$$

Hence $a = U_0^*\pi(a)U_0$, for all $a \in \mathcal{M}$, and therefore $a = U_0^*\pi(a)U_0$, for all $a \in \mathcal{M}$. From this we infer that the map β_1 defined by

$$\beta_1(b) := U_0^*\pi(b)U_0\,, \quad b \in \mathcal{N}$$

is a normal, completely positive map satisfying $\beta_1(\mathcal{N}) \subseteq \mathcal{M}$ and $\beta_1|_{\mathcal{M}} = \mathrm{Id}_{\mathcal{M}}$. Hence β_1 is a normal conditional expectation of $\mathcal{N}$ onto $\mathcal{M}$. This completes the proof of Proposition 2.9. $\square$

By Proposition 2.7 and Proposition 2.9 we obtain immediately the following

Theorem 2.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be properly infinite von Neumann algebras with separable preduals $\mathcal{M}_*$, $\mathcal{N}_*$. If both $\mathcal{M}$ and $\mathcal{N}$ are injective, then the following statements are equivalent:*

1) $\mathcal{M}_* \overset{c.b.}{\cong} \mathcal{N}_*$
2) *$Id_{\mathcal{M}_*}$ admits a cb-factorization through $\mathcal{N}_*$, and $Id_{\mathcal{N}_*}$ admits a cb-factorization through $\mathcal{M}_*$.*
3) *$Id_{\mathcal{M}}$ admits a cb-factorization through $\mathcal{N}$ and $Id_{\mathcal{N}}$ admits a cb-factorization through $\mathcal{M}$, where all four cb-maps involved are normal.*
4) *There exist von Neumann algebras embeddings $i : \mathcal{M} \hookrightarrow \mathcal{N}$, $j : \mathcal{N} \hookrightarrow \mathcal{M}$ and normal conditional expectations $E : \mathcal{N} \to i(\mathcal{M})$, $F : \mathcal{M} \to j(\mathcal{N})$.*

The following result is due to Sakai and Tomiyama (cf. [35] and [42], Theorem 3). For convenience, we include a short proof based on [35].

Lemma 2.11. *([35], [42]) Let $\mathcal{N}$ be a semifinite von Neumann algebra and let $\mathcal{M} \subseteq \mathcal{N}$ be a von Neumann subalgebra of type III. Then there is no normal conditional expectation from $\mathcal{N}$ onto $\mathcal{M}$.*

Proof. Suppose by contradiction that there exists a normal conditional expectation $E : \mathcal{N} \to \mathcal{M}$. Since $\mathcal{N}$ is semifinite, there exists a net $(e_\lambda)_{\lambda\in\Lambda}$ of finite dimensional projections in $\mathcal{N}$ converging in strong operator (s.o.) topology to the identity $1_{\mathcal{N}}$ of $\mathcal{N}$. By normality of E , it follows that $E(e_\lambda) \overset{\text{s.o.}}{\longrightarrow} 1$, where 1 is the identity of $\mathcal{M}$. Hence, there exists a finite projection $e \in \mathcal{N}$ with $E(e) \neq 0$. Moreover, we can choose $\varepsilon > 0$ such that

$$p := 1_{[\varepsilon,\infty)}(E(e)) \neq 0 .$$

We show next that p is a finite projection in $\mathcal{M}$. Set $a := E(e)p + (1 - p)$. Then $a = a^*$ and $a \geq \varepsilon 1$. In particular, a is invertible. Let $(x_\alpha)_{\alpha\in A}$ be a bounded net in $p\mathcal{M}p$, which converges s.o. to 0 . Then, as shown by Sakai (see the proof of Theorem 2.5.6 (2) in [36]), the finiteness of e ensures that the net $(ex^*_\alpha)_{\alpha\in A}$ converges to 0 s.o. Since $x^*_\alpha \in p\mathcal{M}p$, for all $\alpha \in A$, it follows that $ax^*_\alpha = E(e)x^*_\alpha = E(ex^*_\alpha)$. Thus $x^*_\alpha = a^{-1}E(ex^*_\alpha) \overset{\text{s.o.}}{\longrightarrow} 0$. By Theorem 2.5.6 (1) in [36], it follows that $p\mathcal{M}p$ is finite. This implies that p is a finite projection in $\mathcal{M}$, which contradicts the fact that $\mathcal{M}$ is of type III. □

Proof of Theorem 1.1. Let $\mathcal{M}$ and $\mathcal{N}$ be hyperfinite von Neumann algebras with separable preduals, where $\mathcal{M}$ is of type III and $\mathcal{N}$ is semifinite.

Consider first the case when $\mathcal{N}$ is properly infinite, and assume by contradiction that $\mathcal{M}_*$ is cb-isomorphic to a cb-complemented subspace of $\mathcal{N}_*$. Then the statement *ii*) in Proposition 2.9 holds. By Lemma 2.11, this yields a contradiction, and therefore the theorem is proved in this case.

For the general case, note that if $\mathcal{M}_*$ is cb-complemented in $\mathcal{N}_*$, then also $(\mathcal{M} \otimes \mathcal{B}(H))_*$ is cb-complemented in $(\mathcal{N} \otimes \mathcal{B}(H))_*$, where $H = l^2(\mathbb{N})$. But the von Neumann algebras $\mathcal{M} \otimes \mathcal{B}(H)$ and $\mathcal{N} \otimes \mathcal{B}(H)$ are both properly infinite; moreover, the first is of type III, while the second is semifinite. So, by the first part of the proof we obtain a contradiction, and the proof of Theorem 1.1 is complete. □

Remark 2.12. In [27], Oikhberg, Rosenthal and Størmer studied *isometric* embeddings of a predual $\mathcal{M}_*$ into a predual $\mathcal{N}_*$, and show that no such embeddings exist if $\mathcal{M}$ is of type III and $\mathcal{N}$ is semifinite. In this case neither hyperfiniteness, nor separability assumptions are needed.

3. A characterization of type III-factors whose preduals are cb-isomorphic to the predual of the injective type III_1 factor

Our main result in this section is a characterization of type III- factors which admit an invariant normal state on their flow of weights. (See Theorem 3.5 below.) Let $\mathcal{M}$ be a type III-factor with separable predual and let ϕ_0 be a fixed normal faithful state on $\mathcal{M}$. Consider the crossed-product $\mathcal{N} := \mathcal{M} \rtimes_{\sigma^{\phi_0}} \mathbb{R}$, and let $(\widetilde{\theta}_s)_{s\in\mathbb{R}}$ of $\mathbb{R}$ be the dual action of σ^{ψ_0} on $\mathcal{N}$, i.e.,

$$\widetilde{\theta}_s(\pi(x)) = \pi(x), \quad x \in \mathcal{M} \tag{3.1}$$

$$\widetilde{\theta}_s(\lambda(t)) = e^{ist}\lambda(t), \quad t \in \mathbb{R}, \tag{3.2}$$

for all $s \in \mathbb{R}$, where $\pi(\mathcal{M})$ and $(\lambda(t))_{t\in\mathbb{R}}$ are the generators of the crossed product $\mathcal{N}$, as explained in the Introduction. Then the flow of weights of $\mathcal{M}$ is the pair $(Z(\mathcal{N}), (\theta_s)_{s\in\mathbb{R}})$, where $Z(\mathcal{N})$ is the center of $\mathcal{N}$ and θ_s is the restriction of $\widetilde{\theta}_s$ to the center $Z(\mathcal{N})$, for all $s \in \mathbb{R}$. Recall that a state ψ on $Z(\mathcal{N})$ is called *an invariant state on the flow of weights for* $\mathcal{M}$, if ψ is invariant under the dual action $(\theta_s)_{s\in\mathbb{R}}$, i.e.,

$$\psi \circ \theta_s = \psi, \quad s \in \mathbb{R}.$$

Next, let h be the unique positive self-adjoint operator affiliated with $\mathcal{N}$, for which $h^{it} = \lambda(t)$, for all $t \in \mathbb{R}$, and let $\widetilde{\phi}_0$ be the dual weight of ϕ_0 in the sense of Takesaki [39]. Then (cf. [39]), $\widetilde{\phi}_0 = \tau(h\,\cdot)$, for a unique normal, faithful, semifinite trace τ on $\mathcal{N}$, which satisfies

$$\tau \circ \widetilde{\theta}_s = e^{-s}\tau, \quad s \in \mathbb{R}.$$

By [17], we can define a map $\phi \mapsto \widehat{\phi}$ of $S_{\text{nor}}(\mathcal{M})$, the set of normal states on $\mathcal{M}$, into the set $S_{\text{nor}}(Z(\mathcal{N}))$ of normal states on $Z(\mathcal{N})$ in the following way: For $\phi \in S_{\text{nor}}(\mathcal{M})$, let h_ϕ denote the unique positive unbounded operator affiliated with $\mathcal{N}$, such that

$$\widetilde{\phi} = \tau(h_\phi\,\cdot), \tag{3.3}$$

where $\widetilde{\phi}$ is the dual weight of ϕ, and set

$$e_\phi := 1_{(1,\infty)}(h_\phi) \in \mathcal{N}. \tag{3.4}$$

Then $\tau(e_\phi) = 1$, and therefore

$$\widehat{\phi}(z) = \tau(e_\phi z), \quad z \in Z(\mathcal{N})$$

defines a normal state on $Z(\mathcal{N})$ (cf. [17], Def. 3.2). By the proof of Lemma 3.4 in [17],

$$\theta_s(e_\phi) = 1_{[e^s,\infty)}(h_\phi), \quad s \in \mathbb{R}. \tag{3.5}$$

The Main Theorem in [17] states that if $\mathcal{M}$ is a properly infinite von Neumann algebra, then $\phi \mapsto \widehat{\phi}$ maps $S_{\text{nor}}(\mathcal{M})$ onto the set $\{\omega \in S_{\text{nor}}(\mathcal{N}); \omega \circ \theta_s \geq e^{-s}\omega, \forall s \geq 0\}$. Moreover, if $\phi, \psi \in S_{\text{nor}}(\mathcal{M})$, then

$$\|\widehat{\phi} - \widehat{\psi}\| = \inf_{u\in\mathcal{U}(\mathcal{M})} \|u\phi u^* - \psi\|. \tag{3.6}$$

The right hand-side in (3.6) is by definition (cf. [9]) equal to the distance $d([\phi], [\psi])$, where $[\phi]$ denotes the norm-closure of the orbit of ϕ under the action of inner $*$-automorphisms, $\text{Int}(\mathcal{M})$, by $\phi \mapsto u\phi u^*$. The customary notation $u\phi u^*(x) = \phi(u^* x u)$, $\forall x \in \mathcal{M}$ is being used. We write $\phi \sim \psi$ if $d([\phi], [\psi]) = 0$. We now consider the following more general equivalence relation:

Definition 3.1. *Let $m\,,n$ be positive integers. A $*$-isomorphism $\alpha : M_m(\mathcal{M}) \to M_n(\mathcal{M})$ is called* inner *if there exists $u \in M_{n,m}(\mathcal{M})$ satisfying $u^*u = 1_{M_m(\mathcal{M})}$ and $uu^* = 1_{M_n(\mathcal{M})}$, such that*

$$\alpha(x) = uxu^*\,, \quad x \in M_m(\mathcal{M})\,.$$

Let $Int(M_m(\mathcal{M}), M_n(\mathcal{M}))$ denote the set of all inner $*$-isomorphisms $\alpha : M_m(\mathcal{M}) \to M_n(\mathcal{M})$.

If $\phi \in S_{nor}(M_m(\mathcal{M}))$ and $\psi \in S_{nor}(M_n(\mathcal{M}))$, then we write $\phi \sim \psi$ if

$$\inf\{\|\phi - \psi \circ \alpha\|\,; \alpha \in Int(M_m(\mathcal{M}), M_n(\mathcal{M}))\} = 0\,.$$

Clearly $\sim$ is an equivalence relation on $\bigcup_{k=1}^{\infty} S_{nor}(M_k(\mathcal{M}))$.

Proposition 3.2. *Let $\mathcal{M}$ be a properly infinite von Neumann algebra with separable predual. Given $\phi \in S_{nor}(\mathcal{M})$, then $\widehat{\phi}$ is θ-invariant if and only if for all $n \geq 1$,*

$$\phi \sim \phi_n := \frac{1}{n}\phi \otimes Tr_n\,,$$

where Tr_n denotes the non-normalized trace on $M_n(\mathbb{C})$.

Proof. Let $\phi \in S_{\rm nor}(\mathcal{M})$, and let $n \in \mathbb{N}$. Since $\mathcal{M}$ is properly infinite, $\mathcal{M} \cong M_n(\mathcal{M})$. Hence the identity $1_{\mathcal{M}}$ of $\mathcal{M}$ is a sum of n isometries, each equivalent to $1_{\mathcal{M}}$. Therefore $\mathrm{Int}(\mathcal{M}, M_n(\mathcal{M})) \neq \emptyset$. Let $\alpha_n \in \mathrm{Int}(\mathcal{M}, M_n(\mathcal{M}))$, arbitrarily chosen. Set

$$\psi_n := \phi_n \circ \alpha_n\,. \tag{3.7}$$

We will first prove that

$$\widehat{\psi_n} = \widehat{\phi} \circ \theta_{-\log n}\,. \tag{3.8}$$

Further, set

$$\tau_n := \tau \circ \alpha_n^{-1}\,. \tag{3.9}$$

Then τ_n is a trace on $M_n(\mathcal{M})$. More precisely, if $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in M_{n,1}(\mathcal{M})$ is a unitary such that $\alpha_n(x) = uxu^*$, $\forall x \in \mathcal{M}$, then $\tau_n(y) = \tau(u^*yu)$, $\forall y \in M_n(\mathcal{M})$. For $1 \leq i,j \leq n$, let e_{ij} denote the $n \times n$ matrix unit at the (i,j) position. Then, $\tau_n(x \otimes e_{ij}) = \tau(u_i^* x u_j) = \tau(x u_j u_i^*) = \delta_{ij}\tau(x)$, for all $x \in \mathcal{M}$, wherein we have used the fact that $u_j u_i^* = \delta_{ij} 1_{\mathcal{M}}$. Hence

$$\tau_n = \tau \otimes \mathrm{Tr}_n\,. \tag{3.10}$$

Let h_ϕ be defined by (3.3). Then, by uniqueness of the Radon-Nikodym derivative and formula (3.10), it follows that $\widetilde{\phi_n} = \tau_n(h_{\phi_n}\cdot)$, where $h_{\phi_n} = \frac{1}{n}h_\phi \otimes I_n$ and $I_n \in M_n(\mathbb{C})$ denotes the identity $n \times n$ matrix. Then, by setting $e_{\phi_n} := 1_{(1,\infty)}(h_{\phi_n}) \in M_n(\mathcal{N}) = M_n(\mathcal{M} \rtimes_{\sigma^{\phi_0}} \mathbb{R})$, we obtain by (3.5) that

$$e_{\phi_n} = 1_{(1,\infty)}\left(\frac{1}{n}h_\phi \otimes I_n\right) = 1_{(n,\infty)}(h_\phi \otimes I_n) = \theta_{\log n}(e_\phi) \otimes I_n\,, \tag{3.11}$$

where e_ϕ is defined by (3.4). By (3.7) and (3.11) it follows that

$$e_{\psi_n} = \alpha_n^{-1}(e_{\phi_n}) = \alpha_n^{-1}(\theta_{\log n}(e_\phi) \otimes I_n)\,. \tag{3.12}$$

Then, by (3.12), (3.9) and (3.10) we have for all $x \in Z(\mathcal{N})$

$$\begin{aligned}\widehat{\psi}_n(x) \;=\; \tau(e_{\psi_n}x) &= \tau(\alpha_n^{-1}(\theta_{\log n}(e_\phi)\otimes I_n)\cdot \alpha_n^{-1}\alpha_n(x)) \\ &= \tau\circ\alpha_n^{-1}\left((\theta_{\log n}(e_\phi)\otimes I_n)\alpha_n(x)\right) \\ &= (\tau\otimes \mathrm{Tr}_n)((\theta_{\log n}(e_\phi)\otimes I_n)\alpha_n(x)) \\ &\overset{\dagger}{=} n\tau(\theta_{\log n}(e_\phi)x)\,,\end{aligned}$$

where equality (†) follows from the following considerations. Since $\alpha_n(x) = uxu^*$, we have

$$(\theta_{\log n}(e_\phi)\otimes I_n)\alpha_n(x) = \begin{pmatrix} \theta_{\log n}(e_\phi)u_1xu_i^* & \dots & \theta_{\log n}(e_\phi)u_1xu_n^* \\ \vdots & & \vdots \\ \theta_{\log n}(e_\phi)u_nxu_1^* & \dots & \theta_{\log n}(e_\phi)u_nxu_n^* \end{pmatrix}.$$

We infer that

$$(\tau\otimes \mathrm{Tr}_n)((\theta_{\log n}(e_\phi)\otimes I_n)\alpha_n(x)) = \tau\left(\theta_{\log_n}(e_\phi)\sum_{i=1}^n u_ixu_i^*\right).$$

Since $x \in Z(\mathcal{N})$, we have $\sum\limits_{i=1}^n u_ixu_i^* = x\sum\limits_{i=1}^n u_iu_i^* = nx$, and equality (†) is now proved.
Therefore, for all $x \in Z(\mathcal{N})$,

$$\begin{aligned}\widehat{\psi}_n(x) &= n\tau(\theta_{\log n}(e_\phi)x) \\ &= n\tau(\theta_{\log n}(e_\phi)\theta_{\log n}\theta_{-\log n}(x)) \\ &= n(\tau\circ\theta_{\log n})(e_\phi\theta_{-\log n}(x)) \\ &\overset{\ddagger}{=} \tau(e_\phi\theta_{-\log n}(x)) \\ &= (\widehat{\phi}\circ\theta_{-\log n})(x)\,,\end{aligned}$$

where (‡) follows by taking $s = \log n$ in the equality $\tau\circ\theta_s = e^{-s}\tau$. Hence we have proved (3.8).

Now suppose that $\widehat{\phi}$ is θ-invariant. By (3.8) it follows that for all $n \geq 1$, $\widehat{\phi} = \widehat{\psi_n}$. Then, by the main theorem in [17] (cf. (3.6) above), it follows that $\phi \sim \psi_n$. The assertion follows by the definition of ψ_n .

Conversely, assume that $\phi \sim \phi_n := \frac{1}{n}\phi\otimes \mathrm{Tr}_n \in S_{\mathrm{nor}}(M_n(\mathcal{M}))$, for all $n \geq 1$. Then, for any positive integer n, let $\alpha_n \in \mathrm{Int}(\mathcal{M}, M_n(\mathcal{M}))$ be arbitrarily chosen, and set $\psi_n := \phi_n\circ\alpha_n \in S_{\mathrm{nor}}(\mathcal{M})$. By the transitivity property of relation $\sim$, it follows that $\phi \sim \psi_n$. Then, by (3.8) and the main theorem in [17] (cf. (3.6) above) we infer that

$$\widehat{\phi}\circ\theta_{-\log n} = \widehat{\phi}, \quad n \geq 1\,.$$

The group generated by $\{-\log n; n \geq 1\}$ in $(\mathbb{R}, +)$ is $\log(\mathbb{Q}^+)$, which is dense in $(\mathbb{R}, +)$. By the continuity of $(\theta_s)_{s\in\mathbb{R}}$, we conclude that $\widehat{\phi}\circ\theta_s = \widehat{\phi}, \forall s \in \mathbb{R}$, that is, $\widehat{\phi}$ is $(\theta_s)_{s\in\mathbb{R}}$-invariant. The proof is complete. □

Corollary 3.3. *Let $\mathcal{M}$ be a properly infinite von Neumann algebra with separable predual. If $\phi \in S_{nor}(\mathcal{M})$ such that $\phi \sim \frac{1}{n}\phi\otimes Tr_n$ for all positive integers n , then*

$$\phi \sim \begin{pmatrix} a\phi & 0 \\ 0 & (1-a)\phi \end{pmatrix}, \quad 0 < a < 1\,. \tag{3.13}$$

Proof. Assume first that $a \in \mathbb{Q}$. Then $a = \frac{p}{q}$, where p, q are positive integers with $p < q$. By hypothesis, we then have

$$\begin{pmatrix} a\phi & 0 \\ 0 & (1-a)\phi \end{pmatrix} = \frac{1}{q}\begin{pmatrix} p\phi & 0 \\ 0 & (q-p)\phi \end{pmatrix} \sim \frac{1}{q}\begin{pmatrix} \phi \otimes \mathrm{Tr}_p & 0_{p,q-p} \\ 0_{q-p,p} & \phi \otimes \mathrm{Tr}_{q-p} \end{pmatrix} \sim \phi\,,$$

where $0_{p,q-p} \in M_{p,q-p}(\mathbb{C})$ and $0_{q-p,p} \in M_{q-p,p}(\mathbb{C})$ denote matrices of corresponding sizes with all entries equal to zero. By approximation we obtain (3.13) for all $0 < a < 1$. □

Lemma 3.4. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be factors with separable preduals, and assume that $\mathcal{M}_2$ is of type III. Then the following two conditions are equivalent:*

- (a) *There exists a von Neumann algebra embedding $i : \mathcal{M}_1 \hookrightarrow \mathcal{M}_2$ and a normal conditional expectation $E : \mathcal{M}_2 \to \mathcal{M}_1$.*
- (b) *There exists a von Neumann algebra embedding $i : \mathcal{M}_1 \hookrightarrow \mathcal{M}_2$ and a normal faithful conditional expectation $E : \mathcal{M}_2 \to \mathcal{M}_1$.*

Proof. The implication $(b) \Rightarrow (a)$ is trivial. We prove that $(a) \Rightarrow (b)$. For this, we can assume that $\mathcal{M}_1 \subseteq \mathcal{M}_2$, for which there exists a normal conditional expectation $E : \mathcal{M}_2 \to \mathcal{M}_1$. Since E is normal, it has a support projection $p := \mathrm{supp}(E)$, p is the smallest projection in $\mathcal{M}_2$ for which $E(1-p) = 0$. Moreover, $p \in \mathcal{M}_1' \cap \mathcal{M}_2$ by the bimodule property of E. Hence the map ϕ defined by

$$\phi(a) := ap = pa\,, \quad a \in \mathcal{M}_1$$

is a normal $*$-homomorphism of $\mathcal{M}_1$ onto the von Neumann algebra $p\mathcal{M}_1 \subseteq \mathcal{B}(pH)$, where $\mathcal{M}_1 \subseteq \mathcal{B}(H)$. Since $\mathcal{M}_1$ is a factor, the map ϕ is one-to-one, and hence $\mathcal{M}_1 \simeq p\mathcal{M}_1$. Since $\mathcal{M}_2$ is a type III-factor with separable predual and $p \neq 0$, we deduce that $p \sim 1_{\mathcal{M}_2}$, i.e., $p = uu^*$ for an isometry $u \in \mathcal{M}_2$. Hence $\mathcal{M}_2 \simeq p\mathcal{M}_2 p \subseteq \mathcal{B}(pH)$. Therefore, in order to prove (b) it suffices to construct a normal faithful conditional expectation $E' : p\mathcal{M}_1 p \to p\mathcal{M}_2$. Set

$$E'(y) := pE(y) = E(y)p\,, \quad y \in p\mathcal{M}_1 p\,.$$

Then E' is positive, unital and normal. For $a, b \in \mathcal{M}_1$ and $x \in \mathcal{M}_2$, we have

$$\begin{aligned} E'((pa)(pxp)(bp)) &= pE((ap)(pxp)(pb))p \\ &= pE(a(pxp)b)p \\ &= paE(pxp)bp\,. \end{aligned}$$

Hence E' is a normal conditional expectation of $p\mathcal{M}_2 p$ onto $p\mathcal{M}_1 = \mathcal{M}_1 p$. Moreover, if $y \in p\mathcal{M}_2 p$, $y \geq 0$ and $E'(y) = 0$, then $E(y) = \phi^{-1}(E'(y)) = 0$, and hence the support projection of y is less than $1-p$. It follows that $y = 0$, and we have shown that E' is faithful. □

Theorem 3.5. *Let $\mathcal{M}$ be a type III-factor with separable predual. The following statements are equivalent:*

1. *There exists a von Neumann algebra embedding of R_∞ into $\mathcal{M}$ with a normal conditional expectation $E : \mathcal{M} \to R_\infty$.*
2. *For every $0 < \lambda < 1$, there exists a von Neumann algebra embedding of R_λ into $\mathcal{M}$ with a normal conditional expectation $E : \mathcal{M} \to R_\lambda$.*
3. *There exists an invariant normal state on the flow of weights $(Z(\mathcal{N}), (\theta_s)_{s \in \mathbb{R}})$ for $\mathcal{M}$.*

Proof. It was shown by Haagerup, Rosenthal and Sukochev (see [16], Theorem 6.2 (a)) that $(R_\lambda)_*$ is completely isomorphic to $(R_\infty)_*$, for all $0 < \lambda \leq 1$. Then an application of Theorem 2.10 yields the equivalence of statements (1) and (2).

We now prove the implication (1) $\Rightarrow$ (3). Following an argument from [17], we will show a (slightly) more general result. Namely, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be factors of type III with separable preduals such that there is a von Neumann algebra embedding of $\mathcal{M}_1$ into $\mathcal{M}_2$ with a normal conditional expectation $E : \mathcal{M}_2 \to \mathcal{M}_1$. We show that if $\mathcal{M}_1$ is a type III_1-factor, then under the above condition there exists an invariant normal state on the flow of weights for $\mathcal{M}_2$. By Lemma 3.4 we can assume that E is faithful. Let ϕ_1 be a normal, faithful state on $\mathcal{M}_1$ and set $\phi_2 := \phi_1 \circ E$. Then ϕ_2 is a normal faithful state on $\mathcal{M}_2$, and by [38] it follows that

$$\sigma_t^{\phi_2}{}_{|\mathcal{M}_1} = \sigma_t^{\phi_1}\,, \quad t \in \mathbb{R}\,. \tag{3.14}$$

By (3.14) we obtain an embedding of $\mathcal{N}_1 := \mathcal{M}_1 \rtimes_{\sigma^{\phi_1}} \mathbb{R}$ into $\mathcal{N}_2 := \mathcal{M}_2 \rtimes_{\sigma^{\phi_2}} \mathbb{R}$, with a normal faithful conditional expectation $\widetilde{E} : \mathcal{N}_2 \to \mathcal{N}_1$. Moreover, $\widetilde{E}|_{\pi_2(\mathcal{M}_2)} = E$ and

$$\widetilde{E}(\lambda_2(t)) = \lambda_1(t)\,, \quad t \in \mathbb{R}\,, \tag{3.15}$$

where $\pi_i(\mathcal{M}_i)$ and $(\lambda_i(t))_{t\in\mathbb{R}}$ are the generators of $\mathcal{N}_i$, $i = 1, 2$, as explained before. Further, let $\theta^{(1)} := (\theta_s^{(1)})_{s\in\mathbb{R}}$ and $\theta^{(2)} := (\theta_s^{(2)})_{s\in\mathbb{R}}$ denote the dual action of $\mathbb{R}$ on $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. By (3.1) and (3.2), $\theta^{(2)}$ extends $\theta^{(1)}$ and the canonical trace τ_2 on $\mathcal{N}_2$ extends the canonical trace τ_1 on $\mathcal{N}_1$. Also,

$$\widetilde{E}(Z(\mathcal{N}_2)) \subseteq Z(\mathcal{N}_1)\,, \tag{3.16}$$

which can be justified as follows. Given $x \in Z(\mathcal{N}_2)$, then $xy = yx$ for all $y \in \mathcal{N}_1$, and therefore by the bimodule property of conditional expectations, $y\widetilde{E}(x) = \widetilde{E}(yx) = \widetilde{E}(xy) = \widetilde{E}(x)y$.

Since $\mathcal{M}_1$ is a type III_1- factor, then $Z(\mathcal{N}_1) = \mathbb{C}1_{\mathcal{N}_1}$ (cf. **1.6** in the Introduction), where $1_{\mathcal{N}_1}$ is the identity of $\mathcal{N}_1$. By (3.16) we infer that for all $x \in Z(\mathcal{N}_2)$ there exists $\phi(x) \in \mathbb{C}$ such that $\widetilde{E}(x) = \phi(x) \cdot 1_{\mathcal{N}_1}$. It is then easily seen that the correspondence $x \in Z(\mathcal{N}_2) \mapsto \phi(x) \in \mathbb{C}$ defines a normal $\theta^{(2)}$-invariant state on $Z(\mathcal{N}_2)$. Thus assertion (3) is proved.

Now we show the remaining implication (3) $\Rightarrow$ (1). Suppose that $(Z(\mathcal{N}), (\theta_s)_{s\in\mathbb{R}})$ has an invariant normal state ω. Note that, since $\mathcal{M}$ is a factor, the faithfulness of ω is automatic. Indeed, we have that $\theta_s(\mathrm{supp}(\omega)) = \mathrm{supp}(\omega)$, for all $s \in \mathbb{R}$, where $\mathrm{supp}(\omega)$ denotes the support projection of ω. Since $\mathcal{M}$ is a factor, $(\theta_s)_{s\in\mathbb{R}}$ is ergodic, which implies that ω has full support, and hence ω is faithful. Clearly,

$$\omega \circ \theta_s \geq e^{-s}\omega\,, \quad s \geq 0\,.$$

Hence, by the Main Theorem in [17], there exists $\phi \in S_{\mathrm{nor}}(\mathcal{M})$ such that $\widehat{\phi} = \omega$. By Proposition 3.2 and Corollary 3.3, we infer that for all $0 < a < 1$,

$$\phi \sim \begin{pmatrix} a\phi & 0 \\ 0 & (1-a)\phi \end{pmatrix}. \tag{3.17}$$

Let $0 < a_1, a_2 < \frac{1}{2}$ be chosen such that, if λ_1, λ_2 are defined by $\lambda_i := \frac{a_i}{1-a_i}$, $i = 1, 2$, then $\frac{\log \lambda_1}{\log \lambda_2} \notin \mathbb{Q}$. Further, for all integers $k \geq 0$ set $a_{2k+1} := a_1$, $a_{2k+2} := a_2$ and define

$$\phi_n := \mathrm{Tr}_2\left(\begin{pmatrix} a_n & 0 \\ 0 & (1-a_n) \end{pmatrix} \cdot \right), \quad n \geq 1\,.$$

Then, $R_\infty \cong R_{\lambda_1} \otimes R_{\lambda_2} \cong \otimes_{n=1}^{\infty}(M_2(\mathbb{C}), \phi_n)$ (see **1.9.** in the Introduction).

Now let $\varepsilon > 0$. By (3.17), there exists $\alpha \in \mathrm{Int}(\mathcal{M}, M_2(\mathcal{M}))$ such that $\|\phi - (\phi_1 \otimes \phi) \circ \alpha\| < \frac{\varepsilon}{2}$. Hence we can write $\mathcal{M} = M_2(\mathbb{C}) \otimes Q_1$, and choose a normal faithful state ψ_1 on Q_1 so that $(Q_1, \psi_1) \cong (\mathcal{M}, \phi)$ and

$$\|\phi - \phi_1 \otimes \psi_1\| < \frac{\varepsilon}{2}.$$

Similarly, write $Q_1 = M_2(\mathbb{C}) \otimes Q_2$, with a normal faithful state ψ_2 on Q_2 such that $(Q_2, \psi_2) \cong (Q_1, \psi_1) \cong (\mathcal{M}, \phi)$, satisfying, moreover,

$$\|\psi_1 - \phi_2 \otimes \psi_2\| < \frac{\varepsilon}{4}.$$

By continuing in this way, we obtain for all $k \geq 1$ a decomposition $\mathcal{M} = \left(\otimes_{j=1}^{k} M_2(\mathbb{C})\right) \otimes Q_k$ and a normal faithful state ψ_k on Q_k such that $(Q_k, \psi_k) \cong (\mathcal{M}, \phi)$ and, moreover,

$$\|\psi_k - \phi_{k+1} \otimes \psi_{k+1}\| < \frac{\varepsilon}{2^{k+1}}. \tag{3.18}$$

Set $\chi_k := (\phi_1 \otimes \ldots \otimes \phi_k) \otimes \psi_k$, for all $k \geq 1$. By (3.18), we infer that

$$\begin{aligned} \|\chi_{k+1} - \chi_k\| &= \|(\phi_1 \otimes \ldots \phi_k) \otimes (\phi_{k+1} \otimes \psi_{k+1} - \psi_k)\| \\ &= \|\phi_{k+1} \otimes \psi_{k+1} - \psi_k\| \\ &< \frac{\varepsilon}{2^{k+1}}. \end{aligned}$$

This shows that the sequence $(\chi_k)_{k\geq 1}$ converges in $S_{\mathrm{nor}}(\mathcal{M})$. Set $\chi := \lim_{k\to\infty} \chi_k \in S_{\mathrm{nor}}(\mathcal{M})$. We claim that χ is faithful. For this, we first show that $e := \mathrm{supp}(\chi)$ commutes with $P_k := \otimes_{j=1}^{k} M_2(\mathbb{C})$, for all positive integers k.

Let now $k \geq 1$ be fixed. Then, for all $n > k$ we have

$$\chi_n = (\phi_1 \otimes \ldots \otimes \phi_k) \otimes \omega_{k,n},$$

where $\omega_{k,n} := (\phi_{k+1} \otimes \ldots \otimes \phi_n) \otimes \psi_n \in S_{\mathrm{nor}}(Q_k)$. Further, note that for all $n, m > k$,

$$\|\chi_n - \chi_m\| = \|\omega_{k,n} - \omega_{k,m}\|.$$

Since $\chi := \lim_{k\to\infty} \chi_k$, we deduce that $\eta_k := \lim_{n\to\infty} \omega_{k,n}$ exists, $\eta_k \in S_{\mathrm{nor}}(Q_k)$ and

$$\chi = (\phi_1 \otimes \ldots \otimes \phi_k) \otimes \eta_k.$$

Then, since $\phi_1, \ldots, \phi_k$ are faithful states, it follows that

$$e := \mathrm{supp}(\chi) = \mathrm{supp}(\phi_1 \otimes \ldots \otimes \phi_k) \otimes \mathrm{supp}(\eta_k) = 1 \otimes \mathrm{supp}(\eta_k) \in 1 \otimes Q_k = P_k' \cap \mathcal{M},$$

wherein we have used the fact that P_k is a factor. The claim that e commutes with P_k is now proved.

Next set $\chi^{\mathrm{new}} := \chi_{|e\mathcal{M}e}$ and define $P_k^{\mathrm{new}} := eP_k$, $Q_k^{\mathrm{new}} := eQ_ke$ and $\eta_k^{\mathrm{new}} := \eta_{k|_{eQ_ke}}$, for all $k \geq 1$. We obtain a sequence of von Neumann algebras embeddings $P_1^{\mathrm{new}} \hookrightarrow P_2^{\mathrm{new}} \hookrightarrow \ldots \hookrightarrow e\mathcal{M}e$. Also,

$$\chi^{\mathrm{new}} = (\phi_1 \otimes \ldots \otimes \phi_k) \otimes \eta_k^{\mathrm{new}}.$$

Note that χ^{new} is a faithful state on $e\mathcal{M}e$. Since $\mathcal{M}$ is a type III-factor on a separable Hilbert space, all nonzero projections are equivalent, and hence $e\mathcal{M}e \cong \mathcal{M}$. Therefore, from now on we will use $\mathcal{M}$ instead of $e\mathcal{M}e$ and delete the superscript *new*.

Further, let G denote the s.o.t.-closure of $\cup_{k=1}^{\infty} P_k$. It now follows that

$$\left(G, \chi_{|G}\right) \cong \otimes_{k=1}^{\infty} (\mathcal{M}_2(\mathbb{C}), \phi_k) \cong R_\infty$$

(see, e.g., Remark 1.8. in [43]). For all $k \geq 1$ and all $t \in \mathbb{R}$ we have

$$\left(\sigma_t^{\phi_1 \otimes \ldots \otimes \phi_k}\right) \otimes \sigma_t^{\eta_k} = \sigma_t^{\chi} ,$$

which implies that the automorphism group $(\sigma_t^{\chi})_{t \in \mathbb{R}}$ leaves P_k globally invariant for all $k \geq 1$. It follows that $(\sigma_t^{\chi})_{t \in \mathbb{R}}$ leaves G globally invariant. By [38] we deduce that there exists a unique faithful normal conditional expectation E of $\mathcal{M}$ onto G such that $\chi \circ E = \chi$. The proof is complete. □

Remark 3.6. In the special case when $\mathcal{M}$ is hyperfinite, a more elementary proof of the implication (3) ⇒ (1) in Theorem 3.5 can be obtained. The proof below was suggested to us by Georges Skandalis.

Assume that $\mathcal{M}$ is a hyperfinite factor of type III (with separable predual), such that there exists an invariant normal state ϕ on the flow of weights $(A, \theta^{(1)})$ for $\mathcal{M}$. Let $\mathcal{N}_0 := R_\infty \rtimes_{\sigma^\omega} \mathbb{R}$, where ω is a fixed normal faithful state on R_∞ . Then $\mathcal{N}_0$ is the hyperfinite II_∞-factor, and the dual action $\theta^{(0)} := \widetilde{\sigma}^\omega$ of σ^ω satisfies $\tau_0 \circ \theta^{(0)} = e^{-s}\tau_0$, for all $s \in \mathbb{R}$, where τ_0 is a normal faithful trace on $\mathcal{N}_0$. Set $\mathcal{N} := \mathcal{N}_0 \bar{\otimes} A$, $\theta := \theta^{(0)} \otimes \theta^{(1)}$ and $\tau := \tau_0 \otimes \omega$. Then τ is a normal faithful trace on $\mathcal{N}$ satisfying

$$\tau \circ \theta_s = e^{-s}\tau , \quad s \in \mathbb{R} . \tag{3.19}$$

By crossed product theory (cf. [40], Vol. II, Theorem X.2.3 (i)),

$$R_\infty = \mathcal{N}_0^{\theta^{(0)}} := \{x \in \mathcal{N}_0;\, \theta_s^{(0)}(x) = x \,, \text{for all } s \in \mathbb{R}\} .$$

Hence the fixed point algebra $\mathcal{N}^\theta$ for the action of θ on $\mathcal{N}$ satisfies $R_\infty \bar{\otimes} 1 \subseteq \mathcal{N}^\theta \subseteq \mathcal{N}$. Put $E := \mathrm{Id}_{\mathcal{N}_0} \otimes \phi$. Then E is a normal conditional expectation of $\mathcal{N}$ onto $\mathcal{N}_0 \bar{\otimes} 1$ and $E(\mathcal{N}^\theta) \supseteq E(R_\infty \bar{\otimes} 1) = R_\infty \bar{\otimes} 1$. Since ω is $\theta^{(1)}$ invariant, it follows that $E(\mathcal{N}^\theta) \subseteq \mathcal{N}_0^{\theta^{(0)}} = R_\infty \bar{\otimes} 1$, and therefore $E_0 := E_0|_{\mathcal{N}^\theta}$ is a normal conditional expectation of $\mathcal{N}^\theta$ onto $R_\infty \bar{\otimes} 1$. Hence $\mathcal{N}^\theta$ satisfies condition (1) in Theorem 3.5.

We next prove that $\mathcal{M} \simeq \mathcal{N}^\theta$, which will complete the proof of the implication (3) ⇒ (1) in Theorem 3.5 in the hyperfinite case. Note that by (3.19), $(\mathcal{N}, \tau, \theta)$ satisfies condition (i) in Theorem XII.1.1. of [40], Vol. II. Therefore

$$Z(\mathcal{M}_1) = Z(\mathcal{N})^\theta ,$$

where $\mathcal{M}_1 := \mathcal{N} \rtimes_\theta \mathbb{R}$. But since $(Z(\mathcal{N}), \theta|_{Z(\mathcal{N})}) \simeq (A, \theta^{(1)})$, the latter being ergodic, we actually have $Z(\mathcal{M}_1) = \mathbb{C}1$, i.e., $\mathcal{M}_1$ is a factor. Moreover, by Proposition X.2.6 and Lemma XII.1.2. in [40], Vol. II,

$$(\mathcal{N}, \mathbb{R}, \theta) \simeq (\mathcal{M}_2 \rtimes_\alpha \mathbb{R}, \mathbb{R}, \widetilde{\alpha}) ,$$

for some covariant system $(\mathcal{M}_2 \,, \mathbb{R} \,, \alpha)$. By Theorem X.2.3. in [40], Vol. II, it follows that $\mathcal{M}_2 \simeq \mathcal{N}^\theta$ and

$$\mathcal{M}_1 = \mathcal{N} \rtimes_\theta \mathbb{R} \simeq \mathcal{M}_2 \bar{\otimes} \mathcal{B}(L^2(\mathbb{R})) . \tag{3.20}$$

Since $\mathcal{M}_2$ is also a factor, and $R_\infty \bar{\otimes} 1$ is the range of a normal conditional expectation $E_0 : \mathcal{N}^\theta \to R_\infty \bar{\otimes} 1$, it follows by Lemma 2.11 that $\mathcal{M}_2 \simeq \mathcal{N}^\theta$ is of type III. Hence, by (3.20),

$$\mathcal{N}^\theta \simeq \mathcal{M}_2 \simeq \mathcal{M}_1 .$$

By Theorem XII.1.1. in [40], Vol. II, $(\mathcal{N}, \mathbb{R}, \theta)$ is isomorphic to the "noncommutative flow of weights" of $\mathcal{M}_1$ (in the sense of Def. XII.1.3 in [40], Vol. II) and hence $(Z(\mathcal{N}), \theta|_{Z(\mathcal{N})}) \simeq (A, \theta^{(1)})$ is isomorphic to the flow of weights for $\mathcal{M}_1$. Hence $\mathcal{M}$ and $\mathcal{M}_1$ are hyperfinite type III-factors with isomorphic flow of weights, so by **1.7**, **1.8** and **1.9** in the Introduction, $\mathcal{M} \simeq \mathcal{M}_1$, and thus $\mathcal{M} \simeq \mathcal{N}^\theta$. The proof is complete.

Corollary 3.7. *Let $\mathcal{M}$ be a type III-factor with separable predual. If there exists a type III_1-factor which embeds into $\mathcal{M}$ as the range of a normal faithful conditional expectation, then there exists a von Neumann algebra embedding $j : R_\infty \hookrightarrow \mathcal{M}$, with a normal faithful conditional expectation $\widetilde{E} : \mathcal{M} \to j(R_\infty)$.*

Proof. Suppose there is a von Neumann algebra embedding $i : \mathcal{R} \hookrightarrow \mathcal{M}$, where $\mathcal{R}$ is a type III_1- factor, with a normal faithful conditional expectation $E : \mathcal{M} \to i(\mathcal{R})$. By the proof of the implication 1) $\Rightarrow$ 3) in Theorem 3.5, we deduce the existence of an invariant normal state on the flow of weights for $\mathcal{M}$. An application of Theorem 3.5 yields the assertion. □

The following lemma is well-known. For completeness, we include a proof.

Lemma 3.8. *Let $\mathcal{M}$ be a hyperfinite factor with separable predual. Then*

$$M\bar{\otimes}R_\infty \cong R_\infty .$$

Proof. Let $0 < \lambda < 1$. Choose $0 < \mu < \infty$ such that $\frac{\log \lambda}{\log \mu} \notin \mathbb{Q}$. Then $R_\infty \cong R_\mu \bar{\otimes} R_\lambda$ and therefore $R_\infty \bar{\otimes} R_\lambda \cong R_\infty$. We deduce that $(M\bar{\otimes}R_\infty)\bar{\otimes}R_\lambda \cong M\bar{\otimes}R_\infty$. By the definition of the Araki and Woods r_∞-invariant of a factor (see Section 3.6 in [3]), it follows that $\lambda \in r_\infty(M\bar{\otimes}R_\infty)$. By [3], Theorem 3.6.1, we conclude that $\lambda \in S(M\bar{\otimes}R_\infty)$, where S is Connes' S-invariant of a factor. Hence $M\bar{\otimes}R_\infty$ is a type III_1-factor. It is also hyperfinite, and thus the assertion follows (see **1.9**. in the Introduction). □

Proof of Theorem 1.2. This follows now immediately from the equivalence 1) $\Leftrightarrow$ 4) in Theorem 2.10, together with Theorem 3.5 and Lemma 3.4 above, and the fact that if $\mathcal{M}$ is an injective factor with separable predual, then the von Neumann algebra tensor product $\mathcal{M}\bar{\otimes}R_\infty$ is (isomorphic to) R_∞ (cf. Lemma 3.8), which implies that there exists a von Neumann algebra embedding $i : \mathcal{M} \hookrightarrow R_\infty$ with a normal conditional expectation $E : R_\infty \to i(\mathcal{M})$. □

If $\mathcal{M}$ is any factor (not necessarily hyperfinite) of type III_λ, where $0 < \lambda \leq 1$, then there always exists a normal invariant state on the flow of weights for $\mathcal{M}$. Using results of Haagerup and Winsløw (cf. [18]; see also [16], Theorem 6.2), we exhibit in the following an uncountable family of mutually non-isomorphic (in the von Neumann algebras sense) hyperfinite type III_0-factors which admit a normal (faithful) invariant state on their flow of weights:

Example 3.9. Let G be a dense, countable subgroup of $\mathbb{R}$. Further, let ϕ be a normal, faithful state on R_∞ and set $\mathcal{N}_G := R_\infty \rtimes_\alpha G$, where $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ is the restriction of the modular automorphism group $(\sigma_t^\phi)_{t\in\mathbb{R}}$ to G. Then $\mathcal{N}_G$ is an injective type III_0-factor. Moreover, $T(\mathcal{N}_G) = G$, where T is Connes T-invariant. In particular, if $G \neq G'$, then $\mathcal{N}_G$ and $\mathcal{N}_{G'}$ are not von Neumann algebras isomorphic. It is easily checked that there are uncountably many dense, countable subgroups of $\mathbb{R}$. Since $\mathcal{N}_G$ is a crossed product of R_∞ by a discrete group, there exists an embedding of R_∞ into $\mathcal{N}_G$ with a normal, faithful conditional expectation $E : \mathcal{N}_G \to R_\infty$ onto. By Theorem 3.5, we deduce the existence of a normal invariant state on the flow of weights for $\mathcal{N}_G$. Note that $(\mathcal{N}_G)_*$ is cb-isomorphic to $(R_\infty)_*$, as shown in the proof of Theorem 6.2 in [16].

We end this section with the following results concerning the non-hyperfinite case.

Proposition 3.10. *Let $\mathcal{M}$ be any type III-factor with separable predual $\mathcal{M}_*$, such that there exists an invariant normal state on the flow of weights for $\mathcal{M}$. If $\mathcal{N}$ is any semifinite von Neumann algebra with separable predual $\mathcal{N}_*$, then $\mathcal{M}_*$ is not cb-isomorphic to a subspace of $\mathcal{N}_*$. In particular, $\mathcal{M}_*$ and $\mathcal{N}_*$ are not cb-isomorphic.*

Proof. By Theorem 3.5 it follows that there exists an embedding $R_\infty \hookrightarrow \mathcal{M}$ with a normal faithful conditional expectation $E : \mathcal{M} \to R_\infty$. This yields a cb-embedding $E_* : (R_\infty)_* \hookrightarrow \mathcal{M}_*$ such that, moreover, $E_*((R_\infty)_*)$ is cb-complemented into $\mathcal{M}_*$.

Suppose by contradiction that $\mathcal{M}_*$ is cb-isomorphic to a subspace of $\mathcal{N}_*$. This gives rise to a a cb-embedding of $(R_\infty)_*$ into $\mathcal{M}_*$. The crucial point is now the fact that the operator Hilbert space OH is cb-isomorphic to a subspace of $(R_\infty)_*$, as proved by Junge [20]. A different proof that yields to the improved cb-isomorphism constant $\leq \sqrt{2}$ has been recently obtained by the authors (see [15]). However, since $\mathcal{N}$ is semifinite, OH does not cb-embed into the predual $\mathcal{N}_*$, as shown by Pisier [34]. This leads to a contradiction, and the proof is complete. ☐

Examples of non-injective type III_0-factors with an invariant normal state on their flow of weights can be obtained as follows:

Example 3.11. Given a dense countable subgroup G of $\mathbb{R}$, let $\mathcal{N}_G$ be the corresponding injective type III_0-factor constructed in Example 3.9. Let $\mathcal{N}$ be a semifinite non-injective factor, and set $\mathcal{M}_G := \mathcal{N}_G \bar{\otimes} \mathcal{N}$. Then $\mathcal{M}_G$ is a type III_0-factor, since by Corollary 3.2.8 in [3], $S(\mathcal{N}_G \bar{\otimes} \mathcal{N}) = S(\mathcal{N}_G)$. Furthermore, $\mathcal{M}_G$ is clearly non-injective, and, moreover, by Theorem 3.5 it admits an invariant, normal state on the flow of weights.

Remark 3.12. The existence of an invariant normal state on the flow of weights appears in a different context in Connes's paper [6]. Here a certain class class of foliated 3-manifolds (V, F) is considered, and it is proved that if the Godbillon-Vey invariant of such a foliated manifold is non-zero, then the von Neumann algebra $\mathcal{M}$ associated to (V, F) must have an invariant normal state on its flow of weights (see [6], Theorem 0.3. and Theorem 7.14.)

4. CB-isomorphism classes of preduals of injective type III_0 factors

In this section we will prove the following:

Theorem 4.1. *For every $0 \leq t < 2$, there exists a non-transitive ergodic flow $(A^{(t)}, (\theta_s^{(t)})_{s\in\mathbb{R}})$ with separable predual $A_*^{(t)}$, such that for all normal states ω on $A^{(t)}$ we have*

$$\lim_{n\to\infty} \|\omega \circ \theta_{2^n}^{(t)} - \omega\| = t\,.$$

As a consequence we obtain:

Theorem 4.2. *If $(\mathcal{M}^{(t)})_{0\leq t<2}$ are the hyperfinite type III_0- factors with $(A^{(t)}, (\theta_s^{(t)})_{s\in\mathbb{R}})$ from Theorem 4.1 as flow of weights, then for any $0 \leq t_1 < t_2 < 2$, the predual $\mathcal{M}_*^{(t_1)}$ of $\mathcal{M}^{(t_1)}$ is not cb-isomorphic to a cb-complemented subspace of the predual $\mathcal{M}_*^{(t_2)}$ of $\mathcal{M}^{(t_2)}$.*

In particular, $(\mathcal{M}_*^{(t)})_{0\leq t<2}$ is a family of mutually not cb-isomorphic preduals of hyperfinite type III_0 factors.

Proof. Let $0 \le t_1 < t_2 < 2$. Assume by contradiction that $\mathcal{M}_*^{(t_1)}$ is cb-isomorphic to a cb-complemented subspace of $\mathcal{M}_*^{(t_2)}$. By Proposition 2.9 we can then embed $\mathcal{M}^{(t_1)}$ into $\mathcal{M}^{(t_2)}$ as a von Neumann algebra with a normal faithful conditional expectation $E : \mathcal{M}^{(t_2)} \to \mathcal{M}^{(t_1)}$ onto. Let ϕ_1 be a normal faithful state on $\mathcal{M}^{(t_1)}$ and set $\phi_2 := \phi_1 \circ E$. Then ϕ_2 is a normal faithful state on $\mathcal{M}^{(t_2)}$ satisfying (3.14). As explained in the proof of Theorem 3.5, this ensures the existence of an embedding of $\mathcal{N}^{(t_1)} := \mathcal{M}^{(t_1)} \rtimes_{\sigma^{\phi_1}} \mathbb{R}$ into $\mathcal{N}^{(t_2)} := \mathcal{M}^{(t_2)} \rtimes_{\sigma^{\phi_2}} \mathbb{R}$, with a normal faithful conditional expectation $\widetilde{E} : \mathcal{N}^{(t_2)} \to \mathcal{N}^{(t_1)}$, satisfying

$$\widetilde{E}|_{\pi_2(\mathcal{M}^{(t_2)})} = E \tag{4.1}$$

and (3.15), where $\pi_i(\mathcal{M}^{(t_i)})$ and $(\lambda_i(t))_{t\in\mathbb{R}}$ denote the generators of $\mathcal{N}^{(t_i)}$, $i = 1, 2$. By (3.1) and (3.2), we then infer that

$$\widetilde{E} \circ \theta_s^{(t_2)} = \theta_s^{(t_1)} \circ \widetilde{E}, \quad s \in \mathbb{R}.$$

Furthermore, by the bimodule property of conditional expectations, we have as in the proof of (3.16) that

$$\widetilde{E}(A^{(t_2)}) \subseteq A^{(t_1)}.$$

Set $S := \widetilde{E}|_{A^{(t_2)}}$. Then $S : A^{(t_2)} \to A^{(t_1)}$ is a unital, normal, positive mapping satisfying

$$S \circ \theta_s^{(t_2)} = \theta_s^{(t_1)} \circ S, \quad s \in \mathbb{R}. \tag{4.2}$$

Choose a normal state ω_1 on $A^{(t_1)}$ and set $\omega_2 := \omega_1 \circ S$. Then $\omega_2 \in S_{\text{nor}}(A^{(t_2)})$ and by (4.2) we get

$$(\omega_2 \circ \theta_s^{(t_2)} - \omega_2) = (\omega_1 \circ \theta_s^{(t_1)} - \omega_1) \circ S.$$

By positivity we infer that $\|S\| \le 1$ and therefore

$$\|\omega_2 \circ \theta_s^{(t_2)} - \omega_2\| \le \|\omega_1 \circ \theta_s^{(t_1)} - \omega_1\|.$$

Let $s := 2^n$, $n \ge 1$, and pass to the limit as $n \to \infty$. We infer that $t_2 \le t_1$, which is a contradiction. ☐

The proof of Theorem 4.1 will be achieved in several steps. Recall first the action of $\mathbb{Z}$ on $\Omega = \{0,1\}^\infty$ by the *dyadic odometer* transformation (cf. [40], Vol. III, Definition 3.24). Define $g : \Omega \to \Omega$ by:

$$\begin{aligned} g(0, x_2, x_3, x_4, \ldots) &= (1, x_2, x_3, x_4, \ldots) \\ g(1, 0, x_3, x_4, \ldots) &= (0, 1, x_3, x_4, \ldots) \\ &\vdots \\ g(1, 1, \ldots, 1, 0, x_{n+1}, \ldots) &= (0, 0, \ldots, 0, 1, x_{n+1}, \ldots) \\ &\vdots \\ g(1, 1, 1, \ldots) &= (0, 0, 0, \ldots) \end{aligned}$$

Take $0 < a \le \frac{1}{2}$. Define a measure ν_a on Ω by

$$\nu_a := \otimes_{n=1}^\infty \mu_a, \quad \text{where} \ \ \mu_a := a\delta_0 + (1-a)\delta_1.$$

It is easy to check that g preserves the measure class of ν_a, i.e., the image measure $g(\nu_a)$ has the same null-sets as ν_a. Therefore g induces an automorphism σ of $L^\infty(\Omega, \nu_a)$ by

$$\sigma(f)(x) = f(g^{-1}x), \quad f \in L^\infty(\Omega, \nu_a), x \in \Omega. \tag{4.3}$$

It is well-known that σ is ergodic, i.e., if $f \in L^\infty(\Omega, \nu_a)$ satisfies $\sigma(f) = f$ (ν_a-a.e.), then f is ν_a-a.e. equal to a constant function. This can be seen by observing that $(g^n)_{n\in\mathbb{Z}}$ have the same orbits in Ω (up to null-sets) as the natural action of $\mathbb{Z}_2^{(\infty)} = \{(x_1, x_2, x_3 \ldots) \in \Omega; x_i \neq 0 \text{ eventually, as } i \to \infty\}$ on Ω, generated by $(\rho_k)_{k=1}^\infty$, where ρ_k changes x_k to $1 - x_k$ in $x = (x_1, x_2, \ldots) \in \Omega$ and leaves the remaining coordinates of x unchanged. The induced action on $L^\infty(\Omega, \nu_a)$

$$\alpha_\rho(f)(x) = f(\rho^{-1}x), \quad f \in L^\infty(\Omega, \nu_a), \rho \in \mathbb{Z}_2^{(\infty)}, x \in \Omega$$

is ergodic, because the crossed-product $L^\infty(\Omega, \nu_a) \rtimes_\alpha \mathbb{Z}_2^{(\infty)}$ is a factor (cf. [24], Introduction). Therefore, σ is ergodic, as well.

In the following, we will use the symbol ν_a also to denote the normal state on $L^\infty(\Omega, \nu_a)$ given by

$$\nu_a(f) = \int_\Omega f \, d\nu_a .$$

With this notation, we have

Lemma 4.3. *The following equality holds:*

$$\|\nu_a \circ \sigma - \nu_a\| = 2 - 4a . \tag{4.4}$$

Proof. Note that

$$(\nu_a \circ \sigma)(f) = \int_\Omega f(g^{-1}x) \, d\nu_a(x) = \int_\Omega f \, dg^{-1}(\nu_a), \quad f \in L^\infty(\Omega, \nu_a),$$

where $g^{-1}(\nu_a)$ is the image measure of ν_a by the map g^{-1}. Hence

$$\|\nu_a \circ \sigma - \nu_a\|_{L^\infty(\Omega,\nu_a)_*} = \left\| \frac{dg^{-1}(\nu_a)}{d\nu_a} - 1 \right\|_{L^1(\Omega,\nu_a)} .$$

For every $n \geq 1$, let K_n be the set of elements in Ω of the form

$$(\underbrace{1, 1, \ldots, 1}_{n-1 \text{ times}}, 0, x_{n+1}, x_{n+2}, \ldots),$$

where $x_j \in \{0, 1\}$ for $j \geq n+1$, and put $k_0 = (1, 1, 1, \ldots)$. Then $\Omega = (\cup_{n=1}^\infty K_n) \cup \{k_0\}$ (disjoint union). By the definition of g and ν_a it is clear that for every Borel set E in K_n,

$$\nu_a(gE) = \frac{1-a}{a} \left(\frac{a}{1-a} \right)^{n-1} \nu_a(E) .$$

Since $g^{-1}(\nu_a)(E) = \nu_a(gE)$, it follows that

$$\frac{dg^{-1}(\nu_a)}{d\nu_a}(x) = \left(\frac{a}{1-a} \right)^{n-2}, \quad x \in K_n . \tag{4.5}$$

Since $1 - a \geq a$, it follows that

$$\begin{aligned} \left\| \frac{dg^{-1}(\nu_a)}{d\nu_a} - 1 \right\|_{L^1(\Omega,\nu_a)} &= \sum_{n=1}^\infty \left| \left(\frac{a}{1-a} \right)^{n-2} - 1 \right| \nu_a(K_n) \\ &= \left(\frac{1-a}{a} - 1 \right) a + \sum_{n=3}^\infty \left(1 - \left(\frac{a}{1-a} \right)^{n-2} \right) a(1-a)^{n-1} \\ &= 2 - 4a . \end{aligned}$$

The last equality can, of course, be obtained by summation of the infinite sum, but it can be obtained more easily by observing that, since

$$\int_\Omega \frac{dg^{-1}(\nu_a)}{d\nu_a}\, d\nu_a = 1\,,$$

the positive and negative parts of $\frac{dg^{-1}(\nu_a)}{d\nu_a} - 1$ have the same L^1-norm, and therefore

$$\sum_{n=3}^{\infty} \left(1 - \left(\frac{a}{1-a}\right)^{n-2}\right) a(1-a)^{n-1} = \left(\frac{1-a}{a} - 1\right) a = 1 - 2a\,.$$

The proof is complete. □

Note that the action of g on Ω can be considered as binary addition of $(1, 0, 0, \ldots)$ and $(x_1\,, x_2\,, \ldots) \in \Omega$ with carry over to the right. More generally, if $k \in \mathbb{N}$ has the binary representation

$$k = k_1 + k_2 2 + k_3 2^2 + \ldots + k_m 2^{m-1}\,,$$

then the action of g^k on Ω is given by binary addition of $(k_1\,, k_2\,, \ldots\,, k_m\,, 0, 0, \ldots)$ and $(x_1\,, x_2\,, \ldots)$ with carry over to the right. In particular, if $k = 2^n$, for some $n \in \mathbb{N}$, then $m = n + 1$ and

$$(k_1\,, k_2\,, \ldots\,, k_m\,, 0, 0, \ldots) = (\underbrace{0, 0, \ldots, 0}_{n \text{ times}}, 1, 0, \ldots)\,.$$

Hence $g^{2^n}((x_1, \ldots, x_n, x_{n+1}, x_{n+2}\ \ldots)) = (x_1, \ldots, x_n, g((x_{n+1}, x_{n+2}, \ldots)))$. This also implies that for all positive integers n we have $\sigma^{2^n} = (\otimes_{i=1}^n \mathrm{Id}) \otimes \sigma$.

Proposition 4.4. *For all* $\phi \in S_{nor}(L^\infty(\Omega, \nu_a))$*,*

$$\lim_{n\to\infty} \|\phi \circ \sigma^{2^n} - \phi\| = 2 - 4a\,. \tag{4.6}$$

Proof. Let $\phi \in S_{\mathrm{nor}}(L^\infty(\Omega, \nu_a))$. For all $n \geq 1$, let $A_n := \otimes_{k=1}^n l^\infty\{0, 1\} \otimes 1$ and set $\omega_n := \phi|_{A_n}$. Then

$$\phi \circ E_n = \omega_n \otimes \left(\otimes_{k=n+1}^\infty \mu_a\right)\,,$$

where E_n denotes the natural conditional expectation of $L^\infty(\Omega, \nu_a)$ onto A_n . By standard infinite tensor product theory, it follows that for all $\psi \in S_{\mathrm{nor}}(L^\infty(\Omega, \nu_a))$, we have

$$\lim_{n\to\infty} \|\psi \circ E_n - \psi\| = 0\,.$$

Hence it suffices to prove (4.6) for all $n \geq 1$ and all $\phi \in S_{\mathrm{nor}}(L^\infty(\Omega, \nu_a))$ of the form

$$\phi := \omega \otimes \left(\otimes_{k=n+1}^\infty \mu_a\right)\,, \quad \omega \in S_{\mathrm{nor}}(A_n)\,.$$

Fix $n \geq 1$ and consider $\phi \in S_{\mathrm{nor}}(L^\infty(\Omega, \nu_a))$ of this form. Then, for all $m \in \mathbb{N}$ with $m > n$, we have

$$\phi = \omega \otimes \underbrace{\mu_a \otimes \ldots \otimes \mu_a}_{m-n \text{ times}} \otimes \left(\otimes_{k=n+1}^\infty \mu_a\right)\,.$$

Since $\left(\otimes_{k=n+1}^\infty \mu_a\right) = \nu_a$, we conclude by previous considerations that

$$\phi \circ \sigma^{2^m} = \omega \otimes \underbrace{\mu_a \otimes \ldots \otimes \mu_a}_{m-n \text{ times}} \otimes (\nu_a \circ \sigma)\,.$$

By Lemma 4.3 we deduce that

$$\begin{aligned} \|\phi\circ\sigma^{2^m}-\phi\| &= \|\omega\otimes\underbrace{\mu_a\otimes\ldots\otimes\mu_a}_{m-n\text{ times}}\otimes(\nu_a\circ\sigma-\nu_a)\| \\ &= \|\nu_a\circ\sigma-\nu_a\| \\ &= 2-4a\,. \end{aligned}$$

Hence $\lim\limits_{m\to\infty}\|\phi\circ\sigma^{2^m}-\phi\|=2-4a$, and the proof is complete. □

Proof of Theorem 4.1. Let (g,Ω,ν_a) be the transformation space as above. In the following we will construct the associated flow under the constant ceiling function 1. This is a special case of Krieger's construction of the flow under a ceiling function ϕ given in [25], p. 46-47.

Set $\widetilde{\Omega}:=\Omega\times[0,1)$ and $\widetilde{\nu_a}=\nu_a\times dx$, where dx is the Lebesgue measure on $[0,1)$. Given $s\in\mathbb{R}$, define

$$\tilde{g}_s(x,y):=(g^n(x),t')\,,\quad x\in\Omega\,,0\le y<1\,, \tag{4.7}$$

where $s+y=n+y'$, with $n\in\mathbb{Z}$ and $0\le y'<1$.

Then $((\tilde{g}_s)_{s\in\mathbb{R}},\widetilde{\Omega},\widetilde{\nu_a})$ is a one-parameter group of Borel-measurable actions on $\widetilde{\Omega}$ which preserve the measure class of $\widetilde{\nu_a}$. Define now $\widetilde{\sigma}$ as the corresponding action on $L^\infty(\widetilde{\Omega},\widetilde{\nu_a})$, i.e., for all $s\in\mathbb{R}$ and all $f\in L^\infty(\widetilde{\Omega},\widetilde{\nu_a})$ let

$$(\widetilde{\sigma}_s(f))(z):=f(\widetilde{g}_s^{-1}(z))\,,\quad z=(x,y)\in\widetilde{\Omega}\,.$$

For simplicity of notation, set $\widetilde{\sigma}:=(\widetilde{\sigma}_s)_{s\in\mathbb{R}}$.

By the remark following Definition 3.1 in [40] (Vol. II, p. 385), $\widetilde{\sigma}$ is ergodic. Also, g is non-transitive, because every orbit $\{g^nx;n\in\mathbb{Z}\}$ is countable, so all g-orbits have 0 measure. Hence $\widetilde{g}:=(\widetilde{g}_s)_{s\in\mathbb{R}}$ is non-transitive, because $\widetilde{g}$-orbits are of the form $L=L_0\times[0,1)$, where $L\subseteq\Omega$ is an orbit for g, so

$$\widetilde{\nu_a}(L)=\nu_a(L_0)=0\,.$$

We conclude that $(L^\infty(\widetilde{\Omega},\widetilde{\nu_a}),\widetilde{\sigma})$ is an ergodic and non-transitive flow. Hence this flow is the "smooth flow of weights" of a unique (up to von Neumann algebras isomorphism) hyperfinite factor of type III_0 (see **1.4** in the Introduction).

We claim that for all $\phi\in S_{\mathrm{nor}}(L^\infty(\widetilde{\Omega},\widetilde{\nu_a}))$,

$$\lim_{n\to\infty}\|\phi\circ\widetilde{\sigma}_{2^n}-\phi\|=2-4a\,. \tag{4.8}$$

Indeed, let $k\in\mathbb{Z}$. By (4.7) we have

$$\widetilde{g}_k(x,y)=(g^k(x),y)\,,\quad x\in\Omega\,,0\le y<1\,.$$

Hence

$$\widetilde{\sigma}_k=\sigma^k\otimes\mathrm{Id}_{L^\infty([0,1)\,,dx)}\,. \tag{4.9}$$

In particular, we deduce for any $n\ge1$ that $\widetilde{\sigma}_{2^n}=\sigma^{2^n}\otimes\mathrm{Id}_{L^\infty([0,1)\,,dx)}$.

Given a positive integer m, denote by B_m the set of functions which are constant on $\left[\frac{i}{2^m}\,,\frac{i+1}{2^m}\right)$, for all $0\le i\le 2^m$. Note that

$$B_m=\mathrm{Span}\left\{q_i:=1_{\left[\frac{i}{2^m}\,,\frac{i+1}{2^m}\right)};0\le i<2^m\right\}\,.$$

Further, let $F_m : L^\infty([0,1)) \to B_m$ be the natural conditional expectation onto B_m preserving Lebesgue measure dx. Then the mapping $\widetilde{F}_m := \mathrm{Id}_{L^\infty(\Omega,\nu_a)} \otimes F_m$ is a conditional expectation of $L^\infty(\widetilde{\Omega}, \widetilde{\nu_a})$ onto $L^\infty(\Omega,\nu_a) \otimes B_m$ preserving $\widetilde{\nu}_a$. Clearly, for all normal states ϕ on $L^\infty(\widetilde{\Omega}, \widetilde{\nu}_a)$,

$$\lim_{n\to\infty} \|\phi \circ \widetilde{F}_m - \phi\| = 0\,.$$

Therefore, in order to prove (4.8) it suffices to consider states ϕ of the form

$$\phi := \omega \circ \widetilde{F}_m\,, \tag{4.10}$$

where $m \in \mathbb{N}$ (arbitrarily chosen) and ω is a normal state on $L^\infty(\Omega,\nu_a) \otimes B_m$.

Fix now $m \in \mathbb{N}$, and let ϕ be of the form (4.10). For $0 \le i < 2^m$, set

$$\omega_i(f) := \omega(f \otimes q_i)\,, \quad f \in L^\infty(\Omega,\nu_a)\,.$$

Then ω_i are positive linear functionals on $L^\infty(\Omega,\nu_a)$, $\sum\limits_{i=1}^{2^m-1} \omega_i(1) = 1$, and the ω_i's determine ω uniquely. For any $k \in \mathbb{Z}$ we obtain by (4.9) that

$$\|\phi \circ \widetilde{\sigma}_k - \phi\| = \sum_{i=0}^{2^m-1} \|\omega_i \circ \sigma^k - \omega_i\|\,.$$

By Proposition 4.4 we deduce that $\lim\limits_{n\to\infty} \|\chi \circ \sigma^{2^n} - \chi\| = (2-4a)\|\chi\|$, for every positive normal functional χ on $L^\infty(\Omega,\nu_a)$. Hence

$$\lim_{n\to\infty} \left(\sum_{i=0}^{2^m-1} \|\omega_i \circ \sigma^{2^n} - \omega_i\| \right) = (2-4a) \sum_{i=0}^{2^m-1} \|\omega_i\| = (2-4a) \sum_{i=0}^{2^m-1} \omega_i(1) = (2-4a)\,.$$

The proof is complete. □

In the following we will compute Connes' T-invariant for the hyperfinite type III$_0$-factors $\mathcal{M}^{(t)}$, $0 \le t < 2$ constructed above. Recall that, if $\mathcal{M}$ is a von Neumann algebra with a normal, faithful state ϕ, then Connes' T-invariant $T(\mathcal{M})$ defined by

$$T(\mathcal{M}) := \{\tau \in \mathbb{R};\, \sigma_\tau^\phi \in \mathrm{Int}(\mathcal{M})\}$$

is independent of ϕ, since for any normal, faithful state ψ on $\mathcal{M}$,

$$\sigma_\tau^\psi(x) = (D\psi : D\phi)_\tau \sigma_t^\phi(x) (D\psi : D\phi)_\tau^*\,, \quad \tau \in \mathbb{R}\,, x \in \mathcal{M}\,.$$

By [40], Vol. II, Chap. XII, if $\mathcal{M}$ has flow of weights $(Z(\mathcal{N}), \theta)$, then

$$T(\mathcal{M}) = \{\tau \in \mathbb{R};\, \exists\, u \in Z(\mathcal{N}), u \,\text{unitary such that}\, \theta_s(u) = e^{i\tau s}u\,, s \in \mathbb{R}\}\,.$$

Theorem 4.5. *For all* $0 \le t < 2$,

$$T(\mathcal{M}^{(t)}) = \left\{ \frac{2\pi k}{2^n}; k \in \mathbb{Z}\,, n \in \mathbb{N} \right\}\,. \tag{4.11}$$

The proof is based on the following intermediate results:

Lemma 4.6. *For* $0 \le t < 2$, *let* $(A^{(t)}, \theta^{(t)})$ *be the flow constructed in the proof of Theorem 4.1. Then for all* $f \in A^{(t)}$,

$$\lim_{n\to\infty} \|\theta_{2^n}^{(t)}(f) - f\|_2 = 0\,,$$

where the 2-norm is taken with respect to the measure $\widetilde{\nu}_a = \nu_a \otimes dx$, defined in the proof of Theorem 4.1 $(t = 2 - 4a)$.

Proof. Let Ω, ν_a and σ be as defined above (see (4.3)). Then, for all $f \in L^\infty(\Omega, \nu_a)$,

$$\|\sigma(f)\|_2^2 = \int_\Omega |h(g^{-1}x)|^2 d\nu_a(x) = \int_\Omega |h(x)|^2 dg^{-1}(\nu_a)(x) = \int_\Omega |h(x)|^2 \frac{dg^{-1}(\nu_a)}{d\nu_a}(x) d\nu_a(x).$$

By (4.5), we have

$$\frac{dg^{-1}(\nu_a)}{d\nu_a}(x) d\nu_a(x) = \sum_{n=1}^\infty \left(\frac{a}{1-a}\right)^{n-2} 1_{\Omega_n}.$$

Hence $\left\|\frac{dg^{-1}(\nu_a)}{d\nu_a}\right\|_\infty \leq \frac{1-a}{a}$, and therefore

$$\|\sigma(f)\|_2^2 \leq \frac{1-a}{a}\|f\|_2^2. \tag{4.12}$$

Since for all positive integers n, $\sigma^{2^n} = \mathrm{Id}_{A_n} \otimes \sigma^{(n)}$, where $\sigma^{(n)}$ is equal to σ shifted to $\otimes_{k=n+1}^\infty l^\infty\{0,1\}$, we get from (4.12) that

$$\|\sigma^{2^n}(f)\|_2 \leq \left(\frac{1-a}{a}\right)^{\frac{1}{2}} \|f\|_2, \quad f \in L^\infty(\Omega, \nu_a).$$

Moreover, since $\widetilde{\sigma}_{2^n} = \sigma^{2^n} \otimes \mathrm{Id}_{L^\infty([0,1),dx)}$, it also follows that

$$\|\widetilde{\sigma}_{2^n}(f)\|_2 \leq \left(\frac{1-a}{a}\right)^{\frac{1}{2}} \|f\|_2, \quad f \in L^\infty(\widetilde{\Omega}_a, \widetilde{\nu}_a).$$

Since $\cup_{n=1}^\infty A_n$ is dense in $L^\infty(\Omega_a, \nu_a)$, it follows that the increasing union $\cup_{n=1}^\infty (A_n \otimes L^(([0,1)))$ is dense in $L^\infty(\widetilde{\Omega}_a, \widetilde{\nu}_a)$. Let now $f \in L^\infty(\widetilde{\Omega}_a, \widetilde{\nu}_a)$ and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ and $g \in A_n \otimes L^([0,1))$ such that $\|f - g\|_2 < \varepsilon$. Since $\sigma^{2^n}|_{A_n} = \mathrm{Id}_{A_n}$, we have $\widetilde{\sigma}_{2^n}(g) = (\sigma^{2^n} \otimes \mathrm{Id}_{L^\infty([0,1))})(g) = g$. Therefore,

$$\begin{aligned}
\|\widetilde{\sigma}_{2^n}(f) - f\|_2 &\leq \|\widetilde{\sigma}_{2^n}(f-g)\|_2 + \|g - f\|_2 \\
&\leq \left(\left(\frac{1-a}{a}\right)^{\frac{1}{2}} + 1\right) \|f - g\|_2 \\
&\leq \left(\left(\frac{1-a}{a}\right)^{\frac{1}{2}} + 1\right) \varepsilon.
\end{aligned}$$

This shows that $\lim_{n\to\infty} \|\widetilde{\sigma}_{2^n}(f) - f\|_2 = 0$, and the proof is complete. □

Lemma 4.7. *Let $\tau \in \mathbb{R}$ such that $\lim_{n\to\infty} e^{i\tau 2^n} = 1$. Then $\tau \in \left\{\frac{2\pi k}{2^n}; k \in \mathbb{Z}, n \in \mathbb{N}\right\}$.*

Proof. Choose $n_0 \in \mathbb{N}$ such that

$$|e^{i\tau 2^n} - 1| < 1, \quad n \geq n_0. \tag{4.13}$$

Assume further that $e^{i\tau 2^{n_0}} \neq 1$. Then by (4.13), it follows that $e^{i\tau 2^{n_0}} = e^{iv}$, for some $v \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \setminus \{0\}$. Hence there exists $k \in \mathbb{N}$ such that $\frac{\pi}{3}2^{-k} \leq |v| < \frac{\pi}{3}2^{1-k}$. But then $e^{i\tau 2^{n_0+k}} = e^{iv2^k}$ and $\frac{\pi}{3} \leq 2^k v < \frac{2\pi}{3}$. Then $|e^{i\tau 2^{n_0+k}} - 1| \geq 1$, which contradicts (4.13). Hence $e^{i\tau 2^{n_0}} = 1$. This yields the conclusion. □

Proof of Theorem 4.5: Fix $0 \leq t < 2$. Let $\tau \in T(\mathcal{M}_t)$. Then there exists a unitary $u \in L^(\Omega, \nu_a)$ such that $\theta_s(u) = e^{i\tau s}(u)$, for all $s \in \mathbb{R}$. Since

$$\theta_{2^n}(u) \xrightarrow{\text{s.o.}} u\,, \quad \text{as } n \to \infty\,,$$

it follows by Lemma 4.6 that $\lim\limits_{n\to\infty} e^{i\tau 2^n} = 1$. By Lemma 4.7, we conclude that $\tau \in \left\{\frac{2\pi k}{2^n}; k \in \mathbb{Z}, n \in \mathbb{N}\right\}$.

Conversely, let $\tau \in \left\{\frac{2\pi k}{2^n}; k, n \in \mathbb{N}\right\}$. Then there exists $n \in \mathbb{N}$ such that $e^{i\tau 2^n} = 1$. Put, as before, $A_n = \otimes_{k=1}^n l^\infty\{0,1\} \otimes 1$. Note that $\dim(A_n) = 2^n$. For $0 \leq j \leq 2^{n-1}$, put

$$G_j := \{(k_1^{(j)}, k_2^{(j)}, \ldots, k_n^{(j)}, x_{n+1}, x_{n+2}, \ldots); x_j \in \{0,1\} \text{ for } j \geq n+1\}\,,$$

where $j = k_1^{(j)} + k_2^{(j)}2 + \ldots + k_n^{(j)}2^{n-1}$ is the unique binary representation of j ($k_i^j \in \{0,1\}$). Using again the fact that the action of g on Ω is given by the binary addition of $(1,0,0,\ldots)$ and $(x_1, x_2, \ldots) \in \Omega$ with carry over to the right, it follows that

$$g(G_j) = G_{j+1}\,, \quad 0 \leq j \leq 2^{n-1}\,,$$

where $G_{2^n} = G_0$. Hence $p_j := 1_{G_j}$ $(0 \leq j \leq 2^{n-1})$ are orthogonal projections in A_n with sum equal to 1, satisfying $\sigma(p_j) = p_{j+1}$ $(0 \leq j \leq 2^{n-1})$, where indices are calculated modulo 2^n. Set now

$$u_0 := p_0 + e^{-i\tau}p_1 + \ldots + e^{-i(2^n-1)\tau}p_{2^n-1}\,.$$

Then $u_0 \in A_n$ is unitary and satisfies $\sigma(u_0) = e^{i\tau}u_0$. Next set

$$u(x,y) = u_0(x)e^{-i\tau y}\,, \quad x \in \Omega\,, y \in [0,1)\,.$$

Then u is a unitary in $L^\infty(\widetilde{\Omega}_a, \widetilde{\nu}_a)$. We will check that

$$\widetilde{\theta}_s u = e^{i\tau s}u\,, \quad s \in \mathbb{R}\,, \tag{4.14}$$

which implies that $\tau \in T(\mathcal{M}^{(t)})$. Indeed, for any $s \in \mathbb{R}$,

$$(\widetilde{\theta}_{-s}u)(x,y) = u(\widetilde{g}_s(x,y)) = u(g^n x, y')\,,$$

where $s + y = n + y'$ (integer part and fractional part, respectively, of $s+y$). Hence

$$\begin{aligned}(\widetilde{\theta}_{-s}u)(x,y) &= u_0(g^n x)e^{-i\tau y'} = (\sigma^{-n}u_0)(x)e^{-i\tau y'} = e^{-in\tau}u_0(x)e^{-i\tau(s+y-n)} \\ &= e^{-i\tau s}u_u(x)e^{-i\tau y} \\ &= e^{-i\tau s}u(x,y)\,.\end{aligned}$$

Replace now s by $-s$ to obtain (4.14). This completes the proof of Theorem 4.5. □

References

[1] E. Christensen, A. Sinclair, *Completely bounded isomorophisms of injective von Neumann algebras*, Proc. Edinburgh Math. Soc. **32** (1989), 317-327.

[2] E. Christensen, A. Sinclair, *On von Neumann algebras which are complemented subspaces of $\mathcal{B}(H)$*, J. Funct. Analysis **122** (1994), 91-102.

[3] A. Connes, *Une classification des facteurs de type III*, Ann. Scient. Éc. Norm. Sup., 4e série, tome 6 (1973), 133-252.

[4] A. Connes, *Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$.*, Ann. Math. **104** (1976), 73-115.

[5] A. Connes, *Factors of type III_1, property L'_λ and closure of inner automorphisms*, J. Op. Theory **14** (1985), no. 1, 189–211.

[6] A. Connes, *Cyclic cohomology and the transverse fundamental class of a foliation*, Geom. Methods in Op. Alg. (Kyoto, 1983), 52-144, Pitman Res. Notes Math. Ser., **133**, Longman Sci. Tech., Harlow, 1986.

[7] A. Connes, *On hyperfinite factors of type III_0 and Krieger's factors*, J. Funct. Analysis Vol. 18, No. **3** (1995), 318-327.

[8] A. Connes, U. Haagerup, E. Størmer, *Diameters of state spaces of type III factors*, Operator algebras and their connection with topology and ergodic theory, Lect. Notes Math. **1132**, 91-116, Springer Verlag, 1985.

[9] A. Connes, E. Størmer, *Homogeneity of the state space of factors of type III_1*, J. Funct. Analysis **28** (1978), 187-196.

[10] A. Connes, M. Takesaki, The flow of weights on factors of type III, Tôhoku Math. J. **29** (1977), 473-575.

[11] E. Effros and Z.-J. Ruan, *Operator Spaces*, London Math. Soc. Monographs New Series **23**, Oxford University Press, 2000.

[12] U. Haagerup, *A new proof of the equivalence of injectivity and hyperfiniteness for factors on a separable Hilbert space*, J. Funct. Analysis, Vol. 62, No. 2 (1985), 160-201.

[13] U. Haagerup, *Injectivity and decomposition of completely positive maps*, Lect. Notes in Math. **1132** (1985), 170-222.

[14] U. Haagerup, *Connes' bicentralizer problem and uniqueness of the injective factor of type III_1*, Acta Math. **158** (1987), no.1-2, 95-148.

[15] U. Haagerup, M. Musat, *On the best constants in noncommutative Khintchine-type inequalities*, J. Funct. Analysis, to appear.

[16] U. Haagerup, H. P. Rosenthal and F. A. Sukochev, *Banach Embedding Properties of Non-commutative L^p-spaces*, Memoirs of the Amer. Math. Soc., Vol. **163**, No. 776, May 2003.

[17] U. Haagerup, E. Størmer, *Equivalence of normal states on von Neumann algebras and the flow of weights*, Adv. in Math. Vol. 83, No. 2 (1990), 180-262.

[18] U. Haagerup, C. Winsløw, *The Effros-Marechal topology in the space of von Neumann algebras, II*, J. Funct. Analysis **171** (2000), 401-431.

[19] M. Junge, *Embedding of the operator space OH and the logarithmic "little Grothendieck inequality"*, Invent. Math. **161** (2005), 389-406.

[20] M. Junge, *Operator spaces and Araki-Woods factors-A quantum probabilistic approach*, Int. Math. Res. Pap. 2006, Art. ID 76978, 87 pp.

[21] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras I, II*, Academic Press, 1986.

[22] G.G. Kasparov, *Hilbert C^*-modules: Theorems of Stinespring and Voiculescu*, J. Op. Theory **4** (1980), 133-150.

[23] E. Kirchberg, *On nonsemisplit extensions, tensor products and exactness of group C^*-algebras*, Invent. Math **112** (1993), no. 3, 449-489.

[24] W. Krieger, *On a class of hyperfinite factors that arise from null-recurrent Markov chains*, J. Funct. Analysis **7** (1971), 27-42.

[25] W. Krieger, *On ergodic flows and the isomorphism of factors*, Math. Ann. **223** (1976), 19-70.

[26] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, Sequence Spaces*, Ergebnisse, Vol. **92**, Springer-Verlag, 1992.

[27] T. Oikhberg, H. P. Rosenthal and E. Størmer, *A predual characterization of semi-finite von Neumann algebras*, *Advances in Quantum dynamics* (South Haley, MA, 2002), 243-245, Contemp. Math. **335**, Amer. Math. Soc., Providence, RI, 2003.

[28] V. I. Paulsen, *Every completely polynomially bounded operator is similar to a contraction*, J. Funct. Analysis **55** (1984), 1-17.

[29] V. I. Paulsen, *Completely Bounded Maps and Dilations*, Pitman Res. Notes, Longman Sci. Tech., London, 1986.

[30] A. Pelczynski, *Projections in certain Banach spaces*, Studia Math. **29** (1969), 209-227.

[31] G. Pisier, *The Operator Hilbert Space OH, Complex Interpolation and Tensor Norms*, Mem. Amer. Math. Soc. Number **585**, Vol. 122, Providence, RI, 1996.

[32] G. Pisier, *An Introduction to the Theory of Operator Spaces*, London Math. Soc. Lect. Notes Series **294**, Cambridge University Press, Cambridge 2003.

[33] G. Pisier, *Completely bounded maps into certain Hilbertian operator spaces*, Int. Math. Res. Not. (2004), no.74, 3983-4018.

[34] G. Pisier, *The operator Hilbert space OH and type III von Neumann algebras*, Bull. London Math. Soc. **36** (2004), no.4, 455-459.

[35] S. Sakai, *On topological properties of W^*-algebras*, Proc. Japan Acad. **33** (1957), 439-444.

[36] S. Sakai, *C^*-algebras and W^*-algebras*, Springer Verlag Berlin, Heidelberg, 1998.

[37] J. Schwartz, *Two finite, non-hyperfinite, non-isomorphic factors*, Comm. Pure Appl. Math. **16** (1963), 19-26.

[38] M. Takesaki: *Conditional expectations in von Neumann algebras*, J. Funct. Analysis **9** (1972), 306-321.

[39] M. Takesaki: *Duality for crossed products and the structure of von Neumann algebras of type III*, Acta Mathematica **131** (1973), 249-310.

[40] M. Takesaki, *Theory of Operator Algebras I, II, III*, Springer-Verlag, New-Yord, 1979.

[41] M. Terp, *L^p-spaces associated with von Neumann algebras I and II*, Copenhagen Univ., 1981.

[42] J. Tomiyama, *On the projections of norm one in W^*-algebras III*, Tohoku Math. J. (**2**) 11 (1959), 125-129.

[43] D.V. Voiculescu, *Circular and semicircular elements and free product factors*, Operator algebras, Unitary Representations, Enveloping Algebras and Invariant Theory, Progress in Mathematics, vol. 92, Birkhäuser, Boston, 1990.

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