

Proper metrics on locally compact groups, and proper affine isometric actions on Banach spaces.

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Abstract

In this article it is proved, that every locally compact, second countable group has a left invariant metric d , which generates the topology on G , and which is proper, ie. every closed d -bounded set in G is compact. Moreover, we obtain the following extension of a result due to N. Brown and E. Guentner [BG05]: Every locally compact, second countable G admits a proper affine action on the reflexive and strictly convex Banach space

$$\bigoplus_{n=1}^{\infty} L^{2n}(G, d\mu),$$

where the direct sum is taken in the l^2 -sense.

1 Introduction

We consider a special class of metrics on second countable, locally compact groups, namely proper left invariant metrics which generate the given topology on G , and we will denote such a metric a **plig** metric.

It is fairly easy to show, that if a locally compact group G admits a **plig** metric, then G is second countable. Moreover, any two **plig** metrics d_1 and d_2 on G are coarsely equivalent, ie. the identity map on G defines a coarse equivalence between (G, d_1) and (G, d_2) in the sense of [Roe95].

In [LMR00, pp. 14-16] Lubotzky, Moser and Ragunathan shows, that every compactly generated second countable group has a **plig** metric. Moreover, Tu showed in [Tu01, lemma 2.1], that every countable discrete group has a **plig** metric.

The main result of this paper is that every locally compact, second countable group G admits a **plig** metric. Moreover a **plig** metric d can be chosen, such that the d -balls have exponentially controlled growth in the sense that

$$\mu(B_d(e, n)) \leq \beta \cdot e^{\alpha n}, \quad n \in \mathbb{N},$$

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for suitable constants α and β . Here μ denotes the Haar measure on G .

In [BG05], Brown and Guentner proved that for every countable discrete group Γ there exists a sequence of numbers $p_n \in (1, \infty)$ converging to ∞ for $n \rightarrow \infty$, such that Γ has a proper affine action on the reflexive and strictly convex Banach space

$$X_0 = \bigoplus_{n=1}^{\infty} l^{p_n}(\Gamma),$$

where the direct sum is in the l^2 -sense.

By similar methods, we prove that every second countable group has a proper affine action on the reflexive and strictly convex Banach space

$$X = \bigoplus_{n=1}^{\infty} L^{2n}(G, d\mu),$$

where the sum is taken in the l^2 -sense. However, in order to prove, that the exponents $(p_n)_{n=1}^{\infty}$ can be chosen to be $p_n = 2n$, it is essential to work with a **plig** metric on G for which the d -balls have exponentially controlled growth. As a corollary we obtain, that a second countable locally compact group G has a uniform embedding in the above Banach space X

Note, that the Banach spaces X and X_0 above are not uniformly convex. Kasparov and Yu have recently proved, that the Novikov conjecture holds for every discrete countable group, which has a uniform embedding in a uniformly convex Banach space (cf. [KY05]).

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2 Coarse geometry and plig metrics

First, let us fix notation and basic definitions:

Definition 2.1. Let G be a topological group, ie. a group equipped with a Hausdorff topology, such that the map $(x, y) \rightarrow x \cdot y^{-1}$ is continuous. If G is equipped with a metric d , we put $B_d(x, R) = \{y \in G : d(x, y) < R\}$ and $D_d(x, R) = \{y \in G : d(x, y) \leq R\}$.

- G is *locally compact* if every $x \in G$ has a relative compact neighbourhood.
- We say that the metric d *induces the topology* of G if the topology generated by the metric τ_d coincides with the original topology τ .
- The metric d is said to be *left invariant* if for all $g, x, y \in G$ we have that

$$d(x, y) = d(g \cdot x, g \cdot y).$$

- Following [Roe95, p. 10], a metric space is called *proper* if all closed bounded sets are compact. When G is a group, this reduces by the left invariance of the group metric to the requirement, that for every $M > 0$ all the closed balls

$$D(e, M) = \{h \in G : d(e, h) \leq M\}$$

are compact.

We will work with a special class of metrics on locally compact, second countable groups, which we define here:

Definition 2.2. Let G be a topological group. A **plig** metric d on G is a metric on G , which is **Proper**, **Left Invariant** and **Generates the topology**.

M. Gromov started investigating asymptotic invariants of groups, particularly fundamental groups of manifolds.

This lead to the development of coarse geometry - or large scale geometry . Coarse geometry studies global properties of metric spaces, neglecting small (bounded) variations of these spaces. The properties and invariants in coarse geometry are treated in the limit at infinity, as opposed to the traditional world of topology, which focuses on the local structure of the space.

Definition 2.3 ([Roe95]). Let (X, d_X) and (Y, d_Y) be metric spaces.

- A map $f : X \rightarrow Y$ is called *uniformly expansive* if

$$\forall R > 0 \exists S > 0 \quad \text{such that} \quad d_X(x, x') \leq R \Rightarrow d_Y(f(x), f(x')) \leq S.$$

- A map $f : X \rightarrow Y$ is called *metrically proper* if

$$\forall B \subset Y \quad B \text{ is bounded} \Rightarrow f^{-1}(B) \subset X \text{ is bounded.}$$

- A map $f : X \rightarrow Y$ will be called a *coarse map* if it is both metrically proper and uniformly expansive.
- Two coarse maps $h_0, h_1 : X \rightarrow Y$ are said to be *coarsely equivalent* when

$$\exists C > 0 \forall x \in X : \quad d_Y(h_0(x), h_1(x)) < C.$$

We denote the relation of coarse equivalence by ' \sim_c '.

- The spaces X and Y are said to be *coarsely equivalent* if there exist coarse maps $f : X \rightarrow Y$, and $g : Y \rightarrow X$ such that

$$f \circ g \sim_c \text{Id}_Y \quad \text{and} \quad g \circ f \sim_c \text{Id}_X.$$

- The *coarse structure* of X means the coarse equivalence class of the given metric, $[d_X]_{\sim}$.

Example 2.4. It is well known that

$$(0, 1) \sim_h \mathbb{R} \quad \text{and} \quad \mathbb{Z} \not\sim_h \mathbb{R},$$

where \sim_h means that the two spaces in question are homeomorphic.

It is easy to see, that in the coarse case we have:

$$(0, 1) \not\sim_c \mathbb{R} \quad \text{and} \quad \mathbb{Z} \sim_c \mathbb{R}.$$

The reason for working with coarse geometry is, that many geometric group theory properties are coarse invariants. Examples of coarse invariants are: property A [HR00], asymptotic dimension [BD01] and change of generators for a finitely generated group.

M. Gromov has suggested in [Gro95] to solve the Novikov conjecture by considering classes of groups admitting uniform embeddings into Banach spaces with various restraints.

G. Yu proved in [Yu00], that the Novikov Conjecture is true for a space that admits a uniform embedding into a Hilbert space. This result was strengthened in [KY05], where it is shown that the Novikov Conjecture is true for a space that admits a uniform embedding into a uniformly convex Banach space.

Together with the fact, that exact groups admit a uniform embedding into a Hilbert space, see [GK02], makes uniform embedding extremely interesting to study.

The idea of a uniform embedding is to map a metric space (X, d_X) into a metric space (Y, d_Y) in such a way that the large-scale geometry of X is preserved.

This means for instance that we are not allowed to “squeeze” unbounded segments into a point, and we are not allowed to “blow up” bounded segments to unbounded - the limits at infinity must be preserved.

Definition 2.5. A map $f : X \rightarrow Y$ will be called a *uniform embedding* if there exist non-decreasing functions $\rho_-, \rho_+ : [0, \infty) \rightarrow [0, \infty)$ such that

$$\lim_{t \rightarrow \infty} \rho_-(t) = \lim_{t \rightarrow \infty} \rho_+(t) = \infty$$

and

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y)) \quad (1)$$

When $f : X \rightarrow Y$ is a coarse map, we will denote a map $\phi : f(X) \rightarrow X$ a section of f if it fullfills that

$$f \circ \phi = id_{f(X)}.$$

The set of sections of the map f will be denoted by $\text{Inv}(f)$.

Example 2.6. It is easy to see that we can not have a uniform embedding of the free group F_2 in any \mathbb{R}^n . On the other hand it was shown in [Haa79], [dlHV89, p. 63] that F_2 has a uniform embedding in the infinite dimensional Hilbert space \mathbb{H} .

The following is a folklore lemma, as different definitions of uniform embedding are used in the literature, see [Gro93], [HR00] and [GK02], see [Prz05] for a detailed proof.

Lemma 2.7. *Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \rightarrow Y$ a map. The following are equivalent:*

1. $f : X \rightarrow Y$ is a uniform embedding.

2. (Guentner and Kaminker version) f is uniformly expansive, and

$$\forall_{S>0} \exists_{R>0} \quad d_X(x, y) \geq R \Rightarrow d_Y(f(x), f(y)) \geq S. \quad (2)$$

3. (Higson and Roe version) The map f is uniformly expansive, and so is any $\phi \in \text{Inv}(f)$.

4. f is a coarse equivalence between X and $f(X)$ and any section $\phi \in \text{Inv}(f)$ is a coarse map.

Theorem 2.8. *Let G be a locally compact, second countable group. Assume that d_1 and d_2 are **plig** metrics on G .*

Then the identity map

$$\text{Id} : (G, d_1) \rightarrow (G, d_2)$$

is a coarse equivalence.

Proof. To establish the coarse equivalence of the metric spaces, it is by the 3rd case of lemma (2.7), enough to show that $\text{Id} : (G, d_1) \rightarrow (G, d_2)$ and $\text{Id} : (G, d_2) \rightarrow (G, d_1)$ are both uniformly expansive, that is:

$$\begin{aligned} \forall_{R>0} \exists_{S>0} \quad d_1(x, y) \leq R &\Rightarrow d_2(x, y) \leq S. \\ \forall_{R>0} \exists_{S>0} \quad d_2(x, y) \leq R &\Rightarrow d_1(x, y) \leq S. \end{aligned}$$

Since both d_1 and d_2 generates the topology of G , the identity map

$$\text{Id} : (G, d_1) \rightarrow (G, d_2)$$

is a homeomorphism, and the maps $\phi_i : (G, d_i) \rightarrow \mathbb{R}_+$ given by

$$\phi_i(x) = d_i(e, x), \quad x \in G$$

are continuous.

Let $R > 0$. Since the closed d_1 -ball

$$D_1(x, R) = \{x \in G : d_1(x, e) \leq R\}$$

is compact, ϕ_2 attains a maximum values S on $D_1(x, R)$. Moreover

$$d_1(e, x) \leq R \Rightarrow d_2(e, x) \leq S.$$

Hence, by the left invariance of d_1 and d_2

$$d_1(x, y) \leq R \Rightarrow d_2(x, y) \leq S, \quad \forall x, y \in G.$$

Since R was arbitrary, this shows the uniform expansiveness of Id .

By reversing the roles of d_1 and d_2 in the last argument, we also obtain that the inverse map is uniformly expansive. □

Remark 2.9. In the special case of a countable discrete group, Theorem (2.8) was proved by J. Tu (cf. the uniqueness part of [Tu01, lemma 2.1]).

3 Bounded geometry on locally compact groups.

The purpose of this section is to show, that a **plig** metric implies bounded geometry on a locally compact, second countable group. Let us begin by defining bounded geometry, which is a concept from the world of coarse geometry.

Definition 3.1. Following [Roe95, p. 13], the metric space (G, d) is said to have *bounded geometry* if it is coarsely equivalent to a discrete space (Q, d_Q) , where for every $M > 0$ there exists constants Γ_M such that

$$\forall q \in Q \quad |D(q, M)| = |\{p \in Q : d(q, p) \leq M\}| \leq \Gamma_M.$$

Note that (G, d) is a finitely generated group, and d a word length metric. Then G has bounded geometry.

Definition 3.2. Let (Y, d) be a metric space. We say that a discrete space $X \subset Y$ is a *coarse lattice*, if

$$\exists \lambda \in \mathbb{R} \forall y \in Y \exists x_y \in X \quad d(x_y, y) \leq \lambda. \quad (3)$$

Lemma 3.3. *Let G be a locally compact group, and let d be a **plig** metric on G . Then (G, d) is second countable and has bounded geometry.*

Proof. First we observe that the conditions on (G, d) imply, that G is second countable. We can write G as a union of compact metric spaces, namely:

$$G = \bigcup_{n=1}^{\infty} D_d(e, n),$$

where we have that each $D_d(e, n)$ is a compact metric space, since d is proper.

Now, since every compact metric space is separable, see [Eng89, Theorem 4.1.17, page 297], we can conclude that every $D_d(e, n)$ has a countable dense subset. Hence G is separable, and since for any metric space second countability is equivalent to separability, see [Wil70, Theorem 16.11, page 112], it follows that G is second countable.

We will now show that G has bounded geometry by constructing a countable coarse lattice $X \subset G$ such that X has bounded geometry.

Let $X = \{x_i\}_{i \in I}$ be a maximal family of elements from G , such that $d(x_i, x_j) \geq 1, i \neq j$. Since G is separable, the index set I is at most countable.

By maximality of X , we have that

$$G = \bigcup_{i \in I} B_d(x_i, 1).$$

If we had that $|I| < \infty$, then $G = \bigcup_{i \in I} \overline{B_d(x_i, 1)}$ would be a finite union of compact sets, and therefore compact, and hence bounded. Therefore G is coarsely equivalent to $\{\bullet\}$ if I is finite, and therefore G has bounded geometry.

Let us therefore assume, that $|I| = \infty$, and we can use \mathbb{N} instead of the index set I . Let us construct the coarse equivalences between X and G :

Start by setting

$$\begin{aligned} A_1 &= B_d(x_1, 1) \\ A_2 &= B_d(x_2, 1) \setminus A_1 \\ &\dots \\ A_n &= B_d(x_n, 1) \setminus \bigcup_{i=1}^{n-1} A_i \end{aligned}$$

We have now that the family $\{A_n\}_{n=1}^{\infty}$ in G fulfills the following:

$$\begin{cases} A_n \cap A_m = \emptyset & \text{if } m \neq n \\ x_n \in A_n \subset B_d(x_n, 1) & \text{for all } n \in \mathbb{N} \\ \bigcup_{n=1}^{\infty} A_n = G \end{cases}$$

Now equip the set $X = \{x_i\}_{i \in I}$ with the metric inherited from (G, d) . Define:

$$\begin{aligned} \phi : G &\rightarrow X, & \text{by } \phi(x) = x_n & \text{when } x \in A_n \\ \psi : X &\rightarrow G, & \text{by } \psi(x_n) = x_n & \text{for } n \in \mathbb{N} \end{aligned}$$

Remark that both ψ and ϕ are coarse maps. We have from the construction of ϕ and ψ that

$$\phi \circ \psi = \text{Id}_X$$

and

$$\forall x \in G \quad d(\psi \circ \phi(x), x) \leq 1$$

and therefore we see, that the spaces X and G are coarsely equivalent.

Now, we have to show that the set X indeed has bounded geometry. Let $M > 0$ be given, and let us look at the disks of radius M in X :

$$D_X(q, M) = \{x_n \in X : d(q, x_n) \leq M, \quad n \in \mathbb{N}\}. \quad (4)$$

For every $M > 0$ we need to find a constant Γ_M such that

$$\sup_{q \in X} |D_X(q, M)| \leq \Gamma_M.$$

Since $d(x_n, x_m) \geq 1$ when $n \neq m$, the balls $B(x_n, \frac{1}{2})$ are disjoint. Moreover, we have that

$$\bigcup_{x_n \in D_X(q, M)} B(x_n, \frac{1}{2}) \subset B(q, M + \frac{1}{2}).$$

Let μ denote the Haar measure on G , then we have that

$$\sum_{x_n \in D_X(q, M)} \mu(B(x_n, \frac{1}{2})) \leq \mu(B(q, M + \frac{1}{2}))$$

Since the number of terms in the sum above is equal to $|D(q, M)|$, we get by the left invariance of the Haar measure, that

$$|D(q, M)| \cdot \mu(B(e, \frac{1}{2})) \leq \mu(B(e, M + \frac{1}{2})).$$

Hence

$$\sup_{q \in X} |D(q, M)| \leq \frac{\mu(B(e, M + \frac{1}{2}))}{\mu(B(e, \frac{1}{2}))} < \infty.$$

Therefore we see that (X, d) has bounded geometry, and we can conclude that (G, d) also has bounded geometry. \square

Example 3.4. Remark, that lemma (3.3) is not true for general metric spaces, as we can find an example of a metric space X that is proper, but does not have bounded geometry:

Consider the triple (D_n, d_n, x_n) , where D_n is the discrete space with n points, equipped with the discrete metric:

$$d_n(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases},$$

and where $x_n \in D_n$.

Let $X = \coprod_{n \in \mathbb{N}} D_n$, and equip X with the following metric:

$$d(z, y) = d_{j(z)}(z, x_{j(z)}) + |j(z) - j(y)| + d_{j(y)}(y, x_{j(y)}) \text{ where } j(x) = n \Leftrightarrow x \in D_n.$$

It is easy to see that X is proper, but on the other hand X does not have bounded geometry since the number of elements in $D_d(x_n, 1)$ tends to infinity for $n \rightarrow \infty$.

4 Construction of a plig metric on G .

In this and the section, we will prove that every locally compact, second countable group has a **plig** metric d . Together with theorem (2.8) we have therefore that a locally compact, second countable group has exactly one coarse equivalence class of **plig** metrics.

Definition 4.1. We will call a map $l : G \rightarrow \mathbb{R}_+$ for a *length function*, if it satisfies the following conditions (5)–(7).

$$\forall_{g \in G} \quad l(g) = 0 \iff g = e. \quad (5)$$

$$\forall_{g \in G} \quad l(g) = l(g^{-1}). \quad (6)$$

$$\forall_{g, h \in G} \quad l(g \cdot h) \leq l(g) + l(h). \quad (7)$$

Lemma 4.2. 1. If $l : G \rightarrow \mathbb{R}_+$ is a length function, then

$$d(x, y) = l(y^{-1}x), \quad x, y \in G \quad (8)$$

is a left invariant metric on G .

2. Conversely, if $d : G \times G \rightarrow \mathbb{R}_+$ is a left invariant metric on G , then

$$l(x) = d(x, e), \quad x \in G$$

is a length function on G , and d is the metric obtained from l by (8).

Moreover, if l is a length function on G and $d(x, y) = l(y^{-1}x)$ is the associated left invariant metric, then d generates the topology on G if and only if

$$\{l^{-1}[0, r) : r > 0\} \text{ is a basis for the neighborhoods of } e \in G_0. \quad (9)$$

Moreover, d is proper if and only if:

$$\forall_{r > 0} \quad l^{-1}([0, r]) \text{ is compact.} \quad (10)$$

Proof. The proof is elementary, and will be left to the reader. \square

Remark 4.3. If l is a length function on G satisfying (9), then the associated metric

$$d(x, y) = l(y^{-1}x), \quad x, y \in G$$

generates the given topology on G . Therefore

$$l^{-1}([0, r]) = \{x \in G : d(x, e) \leq r\}$$

is closed in G . Hence if we assume (9), then (10) is equivalent to that $l^{-1}([0, r])$ is relatively compact for all $r > 0$, which again is equivalent to that

$$B_d(e, n) = l^{-1}([0, n])$$

is relatively compact for all $n \in \mathbb{N}$.

Lemma 4.4. *Let G be a locally compact, second countable group. Assume that the topology on G is generated by a left invariant metric δ , for which*

$$U = B_\delta(e, 1) \text{ is relatively compact} \quad (11)$$

$$G = \bigcup_{k=1}^{\infty} U^k \quad (12)$$

Then G admits a left invariant metric d generating the topology on G , for which $B_d(e, 1) = U$, and

$$\forall_{n \in \mathbb{N}} \quad B(e, n) \subset B(e, 1)^{2n-1}. \quad (13)$$

*Moreover d is a **plig** metric on G .*

Proof. Let $l_\delta(g) = \delta(g, e)$ be the length function associated to δ , and define a function $l : G \rightarrow \mathbb{R}_+$ by

$$l(g) = \inf \left\{ \sum_{i=1}^k l_\delta(g_i) : g = g_1 \cdot \dots \cdot g_k, \text{ where } g_i \in U, \quad i = 1, \dots, k, k \in \mathbb{N} \right\} \quad (14)$$

Clearly, $l(g) \geq 0$ for all $g \in G$, and by the assumption (12), we have that $l(g) < \infty$. Moreover, one checks easily that

$$l(g \cdot h) \leq l(g) + l(h), \quad g, h \in G. \quad (15)$$

Since $U = U^{-1}$, we have also that

$$l(g^{-1}) = l(g), \quad g \in G. \quad (16)$$

Moreover

$$l(g) \geq l_\delta(g), \quad g \in G \quad (17)$$

and

$$l(g) = l_\delta(g), \quad g \in U \quad (18)$$

By (17) and (18), we have that

$$l(g) = 0 \Leftrightarrow g = e.$$

Hence by lemma (4.2), l is a length function on G . Let

$$d(g, h) = l(g^{-1} \cdot h), \quad g, h \in G \quad (19)$$

be the associated left invariant metric on G . By (18) and (19) we have that

$$B_d(x, r) = B_\delta(x, r), \quad r \leq 1 \quad (20)$$

for all $g \in G$. Hence d generates the same topology on G as δ does. Moreover, we have that

$$B_d(e, 1) = B_\delta(e, 1) = U. \quad (21)$$

We next turn to the proof of (13). Let $n \in \mathbb{N}$, and let $g \in B_d(e, n)$. Then there exists a $k \in \mathbb{N}$, and $g_1, \dots, g_k \in U$ such that

$$g = g_1 \dots g_k \quad \text{and} \quad \sum_{i=1}^k l_\delta(g_i) < n.$$

Further we may assume, that $k \in \mathbb{N}$ is minimal among all the numbers for which such a representation is possible.

We claim, that in this case

$$l_\delta(g_i) + l_\delta(g_{i+1}) \geq 1, \quad i = 1, \dots, k-1. \quad (22)$$

Assume namely, that $l_\delta(g_i) + l_\delta(g_{i+1}) < 1$ for some i , where $1 \leq i \leq k-1$, then we have

$$l_\delta(g_i \cdot g_{i+1}) \leq l_\delta(g_i) + l_\delta(g_{i+1}) < 1, \quad (23)$$

and thus we have that $g_i \cdot g_{i+1} \in U$. Hence g can be written as a product of $k-1$ elements from U :

$$g = g_1 \dots g_i \cdot g_{i-1}(g_i \cdot g_{i+1})g_{i+2} \dots g_k \quad (24)$$

for which

$$\sum_{j=1}^{i-1} l_\delta(g_j) + l_\delta((g_i \cdot g_{i+1})) + \sum_{j=i+2}^k l_\delta(g_j) \leq \sum_{j=1}^k l_\delta(g_j) < n. \quad (25)$$

This contradicts the minimality of k , and hence (22) must hold.

Let $\lfloor r \rfloor$ as usual denote the largest integer such that

$$\lfloor r \rfloor \leq r$$

From (22) we get that

$$\lfloor \frac{k}{2} \rfloor \leq \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (l_\delta(g_{2j-1}) + l_\delta(g_{2j})) \leq \sum_{j=1}^k l_\delta(g_j) < n \quad (26)$$

Since both $\lfloor \frac{k}{2} \rfloor$ and n are integers, we have that $\lfloor \frac{k}{2} \rfloor \leq n-1$, and therefore

$$k \leq 2 \cdot \lfloor \frac{k}{2} \rfloor + 1 \leq 2n-1, \quad (27)$$

and hence we get that

$$g \in U^k \subset U^{2n-1} = B_d(e, 1)^{2n-1}, \quad (28)$$

which proves (13).

Since $U^{2n-1} \subset \overline{U}^{2n-1}$, where the latter set is compact by assumption (11), we have that $B_d(e, n)$ is relatively compact for all $n \in \mathbb{N}$. Hence, by lemma (4.2) and remark (4.3) d is a **plig** metric on G . □

We are now ready to prove, that there exists a **plig** metric on every locally compact, second countable group.

Theorem 4.5. *Every locally compact, second countable group G has a **plig** metric d .*

Proof. Let G be locally compact, second countable group. By Remark [Wil70, Theorem 1.22, page 34], we can choose a left invariant metric δ_0 on G , which generates the topology on G . Moreover, since G is locally compact, there exists an $r > 0$ such that the open ball $B_{\delta_0}(e, r)$ is relatively compact. Put now $\delta = \frac{1}{r}\delta_0$. Then δ is again a left invariant metric on G , which generates the topology. Moreover,

$$U = B_\delta(e, 1) \tag{29}$$

is relatively compact.

Put

$$G_0 = \bigcup_{k=1}^{\infty} U^k. \tag{30}$$

Then G_0 is an open and closed subgroup of G . Since G is second countable, it follows that the space $Y = G/G_0$ of left G_0 -cosets in G is a countable set. In the following we will assume that $|Y| = \infty$. The proof in the case $|Y| < \infty$ can be obtained by the same method with elementary modifications.

We can choose a sequence $\{x_n\}_{n=1}^{\infty} \subset G$, such that $x_0 = e$ and such that is a disjoint union of cosets:

$$G = \bigcup_{n=0}^{\infty} x_n \cdot G_0. \tag{31}$$

By lemma (4.4), there is a **plig** metric d_0 on G_0 , such that

$$B_{d_0}(e, 1) = U \tag{32}$$

$$B_{d_0}(e, n) \subset U^{2n-1}. \tag{33}$$

In particular, d_0 is proper. Let

$$l_0(h) = d_0(h, e), \quad h \in G_0 \tag{34}$$

be the length function associated with d_0 .

Put

$$S = \{x_1, x_2, x_3 \dots\} \subset G \tag{35}$$

and define $l_1 : S \rightarrow \mathbb{N}$ by

$$l_1(x_n) = n. \quad (36)$$

Define furthermore a function $\tilde{l} : G \rightarrow [0, \infty[$ by setting

$$\tilde{l}(g) = \inf \left\{ l_0(h_0) + \sum_{i=1}^k (l_1(s_i) + l_0(h_i)) \right\}, \quad (37)$$

where the infimum is taken over all the representations of g of the form

$$\begin{cases} g = h_0 \cdot s_1 \cdot h_1 \cdot s_2 \cdot h_2 \cdots h_k \\ k \in \mathbb{N} \cup \{0\}, h_0, \dots, h_k \in G_0 \\ s_1, \dots, s_k \in S \end{cases} \quad (38)$$

Note that

$$G = G_0 \cup \bigcup_{n=1}^{\infty} x_n \cdot G_0 = G_0 \cup S \cdot G_0 \subset G_0 \cup G_0 \cdot S \cdot G_0, \quad (39)$$

so that every $g \in G$ has a representation of the form (38) with $k = 0$ or $k = 1$.

Next, put

$$l(g) = \max \left\{ \tilde{l}(g), \tilde{l}(g^{-1}) \right\}, \quad g \in G \quad (40)$$

We will show, that $l(g)$ is a length function on G , and that the associated metric

$$d(g, h) = l(g^{-1} \cdot h), \quad g, h \in G \quad (41)$$

is a **plig** metric on G .

It is easily checked that

$$\tilde{l}(g \cdot h) \leq \tilde{l}(g) + \tilde{l}(h), \quad g, h \in G \quad (42)$$

and hence also

$$l(g \cdot h) \leq l(g) + l(h), \quad g, h \in G. \quad (43)$$

Moreover, by (38)

$$l(g^{-1}) = l(g), \quad g \in G. \quad (44)$$

If $g \in G$ and $l(g) < 1$, then $\tilde{l}(g) \leq l(g) < 1$. Thus for every $\epsilon > 0$, g has a representation of the form (38), such that

$$l_0(h_0) + \sum_{i=1}^k (l_1(s_i) + l_0(h_i)) < l(g) + \epsilon \quad (45)$$

and for sufficiently small ϵ , we have that

$$l(g) + \epsilon < 1,$$

which implies that $k = 0$, because

$$\forall_{s \in S} \quad l_1(s) \geq 1.$$

Hence $g = h_0 \in G_0$, and

$$l_0(g) = l_0(h_0) < l(g) + \epsilon.$$

Since ϵ can be chosen arbitrarily small, we have shown that

$$g \in G \text{ and } l(g) < 1 \Rightarrow g \in G_0 \text{ and } l_0(g) \leq l(g). \quad (46)$$

In particular, $l(g) = 0$ implies that $g = e$, which together with (43) and (44) shows, that l is a length function on G . Hence by lemma (4.2)

$$d(g, h) = l(g^{-1} \cdot h), \quad g, h \in G$$

is a left invariant metric on G .

From (46) we have

$$g \in B_d(e, 1) \Rightarrow g \in G_0 \quad \text{and} \quad l_0(g) \leq l(g). \quad (47)$$

Conversely, if $g \in G_0$, then using (37) with $k = 0$ and $h_0 = g$, we get that

$$\tilde{l}(g) \leq l_0(g)$$

and therefore

$$l(g) \leq \max \left\{ l_0(g), l_0(g^{-1}) \right\} = l_0(g), \quad g \in G_0. \quad (48)$$

By (47) and (48), we have

$$B_d(e, r) = B_{d_0}(e, r), \quad 0 < r \leq 1, \quad (49)$$

and since G_0 is open in G , the sets

$$B_{d_0}(e, r), \quad 0 < r \leq 1$$

form a basis of neighbourhoods for e in G . Hence d generates the original topology on G .

It remains to be proved, that d is proper, ie. that $B_d(e, r)$ is relatively compact for all $r > 0$. Note that it is sufficient to consider the case, where $r = n \in \mathbb{N}$.

Let $n \in \mathbb{N}$, and let $g \in B_d(e, n)$. Then we have that

$$\tilde{l}(g) \leq l(g) < n.$$

Hence by (38), we see that

$$g = h_0 \cdot s_1 \cdot h_1 \cdot s_2 \cdot h_2 \cdots s_k \cdot h_k, \quad (50)$$

where

$$\begin{cases} k \in \mathbb{N} \cup \{0\} \\ h_0, \dots, h_k \in G_0 \\ s_1, \dots, s_k \in S \\ l_0(h_0) + \sum_{i=1}^k (l_1(s_i) + l_0(h_i)) < n \end{cases} \quad (51)$$

Since

$$l_1(s) \geq 1 \quad \forall s \in S,$$

we have that $k \leq n - 1$. Moreover, since $l_1 : S \rightarrow \mathbb{N}$ is defined by

$$l_1(x_m) = m, \quad m = 1, 2, \dots$$

we have

$$s_i \in \{x_1, x_2, \dots, x_{n-1}\}, \quad 1 \leq i \leq k. \quad (52)$$

Moreover

$$h_i \in B_{d_0}(e, n), \quad 0 \leq i \leq k \quad (53)$$

because $l_0(h_i) < n$ by (51). Put

$$T(n) = \{x_1, \dots, x_{n-1}\} \cup \{e\}$$

Then by (50), (51), (52) and (53) we have

$$g \in B_{d_0}(e, n) \left(T(n) \cdot B_{d_0}(e, n) \right)^k \subset \left(T(n) \cdot B_{d_0}(e, n) \right)^{k+1} \subset \left(T(n) \cdot B_{d_0}(e, n) \right)^n$$

where the last inclusion follows from the inequality $k \leq n - 1$. Since $g \in B_d(e, n)$ was chosen arbitrarily, we have shown that

$$B_d(e, n) \subset \left(T(n) \cdot B_{d_0}(e, n) \right)^n.$$

But d_0 is a proper metric on G_0 , and since $T(n)$ is a finite set, it follows, that

$$\left(T(n) \cdot \overline{B_{d_0}(e, n)} \right)^n$$

is compact.

Hence $B_d(e, n)$ is relatively compact for all $n \in \mathbb{N}$, and therefore d is a proper metric on G , cf. remark (4.3). □

Remark 4.6. As mentioned in the introduction, Theorem (4.5) has previously been obtained in two important special cases, the compactly generated case [LMR00] and the countable, discrete case [Tu01].

Example 4.7. In this example, we give an explicit formula for a **plig** metric on $GL(n, \mathbb{R})$. The same formula will also define a **plig** metric on every closed subgroup of $GL(n, \mathbb{R})$. Recall that

$$GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\} \quad (54)$$

and the topology on $GL(n, \mathbb{R})$ is inherited from the topology of $M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$. We equip $M_n(\mathbb{R})$ with the operator norm

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^n, \|x\| \leq 1\},$$

where $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the Euclidian norm on \mathbb{R}^n .

Define a function on $GL(n, \mathbb{R})$ by

$$l(A) = \max\{\ln(1 + \|A - I\|), \ln(1 + \|A^{-1} - I\|)\}. \quad (55)$$

We claim that l is a length function on $GL(n, \mathbb{R})$, and that the associated metric

$$d(A, B) = l(B^{-1}A), \quad A, B \in GL(n, \mathbb{R})$$

is a **plig** metric on $GL(n, \mathbb{R})$. We prove first, that l is a length function. Clearly, $l(A) = l(A^{-1})$ and $l(A) = 0 \Leftrightarrow A = I$.

Let $A, B \in GL(n, \mathbb{R})$. Then

$$\|A - I\| \leq e^{l(A)} - 1, \quad \|B - I\| \leq e^{l(B)} - 1.$$

Put $X = A - I$ and $Y = B - I$. Then

$$\begin{aligned} \|AB - I\| &= \|XY + X + Y\| \leq \|X\| \cdot \|Y\| + \|X\| + \|Y\| \\ &= (\|X\| + 1)(\|Y\| + 1) - 1 \leq e^{l(A)} e^{l(B)} - 1 \end{aligned}$$

and hence

$$\ln(1 + (\|AB - I\|)) \leq l(A) + l(B). \quad (56)$$

Substituting (A, B) with (B^{-1}, A^{-1}) in this inequality, we get

$$\ln(1 + (\|(AB)^{-1} - I\|)) \leq l(B^{-1}) + l(A^{-1}) = l(A) + l(B), \quad (57)$$

and by (56) and (57) it follows that $l(A + B) \leq l(A) + l(B)$. Hence l is a length function on $GL(n, \mathbb{R})$.

To prove that d is a **plig** metric on $GL(n, \mathbb{R})$, it suffices to check, that the conditions (10) and (9) in lemma (4.2) are fulfilled.

Since $A \rightarrow A^{-1}$ is a homeomorphism of $GL(n, \mathbb{R})$ onto itself, (9) is clearly fulfilled. To prove (10) we let $r \in (0, \infty)$ and put $M = e^r$. Since $\|C\| \leq 1 + \|C - I\|$ for $C \in GL(n, \mathbb{R})$, we see that $l^{-1}([0, r])$ is a closed subset of

$$K = \{A \in GL(n, \mathbb{R}) : \|A\| \leq M, \|A^{-1}\| \leq M\}.$$

Denote

$$L = \{(A, B) \in M_n(\mathbb{R}) \times M_n(\mathbb{R}) : AB = BA = I, \|A\| \leq M, \|B\| \leq M\},$$

then L is a compact subset of $M_n(\mathbb{R})^2$, and K is the range of L by the continuous map $\pi : (A, B) \rightarrow A$ of $M_n(\mathbb{R})^2$ onto $M_n(\mathbb{R})$. Hence K is compact, and therefore $l^{-1}([0, r])$ is also compact. This proves (10), and therefore d is a **plig** metric on $GL(n, \mathbb{R})$.

Example 4.8. Let G be a connected Lie group. Then we can choose a left invariant Riemannian structure on G . Let $(g_p)_{p \in G}$ denote the corresponding inner product on the spaces $(T_p)_{p \in G}$. The path length metric on G corresponding to the Riemannian structure is

$$d(g, h) = \inf_{\gamma} L(\gamma)$$

where

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt$$

is the path length of a piecewise smooth path γ in G , and where the infimum is taken over all such paths, that starts in $\gamma(a) = g$ and ends in $\gamma(b) = h$. Then d is a left invariant metric on G which induces the given topology on G , cf. [Hel01, p.51-52].

We claim that d is a proper metric on G . To prove this, it is sufficient to prove, that $B_d(e, r)$ is relatively compact for all $r > 0$. Let $r > 0$, and let $g \in B(e, r)$. Then e and g can be connected with a piecewise smooth path γ of length $L(\gamma) < r$.

Now γ can be divided in two paths each of length $\frac{1}{2}L(\gamma)$. Let h denote the endpoint of the first path. Then

$$d(e, h) \leq \frac{L(\gamma)}{2} \quad \text{and} \quad d(h, g) \leq \frac{L(\gamma)}{2}.$$

Hence $g = h(h^{-1}g)$, where $d(e, r) < \frac{r}{2}$ and $d(h^{-1}g, e) = d(g, h) < \frac{r}{2}$. This shows, that

$$B(e, r) \subset B(e, \frac{r}{2})^2$$

and hence

$$B(e, r) \subset B(e, r \cdot 2^{-k})^{2^k}$$

for all $k \in \mathbb{N}$. Since G is locally compact, we can choose a $k \in \mathbb{N}$ such that $B(e, r2^{-k})$ is relatively compact. Hence $B(e, r)$ is contained in the compact set

$$\overline{B(e, r2^{-k})}^{2^k}.$$

This shows that d is proper, and therefore d is a **plig** metric on G .

5 Exponentially controlled growth of the d -balls

Definition 5.1. Let (G, d) be a locally compact, second countable group with a **plig** metric, and let μ denote the Haar measure on G . Then we say that *the d -balls have exponentially controlled growth* if there exists constants $\alpha, \beta > 0$, such that

$$\mu(B_d(e, n)) \leq \beta e^{\alpha n}, \quad \forall n \in \mathbb{N}. \quad (58)$$

Note, that if (58) holds, then

$$\mu(B_d(e, r)) \leq \beta' \cdot e^{c_2 r}, \quad r \in [1, \infty), \quad (59)$$

where $\beta' = \beta \cdot e^\alpha$.

We now turn to the problem of constructing a **plig** metric on G , for which the d -balls have exponentially controlled growth. We first prove the following simple combinatorial lemma:

Lemma 5.2. *Let $n \in \mathbb{N}$, and let $k \in \{1, \dots, n\}$.*

Put

$$N_{n,k} = \left\{ (n_1, n_2, \dots, n_k) \in \mathbb{N}^k : \sum_{i=1}^k n_i \leq n \right\} \quad (60)$$

Then the number of elements in $N_{n,k}$ is

$$|N_{n,k}| = \binom{n}{k}.$$

Proof. The map

$$(n_1, \dots, n_k) \rightarrow \{n_1, n_1 + n_2, \dots, n_1 + n_2 + \dots + n_k\}$$

is a bijection from $N_{n,k}$ onto the set of subsets of $\{1, \dots, n\}$ with k elements, and the latter set has of course $\binom{n}{k}$ elements. \square

Having established lemma (5.2), we can now turn to giving a proof of the main theorem of this section:

Theorem 5.3. *Every locally compact, second countable group G has a **plig** metric d , for which the d -balls have exponentially controlled growth.*

Proof. The result is obtained, by modifying the construction of a **plig** metric on G from the proof of theorem (4.5).

Let U, G_0, d_0, d and $S = \{x_1, x_2, \dots\}$ be as in the proof of theorem (4.5), and note that by (49) we have

$$B_d(e, 1) = B_{d_0}(e, 1) = U.$$

For each $i \in \mathbb{N}$ the set $\overline{U} \cdot x_i$ is compact in G , and can therefore be covered by finitely many left translates of U :

$$U \cdot x_i \subset \overline{U} \cdot x_i \subset \bigcup_{j=1}^{p(i)} y_{i,j} \cdot U. \quad (61)$$

Define $l_1^* : S \rightarrow [1, \infty[$ by

$$l_1^*(x_i) = i + \log_2(p(i)), \quad (62)$$

and note that

$$l_1^*(x_i) \geq i = l_1(x_i), \quad x_i \in S,$$

where $l_1 : S \rightarrow \mathbb{N}$ is the map defined in (36).

We will now repeat the construction of the left invariant metric d in the proof of theorem (4.5), with l_1 replaced by l_1^* , ie. we first define a function $\tilde{l}^* : G \rightarrow [0, \infty[$ by

$$\tilde{l}^*(g) = \inf \left\{ l_0(h_0) + \sum_{i=1}^k (l_1^*(s_i) + l_0(h_i)) \right\} \quad (63)$$

where the infimum is taken over all representations of g of the form

$$\begin{cases} g = h_0 \cdot s_1 \cdot h_1 \cdot s_2 \cdot h_2 \cdots s_k \cdot h_k \\ k \in \mathbb{N} \cup \{0\} \\ h_0, \dots, h_k \in G_0 \\ s_1, \dots, s_k \in S \end{cases} \quad (64)$$

Next, we put

$$l^*(g) = \max \left\{ \tilde{l}^*(g), \tilde{l}^*(g^{-1}) \right\}, \quad g \in G \quad (65)$$

and

$$d^*(g, h) = l^*(g^{-1} \cdot h), \quad g, h \in G. \quad (66)$$

Then, exactly as for the metric d in the proof of theorem (4.5) we get that d^* is a left invariant metric on G , which generates the given topology on G , and which satisfies the following

$$B_{d^*}(e, 1) = B_{d_0}(e, 1) = U,$$

(cf. the proof of theorem (4.5)).

Moreover, since $l_1^* \geq l_1$, we have that $d^* \geq d$, so the properness of d implies, that d^* is also proper.

Let $n \in \mathbb{N}$. Since

$$l^*(g) \leq \tilde{l}^*(g), \quad g \in G,$$

we get from (63) and (64) that

$$B_{d^*}(e, n) \subset \left\{ g \in G : \tilde{l}^*(g) < n \right\}$$

Hence every $g \in B_{d^*}(e, n)$ can be written on the form (64) with

$$l_0(h_0) + \sum_{i=1}^k (l_1^*(s_i) + l_0(h_i)) < n.$$

Note, that since

$$l_1^*(s) \geq l_1(s) \geq 1, \quad s \in S,$$

we have that $k \leq n - 1$.

Choose next natural numbers m_0, \dots, m_k , such that

$$l_0(h_i) < m_i \leq l_0(h_i) + 1, \quad i = 0, \dots, k.$$

Then we have that

$$h_i \in B_{d^*}(e, m_i), \quad i = 0, \dots, k,$$

and

$$m_0 + \sum_{i=1}^k (l_1^*(s_i) + m_i) < n + (k + 1) \leq 2n + 1.$$

Hence

$$B_{d^*}(e, n) \subset \bigcup_M B_{d_0}(e, m_0) \cdot x_{n_1} \cdot B_{d_0}(e, m_1) \cdot x_{n_2} \cdots x_{n_k} \cdot B_{d_0}(e, m_k), \quad (67)$$

where M is the set of tuples

$$\left(k, m_0, m_1, \dots, m_k, n_1, \dots, n_k \right) \quad (68)$$

for which

$$\begin{cases} k \in \{0, \dots, n - 1\} \\ n_1, \dots, n_k, m_0, \dots, m_k \in \mathbb{N} \\ \sum_{i=0}^k m_i + \sum_{i=1}^k l_1^*(x_{n_i}) < 2n + 1 \end{cases}$$

By (62), the latter condition can be rewritten as

$$\sum_{i=0}^k m_i + \sum_{i=1}^k (n_i + \log_2(p(n_i))) < 2n + 1, \quad (69)$$

where $p(n_i) \in \mathbb{N}$ are given by formula (61). Since $m_i, n_i \in \mathbb{N}$ and $p(n_i) \geq 1$, it follows that

$$\sum_{i=0}^k m_i + \sum_{i=1}^k n_i \leq 2n.$$

Hence $M \subset \bigcup_{k=0}^{n-1} M_k$, where M_k is the set of $2k + 1$ -tuples $(m_0, \dots, m_k, n_1, \dots, n_k)$ of natural numbers for which

$$\sum_{i=0}^k m_i + \sum_{i=1}^k n_i \leq 2n.$$

Therefore, by lemma (5.2) we have that $|M_k| = \binom{2n}{2k+1}$, and thus

$$|M| \leq \sum_{k=0}^{n-1} \binom{2k}{2k+1} \leq \sum_{j=0}^{2n} \binom{2n}{j} = 2^{2n}. \quad (70)$$

From (33), we have

$$B_{d_0}(e, m_i) \subset U^{2m_i-1} \subset U^{2m_i}.$$

Hence by (67), we have

$$B_{d^*}(e, n) \subset \bigcup_M U^{2m_0} \cdot x_{n_1} \cdot U^{2m_1} \cdot x_{n_2} \cdots x_{n_k} \cdot U^{2m_k}, \quad (71)$$

where $|M| \leq 2^{2n}$ and where (69) holds for each $(k, m_0, \dots, m_k, n_1, \dots, n_k) \in M$. Since \bar{U}^2 is compact, it can be covered by finitely many left translates of U , ie.

$$U^2 \subset \bar{U}^2 \subset \bigcup_{i=1}^q z_i \cdot U, \quad z_1, \dots, z_q \in G.$$

It now follows that for every $k \in \mathbb{N}$, the set U^k can be covered by q^{k-1} translates of U , namely

$$U^k \subset \bigcup_{i_1=\dots=i_{k-1}=1}^q z_{i_1} \cdots z_{i_{k-1}} \cdot U, \quad z_1, \dots, z_q \in G. \quad (72)$$

We can now use (61) and (72) to control the Haar measure of each of the sets

$$U^{2m_0} \cdot x_{n_1} \cdot U^{2m_1} \cdot x_{n_2} \cdots x_{n_k} \cdot U^{2m_k}, \quad (73)$$

from (71). By (72) we see that U^{2m_0} can be covered by q^{2m_0-1} left translations of U .

Combined with (61), we get that $U^{2m_0} \cdot x_{n_1}$ can be covered by $q^{2m_0-1} \cdot p(n_1)$ left translates of U . Hence

$$U^{2m_0} x_{n_1} U^{2m_1} = \bigcup_{w \in A_1} w \cdot U^{2m_1+1},$$

where $|A_1| \leq 2^{2m_0-1} p(n_1)$.

Again by (72), we see that U^{2m_1+1} can be covered by q^{2m_1} left translations of U , so altogether we see that the set

$$U^{2m_0} x_{n_1} U^{2m_1}$$

can be covered by $q^{2m_0+2m_1-1}p(n_1)$ left translations of U . Continuing this procedure, we get that the set in (73) can be covered by

$$q^{2m_0+2m_1+\dots+2m_k-1}p(n_1) \cdot p(n_2) \cdots p(n_k)$$

left translates of U , and hence the Haar measure of the set satisfies that

$$\begin{aligned} \mu\left(U^{2m_0} \cdot x_{n_1} \cdot U^{2m_1} \cdot x_{n_2} \cdots x_{n_k} \cdot U^{2m_k}\right) \\ \leq q^{2 \cdot \sum_{i=0}^k m_i} \cdot \prod_{i=1}^k p(n_i) \mu(U) \\ \leq q^{2 \cdot \sum_{i=0}^k m_i} \cdot 2^{\sum_{i=1}^k \log_2(p(n_i))} \mu(U) \end{aligned}$$

By (69), we have that

$$\begin{cases} \sum_{i=0}^k m_i \leq 2n + 1 \\ \sum_{i=1}^k \log_2(p(n_i)) \leq 2n + 1 \end{cases} .$$

Hence, we have that

$$\mu\left(U^{2m_0} \cdot x_{n_1} \cdot U^{2m_1} \cdot x_{n_2} \cdots x_{n_k} \cdot U^{2m_k}\right) \leq (2q^2)^{2n+1} \cdot \mu(U).$$

This holds for all tuples $(k, m_0, \dots, m_k, n_1, \dots, n_k) \in M$, and since we have shown that $|M| \leq 2^{2n}$, we get from (67) that

$$\mu(B_{d^*}(e, n)) \leq (4q^2)^{2n+1} \mu(U), \quad n \in \mathbb{N}, \quad (74)$$

which shows that the d^* -balls have exponentially controlled growth. \square

Example 5.4. In this example, we will give a more direct proof of Theorem (5.3) in the case of a countable discrete group Γ . If Γ is finitely generated, it is elementary to check, that the word length metric d is proper and that the d -balls have exponentially controlled growth, so we can assume that Γ is generated by an infinite, symmetric set S , such that $e \notin S$.

We can write S as a disjoint union

$$S = \bigcup_{n=1}^{\infty} Z_n,$$

where each Z_n is of the form $\{x_n, x_n^{-1}\}$. Note that $|Z_n| = 2$ if $x_n \neq x_n^{-1}$, and $|Z_n| = 1$ if $x_n = x_n^{-1}$. Define a function

$$l_0 : S \rightarrow \mathbb{N}$$

by

$$l_0(x_n) = l_0(x_n^{-1}) = n.$$

Next, define a function

$$l : \Gamma \rightarrow \mathbb{N} \cup \{0\}$$

by

$$l(g) = \begin{cases} \inf\{\sum_{k=1}^n l(g_k)\} & g \neq e \\ 0 & g = e \end{cases} \quad (75)$$

where the infimum is taken over all representations of g of the form

$$g = g_1 \cdots g_n, \quad g_i \in S, n \in \mathbb{N}.$$

Then it is easy to check, that l is a length function on Γ , and since

$$l(g) \geq 1 \quad \text{for } g \in \Gamma \setminus \{e\}$$

the associated left invariant metric

$$d(g, h) = l(g^{-1}h), \quad g, h \in \Gamma \quad (76)$$

generates the discrete topology on Γ . Put

$$D(e, n) = \{g \in \Gamma : d(g, e) \leq n\}. \quad (77)$$

We will next show, that

$$|D(e, n)| \leq 3^n, \quad n \in \mathbb{N}, \quad (78)$$

which clearly implies, that the d -balls have exponentially controlled growth. In order to prove (78), we will show by induction in $n \in \mathbb{N}$, that the set

$$\partial(e, n) = \{g \in \Gamma : d(g, e) = n\} \quad (79)$$

satisfies

$$|\partial(e, n)| \leq 2 \cdot 3^{n-1}, \quad n \in \mathbb{N}. \quad (80)$$

Since $l_0(s) \geq 2$ for $s \in S \setminus \{x_1, x_1^{-1}\}$, we have for $g \in \Gamma$, that

$$l(g) = 1 \Leftrightarrow g \in \{x_1, x_1^{-1}\}. \quad (81)$$

Hence

$$|\partial(e, 1)| = |\{x_1, x_1^{-1}\}| \leq 2 \quad (82)$$

which proves (80) for $n = 1$. Let now $n \geq 2$ and assume as induction hypothesis, that

$$|\partial(e, i)| \leq 2 \cdot 3^{i-1}, \quad i = 1, \dots, n-1. \quad (83)$$

We shall then show, that

$$|\partial(e, n)| \leq 2 \cdot 3^{n-1}.$$

We claim, that

$$\partial(e, n) \subset \bigcup_{k=1}^n Z_k \cdot \partial(e, n - k) \quad (84)$$

To prove (83), let $g \in \partial(e, n)$. Then there exists a $m \in \mathbb{N}$ and $g_1, \dots, g_m \in S$ such that

$$g = g_1 \cdots g_m$$

and

$$\sum_{i=1}^m l_0(g_i) = n$$

Put $k = l_0(g_1)$. Then $k \in \mathbb{N}$ and $k \leq n$. Now

$$g = g_1 \cdot (g_2 \cdots g_m),$$

where

$$l(g_1) \leq l_0(g_1) = k \quad (85)$$

$$l(g_2 \cdots g_m) \leq \sum_{i=2}^m l_0(g_i) = n - k. \quad (86)$$

But, since

$$n = l(g) \leq l(g_1) + l(g_2 \cdots g_m)$$

equality holds in both (85) and (86). Hence $g_2 \cdots g_m \in \partial(e, n - k)$, which proves (84). By (84) we have

$$|\partial(e, n)| \leq \sum_{k=1}^n |Z_k| \cdot |\partial(e, n - k)| \leq 2 \sum_{k=1}^n |\partial(e, n - k)| = 2 \sum_{i=1}^{n-1} |\partial(e, i)|,$$

Since $|\partial(e, 0)| = |\{e\}| = 1$, we get by the induction hypothesis (83), that

$$|\partial(e, n)| \leq \sum_{i=0}^{n-1} 2|\partial(e, i)| \leq 2(1 + \sum_{i=1}^{n-1} 2 \cdot 3^{i-1}) = 2(1 + 3^{n-1} - 1) = 2 \cdot 3^{n-1}. \quad (87)$$

This completes the proof of the induction step. Hence (80) holds for all $n \in \mathbb{N}$. Since l only takes integer values, we have

$$D(e, n) = \bigcup_{i=0}^n \partial(e, i).$$

Therefore

$$|D(e, n)| = \sum_{i=0}^n |\partial(e, i)| \leq 1 + \sum_{i=1}^n 2 \cdot 3^{i-1} = 3^n. \quad (88)$$

This proves (78), and it follows that the d -balls have exponentially controlled growth.

6 Affine actions on Banach spaces.

We have shown in theorem 4.5 that for any locally compact, second countable group G there exists a **plig** metric d , and we have shown in theorem 5.3 that d can be chosen so that the d -balls have exponentially controlled growth. We will now construct an affine action of G on the reflexive separable strictly convex Banach space

$$\bigoplus_{n=1}^{\infty} L^{2n}(G, \mu) \text{ (in the } l^2 \text{ sense).}$$

Gromov suggested in [Gro95], that it is purposeful to attack the Baum Connes Conjecture by considering proper affine isometric actions on various Banach spaces.

It was shown by N. Higson and G. Kasparov in [HK01] that the Baum-Connes Conjecture holds for discrete countable groups that admit a proper affine isometric action on a Hilbert space. In particular, this holds for all discrete amenable groups. Moreover, Yu proved in [Yu05], that a word hyperbolic group Γ has a proper affine action on the uniform convex Banach space $l^p(\Gamma \times \Gamma)$ for some $p \in [2, \infty)$.

Therefore, it is interesting to study what kind of proper affine isometric actions on Banach spaces a given locally compact, second countable group admits.

Definition 6.1. The group of affine actions on G : Let X be a normed vector space, then the affine group of X is:

$$\text{Aff}(X) = \{\phi : X \rightarrow X \quad : \phi(x) = Ax + b; A \in GL(X), b \in X\}.$$

We say that G has an affine action on X , if there exists a group homomorphism of G on $\text{Aff}(X)$, ie:

$$\alpha : G \curvearrowright \text{Aff}(X), \tag{89}$$

such that

$$\forall_{g,h \in G} \quad \alpha(g \cdot h) = \alpha(g) \circ \alpha(h). \tag{90}$$

Let π_g denote the linear part of $\alpha(g)$, and denote the translation part by $b(g)$. We say that $\alpha(g)$ is isometric if the linear part π_g is isometric, i.e:

$$\forall_{\xi \in X} \quad \|\pi_g \xi\| = \|\xi\|.$$

Moreover, we say that the action is proper, if

$$\forall_{\xi \in X} \quad \lim_{g \rightarrow \infty} \|\alpha(g)\xi\| = \infty.$$

Remark 6.2. Since α is a homomorphism of the group G into $\text{Aff}(X)$, we have that:

$$\begin{aligned} \forall_{\xi \in X} \quad \alpha(st)\xi &= \alpha(s)(\alpha(t)\xi) \Leftrightarrow \\ \forall_{\xi \in X} \quad \pi_{st}\xi + b(st) &= \pi_s \pi_t \xi + \pi_s b(t) + b(s) \Leftrightarrow \\ \pi_{st} = \pi_s \circ \pi_t \quad \text{and} \quad b(st) &= \pi_s(b(t)) + b(s). \end{aligned} \tag{91}$$

The formula for $b(st)$ is called the *cocycle condition with respect to π* .

And we also need to know what a strictly convex space is – and we will use the opportunity to define a uniformly convex space as well:

Definition 6.3. Let X be a normed vector space, denote the unit ball by S_X .

1. The following two conditions are equivalent (see [Meg98, Prop. 5.1.2]). If X satisfies any of them, it is called *strictly convex*.

$$(a) \quad \forall_{x \neq y \in S_X, 1 > t > 0} \quad \|tx + (1-t)y\| < 1. \quad (92)$$

$$(b) \quad \forall_{x \neq y \in S_X} \quad \left\| \frac{1}{2}(x+y) \right\| < 1. \quad (93)$$

2. X is called *uniformly convex* if

$$\forall_{\epsilon > 0} \exists_{\delta > 0} \forall_{x, y \in S_X} \quad \|x - y\| \geq \epsilon \Rightarrow \left\| \frac{1}{2}(x+y) \right\| \leq 1 - \delta \quad (94)$$

Remark 6.4. Every space that is uniformly convex is also strictly convex (see [Meg98, Proposition 5.2.6]). Examples of uniformly convex Banach spaces include

$$l_p, l_p^n, \quad \infty > p > 1, n \geq 1$$

(this follows from Milman-Pettis theorem, see [Meg98, Theorem 5.2.15]).

A uniformly convex Banach space is necessarily reflexive (see [Meg98, Theorem]). There are spaces that are strictly convex, but not uniformly convex, and also spaces that are strictly convex and not reflexive. An example of a strictly convex but not uniformly convex Banach space is:

$$\bigoplus_{i=1}^{\infty} l_{p_n}^n, \quad \text{where } p_n = 1 + \frac{1}{n}$$

(with l^2 norm on the direct sum).

As an application of the construction of a **plig** metric with the d -balls have exponentially controlled growth on a given locally compact, second countable group in theorem (5.3), we will construct a proper isometric action on the Banach space

$$\bigoplus_{n=1}^{\infty} L^{2n}(G, \mu) \text{ (in the } l^2 \text{ sense).}$$

In [BG05] a proper isometric action is constructed for a discrete group Γ into the Banach space $\bigoplus_{p=1}^{\infty} L^{p_n}(G, d\mu)$, where p_n is an unbounded sequence. We have generalized this result as follows:

Theorem 6.5. *Let G be a locally compact, second countable group, and let μ denote the Haar measure. Then there exists a proper affine isometric action α of G on the separable, strictly convex, reflexive Banach space*

$$X = \bigoplus_{n=1}^{\infty} L^{2n}(G, \mu) \text{ (in the } l^2 \text{ sense).}$$

Proof. Let G be as in the statement of the theorem, then according to theorem (4.5) and theorem (5.3) we can choose a **plig** metric d on G where the d -balls have exponentially controlled growth, ie.

$$\exists_{\alpha>0} \quad \mu(B_d(e, n)) \leq \beta \cdot e^{\alpha n}, \quad (95)$$

for some constants $\alpha, \beta > 0$. We can without loss of generality assume that $\beta \geq 1$.

Consider the functions $\phi_x^n : G \rightarrow \mathbb{R}$ given by:

$$\phi_x^n(y) = \begin{cases} 1 - \frac{d(x,y)}{n} & \text{when } d(x,y) \leq n \\ 0 & \text{when } d(x,y) \geq n \end{cases} \quad (96)$$

It is easy to check, that ϕ_x^n is $\frac{1}{n}$ -Lipschitz:

$$|\phi_x^n(y) - \phi_x^n(z)| \leq \frac{d(y,z)}{n}. \quad (97)$$

Assume that $x \in B_d(e, \frac{n}{2})$:

$$\phi_e^n(x) = 1 - \frac{d(x,e)}{n} \geq 1 - \frac{\frac{n}{2}}{n} = \frac{1}{2} \cdot 1_{B_d(e, \frac{n}{2})}(x), \quad (98)$$

Let $C_{\text{ucb}}(G)$ denote the set of uniformly continuous bounded functions from G to \mathbb{R} . Define $b^n : G \rightarrow C_{\text{ucb}}(G)$ by:

$$b^n(g) = \lambda(g)\phi_e^n - \phi_e^n \Rightarrow b^n(g)(h) = \phi_e^n(g^{-1}h) - \phi_e^n(h). \quad (99)$$

Since $d(g, e) = d(g^{-1}, e)$, we have that

$$\phi_e^n(g) = \phi_e^n(g^{-1}), \quad g \in G.$$

Hence, by (99) and (97) we have

$$|b^n(g)| \leq |\phi_e^n(g^{-1}h) - \phi_e^n(h)| = |\phi_e^n(h^{-1}g) - \phi_e^n(h^{-1})| \leq \frac{d(h^{-1}g, h^{-1})}{n} \leq \frac{d(e, g)}{n} \quad (100)$$

Since $b_n(g) = 0$, when $x \notin B_d(e, n) \cup B_d(g, n)$, it follows that

$$|b^n(g)| \leq \frac{d(e, g)}{n} \cdot 1_{B_d(e, n) \cup B_d(g, n)}.$$

Hence $b^n(g) \in L^2(G, \mu)$ and

$$\|b^n(g)\|_{2n}^{2n} \leq \left(\frac{d(e, g)}{n}\right)^{2n} (\mu(B_d(e, n)) + \mu(B_d(g, n))).$$

Therefore, by (95) and the left invariance of μ , we have that

$$\|b^n(g)\|_{2n}^{2n} \leq \left(\frac{d(e, g)}{n}\right)^{2n} \cdot 2\beta e^{\alpha n}.$$

Using now, that $\beta \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < 2$, we get

$$\sum_{n=1}^{\infty} \|b^n(g)\|_{2n}^2 \leq \sum_{n=1}^{\infty} \frac{d(e, g)^2}{n^2} (2\beta)^{\frac{1}{n}} e^\alpha \leq 4\beta e^\alpha d(g, e)^2$$

Hence

$$b(g) = \bigoplus_{n=1}^{\infty} b^n(g) \in X$$

and

$$\|b(g)\|_X \leq 2\sqrt{\beta} e^{\frac{\alpha}{2}} d(g, e).$$

Let $\tilde{\lambda}$ denote the left regular representation of G on $X = \bigoplus_{n=1}^{\infty} L^{2n}(G, \mu)$ (in the l^2 sense). Clearly $\tilde{\lambda}(g)$ is an isometry of X for every $g \in G$. We show next, that $b(g)$ fulfills the cocycle condition

$$b(st) = \tilde{\lambda}(s)b(t) + b(s), \quad s, t \in G \quad (101)$$

and (101) follows from

$$\begin{aligned} b^n(st) &= \lambda(st)\phi_e^n - \phi_e^n \\ &= \lambda(s)(\lambda(t)\phi_e^n - \phi_e^n) + (\lambda(s)\phi_e^n - \phi_e^n) = \lambda(s)b^n(t) + b^n(s), \quad s, t \in G, \end{aligned}$$

for alle $n \in \mathbb{N}$. By (101) we can define a continuous affine action α of G on X by

$$\alpha(g)\xi = \tilde{\lambda}(g)\xi + b(g), \quad x \in X, g \in G. \quad (102)$$

The last thing to show is that the action is metrically proper. For $\xi \in X$ and $g \in G$, we have

$$\|\alpha(g)\xi\| = \|\tilde{\lambda}(g)\xi + b(g)\| \geq \|b(g)\| - \|\tilde{\lambda}(g)\xi\| = \|b(g)\| - \|\xi\|.$$

Hence, we only have to check, that

$$\|b(g)\| \rightarrow \infty \quad \text{when} \quad d(g, e) \rightarrow \infty.$$

Let $g \in G$ and assume that $d(g, e) > 2$. Moreover, let $N(g) \in \mathbb{N}$ denote the integer for which

$$\frac{d(g, e)}{2} - 1 \leq N(g) < \frac{d(g, e)}{2}.$$

For $n = 1 \dots, N(g)$, we have that

$$d(g, e) > 2N(g) \geq 2n.$$

Hence

$$\overline{B(e, n)} \cap \overline{B(g, n)} = \emptyset,$$

which implies that ϕ_e^n and ϕ_g^n have disjoint supports. Therefore

$$\|b^n(g)\|_{2n}^{2n} = \|\phi_g^n - \phi_e^n\|_{2n}^{2n} = \|\phi_g^n\|_{2n}^{2n} + \|\phi_e^n\|_{2n}^{2n} \geq \|\phi_e^n\|_{2n}^{2n}$$

Since we have that

$$\phi_e^n \geq \frac{1}{2} \cdot 1_{B(e, \frac{n}{2})}$$

it follows that

$$\|b^n(g)\|_{2n}^{2n} \geq 2^{-2n} \mu(B(e, \frac{n}{2})) \geq 2^{-2n} \mu(B(e, \frac{1}{2})).$$

Hence

$$\begin{aligned} \|b(g)\|^2 &\geq \sum_{n=1}^{N(g)} \|b^n(g)\|_{2n}^2 \geq \frac{1}{4} \sum_{n=1}^{N(g)} \mu(B(e, \frac{1}{2}))^{\frac{1}{n}} \\ &\geq \frac{N(g)}{4} \cdot \min \{ \mu(B(e, \frac{1}{2})), 1 \}. \end{aligned} \quad (103)$$

Since

$$N(g) \geq \frac{d(g, e)}{2} - 1$$

it follows that

$$\|b(g)\| \rightarrow \infty \text{ for } d(g, e) \rightarrow \infty.$$

□

Corollary 6.6. *Let G be a locally compact, 2-nd countable group. Then G has a uniform embedding into the separable, strictly convex Banach space*

$$\bigoplus_{n=1}^{\infty} L^{2n}(G, \mu) \text{ (in the } l^2 \text{ sense).}$$

Proof. We will show, that the map $b : G \rightarrow X$ constructed in the proof of theorem (6.5) is a uniform embedding. By the proof of theorem (6.5), we have that

$$c_1 \sqrt{d(g, e)} \leq \|b(g)\|_X \leq c_2 d(g, e), \quad (104)$$

when $d(g, e) \geq c_3$ for some positive constants c_1, c_2, c_3 . By the cocycle condition (91), we get for $g, h \in G$ that

$$b(g) = b(h(h^{-1}g)) = \tilde{\lambda}(h)b(h^{-1}g) + b(h).$$

Hence

$$\|b(g) - b(h)\|_X = \|\tilde{\lambda}(h)b(h^{-1}g)\|_X = \|b(h^{-1}g)\|_X.$$

Since $d(h^{-1}g, e) = d(g, h)$, we obtain by applying (104) to $b(h^{-1}g)$, that

$$c_1 \sqrt{d(g, h)} \leq \|b(g) - b(h)\|_X \leq c_2 d(g, h),$$

when $d(g, h) \geq c_3$. This proves, that b is a uniform embedding.

□

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