Random Matrices with Complex Gaussian Entries

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Introduction

Random matrices has been an important tool in statistics since 1928 and in physics since 1955 starting with the pioneering works of Wishart [Wis] and Wigner [Wig1]. In the last 12 years random matrices have also played a key role in operator algebra theory and free probability theory starting with Voiculescu's random matrix model for a free semi-circular system (cf. [Vo]). Many results on eigenvalue distributions for random matrices are obtained by complicated combinatorial methods, and the purpose of this paper is to give more easily accessible proofs, by analytic methods, for those results on random matrices, which are of most interest to people working in operator algebra theory and free probability theory.

We will study two classes of random matrices. The first class is the Gaussian unitary ensemble (GUE) (cf. [Meh, Ch.5]), and the second class is the complex Wishart ensemble, which is also called the Laguerre ensemble (cf. [Go], [Kh] and [Fo]). Our new approach is based on the derivation of an explicit formula for the moment generating function:

$$s \mapsto \mathbb{E}(\mathrm{Tr}(\exp(sZ))),$$

where Z is either a GUE random matrix or a complex Wishart matrix. These two formulas are then used to reprove classical results on the asymptotic behavior of the eigenvalue distribution of Z. In particular, we shall study the asymptotic behavior of the largest and smallest eigenvalues of Z in those two cases. The above mentioned explicit formulas also give a new proof of the Harer-Zagier recursion formula for the moments $\mathbb{E}(\mathrm{Tr}(Z^p))$, $p = 1, 2, \ldots$, in the GUE case, and we derive a similar recursion formula for the moments in the complex Wishart case.

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Preliminaries and statement of results

The first class of random matrices studied in this paper is the class of complex selfadjoint random matrices $A = (a_{ij})_{i,j=1}^n$, for which

$$(a_{ii})_{i=1}^n, \quad (\sqrt{2}\mathrm{Re}a_{ij})_{i<j}, \quad (\sqrt{2}\mathrm{Im}a_{ij})_{i<j}$$

form a set of n^2 independent real Gaussian distributed random variables all with mean value 0 and variance σ^2. We denote this class of random matrices $\mathrm{SGRM}(n,\sigma^2)$. If $\sigma^2 = \frac{1}{2}$ one gets the Gaussian unitary ensemble (GUE) from Mehta's book (cf. [Meh, Sect.5]) and the value $\sigma^2 = \frac{1}{n}$ gives the normalization used in Voiculescu's random matrix paper [Vo]. In [Meh, Section 5], it is proved that the "mean density" of the eigenvalues of a random matrix A from the class $\mathrm{SGRM}(n, \frac{1}{2})$ is given by

$$\frac{1}{n}\sum_{k=0}^{n-1}\varphi_k(x)^2 \tag{0.1}$$

where $\varphi_0, \varphi_1, \varphi_2, \dots$ is the sequence of Hermite functions. In Section 3 we derive from (0.1) that for A in $\mathrm{SGRM}(n,\sigma^2)$:

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big) = n\cdot\exp(\tfrac{\sigma^2 s^2}{2})\cdot\Phi(1-n,2;-\sigma^2 s^2), \tag{0.2}$$

where Tr_n is the usual unnormalized trace on $M_n(\mathbb{C})$, and Φ is the confluent hypergeometric function (cf. formula (2.9) in Section 2). From (0.2), we obtain a simple proof of Wigner's Semi-circle Law in the sense of "convergence in moments", i.e., for a sequence (X_n) of random matrices, such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$ for all n,

$$\lim_{n\to\infty}\mathbb{E}\big(\mathrm{tr}_n[X_n^p]\big) = \tfrac{1}{2\pi}\int_{-2}^{2} x^p\sqrt{4-x^2}\,dx, \qquad (p\in\mathbb{N}), \tag{0.3}$$

where $\mathrm{tr}_n = \frac{1}{n}\mathrm{Tr}_n$ is the normalized trace on $M_n(\mathbb{C})$.

In Section 3, we apply (0.2) to show, that if (X_n) is a sequence of random matrices, defined on the same probability space, and such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$ for all n, then

$$\lim_{n\to\infty}\lambda_{\max}\big(X_n(\omega)\big) = 2, \qquad \text{for almost all } \omega, \tag{0.4}$$

$$\lim_{n\to\infty}\lambda_{\min}\big(X_n(\omega)\big) = -2, \qquad \text{for almost all } \omega, \tag{0.5}$$

where $\lambda_{\max}\big(X_n(\omega)\big)$ and $\lambda_{\min}\big(X_n(\omega)\big)$ denote the largest and smallest eigenvalues of $X_n(\omega)$, for each ω in the underlying probability space Ω. This result was proved by combinatorial methods for a much larger class of random matrices by Bai and Yin in [BY1] in 1988. Only random matrices with real entries are considered in [BY1], but the proofs also work in the complex case with only minor modifications (cf. [Ba, Thm. 2.12]). In Section 5 we apply (0.2) to give a new proof of a recursion formula due to Harer and Zagier, [HZ], namely the numbers

$$C(p,n) = \mathbb{E}(\mathrm{Tr}_n[A^{2p}]), \quad p = 0, 1, 2, \dots$$

for A in SGRM$(n,1)$ satisfies

$$C(p+1,n) = n \cdot \tfrac{4p+2}{p+2} \cdot C(p,n) + \tfrac{p(4p^2-1)}{p+2} \cdot C(p-1,n), \tag{0.6}$$

In Sections 6–9 we consider random matrices of the form B^*B where B is in the class GRM(m,n,σ^2) consisting of all $m \times n$ random matrices of the form $B = (b_{jk})_{j,k}$ where $\{b_{jk} \mid 1 \le j \le m, 1 \le k \le n\}$ is a set of mn independent, complex Gaussian random variables, each with density $\pi^{-1}\sigma^{-2}\exp(-|z|^2/\sigma^2)$, $z \in \mathbb{C}$. The distribution of B^*B is known as the complex Wishart distribution or the Laguerre ensemble (cf. [Go], [Kh] and [Fo]). In analogy with (1.1), the "mean density" of the eigenvalues of B^*B in the case $\sigma^2 = 1$ is given by

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi_k^{m-n}(x)^2 \tag{0.7}$$

where the sequence of functions $(\varphi_k^\alpha)_{k=0}^\infty$ can be expressed in terms of the Laguerre polynomials $L_n^\alpha(x)$:

$$\varphi_k^\alpha(x) = \Big[\tfrac{k!}{\Gamma(k+\alpha+1)} x^\alpha \exp(-x)\Big]^{\frac{1}{2}} L_k^\alpha(x). \tag{0.8}$$

From (0.7) and (0.8) we derive in Section 6 the following two formulas:

If $m \ge n$, $B \in$ GRM$(m,n,1)$, and $s \in \mathbb{C}$ such that $\mathrm{Re}(s) < n$,

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(sB^*B)]\big) = \sum_{k=1}^{n} \frac{F(k-m, k-n, 1; s^2)}{(1-s)^{m+n+1-2k}}, \tag{0.9}$$

$$\mathbb{E}\big(\mathrm{Tr}_n[B^*B\exp(sB^*B)]\big) = mn\frac{F(1-m, 1-n, 2; s^2)}{(1-s)^{m+n}}, \tag{0.10}$$

where $F(a,b,c;z)$ is the hyper-geometric function (cf. formula (6.8) in Section 6).

In Section 7, we use (0.10) to give a new proof of the following result originally due to Marchenko and Pastur [MP]:

Let (Y_n) be a sequence of random matrices, such that for all n, $Y_n \in$ GRM$(m(n), n, \frac{1}{n})$, where $m(n) \ge n$. Then, if $\lim_{n\to\infty} \frac{m(n)}{n} = c$, the mean distribution of the eigenvalues of $Y_n^*Y_n$ converges in moments to the probability measure μ_c on $[0,\infty[$ with density

$$\frac{d\mu_c(x)}{dx} = \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \cdot 1_{[a,b]}(x), \tag{0.11}$$

where $a = \left(\sqrt{c}-1\right)^2$ and $b = \left(\sqrt{c}+1\right)^2$. Specificly,

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[(Y_n^*Y_n)^p]\big) = \int_a^b x^p \; d\mu_c(x), \qquad (p \in \mathbb{N}), \tag{0.12}$$

Since Marchenko and Pastur's proof from 1967, many other proofs of (0.12) have been given both for the real and complex Wishart case (cf. [Wa], [GS], [Jo], [Ba] and [OP]).

In Section 8 we use (0.9) to prove that if $(Y_n)_{n=1}^{\infty}$ is a sequence of random matrices defined on the same probability space, such that $Y_n \in \mathrm{GRM}(m(n), n, \frac{1}{n})$ and $m(n) \geq n$ for all $n \in \mathbb{N}$, then if $\lim_{n\to\infty} \frac{m(n)}{n} = c$, one has

$$\lim_{n\to\infty} \lambda_{\max}(Y_n^*Y_n) = \left(\sqrt{c}+1\right)^2, \qquad \text{almost surely,} \tag{0.13}$$

$$\lim_{n\to\infty} \lambda_{\min}(Y_n^*Y_n) = \left(\sqrt{c}-1\right)^2, \qquad \text{almost surely.} \tag{0.14}$$

Again, this is not a new result. (0.13) was proved in 1980 by Geman [Gem] and (0.14) was proved in 1985 by Silverstein [Si]. Only the real Wishart case is considered in [Gem] and [Si], but the proofs can easily be generalized to the complex case. Moreover, (0.13) and (0.14) can be extended to a much larger class of random matrices (cf. [BY2] and [Ba]).

Finally, in Section 8, we use (0.10) combined with the differential equation for the hypergeometric function, to derive a recursion formula for the numbers:

$$D(p, m, n) = \mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big), \qquad (B \in \mathrm{GRM}(m, n, 1),\ p \in \mathbb{N}),$$

analogous to (0.6), namely

$$D(p+1, m, n) = \tfrac{(2p+1)(m+n)}{p+2} \cdot D(p, m, n) + \tfrac{(p-1)(p^2-(m-n)^2)}{p+2} \cdot D(p-1, m, n). \tag{0.15}$$

It would be interesting to know the counterparts of the explicit formulas (0.2), (0.6), (0.9), (0.10) and (0.15), for random matrices with real or symplectic Gaussian entries. The real and symplectic counterparts of the density (0.1) are computed in Mehta's book [Meh, Chap.6 and 7], and the real and symplectic counterpart of (0.7) can be found in Forrester's book manuscript [Fo, Chap.5]. However, the formulas for these densities are much more complicated than in the complex case.

1 Selfadjoint Gaussian Random Matrices

1.1 Definition. By $\mathrm{SGRM}(n, \sigma^2)$ we denote the class of $n \times n$ complex random matrices $A = (a_{jk})_{j,k=1}^n$, where $\bar{a}_{jk} = a_{jk}$, $j, k = 1, \ldots, n$ and

$$a_{jj}, \quad (\sqrt{2}\mathrm{Re}a_{jk})_{j<k}, \quad (\sqrt{2}\mathrm{Im}a_{jk})_{j<k}, \tag{1.1}$$

is a family of n^2 independent identically distributed real Gaussian random variables with mean value 0 and variance σ^2 (on the same probability space $(\Omega, \mathcal{F}, P)$). $\square$

The density of a Gaussian random variable with mean value 0 and variance σ^2 is given by

$$(2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\tfrac{x^2}{2\sigma^2}\right).$$

For a selfadjoint matrix $H = (h_{jk})_{j,k=1}^n$

$$\mathrm{Tr}_n(H^2) = \sum_j h_{jj}^2 + 2\sum_{j<k} |h_{jk}|^2.$$

Therefore, the distribution μ of a random matrix $A \in \mathrm{SGRM}(n, \frac{1}{n})$ (considered as a probability distribution on $M_n(\mathbb{C})_{\mathrm{sa}}$) is given by

$$d\mu(H) = c_1 \exp\left(-\frac{1}{2\sigma^2}\mathrm{Tr}_r(H^2)\right) dH, \tag{1.2}$$

where dH is the Lebesgue measure on $M_n(\mathbb{C})_{\mathrm{sa}}$:

$$dH = \prod_{j=1}^n dh_{jj} \prod_{1\le j<k\le n} d\mathrm{Re}(h_{jk}) d\mathrm{Im}(h_{jk}) \tag{1.3}$$

and

$$c_1 = \left(2^{k/2}(\pi\sigma^2)^{k^2/2}\right)^{-1}.$$

For a selfadjoint matrix $H \in M_n(\mathbb{C})_{\mathrm{sa}}$ we let $\lambda_1(H) \le \lambda_2(H) \le \cdots \le \lambda_n(H)$ denote the ordered list of eigenvalues. Put

$$\Lambda = \left\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n\right\}$$

and let $\eta\colon M_n(\mathbb{C})_{\mathrm{sa}} \to \Lambda$ denote the map

$$\eta(H) = (\lambda_1(H), \dots, \lambda_n(H)), \quad H \in M_n(\mathbb{C})_{\mathrm{sa}}.$$

Then the image measure $\eta(d\mu)$ of the measure $d\mu$, given by (1.2), is equal to

$$\eta(d\mu) = c_2 \prod_{1\le j<k\le n} (\lambda_j - \lambda_k)^2 \exp\left(-\frac{1}{2\sigma^2}\sum_{k=1}^n \lambda_k^2\right) d\lambda_1, \dots, d\lambda_n$$

for $(\lambda_1, \dots, \lambda_n) \in \Lambda$, where $c_2 > 0$ is a new normalization constant:

$$c_2 = \left(\pi^{n(n-1)/2} \prod_{j=1}^{n-1} j!\right)^{-1}$$

(cf. [Meh, Chap.5] or [De, Sect. 5.3]). Hence, after averaging over all permutations of $(\lambda_1, \dots, \lambda_n)$, we get that for any symmetric Borel function $\varphi\colon \mathbb{R}^n \to \mathbb{C}$ one has

$$\int_{M_n(\mathbb{C})_{\mathrm{sa}}} \varphi(\lambda_1(H), \dots, \lambda_n(H))\, d\mu(H) = \int_{\mathbb{R}^n} \varphi(\underline{\lambda}) g(\underline{\lambda})\, d\lambda_1, \dots, d\lambda_n \tag{1.4}$$

where $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ and

$$g(\lambda_1, \dots, \lambda_n) = \frac{c_2}{n!} \prod_{j<k} (\lambda_j - \lambda_k)^2 \exp\big(-\frac{1}{2\sigma^2} \sum_{j=1}^{n} \lambda_j^2\big) \tag{1.5}$$

provided the integrals on both sides of (1.4) are defined. The marginal density h corresponding to (1.5):

$$h(\lambda) = \int_{\mathbb{R}^{n-1}} g(\lambda, \lambda_2, \dots, \lambda_n)\ d\lambda_2, \dots, d\lambda_n, \quad (\lambda \in \mathbb{R}), \tag{1.6}$$

can be computed explicitly. For $\sigma^2 = \frac{1}{2}$, one gets by [Meh, Formulas 5.1.2 and 5.2.16] that h is equal to $\sum_{k=1}^{n} \varphi_k(x)^2$, where $(\varphi_k)_{k=0}^{\infty}$ is the sequence of Hermite functions:

$$\varphi_k(x) = \tfrac{1}{(2^k k! \sqrt{\pi})^{1/2}} H_k(x) \exp(-\tfrac{x^2}{2}), \qquad (k \in \mathbb{N}_0), \tag{1.7}$$

and $H_0, H_1, H_2, \dots$, are the Hermite polynomials:

$$H_k(x) = (-1)^k \exp(x^2) \cdot \Big(\frac{d^k}{dx^k} \exp(-x^2)\Big), \qquad (k \in \mathbb{N}_0), \tag{1.8}$$

(cf. [HTF, Vol. 2, p.193, formula (7)]). Hence, by a simple scaling argument, one gets that for general $\sigma^2 > 0$,

$$h(\lambda) = \frac{1}{n\sigma\sqrt{2}} \sum_{k=0}^{n-1} \varphi_k\big(\tfrac{\lambda}{\sigma\sqrt{2}}\big)^2, \quad \lambda \in \mathbb{R}. \tag{1.9}$$

From the above, one easily gets:

1.2 Proposition. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Borel function, and let $a \mapsto f(a)$ be the map from $M_n(\mathbb{C})_{\rm sa}$ into itself, obtained by the usual function calculus for selfadjoint operators on Hilbert space. Consider furthermore the function h given by (1.9). Then for any element A of $\mathrm{SGRM}(n, \sigma^2)$, we have that*

$$\mathbb{E}\big(\mathrm{Tr}_n[f(A)]\big) = n \int_{\mathbb{R}} f(\lambda) h(\lambda)\ d\lambda, \tag{1.10}$$

provided that the integral on the right hand side of (1.10) is well-defined (i.e., $f \geq 0$ or $\int_{\mathbb{R}} |f(\lambda)| h(\lambda)\ d\lambda < \infty$).

Proof. Assume first that $f \geq 0$. Since

$$\mathrm{Tr}_n[f(A)] = f(\lambda_1(A)) + \cdots + f(\lambda_n(A)),$$

is a symmetric function of the eigenvalues $\lambda_1(A), \dots, \lambda_n(A)$, it follows from (1.4) that

$$\mathbb{E}\big(\mathrm{Tr}_n[f(A)]\big) = \int_{\mathbb{R}^n} \big(\textstyle\sum_{j=1}^{n} f(\lambda_j)\big) \cdot g(\lambda_1, \dots, \lambda_n)\ d\lambda_1, \dots, d\lambda_n.$$

Using then that g is invariant under permutations of $\lambda_1,\dots,\lambda_n$, it follows that

$$\begin{aligned}\mathbb{E}\big(\mathrm{Tr}_n[f(A)]\big) &= n\cdot\int_{\mathbb{R}^n} f(\lambda_1)\cdot g(\lambda_1,\dots,\lambda_n)\ d\lambda_1,\dots,d\lambda_n\\ &= n\int_{\mathbb{R}} f(\lambda)h(\lambda)\ d\lambda,\end{aligned}$$

which proves that (1.10) holds whenever $f\geq 0$. For general, complex-valued Borel functions f, satisfying that $\int_{\mathbb{R}}|f(\lambda)|h(\lambda)\ d\lambda<\infty$, (1.10) follows then from the positive case, and the standard decomposition:

$$f=(\mathrm{Re}f)^+-(\mathrm{Re}f)^-+i\big((\mathrm{Im}f)^+-(\mathrm{Im}f)^-\big). \qquad \blacksquare$$

1.3 Remark. Let $A\in\mathrm{SGRM}(n,\sigma^2)$, and let $\lambda_1(A)\leq\cdots\leq\lambda_n(A)$ be the ordered eigenvalues of A considered as random variables on the underlying probability space Ω. Let, further, ν_k be the probability distribution (on $\mathbb{R}$) of $\lambda_k(A)$, $k=1,\dots,n$. Then

$$h(\lambda)\ d\lambda=\frac{1}{n}\sum_{k=1}^n\nu_k.$$

For that reason, $h(\lambda)$ is called the "mean density" of the eigenvalue distribution of A. $\square$

2 The moment generating function for GUE random matrices

If $A\in\mathrm{SGRM}(n,\sigma^2)$, then $\frac{1}{\sigma\sqrt{2}}A\in\mathrm{SGRM}(n,\frac{1}{2})$, which is the Gaussian, unitary ensemble (GUE) in [Meh, Chapter 5]. Hence, up to a scaling factor, $\mathrm{SGRM}(n,\sigma^2)$ is the same as the GUE-case. In this section we will prove formula (1.2) (cf. Theorem 2.5 below) and use it to give a new proof of the Wigner semi-circle law in the GUE-case. We start by quoting a classical result from probability theory:

2.1 Proposition. *Let $\mu,\mu_1,\mu_2,\mu_3,\dots$, be probability measures on $\mathbb{R}$, and consider the corresponding distribution functions:*

$$F(x)=\mu\big(]-\infty,x]\big),\quad F_n(x)=\mu_n\big(]-\infty,x]\big),\qquad (x\in\mathbb{R},\ n\in\mathbb{N}).$$

Let $C_0(\mathbb{R})$ and $C_b(\mathbb{R})$ denote the set of continuous functions on $\mathbb{R}$ that vanish at $\pm\infty$, respectively the set of continuous, bounded functions on $\mathbb{R}$.

Then the following conditions are equivalent:

(i) $\lim_{n\to\infty}F_n(x)=F(x)$ *for all points x of $\mathbb{R}$ in which F is continuous.*

(ii) $\forall f\in C_0(\mathbb{R})\colon\ \lim_{n\to\infty}\int_{\mathbb{R}}f\ d\mu_n=\int_{\mathbb{R}}f\ d\mu.$

(iii) $\forall f \in C_b(\mathbb{R})$: $\lim_{n\to\infty} \int_{\mathbb{R}} f \, d\mu_n = \int_{\mathbb{R}} f \, d\mu$.

(iv) $\forall t \in \mathbb{R}$: $\lim_{n\to\infty} \int_{\mathbb{R}} \exp(itx) \, d\mu_n(x) = \int_{\mathbb{R}} \exp(itx) \, d\mu(x)$.

Proof. Cf. [Fe, Chapter VIII: Criterion 1,Theorem 1,Theorem 2 and Chapter XV: Theorem 2]. ■

2.2 Definition. Let Prob($\mathbb{R}$) denote the set of probability measures on $\mathbb{R}$. Following standard notation, we say that a sequence $(\mu_n)_{n=1}^{\infty}$ in Prob($\mathbb{R}$) converges *weakly* to $\mu \in$ Prob($\mathbb{R}$) if the above equivalent conditions (i)–(iv) hold. Moreover, we say that μ_n converges to μ *in moments*, if μ_n and μ have moments of all orders, i.e.,

$$\int_{\mathbb{R}} |x|^p \, d\mu(x) < \infty, \quad \text{and} \quad \int_{\mathbb{R}} |x|^p \, d\mu_n(x) < \infty, \qquad (p, n \in \mathbb{N})$$

and the following holds

$$\lim_{n\to\infty} \int_{\mathbb{R}} x^p \, d\mu_n(x) = \int_{\mathbb{R}} x^p \, d\mu(x), \qquad (p \in \mathbb{N}). \qquad \square$$

In general convergence in moments do not imply weak convergence or visa versa. However, if μ is uniquely determined by its moments, then $\mu_n \to \mu$ in moments implies that $\mu_n \to \mu$ weakly (cf. [Bre, Thm. 8.48]). In particular this holds if the limit measure μ has compact support.

Let $(H_k)_{k=0}^{\infty}$ and $(\varphi_k)_{k=0}^{\infty}$ denote the sequences of Hermite polynomials and Hermite functions given by (1.8) and (1.7). Then $H_0, H_1, \ldots$ satisfy the orthogonality relations

$$\int_{-\infty}^{\infty} H_k(x) H_\ell(x) e^{-x^2} dx = \begin{cases} \sqrt{\pi} 2^k k!, & k = \ell \\ 0, & k \neq \ell \end{cases} \tag{2.1}$$

(cf. [HTF, Vol.2, p.164 and p.193 formula (4)]). Hence

$$\int_{-\infty}^{\infty} \varphi_k(x) \varphi_\ell(x) dx = \begin{cases} 1, & k = \ell \\ 0, & k \neq \ell \end{cases} \tag{2.2}$$

2.3 Lemma. *Let (φ_n) denote the sequence of Hermite functions given by* (1.7). *We then have*

$$\varphi_0'(x) = -\tfrac{1}{\sqrt{2}} \varphi_1(x), \tag{2.3}$$

$$\varphi_n'(x) = \sqrt{\tfrac{n}{2}} \varphi_{n-1}(x) - \sqrt{\tfrac{n+1}{2}} \varphi_{n+1}(x), \qquad (n \in \mathbb{N}), \tag{2.4}$$

$$\frac{d}{dx}\Big(\sum_{k=0}^{n-1} \varphi_k(x)^2 \Big) = -\sqrt{2n} \varphi_n(x) \varphi_{n-1}(x), \qquad (n \in \mathbb{N}). \tag{2.5}$$

Proof. The equations (2.3) and (2.4) follow from (1.7) and the elementary formulas

$$x H_n(x) = \tfrac{1}{2} H_{n+1} + n H_{n-1}(x), \tag{2.6}$$

$$H_n'(x) = 2n H_{n-1}(x), \tag{2.7}$$

(cf. [HTF, Vol. 2, p. 193, formulas (10) and (14)]). Moreover, (2.5) is easily derived from (2.3) and (2.4). ■

For any non-negative integer n, and any complex number w, we apply the notation

$$(w)_n = \begin{cases} 1, & \text{if } n = 0, \\ w(w+1)(w+2)\cdots(w+n-1), & \text{if } n \in \mathbb{N}. \end{cases} \tag{2.8}$$

Recall then, that the confluent hyper-geometric function $(a, c, x) \mapsto \Phi(a, c; x)$ is defined by the expression:

$$\Phi(a, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(c)_n n!} = 1 + \frac{a}{c}\frac{x}{1} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2} + \cdots, \tag{2.9}$$

for a, c, x in $\mathbb{C}$, such that $c \notin \mathbb{Z} \setminus \mathbb{N}$ (cf. [HTF, Vol. 1, p.248]). Note, in particular, that if $a \in \mathbb{Z} \setminus \mathbb{N}$, then $x \mapsto \Phi(a, c; x)$ is a polynomial in x of degree $-a$, for any permitted c.

2.4 Lemma. *For any s in $\mathbb{C}$ and k in $\mathbb{N}_0$,*

$$\begin{aligned} \int_{\mathbb{R}} \exp(sx)\varphi_k(x)^2 \, dx &= \exp(\tfrac{s^2}{4})\Phi(-k, 1; -\tfrac{s^2}{2}) \\ &= \exp(\tfrac{s^2}{4})\sum_{j=0}^{k} \frac{k(k-1)\cdots(k+1-j)}{(j!)^2}\Big(\frac{s^2}{2}\Big)^j, \end{aligned} \tag{2.10}$$

and for s in $\mathbb{C}$ and n in $\mathbb{N}$,

$$\begin{aligned} &\int_{\mathbb{R}} \exp(sx)\big(\textstyle\sum_{k=0}^{n-1} \varphi_k(x)^2\big) \, dx \\ &\qquad = n \cdot \exp(\tfrac{s^2}{4})\Phi(1-n, 2; -\tfrac{s^2}{2}) \\ &\qquad = n \cdot \exp(\tfrac{s^2}{4})\sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!}\Big(\frac{s^2}{2}\Big)^j. \end{aligned} \tag{2.11}$$

Proof. For l, m in $\mathbb{N}_0$ and s in $\mathbb{R}$, we have that

$$\int_{\mathbb{R}} \exp(sx)\varphi_l(x)\varphi_m(x) \, dx = \frac{1}{(2^{l+m} l! m! \pi)^{1/2}} \int_{\mathbb{R}} \exp(sx - x^2) H_l(x) H_m(x) \, dx. \tag{2.12}$$

By the substitution $y = x - \frac{s}{2}$, the integral on the right hand side of (2.12) becomes

$$\exp(\tfrac{s^2}{4})\int_{\mathbb{R}} \exp(-y^2) H_l(y + \tfrac{s}{2}) H_m(y + \tfrac{s}{2}) \, dy. \tag{2.13}$$

Note here, that by (1.8) we have for a in $\mathbb{R}$ and k in $\mathbb{N}_0$,

$$\begin{aligned} H_k(x+a) &= (-1)^k \exp((x+a)^2) \cdot \Big(\frac{d^k}{dx^k}\exp(-(x+a)^2)\Big) \\ &= (-1)^k \exp(x^2 + 2ax) \sum_{j=0}^{k} \binom{k}{j} \Big(\frac{d^j}{dx^j}\exp(-x^2)\Big)\Big(\frac{d^{k-j}}{dx^{k-j}}\exp(-2ax)\Big), \end{aligned}$$

which can be reduced to

$$H_k(x+a) = \sum_{j=0}^{k} \binom{k}{j} (2a)^{k-j} H_j(x). \tag{2.14}$$

It follows thus that the quantity in (2.13) equals

$$\exp(\tfrac{s^2}{4}) \int_{\mathbb{R}} \exp(-y^2) \Big(\sum_{j=0}^{l} \binom{l}{j} (s)^{l-j} H_j(y) \Big) \Big(\sum_{j=0}^{m} \binom{m}{j} (s)^{m-j} H_j(y) \Big)\ dy,$$

which by the orthogonality relations (2.1) can be reduced to

$$\exp(\tfrac{s^2}{4}) \sum_{j=0}^{\min\{l,m\}} \binom{l}{j} \binom{m}{j} 2^j j! \sqrt{\pi}\ s^{l+m-2j}.$$

Altogether, we have shown that for m, l in $\mathbb{N}_0$ and s in $\mathbb{R}$,

$$\int_{\mathbb{R}} \exp(sx) \varphi_l(x) \varphi_m(x)\ dx = \frac{\exp(\frac{s^2}{4})}{\sqrt{l! m!}} \sum_{j=0}^{\min\{l,m\}} j! \binom{l}{j} \binom{m}{j} \Big(\frac{s}{\sqrt{2}} \Big)^{l+m-2j}. \tag{2.15}$$

But since both sides of (2.15) are analytic functions of $s \in \mathbb{C}$, the formula (2.15) holds for all s in $\mathbb{C}$.

Putting now $l = m = k$, and substituting j by $k - j$, (2.15) becomes

$$\begin{aligned} \int_{\mathbb{R}} \exp(sx) \varphi_k(x)^2\ dx &= \frac{\exp(\frac{s^2}{4})}{k!} \sum_{j=0}^{k} (k-j)! \binom{k}{j}^2 \Big(\frac{s}{\sqrt{2}} \Big)^{2j} \\ &= \exp(\tfrac{s^2}{4}) \sum_{j=0}^{k} \frac{k(k-1)\cdots(k+1-j)}{(j!)^2} \Big(\frac{s^2}{2} \Big)^j, \end{aligned}$$

and this proves (2.10).

The formula (2.11) is trivial in the case $s = 0$, because of the orthogonality relations (2.2). If $s \in \mathbb{C} \setminus \{0\}$, then by (2.5) and partial integration, we get that

$$\int_{\mathbb{R}} \exp(sx) \big(\textstyle\sum_{k=0}^{n-1} \varphi_k(x)^2 \big)\ dx = \frac{\sqrt{2n}}{s} \displaystyle\int_{\mathbb{R}} \exp(sx) \varphi_n(x) \varphi_{n-1}(x)\ dx.$$

Using now (2.15) in the case $l = n,\ m = n-1$, we get, after substituting j by $n - 1 - j$, that

$$\begin{aligned} \frac{\sqrt{2n}}{s} \int_{\mathbb{R}} \exp(sx) \varphi_n(x) \varphi_{n-1}(x)\ dx &= \frac{\sqrt{2} \exp(\frac{s^2}{4})}{s(n-1)!} \sum_{j=0}^{n-1} (n-1-j)! \binom{n}{j+1} \binom{n-1}{j} \Big(\frac{s}{\sqrt{2}} \Big)^{2j+1} \\ &= n \exp(\tfrac{s^2}{4}) \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!} \Big(\frac{s^2}{2} \Big)^j, \end{aligned}$$

and (2.11) follows. ■

2.5 Theorem. (i) *For any element A of* $\mathrm{SGRM}(n,\sigma^2)$ *and any s in $\mathbb{C}$, we have that*

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big) = n\cdot\exp(\tfrac{\sigma^2 s^2}{2})\cdot\Phi(1-n,2;-\sigma^2 s^2). \tag{2.16}$$

(ii) *Let (X_n) be a sequence of random matrices, such that $X_n \in \mathrm{SGRM}(n,\frac{1}{n})$ for all n in $\mathbb{N}$. Then for any s in $\mathbb{C}$, we have that*

$$\lim_{n\to\infty}\mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big) = \tfrac{1}{2\pi}\int_{-2}^{2}\exp(sx)\sqrt{4-x^2}\,dx, \tag{2.17}$$

and the convergence is uniform on compact subsets of $\mathbb{C}$.

Proof. From (1.9) and Proposition 1.2 we have

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big) = \frac{1}{n\sigma\sqrt{2}}\int_{-\infty}^{\infty}\exp(s\lambda)\sum_{k=0}^{n-1}\varphi_k\big(\tfrac{\lambda}{\sigma\sqrt{2}}\big)^2 d\lambda. \tag{2.18}$$

Hence (2.16) follows from (2.11) by substituting $x=\frac{\lambda}{\sigma\sqrt{2}}$ in (2.18). This proves (i). By application of (i), it follows then, that for X_n from $\mathrm{SGRM}(n,\frac{1}{n})$ and s in $\mathbb{C}$, we have that

$$\begin{aligned}\mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big) &= \exp(\tfrac{s^2}{2n})\cdot\Phi(1-n,2;-\tfrac{s^2}{n})\\ &= \exp(\tfrac{s^2}{2n})\sum_{j=0}^{n-1}\frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!}\Big(\frac{s^2}{n}\Big)^j.\end{aligned} \tag{2.19}$$

By Lebesgue's Theorem on Dominated Convergence, it follows thus that

$$\lim_{n\to\infty}\mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big) = \sum_{j=0}^{\infty}\frac{s^{2j}}{j!(j+1)!}.$$

The even moments of the standard semi-circular distribution are:

$$\tfrac{1}{2\pi}\int_{\mathbb{R}}x^{2p}\sqrt{4-x^2}\,dx = \tfrac{1}{p+1}\binom{2p}{p}, \qquad (p\in\mathbb{N}_0),$$

and the odd moments vanish. Hence, using the power series expansion of $\exp(sx)$, we find that

$$\tfrac{1}{2\pi}\int_{-2}^{2}\exp(sx)\sqrt{4-x^2}\,dx = \sum_{j=0}^{\infty}\frac{s^{2j}}{(2j)!(j+1)}\binom{2j}{j} = \sum_{j=0}^{\infty}\frac{s^{2j}}{j!(j+1)!}.$$

Therefore,

$$\lim_{n\to\infty}\mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big) = \tfrac{1}{2\pi}\int_{-2}^{2}\exp(sx)\sqrt{4-x^2}\,dx, \qquad (s\in\mathbb{C}). \tag{2.20}$$

Note next, that by (2.19), we have that

$$\big|\mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big)\big| \le \sum_{j=0}^{\infty}\frac{|s|^{2j}}{j!(j+1)!}, \qquad (s\in\mathbb{C}),$$

so the functions $s \mapsto \mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big)$, $(n \in \mathbb{N})$, are uniformly bounded on any fixed bounded subset of $\mathbb{C}$. Hence by a standard application of Cauchy's Integral Formula and Lebesgue's theorem on Dominated Convergence, it follows that the convergence in (2.20) is uniform on compact subsets of $\mathbb{C}$. ■

Wigner's semi-circle law for the GUE-case (cf. [Wig1], [Wig2] and [Meh, Chap.5]) is now a simple consequence of Theorem 2.5. We formulate it both in the sense of convergence in moments and in the sense of weak convergence (cf. Definition 2.2):

2.6 Corollary. *(cf. [Wig1], [Wig2], [Meh]) Let (X_n) be a sequence of random matrices, such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$ for all n. We then have*

(i) *For any p in $\mathbb{N}$,*

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[X_n^p]\big) = \tfrac{1}{2\pi}\int_{-2}^{2} x^p\sqrt{4-x^2}\ dx. \tag{2.21}$$

(ii) *For every continuous bounded function $f\colon \mathbb{R} \to \mathbb{C}$,*

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[f(X_n)]\big) = \tfrac{1}{2\pi}\int_{-2}^{2} f(x)\sqrt{4-x^2}\ dx.$$

Proof. Let $h_n(\lambda)$ denote the function $h(\lambda)$ in (1.9) for the special case $\sigma^2 = \frac{1}{n}$. By Proposition 1.2 and Theorem 2.5(ii),

$$\lim_{n\to\infty}\int_{\mathbb{R}} \exp(sx)h_n(x)dx = \tfrac{1}{2\pi}\int_{-2}^{2} \exp(sx)\sqrt{4-x^2}\ dx$$

for all $s \in \mathbb{C}$ and the convergence is uniform in s on compact subsets of $\mathbb{C}$. Hence, by Cauchy's integral formulas, we have

$$\lim_{n\to\infty} \frac{d^p}{ds^p}\Big(\int_{\mathbb{R}} \exp(sx)h_n(x)\ dx\Big) = \frac{d^p}{ds^p}\Big(\tfrac{1}{2\pi}\int_{-2}^{2} \exp(sx)\sqrt{4-x^2}\ dx\Big),$$

for all s in $\mathbb{C}$. Putting $s = 0$, it follows that

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[X_n^p]\big) = \lim_{n\to\infty}\Big(\int_{\mathbb{R}} x^p h_n(x)\ dx\Big) = \tfrac{1}{2\pi}\int_{-2}^{2} x^p\sqrt{4-x^2}\ dx,$$

which proves (i). Putting $s = it$ in Lemma 2.5(ii), it follows that for any t in $\mathbb{R}$,

$$\lim_{n\to\infty}\int_{\mathbb{R}} \exp(itx)h_n(x)\ dx = \tfrac{1}{2\pi}\int_{-2}^{2} \exp(itx)\sqrt{4-x^2}\ dx. \tag{2.22}$$

Hence by Proposition 2.1,

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[f(X_n)]\big) = \lim_{n\to\infty}\int_{\mathbb{R}} f(x)h_n(x)\ dx = \tfrac{1}{2\pi}\int_{-2}^{2} f(x)\sqrt{4-x^2}\ dx,$$

for any continuous bounded function f on $\mathbb{R}$, and this proves (ii). ■

2.7 Remark. Arnold's strengthening of Wigner's Semi-circle Law to a result about almost sure convergence of the empirical distributions of the eigenvalues (cf. [Ar]), will be taken up in Section 3 (see Proposition 3.6). A very good survey of the history of Wigner's Semi-circle Law is given by Olson and Uppuluri in [OU]. □

3 Almost Sure Convergence of the Largest and Smallest Eigenvalues in the GUE case

Bai and Yin proved in [BY1] that for a large class of selfadjoint random matrices for which Wigner's semi-circle law holds, one also gets that the largest (resp. smallest) eigenvalue converges almost surely to 2 (resp. -2) as $n \to \infty$. In [BY1] only random matrices with real entries are considered, but the proof can easily be extended to the complex case (cf. [Ba, Thm. 2.12]). In this section we will give a simple proof of Bai's and Yin's result in the special case of GUE random matrices, based on Theorem 2.5 (cf. Theorem 3.1 below).

Thanks to results of Tracy and Widom ([TW1], [TW2]), one now has much more precise information on the asymptotic behavior of the largest (and smallest) eigenvalue in the GUE case, as well as in the corresponding real and symplectic cases (GUE and GSE). These results, however, lie outside the scope of the present paper.

3.1 Theorem. *(cf. [BY1] and [Ba]) Let (X_n) be a sequence of random matrices, defined on the same probability space $(\Omega, \mathcal{F}, P)$, and such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$, for each n in $\mathbb{N}$. For each ω in Ω and n in $\mathbb{N}$, let $\lambda_{\max}\big(X_n(\omega)\big)$ and $\lambda_{\min}\big(X_n(\omega)\big)$ denote the largest respectively the smallest eigenvalue of $X_n(\omega)$. We then have*

$$\lim_{n\to\infty} \lambda_{\max}(X_n) = 2, \qquad \textit{almost surely,} \tag{3.1}$$

and

$$\lim_{n\to\infty} \lambda_{\min}(X_n) = -2, \qquad \textit{almost surely.} \tag{3.2}$$

For the proof of Theorem 3.1, we need some lemmas:

3.2 Lemma. (Borel-Cantelli) *Let $F_1, F_2, F_3, \dots,$ be a sequence of measurable subsets of Ω, and assume that $\sum_{n=1}^{\infty} P(\Omega \setminus F_n) < \infty$. Then $P(F_n \text{ eventually}) = 1$, where*

$$(F_n \text{ eventually}) = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} F_n,$$

i.e., for almost all ω in Ω, $\omega \in F_n$ eventually as $n \to \infty$.

Proof. Cf. [Bre, Lemma 3.14]. ■

3.3 Lemma. *Let (X_n) be a sequence of random matrices, defined on the same probability space $(\Omega, \mathcal{F}, P)$, and such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$ for all n in $\mathbb{N}$. We then have,*

$$\limsup_{n\to\infty} \lambda_{\max}(X_n) \leq 2, \qquad \textit{almost surely,} \tag{3.3}$$

and

$$\liminf_{n\to\infty} \lambda_{\min}(X_n) \geq -2, \qquad \textit{almost surely.} \tag{3.4}$$

Proof. By (2.19), we have for any n in $\mathbb{N}$, that

$$\begin{aligned}
\mathbb{E}\big(\mathrm{Tr}_n[\exp(tX_n)]\big) &= n \cdot \exp(\tfrac{t^2}{2n})\sum_{j=0}^{\infty} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!}\Big(\frac{t^2}{n}\Big)^j \\
&\le n \cdot \exp(\tfrac{t^2}{2n})\sum_{j=0}^{\infty} \frac{t^{2j}}{j!(j+1)!} \\
&\le n \cdot \exp(\tfrac{t^2}{2n})\Big[\sum_{j=0}^{\infty} \frac{t^j}{j!}\Big]^2.
\end{aligned}$$

It follows thus, that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tX_n)]\big) \le n \cdot \exp(\tfrac{t^2}{2n} + 2t), \qquad (t \in \mathbb{R}_+). \tag{3.5}$$

Note here, that since all eigenvalues of $\exp(tX_n)$ are positive, we have that

$$\mathrm{Tr}_n[\exp(tX_n)] \ge \lambda_{\max}(\exp(tX_n)) = \exp(t\lambda_{\max}(X_n)),$$

and hence by (3.5) and integration,

$$\mathbb{E}\big(\exp(t\lambda_{\max}(X_n))\big) \le n \cdot \exp(\tfrac{t^2}{2n} + 2t), \qquad (t \in \mathbb{R}_+). \tag{3.6}$$

It follows thus, that for any ϵ in $]0, \infty[$,

$$\begin{aligned}
P(\lambda_{\max}(X_n) \ge 2 + \epsilon) &= P\big(\exp(t\lambda_{\max}(X_n) - t(2+\epsilon)) \ge 1\big) \\
&\le \mathbb{E}\big(\exp(t\lambda_{\max}(X_n) - t(2+\epsilon))\big) \\
&\le \exp(-t(2+\epsilon))\mathbb{E}\big(\exp(t\lambda_{\max}(X_n))\big),
\end{aligned}$$

and hence by (3.6),

$$P(\lambda_{\max}(X_n) \ge 2 + \epsilon) \le n \cdot \exp(\tfrac{t^2}{2n} - \epsilon t), \qquad (t \in \mathbb{R}_+). \tag{3.7}$$

As a function of $t \in \mathbb{R}_+$, the right hand side of (3.7) attains its minimum when $t = n\epsilon$. For this value of t, (3.7) becomes,

$$P(\lambda_{\max}(X_n) \ge 2 + \epsilon) \le n \cdot \exp(\tfrac{-n\epsilon^2}{2}).$$

Hence by the Borel-Cantelli Lemma (Lemma 3.2),

$$\limsup_{n\to\infty} \lambda_{\max}(X_n) \le 2 + \epsilon, \qquad \text{almost surely.}$$

Since this holds for arbitrary positive ϵ, we have proved (3.3). We note finally that (3.4) follows from (3.3), since the sequence $(-X_n)$ of random matrices also satisfies that $-X_n \in \mathrm{SGRM}(n, \frac{1}{n})$ for all n. ∎

To complete the proof of Theorem 3.1, we shall need an "almost sure convergence version" of Wigner's semi-circle law. This strengthened version of the semi-circle law was proved by Arnold in [Ar]. Arnold's result is proved for real symmetric random matrices, with rather general conditions on the entries. His proof is combinatorial and can easily be generalized to the complex case. For convenience of the reader, we include below a short proof of Arnold's result in the GUE case (cf. Proposition 3.6 below). The proof relies on the following lemma, due to Pisier (cf. [Pi, Theorem 4.7]), which is related to the "concentration of measure phenomenon" (cf. [Mi]).

3.4 Lemma. ([Pi]) *Let $G_{N,\sigma}$ denote the Gaussian distribution on $\mathbb{R}^N$ with density*

$$\frac{dG_{N,\sigma}(x)}{dx} = (2\pi\sigma^2)^{-N/2}\exp(-\tfrac{\|x\|^2}{2\sigma^2}), \tag{3.8}$$

where $\|x\|$ is the Euclidean norm of x. Furthermore, let $F\colon \mathbb{R}^N \to \mathbb{R}$ be a function that satisfies the Lipschitz condition

$$|F(x) - F(y)| \le c\|x - y\|, \qquad (x, y \in \mathbb{R}^N), \tag{3.9}$$

for some positive constant c. Then for any positive number t, we have that

$$G_{N,\sigma}\big(\big\{x \in \mathbb{R}^N \bigm| |F(x) - \mathbb{E}(F)| > t\big\}\big) \le 2\exp(-\tfrac{Kt^2}{c^2\sigma^2}),$$

where $\mathbb{E}(F) = \int_{\mathbb{R}^N} F(x)\, dG_{N,\sigma}(x)$, and $K = \frac{2}{\pi^2}$.

Proof. For $\sigma = 1$, this is proved in [Pi, Theorem 4.7], and the general case follows easily from this case, by using that $G_{N,\sigma}$ is the range measure of $G_{N,1}$ under the mapping $x \mapsto \sigma x\colon \mathbb{R}^N \to \mathbb{R}^N$, and that the composed function $x \mapsto F(\sigma x)$, satisfies a Lipschitz condition with constant $c\sigma$. ∎

The following result is also well-known:

3.5 Lemma. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that satisfies the Lipschitz condition*

$$|f(s) - f(t)| \le c|s - t|, \qquad (s, t \in \mathbb{R}). \tag{3.10}$$

Then for any n in $\mathbb{N}$, and all matrices A, B in $M_n(\mathbb{C})_{\rm sa}$, we have that

$$\|f(A) - f(B)\|_{\rm HS} = c\|A - B\|_{\rm HS},$$

where $\|\cdot\|_{\rm HS}$ is the Hilbert-Schmidt norm, i.e., $\|C\|_{\rm HS} = \mathrm{Tr}_n(C^2)^{1/2}$, for all C in $M_n(\mathbb{C})_{\rm sa}$.

Proof. The proof can be extracted from the proof of [Co, Proposition 1.1]: Note first that we may write,

$$A = \sum_{i=1}^n \lambda_i E_i, \quad B = \sum_{i=1}^n \mu_i F_i,$$

where $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_n$ are the eigenvalues of A and B respectively, and where $E_1 \dots, E_n$ and $F_1, \dots, F_n$ are two families of mutually orthogonal one-dimensional projections (adding up to $\mathbf{1}_n$). Using then that $\mathrm{Tr}_n(E_iF_j) \geq 0$ for all i,j, we find that

$$\begin{aligned}\|f(A)-f(B)\|_{\mathrm{HS}}^2 &= \mathrm{Tr}_n(f(A)^2) + \mathrm{Tr}_n(f(B)^2) - 2\mathrm{Tr}_n(f(A)f(B)) \\ &= \sum_{i,j=1}^n (f(\lambda_i) - f(\mu_j))^2 \cdot \mathrm{Tr}_n(E_iF_j) \\ &\leq c^2 \cdot \sum_{i,j=1}^n (\lambda_i - \mu_j)^2 \cdot \mathrm{Tr}_n(E_iF_j) \\ &= c^2\|A-B\|_{\mathrm{HS}}^2. \qquad \blacksquare\end{aligned}$$

3.6 Proposition. (cf. [Ar]) *Let (X_n) be a sequence of random matrices, defined on the same probability space $(\Omega, \mathcal{F}, P)$, and such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$, for each n in $\mathbb{N}$. For each ω in Ω, let $\mu_{n,\omega}$ denote the empirical distribution of the ordered eigenvalues $\lambda_1(X_n(\omega)) \leq \lambda_2(X_n(\omega)) \leq \cdots \leq \lambda_n(X_n(\omega))$, of $X_n(\omega)$, i.e., with the usual Dirac measure notation,*

$$\mu_{n,\omega} = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i(X_n(\omega))}. \tag{3.11}$$

Then for almost all ω in Ω, $\mu_{n,\omega}$ converges weakly to the standard semi-circular distribution γ, with density $x \mapsto \frac{1}{2\pi}\sqrt{4-x^2} \cdot 1_{[-2,2]}(x)$.

Hence, for any interval I in $\mathbb{R}$, and almost all ω in Ω, we have that

$$\lim_{n\to\infty} \Big(\tfrac{1}{n} \cdot \mathrm{card}\big(\mathrm{sp}[X_n(\omega)] \cap I\big)\Big) = \gamma(I).$$

Proof. Note first that for any f in $C_0(\mathbb{R})$, we have that

$$\int_{\mathbb{R}} f(x)\, d\mu_{n,\omega}(x) = \mathrm{tr}_n\big[f(X_n(\omega))\big],$$

for all ω in Ω. Hence by Proposition 2.1, it suffices to show, that for almost all ω in Ω, we have that

$$\lim_{n\to\infty} \mathrm{tr}_n\big[f(X_n(\omega))\big] = \int_{\mathbb{R}} f\, d\mu, \qquad \text{for all } f \text{ in } C_0(\mathbb{R}). \tag{3.12}$$

By separability of the Banach space $C_0(\mathbb{R})$, it is enough to check that (3.12) holds almost surely for each fixed f in $C_0(\mathbb{R})$ or for each fixed f in some dense subset of $C_0(\mathbb{R})$. In the following we shall use, as such a dense subset, $C_c^1(\mathbb{R})$, i.e., the set of continuous differentiable functions on $\mathbb{R}$ with compact support. So consider a function f from $C_c^1(\mathbb{R})$, and put

$$F(A) = \mathrm{tr}_n\big[f(A)\big], \qquad (X \in M_n(\mathbb{C})_{\mathrm{sa}}).$$

Then for any A, B in $M_n(\mathbb{C})_{\mathrm{sa}}$, we have that

$$|F(A) - F(B)| \leq \tfrac{1}{n}\big|\mathrm{Tr}_n\big[f(A)\big] - \mathrm{Tr}_n\big[f(B)\big]\big| \leq \tfrac{1}{\sqrt{n}}\|f(A) - f(B)\|_{\mathrm{HS}},$$

and since f is Lipschitz with constant $c = \sup_{x\in\mathbb{R}}|f'(x)| < \infty$, it follows then by Lemma 3.5, that

$$|F(A) - F(B)| \leq \tfrac{c}{\sqrt{n}}\|A - B\|_{\mathrm{HS}}, \qquad (A, B \in M_n(\mathbb{C})_{\mathrm{sa}}). \tag{3.13}$$

The linear bijection $\Phi\colon M_n(\mathbb{C})_{\rm sa} \to \mathbb{R}^{n^2}$, given by

$$\Phi(A) = \big((a_{ii})_{1\le i\le n}, (\sqrt{2}\mathrm{Re}(a_{ij}))_{1\le i<j\le n}, (\sqrt{2}\mathrm{Im}(a_{ij}))_{1\le i<j\le n}\big), \qquad (A=(a_{ij})\in M_n(\mathbb{C})_{\rm sa}),$$

maps the distribution on $M_n(\mathbb{C})_{\rm sa}$ of an element of $\mathrm{SGRM}(n,\frac{1}{n})$ (cf. Definition 1.1) onto the joint distribution of n^2 independent, identically distributed random variables with distribution $N(0,1)$, i.e., the distribution $G_{n^2,n^{-1/2}}$ on $\mathbb{R}^{n^2}$ with density

$$\frac{dG_{n^2,n^{-1/2}}(x)}{dx} = \Big(\frac{n}{2\pi}\Big)^{-n^2/2}\exp(-\tfrac{n\|x\|^2}{2}), \qquad (x\in\mathbb{R}^{n^2}).$$

Moreover, the Euclidean norm on $\mathbb{R}^{n^2}$ corresponds, via the mapping Φ, to the Hilbert-Schmidt norm on $M_n(\mathbb{C})_{\rm sa}$. Hence by (3.13) and Lemma 3.4, we get for any positive t, that

$$P\big(\{\omega\in\Omega \mid |F(X_n(\omega)) - \mathbb{E}(F(X_n))| > t\}\big) \le \exp(-\tfrac{n^2Kt^2}{c^2}),$$

where $K=\frac{2}{\pi^2}$. Hence by the Borel-Cantelli Lemma, it follows that

$$\big|\mathrm{tr}_n\big[f(X_n(\omega))\big] - \mathbb{E}\big(\mathrm{tr}_n[f(X_n)]\big)\big| \le t, \qquad \text{eventually},$$

for almost all ω. Since $t>0$ was arbitrary, we get by Corollary 2.6 that

$$\lim_{n\to\infty}\mathrm{tr}_n\big[f(X_n(\omega))\big] = \lim_{n\to\infty}\mathbb{E}\big(\mathrm{tr}_n[f(X_n)]\big) = \tfrac{1}{2\pi}\int_{-2}^{2} f(x)\sqrt{4-x^2}\,dx,$$

for almost all ω. The last assertion in the proposition follows by Proposition 2.1(i) and Definition 2.2. This completes the proof. ■

Proof of Theorem 3.1. By Lemma 3.3, we have that

$$\limsup_{n\to\infty}\lambda_{\max}(X_n(\omega)) \le 2, \qquad \text{for almost all } \omega \text{ in } \Omega.$$

On the other hand, given any positive ϵ, it follows from Proposition 3.6, that

$$\mathrm{card}\big(\mathrm{sp}[X_n(\omega)]\cap[2-\epsilon,\infty[\big) \to \infty, \quad \text{as } n\to\infty, \qquad \text{for almost all } \omega \text{ in } \Omega,$$

and hence that

$$\liminf_{n\to\infty}\lambda_{\max}(X_n(\omega)) \ge 2-\epsilon, \qquad \text{for almost all } \omega \text{ in } \Omega.$$

Since this is true for any positive ϵ, it follows that (3.1) holds, and (3.2) follows from (3.1) by considering the sequence $(-X_n)$. ■

4 The Harer-Zagier Recursion Formula

In [HZ, Section 4, Proposition 1], Harer and Zagier considered the numbers:

$$C(p,n) = 2^{-k/2}\pi^{-k^2/2}\int_{M_n(\mathbb{C})_{\rm sa}} \mathrm{Tr}_n(A^{2p})\exp(-\tfrac{1}{2}\mathrm{Tr}_n(A^2))\,dA, \qquad (n\in\mathbb{N}, p\in\mathbb{N}_0),$$

where $dA = \prod_{i=1}^n da_{ii} \prod_{i<j} d(\mathrm{Re}(a_{i,j}))\ d(\mathrm{Im}(a_{i,j}))$.

Comparing with Section 1, it follows, that if $A \in \mathrm{SGRM}(n,1)$, then for all p in $\mathbb{N}_0$,

$$C(p,n) = \mathbb{E}\big(\mathrm{Tr}_n[A^{2p}]\big).$$

Harer and Zagier proved that

$$C(p,n) = \sum_{j=1}^{[\frac{p}{2}]} \varepsilon_j(p) n^{p+1-2j}, \qquad (n,p \in \mathbb{N}),$$

where the coefficients $\varepsilon_j(p)$ satisfy the following recursion formula:

$$(p+2)\varepsilon_j(p+1) = (4p+2)\varepsilon_j(p) + p(4p^2-1)\varepsilon_{j-1}(p-1),$$

(cf. [HZ, p. 460, line 3], with (n,g) substituted by $(p+1,j)$).

Below we give a new proof of the above recursion formula, based on Theorem 2.5 and the differential equation for the confluent hyper-geometric function $x \mapsto \Phi(a,c;x)$. Another treatment of this result of Harer and Zagier can be found in [Meh, pp. 117-120].

4.1 Theorem. *Let A be an element of* $\mathrm{SGRM}(n,1)$, *and define*

$$C(p,n) = \mathbb{E}\big(\mathrm{Tr}_n[A^{2p}]\big), \qquad (p \in \mathbb{N}_0). \tag{4.1}$$

Then $C(0,n) = n$, $C(1,n) = n^2$, and for fixed n in $\mathbb{N}$, the numbers $C(p,n)$ satisfy the recursion formula:

$$C(p+1,n) = n \cdot \tfrac{4p+2}{p+2} \cdot C(p,n) + \tfrac{p(4p^2-1)}{p+2} \cdot C(p-1,n), \qquad (p \geq 1). \tag{4.2}$$

Proof. Let a, c be complex numbers, such that $c \notin \mathbb{Z} \setminus \mathbb{N}$. Then the confluent hyper-geometric function

$$x \mapsto \Phi(a,c;x) = 1 + \tfrac{a}{c}\tfrac{x}{1} + \tfrac{a(a+1)}{c(c+1)}\tfrac{x^2}{2} + \tfrac{a(a+1)(a+2)}{c(c+1)(c+2)}\tfrac{x^3}{3!} + \cdots, \qquad (x \in \mathbb{C}),$$

is an entire function, and $y = \Phi(a,c;x)$ satisfies the differential equation

$$x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay = 0, \tag{4.3}$$

(cf. [HTF, Vol. 1, p.248, formula (2)]). By (2.16) in Lemma 2.5, we have, for any A in $\mathrm{SGRM}(n,1)$, that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big) = n \cdot \exp(\tfrac{s^2}{2}) \cdot \Phi(1-n, 2; -s^2), \qquad (s \in \mathbb{C}).$$

Since A and $-A$ have the same distribution, $\mathbb{E}\big(\mathrm{Tr}_n[A^{2q-1}]\big) = 0$, for any q in $\mathbb{N}$, and consequently

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big) = \sum_{p=0}^{\infty} \frac{s^{2p}}{(2p)!}\mathbb{E}\big(\mathrm{Tr}_n[A^{2p}]\big).$$

It follows thus, that $\frac{C(p,n)}{(2p)!}$ is the coefficient to x^p in the power series expansion of the function

$$\sigma_n(x) = n \cdot \exp(\tfrac{x}{2}) \cdot \Phi(1-n, 2; -x).$$

By (4.3) the function $\rho_n(x) = \Phi(1-n, 2; -x)$, satisfies the differential equation

$$x\rho_n''(x) + (2+x)\rho_n'(x) - (n-1)\rho_n(x) = 0,$$

which implies that $\sigma_n(x) = n \cdot \exp(\frac{x}{2}) \cdot \rho_n(x)$, satisfies the differential equation

$$x\sigma_n''(x) + 2\sigma_n'(x) - (\tfrac{x}{4} + n)\sigma_n(x) = 0. \tag{4.4}$$

We know that σ_n has the power series expansion:

$$\sigma_n(x) = \sum_{p=0}^{\infty} \alpha_p x^p, \qquad \text{where} \quad \alpha_p = \frac{C(p,n)}{(2p)!}, \quad (p \in \mathbb{N}). \tag{4.5}$$

Inserting (4.5) in (4.4), we find that

$$(p+1)(p+2)\alpha_{p+1} - n\alpha_p - \tfrac{1}{4}\alpha_{p-1} = 0, \qquad (p \geq 1), \tag{4.6}$$

and that

$$2\alpha_1 - n\alpha_0 = 0. \tag{4.7}$$

Inserting then $C(p,n) = \frac{C(p,n)}{(2p)!}$, in (4.6), we obtain (4.2). Moreover, it is clear that $C(0,n) = \mathrm{Tr}_n(\mathbf{1}_n) = n$, and thus, by (4.7), $C(1,n) = 2\alpha_1 = n\alpha_0 = n^2$. ■

4.2 Corollary. ([HZ]) *With $C(p,n)$ as introduced in (4.1), we have that*

$$C(p,n) = \sum_{j=0}^{[\frac{p}{2}]} \varepsilon_j(p) n^{p+1-2j}, \qquad (p \in \mathbb{N}_0, n \in \mathbb{N}), \tag{4.8}$$

where the coefficients $\varepsilon_j(p)$, $j, p \in \mathbb{N}_0$, are determined by the conditions

$$\varepsilon_j(p) = 0, \quad \text{whenever } j \geq [\tfrac{p}{2}] + 1, \tag{4.9}$$

$$\varepsilon_0(p) = \tfrac{1}{p+1}\tbinom{2p}{p}, \qquad (p \in \mathbb{N}_0), \tag{4.10}$$

$$\varepsilon_j(p+1) = \tfrac{4p+2}{p+2} \cdot \varepsilon_j(p) + \tfrac{p(4p^2-1)}{p+2} \cdot \varepsilon_{j-1}(p-1), \qquad (p, j \in \mathbb{N}). \tag{4.11}$$

Proof. It is immediate from (4.2) of Theorem 4.1, that for fixed p, $C(p,n)$ is a polynomial in n of degree $p+1$ and without constant term. Moreover, it follows from (4.2), that only $n^{p+1}, n^{p-1}, n^{p-3}$, etc., have non-zero coefficients in this polynomial. Therefore $C(p,n)$ is of the form set out in (4.8) for suitable coefficients

$$\varepsilon_j(p), \qquad p \geq 0, \quad 0 \leq j \leq [\tfrac{p}{2}].$$

Inserting (4.8) in (4.2), and applying the convention (4.9), we obtain (4.11), and also that

$$\varepsilon_0(p+1) = \tfrac{4p+2}{p+2} \cdot \varepsilon_0(p), \qquad (p \geq 1). \tag{4.12}$$

Clearly, $\varepsilon_0(0) = \varepsilon_0(1) = 1$, and thus by induction on (4.12), we obtain (4.10). ■

From Theorem 4.1 or Corollary 4.2, one gets, that for any A in $\mathrm{SGRM}(n,1)$,

$$\begin{aligned}
\mathbb{E}\big(\mathrm{Tr}_n[A^2]\big) &= n^2, \\
\mathbb{E}\big(\mathrm{Tr}_n[A^4]\big) &= 2n^3 + n, \\
\mathbb{E}\big(\mathrm{Tr}_n[A^6]\big) &= 5n^4 + 10n^2, \\
\mathbb{E}\big(\mathrm{Tr}_n[A^8]\big) &= 14n^5 + 70n^3 + 21n, \\
\mathbb{E}\big(\mathrm{Tr}_n[A^{10}]\big) &= 42n^6 + 420n^4 + 483n^2,
\end{aligned}$$

etc. (see [HZ, p. 459] for a list of the numbers $\varepsilon_j(p)$, $p \leq 12$). If, as in Sections 2 and 3, we replace the A above by an element X of $\mathrm{SGRM}(n, \frac{1}{n})$, and Tr_n by tr_n, then we have to divide the above numbers by n^{p+1}. Hence for X in $\mathrm{SGRM}(n, \frac{1}{n})$, we have

$$\begin{aligned}
\mathbb{E}\big(\mathrm{tr}_n[X^2]\big) &= 1, \\
\mathbb{E}\big(\mathrm{tr}_n[X^4]\big) &= 2 + \tfrac{1}{n^2}, \\
\mathbb{E}\big(\mathrm{tr}_n[X^6]\big) &= 5 + \tfrac{10}{n^2}, \\
\mathbb{E}\big(\mathrm{tr}_n[X^8]\big) &= 14 + \tfrac{70}{n^2} + \tfrac{21}{n^4}, \\
\mathbb{E}\big(\mathrm{tr}_n[X^{10}]\big) &= 42 + \tfrac{420}{n^2} + \tfrac{483}{n^4},
\end{aligned}$$

etc. Note that the constant term in $\mathbb{E}\big(\mathrm{tr}_n[X^{2p}]\big)$ is

$$\varepsilon_0(p) = \tfrac{1}{p+1}\tbinom{2p}{p} = \tfrac{1}{2\pi}\int_{-2}^{2} x^{2p}\sqrt{4-x^2}\,dx,$$

in concordance with Wigner's semi-circle law.

5 Rectangular Gaussian Random Matrices and the Complex Wishart Distribution

5.1 Definition. Let $m, n \in \mathbb{N}$ and $\sigma > 0$. We denote by $\mathrm{GRM}(m,n,\sigma^2)$ the class of $m \times n$ random matrices

$$B = (b_{jk})_{1\leq j\leq m, 1\leq k\leq n} \tag{5.1}$$

for which the entries are mn independent complex-valued random variables, such that the distribution of each b_{jk} has density

$$\frac{1}{\pi\sigma^2}\exp\big(-\tfrac{|z|^2}{\sigma^2}\big), \quad z \in \mathbb{C} \tag{5.2}$$

with respect to the Lebesgue measure $d(\mathrm{Re}\, z)d(\mathrm{Im}\, z)$ on $\mathbb{C}$. □

Note that $B \in \mathrm{GRM}(m,n,\sigma^2)$ if and only if $(\mathrm{Re}\, b_{jk})_{j,k}, (\mathrm{Im}\, b_{jk})_{jk}$ form a family of $2mn$ independent real Gaussian random variables, each with mean value 0 and variance $\frac{1}{2}\sigma^2$.

5.2 Definition. If $B \in \mathrm{GRM}(m,n,1)$, then the distribution of the selfadjoint $m \times n$ random matrix B^*B is called the complex Wishart distribution with parameters (m,n).
$\square$

The complex Wishart distribution was first studied by Goodman and Khalid in [Go] and [Kh].

5.3 Proposition. ([Go],[Kh]) *Let $B \in \mathrm{GRM}(m,n,1)$. For $m \geq n$ the distribution $d\nu(S)$ of $S = B^*B$ is given by*

$$d\nu(S) = c_3(\det S)^{m-n} \exp(-\mathrm{Tr}_n(S))\ dS \tag{5.3}$$

for $S \in M_n(\mathbb{C})_+$ (the positive cone in $M_n(\mathbb{C})$), where $c_3 > 0$ is a normalization constant, depending on m and n, and

$$dS = \left(\prod_{j=1}^{n} ds_{jj}\right) \prod_{j<k} d(\mathrm{Re} s_{jk}) d(\mathrm{Im} s_{jk}). \tag{5.4}$$

Moreover, the joint distribution of the ordered eigenvalues $\lambda_1(S) \leq \lambda_2(S) \leq \cdots \leq \lambda_n(S)$ of S is given by

$$c_4 \prod_{j<k}(\lambda_j - \lambda_k)^2 \left(\prod_{j=1}^{k} \lambda_j\right)^{m-n} \exp\big(-\sum_{j=1}^{n} \lambda_j\big)\ d\lambda_1, \dots, d\lambda_n \tag{5.5}$$

on

$$\Lambda^+ = \big\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\big\},$$

where $c_4 > 0$ is again a normalization constant depending on m and n.

Put $\mathbb{R}_+ = [0, \infty[$. Assume $m \geq n$ and let $g\colon \mathbb{R}_+^n \to \mathbb{R}$ denote the function

$$g(\underline{\lambda}) = \frac{c_4}{n!}(\lambda_j - \lambda_k)^2 \big(\prod_{j=1}^{n} \lambda_j\big)^{m-n} \exp\big(-\sum_{j=1}^{n} \lambda_j\big) \tag{5.6}$$

where $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ and c_4 is as described above. Then, by averaging over all permutations of $(\lambda_1, \dots, \lambda_n)$, we get, as in Section 1, that for any symmetric Borel function $\varphi\colon \mathbb{R}_+^n \to \mathbb{C}$,

$$\int_{M_n(\mathbb{C})_+} \varphi(\lambda_1(S), \dots, \lambda_n(S))\ d\nu(S) = \int_{\mathbb{R}_+^n} \varphi(\underline{\lambda}) g(\underline{\lambda})\ d\lambda_1, \dots, \lambda_n. \tag{5.7}$$

The marginal density h corresponding to (5.6),

$$h(\lambda) = \int_{\mathbb{R}_+^{n-1}} g(\lambda, \lambda_2, \dots, \lambda_n)\ d\lambda_2, \dots, d\lambda_n \tag{5.8}$$

can be computed explicitly. This is done in [Bro]. In analogy with (1.9), one gets

$$h(\lambda) = \frac{1}{n}\sum_{j=0}^{n-1} \varphi_k^{m-n}(x)^2 \tag{5.9}$$

where

$$\varphi_k^\alpha(x) = \left[\tfrac{k!}{\Gamma(k+\alpha+1)} x^\alpha \exp(-x)\right]^{1/2} \cdot L_k^\alpha(x), \qquad (k \in \mathbb{N}_0), \tag{5.10}$$

and $(L_k^\alpha)_{k\in\mathbb{N}_0}$ is the sequence of generalized Laguerre polynomials of order α, i.e.,

$$L_k^\alpha(x) = (k!)^{-1} x^{-\alpha} \exp(x) \cdot \frac{d^k}{dx^k}\big(x^{k+\alpha}\exp(-x)\big), \qquad (k \in \mathbb{N}_0). \tag{5.11}$$

5.4 Proposition. *Let B be an element of* $\mathrm{GRM}(m,n,1)$*, let φ_k^α, $\alpha \in \,]0,\infty[$, $k \in \mathbb{N}_0$, be the functions introduced in* (5.10)*, and let $f\colon [0,\infty[\, \to \mathbb{R}$ be a Borel function.*

(i) *If $m \geq n$, we have that*

$$\mathbb{E}\big(\mathrm{Tr}_n[f(B^*B)]\big) = \int_0^\infty f(x)\left[\textstyle\sum_{j=0}^{n-1}\varphi_k^{m-n}(x)^2\right]\,dx,$$

whenever the integral on the right hand side is well-defined.

(ii) *If $m < n$, we have that*

$$\mathbb{E}\big(\mathrm{Tr}_n[f(B^*B)]\big) = (n-m)f(0) + \int_0^\infty f(x)\left[\textstyle\sum_{j=0}^{m-1}\varphi_k^{n-m}(x)^2\right]\,dx,$$

whenever the integral on the right hand side is well-defined.

Proof. (i) The proof of (i) can be copied from the proof of Proposition 1.2, using (5.5)–(5.9) instead of (1.4)–(1.9).

(ii) Assume that $m < n$, and note that $B^* \in \mathrm{GRM}(n,m,1)$. If $T \in M_{m,n}(\mathbb{C})$, then T^*T and TT^* have the same list of non-zero eigenvalues counted with multiplicity, and hence T^*T must have $n-m$ more zeroes in its list of eigenvalues than TT^* has. Combining these facts with (i), we obtain (ii). ■

5.5 Remark. The real, complex and symplectic Wishart distribution has been extensively studied in the literature (see f.inst. [Wis], [Go], [Kh], [Ja], [ABJ], [HSS], and [LM]). Due to the connection to Laguerre polynomials, the complex Wishart distribution is also called the Laguerre ensemble. The book manuscript of Forrester [Fo] gives a self-contained treatment of all the results quoted in this section. The orthogonalization procedure which is used to derive (5.9) from (5.6) is also described in details in Deift's book [De, Section 5.4]. □

6 The Moment Generating Function for the Complex Wishart Distribution

In this section we prove the formulas (0.9) and (0.10) (cf. Theorem 6.4 below). Moreover, we apply the second of these two formulas to give a new proof of Marchenko's and Pastur's result [MP] on the limit distribution of the eigenvalues of B^*B, when $m = m(n)$, and $\lim_{n\to\infty} m(n)/n = c > 0$ (cf. Corollary 6.8 below). As in Section 5, for any real number α in $]-1,\infty[$, we denote by $(L_k^\alpha)_{k\in\mathbb{N}_0}$ the sequence of Laguerre polynomials of order α, i.e.,

$$L_k^\alpha(x) = (k!)^{-1}x^{-\alpha}\exp(x)\frac{d^k}{dx^k}\big(x^{k+\alpha}\exp(-x)\big), \qquad (k\in\mathbb{N}_0,\ x>0), \tag{6.1}$$

and by $(\varphi_k^\alpha)_{k\in\mathbb{N}_0}$ the sequence of functions given by

$$\varphi_k^\alpha(x) = \big(\tfrac{k!}{\Gamma(k+\alpha+1)}x^\alpha\exp(-x)\big)^{1/2}L_k^\alpha(x), \qquad (x>0). \tag{6.2}$$

The Laguerre polynomials satisfy the orthogonality relations:

$$\int_0^\infty L_j^\alpha(x)L_k^\alpha(x)\cdot x^\alpha\exp(-x)\ dx = \begin{cases}\frac{\Gamma(k+\alpha+1)}{k!}, & \text{if } j=k,\\ 0, & \text{if } j\neq k,\end{cases} \tag{6.3}$$

(cf. [HTF, Vol. 2, p.188, formula (2)]), which implies that the sequence of functions $(\varphi_k^\alpha)_{k\in\mathbb{N}_0}$ is an orthonormal sequence in the Hilbert space $L_2([0,\infty[,dx)$, i.e.,

$$\int_0^\infty \varphi_j^\alpha(x)\varphi_k^\alpha(x)\ dx = \begin{cases}1 & \text{if } j=k,\\ 0 & \text{if } j\neq k.\end{cases}$$

6.1 Lemma. *For any n in $\mathbb{N}_0$, we have that*

$$\frac{d}{dx}\Big(x\sum_{j=0}^{n-1}\varphi_j^\alpha(x)^2\Big) = \sqrt{n(n+\alpha)}\cdot\varphi_{n-1}^\alpha(x)\varphi_n^\alpha(x), \qquad (x>0). \tag{6.4}$$

Proof. For each n in $\mathbb{N}$, we define

$$\rho_n(x) = \sum_{j=0}^{n-1}\tfrac{j!}{\Gamma(j+\alpha+1)}L_j^\alpha(x)^2, \qquad (x>0). \tag{6.5}$$

Using [HTF, Volume 2, p.188, formula (7)], we have here that

$$\begin{aligned}\rho_n(x) &= \lim_{y\to x}\sum_{j=0}^{n-1}\tfrac{j!}{\Gamma(j+\alpha+1)}L_j^\alpha(y)L_j^\alpha(x)\\ &= \lim_{y\to x}\frac{n!\big(L_{n-1}^\alpha(y)L_n^\alpha(x)-L_n^\alpha(y)L_{n-1}^\alpha(x)\big)}{(y-x)\cdot\Gamma(n+\alpha)}.\end{aligned}$$

Therefore, it follows that

$$\rho_n(x) = \tfrac{n!}{\Gamma(n+\alpha)}\big((L_{n-1}^\alpha)'(x)L_n^\alpha(x) - (L_n^\alpha)'(x)L_{n-1}^\alpha(x)\big), \tag{6.6}$$

and hence that

$$\rho_n'(x) = \tfrac{n!}{\Gamma(n+\alpha)}\big((L_{n-1}^\alpha)''(x)L_n^\alpha(x) - (L_n^\alpha)''(x)L_{n-1}^\alpha(x)\big). \tag{6.7}$$

By [HTF, Volume 2, p.188, formula (10)], we have that

$$\begin{aligned} x(L_{n-1}^\alpha)''(x) + (\alpha+1-x)(L_{n-1}^\alpha)'(x) &= -(n-1)L_{n-1}^\alpha(x), \\ x(L_n^\alpha)''(x) + (\alpha+1-x)(L_n^\alpha)'(x) &= -nL_n^\alpha(x). \end{aligned}$$

Combining these two formulas with (6.6) and (6.7), we find that

$$\begin{aligned} x\rho_n'(x) + (\alpha+1-x)\rho_n(x) &= \tfrac{n!}{\Gamma(n+\alpha)}\big(-(n-1)L_{n-1}^\alpha(x)L_n^\alpha(x) + nL_n^\alpha(x)L_{n-1}^\alpha(x)\big) \\ &= \tfrac{n!}{\Gamma(n+\alpha)}L_{n-1}^\alpha(x)L_n^\alpha(x). \end{aligned}$$

It follows now, that

$$\begin{aligned} \frac{d}{dx}\Big(x\sum_{j=0}^{n-1}\varphi_j^\alpha(x)^2\Big) &= \frac{d}{dx}\big(\rho_n(x)x^{\alpha+1}\exp(-x)\big) \\ &= \big(x\rho_n'(x) + (\alpha+1-x)\rho_n(x)\big)x^\alpha\exp(-x) \\ &= \tfrac{n!}{\Gamma(n+\alpha)}L_{n-1}^\alpha(x)L_n^\alpha(x)x^\alpha\exp(-x) \\ &= \tfrac{n!}{\Gamma(n+\alpha)}\Big(\tfrac{\Gamma(n+\alpha)}{(n-1)!}\tfrac{\Gamma(n+\alpha+1)}{n!}\Big)^{1/2}\varphi_{n-1}^\alpha(x)\varphi_n^\alpha(x) \\ &= \sqrt{n(n+\alpha)}\cdot\varphi_{n-1}^\alpha(x)\varphi_n^\alpha(x), \end{aligned}$$

which is the desired formula. ■

In order to state the next lemma, we need to introduce the hyper-geometric function F, which is given by the equation (cf. [HTF, Vol. 1, p.56, formula (2)]),

$$F(a,b,c;z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!}z^n, \tag{6.8}$$

with the notation introduced in (2.8). We note that $F(a,b,c;z)$ is well-defined whenever $c \notin \mathbb{Z}\setminus\mathbb{N}$ and $|z| < 1$. If either $-a \in \mathbb{N}_0$ or $-b \in \mathbb{N}_0$, then $F(a,b,c;z)$ becomes a polynomial in z, and is thus well-defined for all z in $\mathbb{C}$.

6.2 Lemma. *Consider α in $]-1,\infty[$ and j,k in $\mathbb{N}_0$. Then for any complex number s, such that $s \neq 0$ and $\mathrm{Re}(s) < 1$, we have that*

$$\int_0^\infty \varphi_j^\alpha(x)\varphi_k^\alpha(x)\exp(sx)\,dx = \gamma(\alpha,j,k)\cdot\frac{s^{j+k}}{(1-s)^{\alpha+j+k+1}}\cdot F(-j,-k,\alpha+1;s^{-2}), \tag{6.9}$$

where

$$\gamma(\alpha,j,k) = \frac{(-1)^{j+k}}{\Gamma(\alpha+1)}\Big(\frac{\Gamma(\alpha+j+1)\Gamma(\alpha+k+1)}{j!k!}\Big)^{1/2}. \tag{6.10}$$

Proof. The formula (6.9) can be extracted from the paper [Ma] by Mayr, but for the readers convenience, we include an elementary proof. Both sides of the equality (6.9) are analytic functions of $s \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 1\}$, so it suffices to check (6.9) for all s in $]-\infty, 1[\setminus\{0\}$. By (6.2), we have that

$$\begin{aligned}
&\int_0^\infty \varphi_j^\alpha(x)\varphi_k^\alpha(x)\exp(sx)\ dx \\
&\quad= \left(\tfrac{j!k!}{\Gamma(j+\alpha+1)\Gamma(k+\alpha+1)}\right)^{1/2}\int_0^\infty L_j^\alpha(x)L_k^\alpha(x)x^\alpha\exp((s-1)x)\ dx \\
&\quad= \left(\tfrac{j!k!}{\Gamma(j+\alpha+1)\Gamma(k+\alpha+1)}\right)^{1/2}\tfrac{1}{(1-s)^{\alpha+1}}\int_0^\infty L_j^\alpha(\tfrac{y}{1-s})L_k^\alpha(\tfrac{y}{1-s})y^\alpha\exp(-y)\ dy,
\end{aligned} \tag{6.11}$$

where, in the last equality, we applied the substitution $y = \frac{x}{1-s}$. We note here, that by [HTF, Volume 2, p.192, formula (40)], we have for any positive number λ, that

$$\begin{aligned}
L_k^\alpha(\lambda x) &= \sum_{r=0}^k \binom{k+\alpha}{r}\lambda^{k-r}(1-\lambda)^r L_{k-r}^\alpha(x) \\
&= \sum_{r=0}^k \binom{k+\alpha}{k-r}\lambda^r(1-\lambda)^{k-r}L_r^\alpha(x).
\end{aligned} \tag{6.12}$$

By application of this formula and the orthogonality relation (6.3) for the Laguerre polynomials, we obtain that

$$\begin{aligned}
&\int_0^\infty L_j^\alpha(x)L_k^\alpha(x)x^\alpha\exp((s-1)x)\ dx \\
&\quad= \frac{1}{(1-s)^{\alpha+1}}\sum_{r=0}^{\min\{j,k\}}\binom{j+\alpha}{j-r}\binom{k+\alpha}{k-r}\Big(\frac{1}{1-s}\Big)^{2r}\Big(1-\frac{1}{1-s}\Big)^{j+k-2r}\frac{\Gamma(\alpha+r+1)}{r!} \\
&\quad= \frac{(-s)^{j+k}}{(1-s)^{\alpha+j+k+1}}\sum_{r=0}^{\min\{j,k\}}\binom{j+\alpha}{j-r}\binom{k+\alpha}{k-r}\frac{\Gamma(\alpha+r+1)}{r!}(-s)^{-2r} \\
&\quad= \frac{(-s)^{j+k}}{(1-s)^{\alpha+j+k+1}}\sum_{r=0}^{\min\{j,k\}}\frac{\Gamma(j+\alpha+1)\Gamma(k+\alpha+1)}{(j-r)!(k-r)!r!\Gamma(\alpha+r+1)}s^{-2r}.
\end{aligned} \tag{6.13}$$

We note here that

$$\frac{j!k!\Gamma(\alpha+1)}{(j-r)!(k-r)!r!\Gamma(\alpha+r+1)} = \frac{(-j)_r(-k)_r}{(\alpha+1)_r r!},$$

and hence it follows that

$$\begin{aligned}
&\int_0^\infty L_j^\alpha(x)L_k^\alpha(x)x^\alpha\exp((s-1)x)\ dx \\
&\quad= \frac{\Gamma(j+\alpha+1)\Gamma(k+\alpha+1)\cdot(-s)^{j+k}}{j!k!\Gamma(\alpha+1)\cdot(1-s)^{\alpha+j+k+1}}\cdot F(-j,-k,\alpha+1;s^{-2}).
\end{aligned} \tag{6.14}$$

Combining (6.11) and (6.14), we obtain (6.9). ∎

6.3 Lemma. *Assume that* $\alpha \in]-1,\infty[$, *and that* $n \in \mathbb{N}$, $k \in \mathbb{N}_0$. *Then for any complex number* s *such that* $\mathrm{Re}(s) < 1$, *we have that*

$$\int_0^\infty \varphi_k^\alpha(x)^2 \exp(sx)\ dx = \frac{F(-k-\alpha,-k,1;s^2)}{(1-s)^{\alpha+2k+1}} \tag{6.15}$$

$$\int_0^\infty \Big(\sum_{j=0}^{n-1} \varphi_j^\alpha(x)^2\Big) x \exp(sx)\ dx = n(n+\alpha)\frac{F(1-n-\alpha,1-n,2;s^2)}{(1-s)^{\alpha+2n}}. \tag{6.16}$$

Proof. By continuity, it suffices to prove (6.15) and (6.16) for all s in $\mathbb{C}\setminus\{0\}$, for which $\mathrm{Re}(s) < 1$. Before doing so, we observe that for j,k in $\mathbb{N}_0$ such that $j \le k$, we have that

$$\begin{aligned}
F(-j,-k,\alpha+1;s^{-2}) &= \sum_{r=0}^{j} \frac{(-j)_r(-k)_r}{(\alpha+1)_r r!} s^{-2r} \\
&= \sum_{r=0}^{j} \frac{j!k!\Gamma(\alpha+1)}{(j-r)!(k-r)!r!\Gamma(\alpha+r+1)} s^{-2r}.
\end{aligned}$$

Replacing now r by $j-r$ in the summation, it follows that

$$\begin{aligned}
F(-j,-k,\alpha+1;s^{-2}) &= \sum_{r=0}^{j} \frac{j!k!\Gamma(\alpha+1)}{r!(k-j+r)!(j-r)!\Gamma(\alpha+j-r+1)} s^{2r-2j} \\
&= \frac{k!\Gamma(\alpha+1)}{(k-j)!\Gamma(\alpha+j+1)} \sum_{r=0}^{j} \frac{(-j)_r(-\alpha-j)_r}{r!(1+k-j)_r} s^{2r-2j}.
\end{aligned}$$

Hence for j,k in $\mathbb{N}_0$ such that $j \le k$, we have that

$$F(-j,-k,\alpha+1;s^{-2}) = \frac{k!\Gamma(\alpha+1)}{(k-j)!\Gamma(\alpha+j+1)} \cdot \frac{F(-j-\alpha,-j,1+k-j;s^2)}{s^{2j}}. \tag{6.17}$$

Returning now to the proof of (6.15) and (6.16), we note that by Lemma 6.2 and (6.17), we have that

$$\begin{aligned}
\int_0^\infty \varphi_k^\alpha(x)^2 \exp(sx)\ dx &= \frac{\Gamma(\alpha+k+1)\cdot s^{2k}}{k!\Gamma(\alpha+1)\cdot(1-s)^{\alpha+2k+1}} \cdot F(-k,-k,\alpha+1;s^{-2}) \\
&= \frac{F(-k-\alpha,-k,1;-s^2)}{(1-s)^{\alpha+2k+1}},
\end{aligned}$$

which proves (6.15). Regarding (6.16), we get by partial integration, Lemma 6.1, Lemma 6.2

and (6.17), that

$$
\begin{aligned}
&\int_0^\infty \Big(\sum_{j=0}^{n-1} \varphi_j^\alpha(x)^2\Big) x \exp(sx)\ dx \\
&\quad = \frac{-1}{s}\int_0^\infty \frac{d}{dx}\Big(x\sum_{j=0}^{n-1}\varphi_j^\alpha(x)^2\Big)\exp(sx)\ dx \\
&\quad = \frac{-\sqrt{n(n+\alpha)}}{s}\int_0^\infty \varphi_{n-1}^\alpha(x)\varphi_n^\alpha(x)\exp(sx)\ dx \\
&\quad = \frac{-\sqrt{n(n+\alpha)}\cdot\gamma(\alpha,n-1,n)\cdot s^{2n-1}}{s(1-s)^{\alpha+2n}}\cdot F(-n+1,-n,\alpha+1;s^{-2}) \\
&\quad = \frac{-\sqrt{n(n+\alpha)}\cdot\gamma(\alpha,n-1,n)\cdot s^{2n-1}\cdot n!\cdot\Gamma(\alpha+1)}{s(1-s)^{\alpha+2n}\cdot\Gamma(\alpha+n)}\cdot\frac{F(-n-\alpha+1,-n+1,2;s^2)}{s^{2n-2}} \\
&\quad = \frac{-\sqrt{n(n+\alpha)}\cdot\gamma(\alpha,n-1,n)\cdot n!\cdot\Gamma(\alpha+1)}{(1-s)^{\alpha+2n}\cdot\Gamma(\alpha+n)}\cdot F(1-n-\alpha,1-n,2;s^2).
\end{aligned}
\tag{6.18}
$$

Recall here from (6.10), that

$$
\gamma(\alpha,n-1,n) = \frac{-1}{\Gamma(\alpha+1)}\Big(\frac{\Gamma(\alpha+n)\Gamma(\alpha+n+1)}{(n-1)!n!}\Big)^{1/2} = \frac{-\Gamma(\alpha+n)}{\Gamma(\alpha+1)n!}\sqrt{n(n+\alpha)},
$$

and inserting this in (6.18), we obtain (6.16). ■

6.4 Theorem. *Assume that* $m,n\in\mathbb{N}$ *and that* $B\in\mathrm{GRM}(m,n,1)$. *Then for any complex number* s, *such that* $\mathrm{Re}(s)<1$, *we have that*

$$
\mathbb{E}\big(\mathrm{Tr}_n[B^*B\exp(sB^*B)]\big) = m\cdot n\cdot\frac{F(1-m,1-n,2;s^2)}{(1-s)^{m+n}}, \tag{6.19}
$$

and that

$$
\mathbb{E}\big(\mathrm{Tr}_n[\exp(sB^*B)]\big) = \sum_{k=1}^{n}\frac{F(k-m,k-n,1;s^2)}{(1-s)^{m+n+1-2k}}, \quad \textit{if } m\ge n, \tag{6.20}
$$

$$
\mathbb{E}\big(\mathrm{Tr}_n[\exp(sB^*B)]\big) = (n-m)+\sum_{k=1}^{m}\frac{F(k-m,k-n,1;s^2)}{(1-s)^{m+n+1-2k}}, \quad \textit{if } m< n. \tag{6.21}
$$

Proof. To prove (6.19), assume first that $m\ge n$. Then by Proposition 5.4(i), we have that

$$
\mathbb{E}\big(\mathrm{Tr}_n[B^*B\exp(sB^*B)]\big) = \int_0^\infty \Big(\sum_{k=0}^{n-1}\varphi_k^{m-n}(x)^2\Big)x\exp(sx)\ dx,
$$

and hence (6.19) follows from (6.16) in Lemma 6.3. The case $m<n$ is proved similarly by application of Proposition 5.4(ii) instead of Proposition 5.4(i).

To prove (6.20), assume that $m \geq n$, and note then that by Proposition 5.4(i) and (6.15) in Lemma 6.3,

$$\begin{aligned}\mathbb{E}\big(\mathrm{Tr}_n[\exp(sB^*B)]\big) &= \int_0^\infty \Big(\sum_{k=0}^{n-1} \varphi_k^{m-n}(x)^2\Big) \exp(sx)\ dx \\ &= \sum_{k=0}^{n-1} \frac{F(-k-m+n,-k,1;s^2)}{(1-s)^{m-n+2k+1}}.\end{aligned}$$

Replacing then k by $n-k$ in this summation, we obtain (6.20).

We note finally that (6.21) is proved the same way as (6.20), by application Proposition 5.4(ii) instead of Proposition 5.4(i). ∎

6.5 Definition. For c in $]0,\infty[$, we denote by μ_c, the measure on $[0,\infty[$ given by the equation

$$\mu_c = \max\{1-c,0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \cdot 1_{[a,b]}(x) \cdot dx,$$

where $a = (\sqrt{c}-1)^2$ and $b = (\sqrt{c}+1)^2$. It is not hard to check that

$$\int_a^b \frac{\sqrt{(x-a)(b-x)}}{2\pi x}\ dx = \begin{cases} 1, & \text{if } c \geq 1, \\ c, & \text{if } c < 1, \end{cases}$$

and this implies that μ_c is a probability measure for all c in $]0,\infty[$.

The measure μ_c is called the *Marchenko-Pastur distribution* (cf. [MP] and [OP]). It is also known as the *free analog of the Poisson distribution* with parameter c (cf. [VDN]). □

6.6 Lemma. *Assume that $c \in]0,\infty[$, and let $(m(n))_n$ be a sequence of positive integers, such that $\lim_{n\to\infty} \frac{m(n)}{n} = c$. Consider furthermore a sequence (Y_n) of random matrices, such that for all n in $\mathbb{N}$, $Y_n \in \mathrm{GRM}(m(n), n, \frac{1}{n})$. We then have*

(i) *For any s in $\mathbb{C}$ and n in $\mathbb{N}$, such that $n > \mathrm{Re}(s)$, we have that*

$$\mathbb{E}\Big(\big|\mathrm{tr}_n[Y_n^* Y_n \exp(sY_n^*Y_n)]\big|\Big) < \infty.$$

(ii) *For any complex number s, we have that*

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[Y_n^*Y_n \exp(sY_n^*Y_n)]\big) = \int_0^\infty x \exp(sx)\ d\mu_c(x), \tag{6.22}$$

and the convergence is uniform on compact subsets of $\mathbb{C}$.

Proof. For each n in $\mathbb{N}$, put $B_n = \sqrt{n} Y_n$, and note that $B_n \in \mathrm{GRM}(m(n), n, 1)$. If $s \in \mathbb{C}$ and $n \in \mathbb{N}$ such that $n > \mathrm{Re}(s)$, then by Theorem 6.4, we have that

$$\begin{aligned}\mathbb{E}\Big(\big|\mathrm{tr}_n[Y_n^*Y_n \exp(sY_n^*Y_n)]\big|\Big) &\leq \mathbb{E}\Big(\mathrm{tr}_n\big[Y_n^*Y_n \exp(\mathrm{Re}(s)Y_n^*Y_n)\big]\Big) \\ &\leq \tfrac{1}{n^2}\mathbb{E}\Big(\mathrm{Tr}_n\big[B_n^*B_n \exp(\tfrac{\mathrm{Re}(s)}{n}B_n^*B_n)\big]\Big) < \infty,\end{aligned}$$

which proves (i). Regarding (ii), Theorem 6.4 yields furthermore (still under the assumption that $n > \mathrm{Re}(s)$), that

$$\begin{aligned}\mathbb{E}\big(\mathrm{tr}_n[Y_n^*Y_n\exp(sY_n^*Y_n)]\big) &= \tfrac{1}{n^2}\mathbb{E}\big(\mathrm{Tr}_n[B_n^*B_n\exp(\tfrac{s}{n}B_n^*B_n)]\big)\\ &= \frac{m(n)\cdot F(1-m(n),1-n,2;\frac{s^2}{n^2})}{n\cdot(1-\frac{s}{n})^{m(n)+n}}.\end{aligned}$$

Here,

$$F(1-m(n),1-n,2;\tfrac{s^2}{n^2}) = \sum_{j=0}^{\infty}\frac{1}{j+1}\binom{m(n)-1}{j}\binom{n-1}{j}\frac{s^{2j}}{n^{2j}},$$

with the convention that $\binom{k}{j} = 0$, whenever $j > k$, $(j,k\in\mathbb{N}_0)$. Since $\lim_{n\to\infty}\frac{m(n)}{n} = c$, it follows that for each fixed j in $\mathbb{N}$,

$$\lim_{n\to\infty}\frac{1}{j+1}\binom{m(n)-1}{j}\binom{n-1}{j}\frac{s^{2j}}{n^{2j}} = \frac{(cs^2)^j}{j!(j+1)!}.$$

Moreover, with $\gamma := \sup_{n\in\mathbb{N}}\frac{m(n)}{n} < \infty$, we have that

$$\Big|\frac{1}{j+1}\binom{m(n)-1}{j}\binom{n-1}{j}\frac{s^{2j}}{n^{2j}}\Big| \le \frac{(\gamma s^2)^j}{j!(j+1)!},$$

for all j,n. Hence by Lebesgue's Theorem on Dominated Convergence (for series), it follows that

$$\lim_{n\to\infty}F(1-m(n),1-n,2;\tfrac{s^2}{n^2}) = \sum_{j=0}^{\infty}\frac{(cs^2)^j}{j!(j+1)!}, \qquad (s\in\mathbb{C}), \tag{6.23}$$

and moreover

$$|F(1-m(n),1-n,2;\tfrac{s^2}{n^2})| \le \sum_{j=0}^{\infty}\frac{(c|s|^2)^j}{j!(j+1)!} \le \exp(\gamma|s|^2), \qquad (s\in\mathbb{C}). \tag{6.24}$$

A standard application of Cauchy's Integral Formula and Lebesgue's Theorem on Dominated Convergence (using (6.24)) now shows that the convergence in (6.23) actually holds uniformly on compact subsets of $\mathbb{C}$.

Recalling next that $\lim_{n\to\infty}(1-\frac{s}{n})^n = \exp(-s)$ for any complex number s, it follows that

$$\lim_{n\to\infty}(1-\tfrac{s}{n})^{m(n)+n} = \exp(-(c+1)s), \qquad (s\in\mathbb{C}). \tag{6.25}$$

Using then that

$$\big|(1-\tfrac{s}{n})^{m(n)+n}\big| \le (1-\tfrac{|s|}{n})^{(\gamma+1)n} \le \exp((\gamma+1)|s|), \qquad (s\in\mathbb{C}),$$

it follows as before, that (6.25) holds uniformly on compact subsets of $\mathbb{C}$.

Taken together, we have verified that,

$$\lim_{n\to\infty} \mathbb{E}\big(\operatorname{tr}_n[Y_n^*Y_n \exp(sY_n^*Y_n)]\big) = c\exp((c+1)s)\sum_{j=0}^{\infty}\frac{(cs^2)^j}{j!(j+1)!}, \qquad (s\in\mathbb{C}), \tag{6.26}$$

and that the convergence is uniform on compact subsets of $\mathbb{C}$.

It remains thus to show that

$$\int_0^{\infty} x\exp(sx)\ d\mu_c(x) = c\exp((c+1)s)\sum_{j=0}^{\infty}\frac{(cs^2)^j}{j!(j+1)!}, \qquad (s\in\mathbb{C}). \tag{6.27}$$

Note for this, that for any c in $]0,\infty[$,

$$\int_0^{\infty} x\exp(sx)\ d\mu_c(x) = \tfrac{1}{2\pi}\int_{c+1-2\sqrt{c}}^{c+1+2\sqrt{c}} \exp(sx)\sqrt{4c-(x-c-1)^2}\ dx,$$

since, in the case where $c<1$, the mass at 0 for μ_c does not contribute to the integral. Applying then the substitution $x = c+1+\sqrt{c}y$, we get that

$$\int_0^{\infty} x\exp(sx)\ d\mu_c(x) = \frac{c\exp((c+1)s)}{2\pi}\int_{-2}^{2}\sqrt{4-y^2}\exp(s\sqrt{c}y)\ dy, \tag{6.28}$$

and here, as we saw in the proof of Theorem 2.5,

$$\tfrac{1}{2\pi}\int_{-2}^{2}\sqrt{4-y^2}\exp(ty)\ dy = \sum_{j=0}^{\infty}\frac{t^{2j}}{j!(j+1)!}, \qquad (t\in\mathbb{C}). \tag{6.29}$$

Combining (6.28) and (6.29), we obtain (6.27). ■

6.7 Theorem. *Assume that $c\in\,]0,\infty[$ and let $(m(n))_n$ be a sequence of positive integers, such that $\lim_{n\to\infty}\frac{m(n)}{n}=c$. Consider furthermore a sequence (Y_n) of random matrices, satisfying that $Y_n\in\mathrm{GRM}(m(n),n,\frac{1}{n})$ for all n. Then for any s in $\mathbb{C}$ and n in $\mathbb{N}$, such that $n>\mathrm{Re}(s)$,*

$$\mathbb{E}\Big(\big|\operatorname{tr}_n[\exp(sY_n^*Y_n)]\big|\Big)<\infty,$$

and moreover

$$\lim_{n\to\infty}\mathbb{E}\big(\operatorname{tr}_n[\exp(sY_n^*Y_n)]\big) = \int_0^{\infty}\exp(sx)\ d\mu_c(x), \qquad (s\in\mathbb{C}), \tag{6.30}$$

with uniform convergence on compact subsets of $\mathbb{C}$.

Proof. Since $\exp(u)\le 1+u\exp(u)$, for any u in $[0,\infty[$, the first statement of (i) follows immediately from Lemma 6.6.

Consider next an element Y of $\mathrm{GRM}(m, n, \frac{1}{n})$, and put $B = \sqrt{n}Y \in \mathrm{GRM}(m, n, 1)$. Then by Proposition 5.4, we have that

$$\begin{aligned}
&\mathbb{E}\big(\mathrm{tr}_n[f(Y^*Y)]\big) \\
&\quad = \tfrac{1}{n}\mathbb{E}\big(\mathrm{Tr}_n[f(\tfrac{1}{n}B^*B)]\big) \\
&\quad = \begin{cases} \displaystyle\int_0^\infty \big(\textstyle\sum_{k=0}^{n-1}\varphi_k^{m-n}(nx)^2\big) f(x)\ dx, & \text{if } m \geq n, \\ (1 - \frac{m}{n})f(0) + \displaystyle\int_0^\infty \big(\textstyle\sum_{k=0}^{m-1}\varphi_k^{n-m}(nx)^2\big) f(x)\ dx, & \text{if } m < n, \end{cases}
\end{aligned} \tag{6.31}$$

for any Borel function $f\colon [0, \infty[\to \mathbb{C}$, for which the integrals on the right hand side make sense.

From this formula, it follows easily that $s \mapsto \mathbb{E}\big(\mathrm{tr}_n[\exp(sY^*Y)]\big)$, is an analytic function in the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) < n\}$, and that

$$\frac{d}{ds}\mathbb{E}\big(\mathrm{tr}_n[\exp(sY^*Y)]\big) = \mathbb{E}\big(\mathrm{tr}_n[Y^*Y\exp(sY^*Y)]\big), \qquad (\mathrm{Re}(s) < n). \tag{6.32}$$

Now for each n in $\mathbb{N}$, define

$$f_n(s) = \mathbb{E}\big(\mathrm{tr}_n[\exp(sY_n^*Y_n)]\big), \qquad (\mathrm{Re}(s) < n),$$

where (Y_n) is as set out in the theorem. Define furthermore,

$$f(s) = \int_0^\infty \exp(sx)\ d\mu_c(x), \qquad (s \in \mathbb{C}).$$

Since μ_c has compact support, f is an entire function, and moreover

$$f'(s) = \int_0^\infty x\exp(sx)\ d\mu_c(x), \qquad (s \in \mathbb{C}).$$

It follows thus by (6.32) and Lemma 6.6, that

$$f_n'(s) \to f'(s), \quad \text{as } n \to \infty, \qquad (s \in \mathbb{C}), \tag{6.33}$$

with uniform convergence on compact subsets of $\mathbb{C}$. Now for fixed s in $\mathbb{C}$, we may choose a smooth path $\gamma\colon [0, 1] \to \mathbb{C}$, such that $\gamma(0) = 0$ and $\gamma(1) = s$. Then since $f_n(0) = 1 = f(0)$ for all n, it follows that

$$f_n(s) - f(s) = \int_\gamma \big(f_n'(z) - f'(z)\big)\ dz,$$

whenever $n > \mathrm{Re}(s)$. Combining this fact with (6.33), it follows readily that $f_n(s) \to f(s)$ for all s in $\mathbb{C}$, and that the convergence is uniform on compact subsets of $\mathbb{C}$. This completes the proof of Theorem 6.7. ∎

Marchenko and Pastur's limit result from [MP] in the complex Wishart case is now an immediate consequence of Theorem 6.7:

6.8 Corollary. *(cf [MP], [Wa], [GS], [Jo], [Ba], [OP].) Assume that* $c \in]0, \infty[$ *and let* $(m(n))_n$ *be a sequence of positive integers such that* $\lim_{n\to\infty} \frac{m(n)}{n} = c$. *Consider, furthermore, a sequence* (Y_n) *of random matrices satisfying that* $Y_n \in \mathrm{GRM}(m(n), n, \frac{1}{n})$ *for all* $n \in \mathbb{N}$. *Then*

(i) For any positive integer p,

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{tr}_n[(Y_n^* Y_n)^p]) = \int_0^\infty x^p \, d\mu_c(x). \tag{6.34}$$

(ii) For any bounded continuous function $f \colon [0, \infty[\to \mathbb{C}$,

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{tr}_n[f(Y_n^* Y_n)]) = \int_0^\infty f(x) \, d\mu_c(x). \tag{6.35}$$

Proof. (i) follows from Theorem 6.7 by repeating the arguments given in the proof of Corollary 2.6. (ii) follows by using (6.30) in the case $s = it$, $t \in \mathbb{R}$, as well as the implication (iv) $\Rightarrow$ (iii) in Proposition 2.1. ∎

6.9 Remark. In [OP, Proposition 1.1], Oravecz and Petz showed that

$$\int_0^\infty x^p \, d\mu_c(x) = \tfrac{1}{p} \sum_{k=1}^{p} \binom{p}{k} \binom{p}{k-1} c^k, \qquad (c > 0, \ p \in \mathbb{N}), \tag{6.36}$$

by solving a recursion formula for the moments of μ_c. It is also possible to derive this formula directly: For p in $\mathbb{N}$, the point-mass at 0 for μ_c (if $c < 1$), does not contribute to the integral on the left hand side of (6.36), and hence

$$\int_0^\infty x^p \, d\mu_c(x) = \tfrac{1}{2\pi} \int_{c+1-2\sqrt{c}}^{c+1+2\sqrt{c}} \sqrt{4c - (x-c-1)^2} x^{p-1} \, dx.$$

Applying now the substitution $x = c + 1 + 2\sqrt{c}\cos\theta$, $\theta \in [0, \pi]$, we get that

$$\begin{aligned} \int_0^\infty x^p \, d\mu_c(x) &= \tfrac{2c}{\pi} \int_0^\pi \sin^2\theta \cdot (c + 1 + 2\sqrt{c}\cos\theta)^{p-1} \, d\theta \\ &= \tfrac{c}{\pi} \int_{-\pi}^\pi \sin^2\theta \cdot (c + 1 + 2\sqrt{c}\cos\theta)^{p-1} \, d\theta. \end{aligned}$$

Consider next the functions,

$$g_p(\theta) = (1 + \sqrt{c}e^{i\theta})^{p-1}, \quad h(\theta) = e^{i\theta} g_p(\theta), \quad \text{and} \quad k(\theta) = e^{-i\theta} g_p(\theta), \qquad (\theta \in [0, \pi]).$$

Using then the formula: $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, we find that

$$\begin{aligned} \int_0^\infty x^p \, d\mu_c(x) &= \tfrac{c}{2\pi} \int_{-\pi}^\pi \mathrm{Re}(1 - e^{i2\theta}) \cdot |g_p(\theta)|^2 \, d\theta \\ &= \tfrac{c}{2\pi} \Big(\int_{-\pi}^\pi |g(\theta)|^2 \, d\theta - \mathrm{Re}\Big(\int_{-\pi}^\pi h_p(\theta) \overline{k_p(\theta)} \, d\theta \Big) \Big). \end{aligned} \tag{6.37}$$

By the binomial formula and Parseval's formula, we have here that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(\theta)|^2 \, d\theta = \sum_{j=0}^{p-1} \binom{p-1}{j}^2 c^j,$$

and that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h_p(\theta)\overline{k_p(\theta)} \, d\theta = \sum_{j=0}^{p-1} \binom{p-1}{j-1}\binom{p-1}{j+1} c^j,$$

where we have put $\binom{p-1}{-1} = \binom{p-1}{p} = 0$. A simple computation shows that

$$\binom{p-1}{j}^2 - \binom{p-1}{j-1}\binom{p-1}{j+1} = \frac{1}{p}\binom{p}{j+1}\binom{p}{j}, \qquad (0 \le j \le p-1). \tag{6.38}$$

Now (6.36) follows by combining (6.37)-(6.38), and substituting j by $j-1$. □

7 Almost Sure Convergence of the Largest and Smallest Eigenvalues in the Complex Wishart case

In the paper [Gem] from 1980, Geman studied a sequence (T_n) of random matrices, such that for all n in $\mathbb{N}$, T_n is an $m(n) \times n$ random matrix, satisfying that the entries $t_{jk}^{(n)}$, $1 \le j \le m(n)$, $1 \le k \le n$, are independent, identically distributed, real valued random variables, with mean 0 and variance 1. Under the assumption that $\lim_{n\to\infty} \frac{m(n)}{n} = c$, and some extra conditions on the growth of the higher order moments of the entries $t_{jk}^{(n)}$, Geman proved that

$$\lim_{n\to\infty} \lambda_{\max}(\tfrac{1}{n}T_n^t T_n) = (\sqrt{c}+1)^2, \qquad \text{almost surely,} \tag{7.1}$$

where $\lambda_{\max}(\frac{1}{n}T_n^t T_n)$ denotes the largest eigenvalue of $\frac{1}{n}T_n^t T_n$. Under the additional assumptions that T_n is Gaussian for all n, (i.e., $t_{jk}^{(n)} \sim N(0,1)$ for all j,k,n), and that $m(n) \ge n$ for all n, Silverstein proved in 1985, that

$$\lim_{n\to\infty} \lambda_{\min}(\tfrac{1}{n}T_n^t T_n) = (\sqrt{c}-1)^2, \qquad \text{almost surely,} \tag{7.2}$$

where $\lambda_{\min}(\frac{1}{n}T_n^t T_n)$ denotes the smallest eigenvalue of $\frac{1}{n}T_n^t T_n$ (cf. [Si]). Both Geman's and Silverstein's conditions have later been relaxed to the condition that the entries of T_n have finite fourth moment, i.e., $\mathbb{E}\big(|t_{jk}^{(n)}|^4\big) < \infty$ (cf. [YBK] and [BY2]). This condition is also necessary for (7.1) (cf. [BSY]).

The above quoted papers consider only real random matrices, but it is not hard to generalize the proofs to the complex case (cf. [Ba]). In this section we give a new proof of (7.1) and (7.2) in the complex Wishart case, by taking a different route, namely by applying the explicit formula for $\mathbb{E}\big(\mathrm{Tr}_n[\exp(B_n^* B_n)]\big)$, $B \in \mathrm{GRM}(m,n,1)$, that we obtained in Section 6. This route is similar to the one taken in Section 3.

7.1 Theorem. *Let c be a strictly positive number, and let $(m(n))_n$ be a sequence of positive integers, such that $\lim_{n\to\infty} \frac{m(n)}{n} = c$. Consider furthermore a sequence (Y_n) of random matrices, defined on the same probability space $(\Omega, \mathcal{F}, P)$, and such that $Y_n \in \mathrm{GRM}(m(n), n, \frac{1}{n})$, for all n. We then have*

$$\lim_{n\to\infty} \lambda_{\max}(Y_n^* Y_n) = (\sqrt{c}+1)^2, \qquad \textit{almost surely}, \tag{7.3}$$

and

$$\lim_{n\to\infty} \lambda_{\min}(Y_n^* Y_n) = \begin{cases} (\sqrt{c}-1)^2, & \textit{if } c > 1, \\ 0, & \textit{if } c \le 1, \end{cases} \qquad \textit{almost surely.} \tag{7.4}$$

We start by proving two lemmas:

7.2 Lemma. *Consider an element B of $\mathrm{GRM}(m, n, 1)$. We then have*

(i) *For any t in $[0, \frac{1}{2}]$,*

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tB^*B)]\big) \le n \exp\big((\sqrt{m}+\sqrt{n})^2 t + (m+n)t^2\big), \tag{7.5}$$

(ii) *If $m \ge n$ and $t \ge 0$, then*

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(-tB^*B)]\big) \le n \exp\big(-(\sqrt{m}-\sqrt{n})^2 t + (m+n)t^2\big). \tag{7.6}$$

Proof. (i) Assume first that $m \ge n$. Then by (6.20) in Theorem 6.4, we have that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tB^*B)]\big) = \sum_{k=1}^{n} \frac{F(m-k, n-k, 1; t^2)}{(1-t)^{n+m+1-2k}}, \qquad (t \in\,]-\infty, 1[). \tag{7.7}$$

For k in $\{1, 2, \dots, n\}$, we have here that

$$\begin{aligned} F(m-k, n-k, 1; t^2) &= \sum_{j=0}^{\infty} \binom{m-k}{j}\binom{n-k}{j} t^{2j} \\ &\le \sum_{j=0}^{\infty} \frac{(m-k)^j (n-k)^j}{(j!)^2} t^{2j} \\ &\le \Big(\sum_{j=0}^{\infty} \frac{(\sqrt{(m-k)(n-k)}|t|)^j}{j!}\Big)^2, \end{aligned}$$

and thus we obtain the estimate

$$F(m-k, n-k, 1; t^2) \le \exp(2\sqrt{(m-k)(n-k)}|t|), \qquad (k \in \{1, 2, \dots, n\}). \tag{7.8}$$

For t in $[0, 1[$ and k in $\mathbb{N}$, we have also that $(1-t)^{2k-1} \le 1$, and hence by (7.7) and (7.8), we get the estimate

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tB^*B)]\big) \le \sum_{k=1}^{n} \frac{\exp(2\sqrt{mn}t)}{(1-t)^{m+n}} = \frac{n\exp(2\sqrt{mn}t)}{(1-t)^{m+n}}, \qquad (t \in [0, 1[). \tag{7.9}$$

Regarding the denominator of the fraction on the right hand side of (7.9), note that for t in $[0,\frac{1}{2}]$, we have that

$$-\log(1-t)=\sum_{n=1}^{\infty}\frac{t^n}{n}\le t+\tfrac{1}{2}(t^2+t^3+t^4+\cdots)\le t+t^2,$$

and hence that $(1-t)^{-1}\le\exp(t+t^2)$. Inserting this inequality in (7.9), we obtain (7.5), in the case where $m\ge n$.

If, conversely, $m<n$, then by application of (6.21) in Theorem 6.4, we get as above, that for t in $[0,1[$,

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tB^*B)]\big)\le(n-m)+\frac{m\exp(2\sqrt{mn}t)}{(1-t)^{m+n}}\le\frac{n\exp(2\sqrt{mn}t)}{(1-t)^{m+n}}.$$

Estimating then the denominator as above, it follows that (7.5) holds for all t in $[0,\frac{1}{2}]$.

(ii) Assume that $m\ge n$, and note then that for k in $\{1,2,\dots,n\}$, we have that

$$\sqrt{(m-k)(n-k)}\le\sqrt{mn}-k.$$

Combining this inequality with (7.7) and (7.8), we get for t in $[0,\infty[$, that

$$\begin{aligned}\mathbb{E}\big(\mathrm{Tr}_n[\exp(-tB^*B)]\big)&=\sum_{k=1}^{n}\frac{F(m-k,n-k,1;t^2)}{(1+t)^{m+n+1-2k}}\\&\le\frac{1}{(1+t)^{m+n+1}}\Big(\sum_{k=1}^{n}\frac{\exp(2(\sqrt{mn}-k)t)}{(1+t)^{-2k}}\Big)\\&\le\frac{\exp(2\sqrt{mn}t)}{(1+t)^{m+n}}\Big(\sum_{k=1}^{n}\big((1+t)\exp(-t)\big)^{2k}\Big).\end{aligned}$$

Here, $(1+t)\exp(-t)\le 1$ for all t in $[0,\infty[$, and hence we see that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(-tB^*B)]\big)\le\frac{n\exp(2\sqrt{mn}t)}{(1+t)^{m+n}},\qquad(t\in[0,\infty[).\tag{7.10}$$

Regarding the denominator of the fraction on the right hand side of (7.10), we note that for any t in $[0,\infty[$, we have by Taylor's formula with remainder term,

$$\log(1+t)=t-\tfrac{t^2}{2}+\tfrac{t^3}{3(1+\xi(t))^3},$$

for some number $\xi(t)$ in $[0,t[$. It follows thus that $\log(1+t)\ge t-\frac{t^2}{2}$, and hence that $(1+t)^{-1}\le\exp(-t+t^2)$, for any t in $[0,\infty[$. Combining this fact with (7.10), we obtain (7.6). ■

7.3 Lemma. *Let c, $(m(n))_n$ and (Y_n) be as set out in Theorem 7.1. We then have*

(i) *For almost all ω in Ω,*

$$\limsup_{n\to\infty}\lambda_{\max}\big(Y_n^*(\omega)Y_n(\omega)\big)\le(\sqrt{c}+1)^2.\tag{7.11}$$

(ii) *If $c > 1$, then for almost all ω in Ω,*

$$\liminf_{n\to\infty} \lambda_{\min}\big(Y_n^*(\omega)Y_n(\omega)\big) \geq (\sqrt{c}-1)^2. \tag{7.12}$$

Proof. For each n in $\mathbb{N}$, we put $c_n = \frac{m(n)}{n}$, and $B_n = \sqrt{n}Y_n \in \mathrm{GRM}(m(n), n, 1)$. By Lemma 7.2, we have then that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tY_n^*Y_n)]\big) \leq n\exp\big((\sqrt{c_n}+1)^2t + \tfrac{1}{n}(c_n+1)t^2\big), \qquad (t \in [0, \tfrac{n}{2}]), \tag{7.13}$$

and that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(-tY_n^*Y_n)]\big) \leq n\exp\big(-(\sqrt{c_n}-1)^2t + \tfrac{1}{n}(c_n+1)t^2\big), \qquad (t \in [0, \infty[), \tag{7.14}$$

Since all the eigenvalues of $\exp(\pm tY_n^*Y_n)$ are positive, we have here for any t in $[0, \infty[$, that

$$\begin{aligned} \mathrm{Tr}_n[\exp(tY_n^*(\omega)Y_n(\omega))] &\geq \lambda_{\max}\big(\exp(tY_n^*(\omega)Y_n(\omega))\big) \\ &= \exp\big(t\lambda_{\max}(Y_n^*(\omega)Y_n(\omega))\big), \qquad (\omega \in \Omega), \end{aligned} \tag{7.15}$$

and that

$$\begin{aligned} \mathrm{Tr}_n[\exp(-tY_n^*(\omega)Y_n(\omega))] &\geq \lambda_{\max}\big(\exp(-tY_n^*(\omega)Y_n(\omega))\big) \\ &= \exp\big(-t\lambda_{\min}(Y_n^*(\omega)Y_n(\omega))\big), \qquad (\omega \in \Omega). \end{aligned} \tag{7.16}$$

For fixed n in $\mathbb{N}$, t in $]0, \frac{n}{2}]$ and ϵ in $]0,1[$, we get now by (7.15) and (7.13),

$$\begin{aligned} P\Big(\lambda_{\max}(Y_n^*Y_n) \geq (\sqrt{c_n}+1)^2+\epsilon\Big) &= P\Big(\exp\big[t\lambda_{\max}(Y_n^*Y_n) - t(\sqrt{c_n}+1)^2 - t\epsilon\big] \geq 1\Big) \\ &\leq \mathbb{E}\Big(\exp\big[t\lambda_{\max}(Y_n^*Y_n) - t(\sqrt{c_n}+1)^2 - t\epsilon\big]\Big) \\ &\leq \exp[-t(\sqrt{c_n}+1)^2 - t\epsilon]\mathbb{E}\big(\mathrm{Tr}_n[\exp(tY_n^*Y_n)]\big) \\ &\leq n\exp\big(-t\epsilon + \tfrac{1}{n}(c_n+1)t^2\big). \end{aligned}$$

For fixed n in $\mathbb{N}$ and ϵ in $]0,\infty[$, the function $t \mapsto -t\epsilon + \frac{1}{n}(c_n+1)t^2$, attains its minimum at $t_0 = \frac{n\epsilon}{2(c_n+1)} \in]0, \frac{n}{2}]$. With this value of t, the above inequality becomes

$$P\big(\lambda_{\max}(Y_n^*Y_n) \geq (\sqrt{c_n}+1)^2+\epsilon\big) \leq n\exp\big(-t_0\epsilon + \tfrac{1}{n}(c_n+1)t_0^2\big) = n\exp\big(\tfrac{-n\epsilon^2}{4(c_n+1)}\big).$$

Since $c_n \to c$ as $n \to \infty$, the sequence (c_n) is bounded, and thus it follows that

$$\sum_{n=1}^{\infty} P\big(\lambda_{\max}(Y_n^*Y_n) \geq (\sqrt{c_n}+1)^2+\epsilon\big) \leq \sum_{n=1}^{\infty} n\exp\big(\tfrac{-n\epsilon^2}{4(c_n+1)}\big) < \infty.$$

Hence the Borel-Cantelli lemma yields, that on a set with probability one, we have that

$$\lambda_{\max}(Y_n^*Y_n) \leq (\sqrt{c_n}+1)^2 + \epsilon, \quad \text{eventually},$$

and consequently that

$$\limsup_{n\to\infty} \lambda_{\max}(Y_n^* Y_n) \le \limsup_{n\to\infty} \left[(\sqrt{c_n}+1)^2+\epsilon\right] = (\sqrt{c}+1)^2+\epsilon.$$

Taken together, we have verified that for any ϵ in $]0,\infty[$, we have that

$$P\Big(\limsup_{n\to\infty} \lambda_{\max}(Y_n^* Y_n) \le (\sqrt{c}+1)^2+\epsilon\Big) = 1,$$

and this proves (7.11). The proof of (7.12) can be carried out in exactly the same way, using (7.16) and (7.14) instead of (7.15) respectively (7.13). We leave the details to the reader. ∎

To conclude the proof of Theorem 7.1, we must, as in Geman's paper [Gem], rely on Wachter's result from [Wa] on almost sure convergence of the empirical distribution of the eigenvalues to the measure μ_c. As mentioned in the beginning of Section 6, the random matrices considered by Wachter have real valued (but not necessarily Gaussian) entries. His method works also for random matrices with complex valued entries, but in the following we shall give a short proof for the case of complex Gaussian random matrices, based on the "concentration of measures phenomenon" in the form of Lemma 3.4.

7.4 Proposition. (cf. [Wa]) *Let c, $(m(n))_n$ and (Y_n) be as in Theorem 7.1, and for all n in $\mathbb{N}$ and ω in Ω, let $\mu_{n,\omega}$ denote the empirical distribution of the eigenvalues of $Y_n^*(\omega)Y_n(\omega)$, i.e.,*

$$\mu_{n,\omega} = \tfrac{1}{n}\textstyle\sum_{j=1}^n \delta_{\lambda_k(Y_n^*(\omega)Y_n(\omega))},$$

where, as usual, $\lambda_1(Y_n^*(\omega)Y_n(\omega)) \le \cdots \le \lambda_n(Y_n^*(\omega)Y_n(\omega))$ are the ordered eigenvalues of $Y_n^*(\omega)Y_n(\omega)$. We then have

(i) *For almost all ω in Ω, $\mu_{n,\omega}$ converges weakly to the measure μ_c introduced in Definition 6.5.*

(ii) *On a set with probability 1, we have for any interval I in $\mathbb{R}$, that*

$$\lim_{n\to\infty} \left(\tfrac{1}{n}\cdot \operatorname{card}\left[\operatorname{sp}(Y_n^* Y_n)\cap I\right]\right) = \mu_c(I).$$

Proof. Note first that (ii) follows from (i), Proposition 2.1 and Definition 2.2.

To prove (i), it suffices, as in the proof of Proposition 3.6, to show that for every fixed function f from $C_c^1(\mathbb{R})$, we have that

$$\lim_{n\to\infty} \operatorname{tr}_n[f(Y_n^* Y_n)] = \int_0^\infty f\, d\mu_c, \qquad \text{almost surely.}$$

So let such an f be given, and define $g\colon \mathbb{R}\to\mathbb{C}$ by the equation: $g(x) = f(x^2)$, $(x\in\mathbb{R})$. Then $g\in C_c^1(\mathbb{R})$, so in particular g is Lipschitz with constant

$$c = \sup_{x\in\mathbb{R}} |g'(x)| < \infty.$$

Consider furthermore fixed m, n in $\mathbb{N}$, and for A, B in $M_{m,n}(\mathbb{C})$, define $\tilde{A}$ and $\tilde{B}$ in $M_{m+n}(\mathbb{C})$ by the equations

$$\tilde{A} = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}.$$

By Lemma 3.5 it follows then that

$$\|g(\tilde{A}) - g(\tilde{B})\|_{\mathrm{HS}} \le c\|\tilde{A} - \tilde{B}\|_{\mathrm{HS}}. \tag{7.17}$$

Note here that

$$\tilde{A}^2 = \begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix}, \qquad \tilde{B}^2 = \begin{pmatrix} B^*B & 0 \\ 0 & BB^* \end{pmatrix}.$$

so that

$$g(\tilde{A}) = \begin{pmatrix} f(A^*A) & 0 \\ 0 & f(AA^*) \end{pmatrix}, \qquad g(\tilde{B}) = \begin{pmatrix} f(B^*B) & 0 \\ 0 & f(BB^*) \end{pmatrix}.$$

Hence, it follows from (7.17) that

$$\|f(A^*A) - f(B^*B)\|_{\mathrm{HS}}^2 + \|f(AA^*) - f(BB^*)\|_{\mathrm{HS}}^2 \le c^2\big(\|A - B\|_{\mathrm{HS}}^2 + \|A^* - B^*\|_{\mathrm{HS}}^2\big).$$

Since $\|A^* - B^*\|_{\mathrm{HS}}^2 = \|A - B\|_{\mathrm{HS}}^2$, the above inequality implies that

$$\|f(A^*A) - f(B^*B)\|_{\mathrm{HS}} \le c\sqrt{2}\|A - B\|_{\mathrm{HS}},$$

and hence, by the Cauchy-Schwarz inequality, that

$$\big|\mathrm{tr}_n[f(A^*A)] - \mathrm{tr}_n[f(B^*B)]\big| \le c\sqrt{\tfrac{2}{n}}\|A - B\|_{\mathrm{HS}}.$$

It follows thus, that the function $F\colon M_{m,n}(\mathbb{C}) \to \mathbb{R}$, given by

$$F(A) = \mathrm{tr}_n[f(A^*A)], \qquad (A \in M_{m,n}(\mathbb{C})), \tag{7.18}$$

satisfies the Lipschitz condition

$$|F(A) - F(B)| \le c\sqrt{\tfrac{2}{n}}\|A - B\|_{\mathrm{HS}}, \qquad (A, B \in M_{m,n}(\mathbb{C})). \tag{7.19}$$

The linear bijection $\Phi\colon M_{m,n}(\mathbb{C}) \to \mathbb{R}^{2mn}$, given by

$$\Phi(A) = \big(\mathrm{Re}(A_{jk}), \mathrm{Im}(A_{jk})\big)_{\substack{1\le j\le m,\\ 1\le k\le n}}, \qquad (A \in M_{m,n}(\mathbb{C})),$$

transforms the distribution on $M_{m,n}(\mathbb{C})$ of an element of $\mathrm{GRM}(m, n, \frac{1}{n})$ onto the joint distribution of $2mn$ independent, identically $N(0, \frac{1}{2n})$-distributed random variables, i.e., the distribution $G_{2mn,(2n)^{-\frac{1}{2}}}$ on $\mathbb{R}^{2mn}$ with density

$$\frac{dG_{2mn,(2n)^{-\frac{1}{2}}}(x)}{dx} = \Big(\frac{n}{\pi}\Big)^{mn} \exp(-n\|x\|^2), \qquad (x \in \mathbb{R}^{2mn}),$$

w.r.t. Lebesque measure on $\mathbb{R}^{2mn}$. Moreover, the Hilbert-Schmidt norm on $M_{m,n}(\mathbb{C})$ corresponds to the Euclidean norm on $\mathbb{R}^{2mn}$ via the mapping Φ. Combining these observations with Lemma 3.4 (in the case $\sigma^2 = \frac{1}{\sqrt{2n}}$) and (7.19), it follows that with (Y_n) as set out in the proposition, we have for any n in $\mathbb{N}$ and t from $]0,\infty[$, that

$$P\big(|F(Y_n) - \mathbb{E}(F(Y_n))| > t\big) \le 2\exp(-\tfrac{n^2Kt^2}{c^2}),$$

where $K = \frac{2}{\pi^2}$. It follows thus by application of the Borel-Cantelli lemma, that

$$\lim_{n\to\infty} |F(Y_n) - \mathbb{E}(F(Y_n))| = 0, \qquad \text{almost surely.}$$

Using then (7.18) and Corollary 6.8(ii), we get that

$$\lim_{n\to\infty} \mathrm{tr}_n[f(Y_n^*Y_n)] = \int_0^\infty f\ d\mu_c, \qquad \text{almost surely,}$$

as desired. ■

Proof of Theorem 7.1. By Lemma 7.3, we only need to show, that for any c from $]0,\infty[$, we have that

$$\liminf_{n\to\infty} \lambda_{\max}(Y_n^*Y_n) \ \ge\ (\sqrt{c}+1)^2, \qquad \text{almost surely,} \tag{7.20}$$

$$\limsup_{n\to\infty} \lambda_{\min}(Y_n^*Y_n) \ \le\ \begin{cases} (\sqrt{c}-1)^2, & \text{if } c > 1, \\ 0, & \text{if } c \le 1, \end{cases} \qquad \text{almost surely.} \tag{7.21}$$

By Proposition 7.4, it follows, that for any strictly positive ϵ and almost all ω from Ω, the numbers of eigenvalues of $Y_n^*(\omega)Y_n(\omega)$ in the intervals $\big[(\sqrt{c}+1)^2-\epsilon, (\sqrt{c}+1)^2\big]$ and $\big[(\sqrt{c}-1)^2, (\sqrt{c}-1)^2+\epsilon\big]$, both tend to ∞, as $n\to\infty$. This proves (7.20) and, when $c \ge 1$, also (7.21). If $c < 1$, then $m(n) < n$ eventually, and this implies that eventually, 0 is an eigenvalue for $Y_n^*(\omega)Y_n(\omega)$, for any ω in Ω. Hence we conclude that (7.21) holds in this case too. ■

8 A Recursion Formula for the Moments of the complex Wishart distribution

In [HSS], Hanlon, Stanley and Stembridge used representation theory of the Lie group $U(n)$ to compute the moments $\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big)$ of B^*B, when $B \in \mathrm{GRM}(m,n,1)$. They derived the following formula (cf. [HSS, Theorem 2.5]):

$$\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big) = \tfrac{1}{p}\sum_{j=1}^{p}(-1)^{j-1}\frac{[m+p-j]_p[n+p-j]_p}{(p-j)!(j-1)!}, \qquad (p\in\mathbb{N}), \tag{8.1}$$

where we apply the notation: $[a]_p = a(a-1)\cdots(a-p+1)$, $(a\in\mathbb{C}, p\in\mathbb{N}_0)$.

By application of the results of Section 6, we can derive another explicit formula for the moments of B^*B:

8.1 Proposition. *Let m, n be positive integers, and let B be an element of* $\mathrm{GRM}(m,n,1)$*. Then for any p in $\mathbb{N}$, we have that*

$$\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big) = mn(p-1)! \sum_{j=0}^{[\frac{p-1}{2}]} \frac{1}{j+1}\binom{m-1}{j}\binom{n-1}{j}\binom{m+n+p-2j-2}{p-2j-1}. \quad (8.2)$$

Proof. In Section 6, we saw that that for any complex number s, such that $\mathrm{Re}(s) < 1$, we have the formula

$$\mathbb{E}\big(\mathrm{Tr}_n[B^*B\exp(sB^*B)]\big) = \frac{m \cdot n \cdot F(1-m, 1-n, 2; s^2)}{(1-s)^{m+n}}, \quad (8.3)$$

(cf. formula (6.19)). Hence, by Taylor series expansion, for any s in $\mathbb{C}$, such that $|s| < 1$, we have that

$$\sum_{p=1}^{\infty} \frac{\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big) \cdot s^{p-1}}{(p-1)!} = \frac{m \cdot n \cdot F(1-m, 1-n, 2; s^2)}{(1-s)^{m+n}}, \quad (8.4)$$

Formula (8.2) now follows by multiplying the two power series

$$F(1-m, 1-n, 2; s^2) = \sum_{j=0}^{\infty} \frac{1}{j+1}\binom{m-1}{j}\binom{n-1}{j} s^{2j},$$

and

$$(1-s)^{-(m+n)} = \sum_{k=0}^{\infty} \binom{m+n+k-1}{k} s^k,$$

and comparing terms in (8.4). ■

We prove next a recursion formula for the moments of B^*B, similar to the Harer-Zagier recursion formula, treated in Section 4.

8.2 Theorem. *Let m, n be positive integers, let B be an element of* $\mathrm{GRM}(m,n,1)$*, and for p in $\mathbb{N}_0$, define*

$$D(p,m,n) = \mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big). \quad (8.5)$$

Then $D(0,m,n) = n$, $D(1,m,n) = mn$, and for fixed m, n, the numbers $D(p,m,n)$ satisfy the recursion formula

$$D(p+1,m,n) = \tfrac{(2p+1)(m+n)}{p+2} \cdot D(p,m,n) + \tfrac{(p-1)(p^2-(m-n)^2)}{p+2} \cdot D(p-1,m,n), \qquad (p \in \mathbb{N}). \quad (8.6)$$

Proof. Recall from Section 6, that the hyper-geometric function F is defined by the formula

$$F(a,b,c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k,$$

for a, b, c, x in $\mathbb{C}$, such that $c \notin \mathbb{Z} \setminus \mathbb{N}_0$, and $|x| < 1$. For fixed a, b, c, the function $u(x) = F(a, b, c; x)$, is a solution to the differential equation

$$x(1-x)\frac{d^2u}{dx^2} + (c - (a+b+1)x)\frac{du}{dx} - abu = 0,$$

(cf. [HTF, Vol. 1, p.56, formula (1)]). In particular, if $a = 1-n$, $b = 1-m$ and $c = 2$, then u satisfies the differential equation

$$x(1-x)\frac{d^2u}{dx^2} + (2 + (m+n-3)x)\frac{du}{dx} - (m-1)(n-1)u = 0. \tag{8.7}$$

Define now, for these a, b, c,

$$v(t) = u(t^2) = F(1-m, 1-n, 2; t^2), \qquad (|t| < 1).$$

Then (8.7) implies that v satisfies the differential equation

$$t(1-t)\frac{d^2v}{dt^2} + (3 + (2m+2n-5)t^2)\frac{dv}{dt} - 4(m-1)(n-1)tv = 0, \qquad (|t| < 1). \tag{8.8}$$

Define next

$$w(t) = \frac{v(t)}{(1-t)^{m+n}} = \frac{F(1-m, 1-n, 2; t^2)}{(1-t)^{m+n}}, \qquad (|t| < 1).$$

A tedious, but straightforward computation, then shows that w satisfies the differential equation

$$\begin{aligned} t(1-t^2)\frac{d^2w}{dt^2} &+ (3 - 2(m+n)t - 5t^2)\frac{dw}{dt} \\ &- (3(m+n) + 4t - (m-n)^2 t)w = 0, \qquad (|t| < 1). \end{aligned} \tag{8.9}$$

Introduce now the power series expansion $w(t) = \sum_{p=0}^{\infty} \alpha_p t^p$, of $w(t)$. Inserting this expansion in (8.9), one finds (after some reductions), that the coefficients α_p satisfy the formulas

$$\alpha_0 = 1, \quad \text{and} \quad \alpha_1 = m+n, \tag{8.10}$$

$$p(p+2)\alpha_p - (2p+1)(m+n)\alpha_{p-1} - (p^2 - (m-n)^2)\alpha_{p-2} = 0, \qquad (p \geq 2). \tag{8.11}$$

On the other hand, inserting the power series expansion of $w(t)$ in (8.4), yields the formula

$$D(p, m, n) = \mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big) = mn(p-1)!\alpha_{p-1}, \qquad (p \in \mathbb{N}). \tag{8.12}$$

Combining this formula with (8.11), it follows that (8.6) holds, whenever $p \geq 2$. Regarding the case $p = 1$, it follows from (8.10) and (8.12), that $D(1, m, n) = mn$, $D(2, m, n) = mn(m+n)$, and hence (8.6) holds in this case too. It remains to note that $D(0, m, n) = \mathbb{E}\big(\mathrm{Tr}_n[\mathbf{1}_n]\big) = n$. ∎

The recursion formula (8.6) is much more efficient than (8.1) and (8.2) to generate tables of the moments of B^*B. For an element B of $\mathrm{GRM}(m,n,1)$, we get

$$\begin{aligned}
\mathbb{E}\big(\mathrm{Tr}_n[B^*B]\big) &= mn \\
\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^2]\big) &= m^2n+mn^2 \\
\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^3]\big) &= (m^3n+3m^2n^2+mn^3)+mn \\
\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^4]\big) &= (m^4n+6m^3n^2+6m^2n^3+mn^4)+(5m^2n+5mn^2) \\
\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^5]\big) &= (m^5n+10m^4n^2+20m^3n^3+10m^2n^4+mn^5) \\
&\qquad\qquad + (15m^3n+40m^2n^2+15mn^3)+8mn.
\end{aligned}$$

For $p \leq 4$, these moments were also computed in [HSS, p.172] by application of (8.1). Note that only terms of homogeneous degree $p+1-2j$, $j \in \{0,1,2,\ldots,[\frac{p-1}{2}]\}$, appear in the above formulas. This is a general fact, which can easily be proved by Theorem 8.2 and induction. If we replace the B from $\mathrm{GRM}(m,n,1)$ considered above by an element Y from $\mathrm{GRM}(m,n,\frac{1}{n})$, and Tr_n by tr_n, then we have to divide the right hand sides of the above formulas by n^{p+1}. Thus with $c=\frac{m}{n}$, we obtain the formulas

$$\begin{aligned}
\mathbb{E}\big(\mathrm{tr}_n[Y^*Y]\big) &= c \\
\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^2]\big) &= c^2+c \\
\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^3]\big) &= (c^3+3c^2+c)+cn^{-2} \\
\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^4]\big) &= (c^4+6c^3+6c^2+c)+(5c^2+5c)n^{-2} \\
\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^5]\big) &= (c^5+10c^4+20c^3+10c^2+c) \\
&\qquad\qquad + (15c^3+40c^2+15c)n^{-2}+8cn^{-4}.
\end{aligned}$$

In general $\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^p]\big)$ is a polynomial of degree $[\frac{p-1}{2}]$ in n^{-2}, for fixed c. By Theorem 8.2, the constant term $\gamma(p,c)$ in this polynomial, satisfies the recursion formula

$$\gamma(p+1,c) = \tfrac{(2p+1)(c+1)}{p+2}\cdot\gamma(p,c) - \tfrac{(p-1)(c-1)^2}{p+2}\cdot\gamma(p-1,c), \qquad (p \in \mathbb{N}),$$

and moreover, $\gamma(0,c)=1$, $\gamma(1,c)=c$. As was proved in [OP], for any c in $]0,\infty[$, the solution to this difference equation is exactly the sequence of moments of the free Poisson distribution μ_c with parameter c, i.e.,

$$\gamma(p,c) = \int_0^\infty x^p \, d\mu_c(x) = \tfrac{1}{p}\sum_{k=1}^{p}\binom{p}{k}\binom{p}{k-1}c^k, \qquad (p \in \mathbb{N}),$$

(cf. [OP, Formula (1.2) and Proposition 1.1]). Thus, if $Y_n \in \mathrm{GRM}(m(n),n,\frac{1}{n})$, for all n in $\mathbb{N}$, and $\frac{m(n)}{n} \to c$, as $n \to \infty$, then we have that

$$\lim_{n\to\infty}\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^p]\big) = \gamma(p,c) = \int_0^\infty x^p \, d\mu_c(x),$$

in concordance with Corollary 6.8(i).

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