Invariant Subspaces for Operators in a General II_1-factor

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Abstract

Let $\mathcal{M}$ be a von Neumann factor of type II_1 with a normalized trace τ. In 1983 L. G. Brown showed that to every operator $T \in \mathcal{M}$ one can in a natural way associate a spectral distribution measure μ_T (now called the Brown measure of T), which is a probability measure in $\mathbb{C}$ with support in the spectrum $\sigma(T)$ of T. In this paper it is shown that for every $T \in \mathcal{M}$ and every Borel set B in $\mathbb{C}$, there is a unique closed T-invariant subspace $\mathcal{K} = \mathcal{K}_T(B)$ affiliated with $\mathcal{M}$, such that the Brown measure of $T|_{\mathcal{K}}$ is concentrated on B and the Brown measure of $P_{\mathcal{K}^\perp} T|_{\mathcal{K}^\perp}$ is concentrated on $\mathbb{C} \setminus B$. Moreover, $\mathcal{K}$ is T-hyperinvariant and the trace of $P_{\mathcal{K}}$ is equal to $\mu_T(B)$. In particular, if $T \in \mathcal{M}$ has a Brown measure which is not concentrated on a singleton, then there exists a non-trivial, closed, T-hyperinvariant subspace. Furthermore, it is shown that for every $T \in \mathcal{M}$ the limit $A := \lim_{n\to\infty}[(T^n)^* T^n]^{\frac{1}{2n}}$ exists in the strong operator topology, and the projection onto $\mathcal{K}_T(\overline{B(0,r)})$ is equal to $1_{[0,r]}(A)$, for every $r > 0$.

1 Introduction

Consider a von Neumann algebra $\mathcal{M}$ acting on the Hilbert space $\mathcal{H}$. A closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ is said to be *affiliated with* $\mathcal{M}$ if the projection of $\mathcal{H}$ onto $\mathcal{H}_0$ belongs to $\mathcal{M}$. $\mathcal{H}_0$ is said to be *non-trivial* if $\mathcal{H}_0 \neq 0$ and $\mathcal{H}_0 \neq \mathcal{H}$. For $T \in \mathcal{M}$, a subspace $\mathcal{H}_0$ of $\mathcal{H}$ is said to be *T-invariant*, if $T(\mathcal{H}_0) \subseteq \mathcal{H}_0$, i.e. if T and the projection $P_{\mathcal{H}_0}$ onto $\overline{\mathcal{H}_0}$ satisfy

$$P_{\mathcal{H}_0} T P_{\mathcal{H}_0} = T P_{\mathcal{H}_0}.$$

$\mathcal{H}_0$ is said to be *hyperinvariant* for T (or *T-hyperinvariant*) if it is S-invariant for every $S \in B(\mathcal{H})$ satisfying $ST = TS$. It is not hard to see that if $\mathcal{H}_0$ is hyperinvariant for T, then $P_{\mathcal{H}_0} \in W^*(T) = \{T\}''$. However, the converse statement does not hold true. In fact, one can find $A \in M_3(\mathbb{C})$ and an A-invariant projection $P \in W^*(A)$ which is not A-hyperinvariant (cf. [D]).

The *invariant subspace problem relative to* $\mathcal{M}$ asks whether every operator $T \in \mathcal{M} \setminus \mathbb{C}\mathbf{1}$ has a non-trivial invariant subspace $\mathcal{H}_0$ affiliated with $\mathcal{M}$, and the *hyperinvariant subspace*

*Supported by The Danish National Research Foundation.

problem asks whether one can always choose such an $\mathcal{H}_0$ to be hyperinvariant for T. Of course, if $\mathcal{M}$ is not a factor, then both of these questions may be answered in the affirmative. Also, if $\mathcal{M}$ is a factor of finite dimension, i.e. $\mathcal{M} \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, then every operator $T \in \mathcal{M} \setminus \mathbb{C}\mathbf{1}$ has a non-trivial eigenspace, and therefore $\mathbb{C}^n$ has a non-trivial T-invariant subspace. In this paper we shall focus on the invariant subspace problem relative to a II_1-factor.

Recall from [KR2, Section 8] that every II_1-factor $\mathcal{M}$ has a unique tracial state τ, and τ is faithful and normal. The *Fuglede-Kadison determinant*, $\Delta : \mathcal{M} \to [0, \infty)$, is given by

$$\Delta(T) = \exp\{\tau(\log |T|)\}, \qquad (T \in \mathcal{M}), \tag{1.1}$$

with $\exp\{-\infty\} := 0$ (cf. [FuKa]). Also recall from [Br] that for fixed $T \in \mathcal{M}$, the function

$$\lambda \mapsto \log \Delta(T - \lambda\mathbf{1})$$

is subharmonic in $\mathbb{C}$, and its Laplacian

$$\mathrm{d}\mu_T(\lambda_1 + \mathrm{i}\lambda_2) := \frac{1}{2\pi}\nabla^2 \log \Delta[T - (\lambda_1 + \mathrm{i}\lambda_2)\mathbf{1}]\, \mathrm{d}\lambda_1\, \mathrm{d}\lambda_2 \tag{1.2}$$

(taken in the distribution sense) defines a probability measure μ_T on $\mathbb{C}$, the *Brown measure of* T, with $\mathrm{supp}(\mu_T) \subseteq \sigma(T)$. Note that if $T \in \mathcal{M}$ is normal, then $\mu_T = \tau \circ E_T$, where $E_T : \mathbb{B}(\mathbb{C}) \to \mathrm{Proj}(\mathcal{M})$ ($\mathbb{B}(\mathbb{C})$ being the Borel σ-algebra of $\mathbb{C}$) is the projection valued measure on $(\mathbb{C}, \mathbb{B}(\mathbb{C}))$ in the spectral resolution of T. That is,

$$T = \int_{\sigma(T)} \lambda\, \mathrm{d}E_T(\lambda).$$

If $\mathcal{M} = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, then the Fuglede-Kadison determinant and the Brown measure are also defined for $T \in \mathcal{M}$, and in this case we have that

$$\Delta(T) = |\mathrm{det} T|^{\frac{1}{n}},$$

and

$$\mu_T = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of T, repeated according to multiplicity. The Brown measure of an operator T has been computed explicitly for a number of special operators (see [HL], [BL] and [DH2]).

The main result of this paper is (cf. Theorem 7.1):

1.1 Main Theorem. *Let $\mathcal{M}$ be a II_1-factor. Then for every $T \in \mathcal{M}$ and every Borelset $B \subseteq \mathbb{C}$ there is a largest closed, T-invariant subspace, $\mathcal{K} = \mathcal{K}_T(B)$, affiliated with $\mathcal{M}$, such that the Brown measure of $T|_{\mathcal{K}}$, $\mu_{T|_{\mathcal{K}}}$, is concentrated on B.*[1] *Moreover, $\mathcal{K}$ is hyperinvariant for T, and if $P = P_T(B) \in \mathcal{M}$ denotes the projection onto $\mathcal{K}$, then*

[1]If $\mathcal{K} = \{0\}$, then we define $\mu_{T|_{\mathcal{K}}} := 0$. If $\mathcal{K} \neq \{0\}$, then $\mu_{T|_{\mathcal{K}}}$ is computed relative to the II_1-factor $P\mathcal{M}P$, where $P \in \mathcal{M}$ denotes the projection onto $\mathcal{K}$.

(i) $\tau(P) = \mu_T(B)$,

(ii) the Brown measure of $P^\perp T P^\perp$*, considered as an element of* $P^\perp \mathcal{M} P^\perp$*, is concentrated on* $\mathbb{C} \setminus B$.

The subspace $\mathcal{K} = \mathcal{K}(B)$ can also be characterized as the unique closed T-invariant subspace affiliated with $\mathcal{M}$ for which the Brown measures of $T|_{\mathcal{K}}$ and $P^\perp T P^\perp$ are concentrated on B and $\mathbb{C} \setminus B$, respectively (cf. Corollary 7.28). It is an easy consequence of the Main Theorem that if $T \in \mathcal{M}$, and if μ_T is not a Dirac measure, then T has a non-trivial hyperinvariant subspace (cf. Corollary 7.2).

A II_1-factor $\mathcal{M}$ on a separable Hilbert space is said to have *the embedding property*, if it embeds in the ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1-factor $\mathcal{R}$ for some free ultrafilter ω on $\mathbb{N}$. In 1976 Connes (cf. [Co]) raised the question whether every II_1-factor on a separable Hilbert space has the embedding property. This problem remains unsolved.

In his unpublished lecture notes [H1] from MSRI 2001 (see also [H2]), the first author proved our Main Theorem in the special case where $\mathcal{M}$ has the embedding property. The contents of sections 2 and 3 are by and large taken from [H1], but for the rest of the paper, we resort to a completely different line of proof in order to treat the general case. Our proof is based on free probability theory (cf. sections 4 and 5) and the Turpin–Waelbroek method of integration in quasinormed spaces (cf. section 6).

This research has been motivated by recent papers on invariant subspaces for operators in II_1-factors (cf. [DH1], [DH2] and [SS]), where results similar to Theorem 1.1 have been obtained for special operators and special Borel sets.

The construction of the spectral subspaces $\mathcal{K}_T(B)$ referred to in the Main Theorem is carried out in several steps. In section 3 we introduce the closed T–hyperinvariant subspaces $E(T,r)$ and $F(T,r)$ in the following way: $E(T,r)$ is defined as the set of vectors $\xi \in \mathcal{H}$, for which there is a sequence $(\xi_n)_{n=1}^\infty$ in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \|\xi_n - \xi\| = 0 \ \text{ and } \ \limsup_{n\to\infty} \|(T - \lambda\mathbf{1})^n \xi_n\|^{\frac{1}{n}} \leq r,$$

and $F(T,r)$ is defined as the set of vectors $\eta \in \mathcal{H}$, for which there is a sequence $(\eta_n)_{n=1}^\infty$ in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \|(T - \lambda\mathbf{1})^n \eta_n - \eta\| = 0 \ \text{ and } \ \limsup_{n\to\infty} \|\eta_n\|^{\frac{1}{n}} \leq \frac{1}{r}.$$

In section 7 we combine the results of sections 2 through 6 and prove that with

$$\mathcal{K}_T(\overline{B(0,r)}) := E(T,r), \qquad r > 0,$$

and

$$\mathcal{K}_T(\mathbb{C} \setminus B(0,r)) := F(T,r), \qquad r > 0,$$

$\mathcal{K}_T(\overline{B(0,r)})$ ($\mathcal{K}_T(\mathbb{C} \setminus B(0,r))$, resp.) satisfies the conditions listed in our Main Theorem in the case $B = \overline{B(0,r)}$ ($B = \mathbb{C} \setminus B(0,r)$, resp.).

Then for a closed subset F of $\mathbb{C}$, $F \neq \mathbb{C}$, we write $\mathbb{C} \setminus F$ as a countable union of open balls:

$$\mathbb{C} \setminus F = \bigcup_{n=1}^{\infty} B(\lambda_n, r_n),$$

and we prove that the subspace

$$\mathcal{K}_T(F) := \bigcap_{n \in \mathbb{N}} F(T - \lambda_n \mathbf{1}, r_n)$$

has the properties mentioned in Theorem 1.1.

Finally, for arbitrary $B \in \mathbb{B}(\mathbb{C})$ we show that

$$\mathcal{K}_T(B) := \overline{\bigcup_{K \subseteq B,\, K \text{ compact}} \mathcal{K}_T(K)}$$

does the job for us.

We now describe the contents of the rest of the paper. The main result of section 2 is that for T in the II_1-factor $\mathcal{M}$, the push-forward measure of $\mu_{(T^*)^n T^n}$ under the map $t \mapsto t^{\frac{1}{n}}$ converges weakly (as $n \to \infty$) to the measure $\nu \in \mathrm{Prob}([0, \infty))$ which is uniquely determined by

$$\nu([0, t^2]) = \mu_T(\overline{B(0,t)}), \qquad (t > 0).$$

In section 3 we define for $T \in \mathcal{M}$ and $r > 0$ the T-hyperinvariant subspaces $E(T, r)$ and $F(T, r)$ mentioned above, and we explore some of their properties. We shall see that these subspaces seem to be good candidates for the desired T-invariant subspaces $\mathcal{K}_T(\overline{B(0,r)})$ and $\mathcal{K}_T(\mathbb{C} \setminus B(0, r))$, respectively. However, in order to prove that $E(T, r)$ and $F(T, r)$ fulfill the requirements listed in Theorem 1.1, some more work has to be done.

We begin by considering the case $r = 1$ and assume that $\mu_T(\partial B(0, 1)) = 0$. The idea of proof is the following: If $\sigma(T) \cap \partial B(0, 1) = \emptyset$, then one can always define an idempotent $e \in \mathcal{M}$ by

$$e = \frac{1}{2\pi \mathrm{i}} \int_{\partial B(0,1)} (\lambda \mathbf{1} - T)^{-1} \, \mathrm{d}\lambda \tag{1.3}$$

- as a Banach space valued integral in $\mathcal{M}$. Then it is a fact that the range projection p of e is T-invariant with

$$\sigma(T|_{p(\mathcal{H})}) \subseteq B(0, 1) \quad \text{and} \quad \sigma((\mathbf{1} - p)T|_{p(\mathcal{H})^\perp}) \subseteq \mathbb{C} \setminus \overline{B(0, 1)}.$$

Hence, the Brown measure of $T|_{p(\mathcal{H})}$ and $(\mathbf{1} - p)T|_{p(\mathcal{H})^\perp}$ are concentrated on $B(0, 1)$ and $\mathbb{C} \setminus \overline{B(0, 1)}$, respectively.

However, in general one can not make sense of the integral in (1.3), but this can be remedied by adding a small perturbation to T. We consider $\mathcal{M}$ as a subfactor of the II_1-factor $\mathcal{N} = \mathcal{M} * L(\mathbb{F}_4)$ (with a tracial state which we also denote by τ), and we note that $\mathcal{N}$ contains a circular system $\{x, y\}$ which is $*$-free from $\mathcal{M}$. Moreover, we can define the unbounded operator $z = xy^{-1}$ which by [HS, Theorem 5.2] belongs to $L^p(\mathcal{N}, \tau)$ for $0 < p < 1$. We will consider the perturbations of T given by

$$T_n = T + \frac{1}{n} z, \qquad n \in \mathbb{N}.$$

Since $T_n \in L^p(\mathcal{N}, \tau)$, $0 < p < 1$, it has a well–defined Fuglede–Kadison determinant $\Delta(T_n)$ and a well–defined Brown measure μ_{T_n} (cf. [Br, Appendix] or [HS, Section 2]). In section 4 we will prove that

$$\Delta(T_n) = \Delta(T^*T + \tfrac{1}{n^2}\mathbf{1})^{\frac{1}{2}}$$

and that

$$\mu_{T_n} \to \mu_T \quad \text{as} \quad n \to \infty$$

in the weak topology on $\mathrm{Prob}(\mathbb{C})$.

In 1968, Turpin and Waelbroek introduced an approach to vector valued integration in quasinormed spaces such as $L^p(\mathcal{N}, \tau)$, $0 < p < 1$ (cf. [TuWa], [Wa], [Ka]). In particular, one can define the integral

$$\int_a^b f(x)\,\mathrm{d}x$$

for every function $f : [a, b] \to L^p(\mathcal{N}, \tau)$ which satisfies the Hölder condition

$$\|f(x) - f(y)\|_p \leq C|x - y|^\alpha$$

with exponent $\alpha > \frac{1}{p} - 1$ (cf. section 10 for a selfcontained proof). Based on results from sections 5 and 6, we prove in section 7 that $(\lambda\mathbf{1} - T_n)^{-1} \in L^p(\mathcal{N}, \tau)$ for $\lambda \in \mathbb{C}$ and $0 < p < 1$. Moreover, if $0 < p < \frac{2}{3}$, then

$$\|(\lambda\mathbf{1} - T_n)^{-1} - (\mu\mathbf{1} - T_n)^{-1}\|_p \leq C_{p,n}|\lambda - \mu|$$

for some constant $C_{p,n} > 0$. It follows that he integral

$$e_n = \frac{1}{2\pi\mathrm{i}} \int_{\partial B(0,1)} (\lambda\mathbf{1} - T_n)^{-1}\,\mathrm{d}\lambda$$

makes sense as a Turpin–Waelbroek integral in $L^p(\mathcal{N}, \tau)$ for $\frac{1}{2} < p < \frac{2}{3}$. Let P_n denote the range projection of e_n. Then

(i) $P_n T_n P_n = T_n P_n$,

(ii) $\mathrm{supp}(\mu_{T_n|_{P_n(\mathcal{H})}}) \subseteq \overline{B(0,1)}$

(iii) $\mathrm{supp}(\mu_{P_n^\perp T_n|_{P_n(\mathcal{H})^\perp}}) \subseteq \mathbb{C} \setminus B(0,1)$

(cf. Theorem 7.3). Then we take a free ultrafilter ω on $\mathbb{N}$ and define $P \in \mathcal{N}^\omega$ to be the image of $(P_n)_{n=1}^\infty$ under the quotient mapping $\rho : \ell^\infty(\mathcal{N}) \to \mathcal{N}^\omega$. Using (i), (ii) and (iii), one can prove that P is T-invariant,

$$\tau_\omega(P) = \mu_T(\overline{B(0,1)}),$$

$$\operatorname{supp}(\mu_{T|_{P(\mathcal{H})}}) \subseteq \overline{B(0,1)},$$

and

$$\operatorname{supp}(\mu_{(\mathbf{1}-P)T|_{P(\mathcal{H})^\perp}}) \subseteq \mathbb{C} \setminus B(0,1).$$

We then prove that $P = P_{E(T,1)}$ (cf. Lemma 7.16), and thus the T-hyperinvariant subspace $E(T,1)$ has the desired properties. The last part of section 7 takes care of a general Borel set $B \subseteq \mathbb{C}$, as was outlined above.

In section 8 we realize the $E(T,r)$ and $F(T,r)$ as spectral projections of the positive operators $SO-\lim_{n\to\infty}((T^*)^n T^n)^{\frac{1}{2n}}$ and $SO-\lim_{n\to\infty}(T^n (T^*)^n)^{\frac{1}{2n}}$, respectively. In particular, we prove that these two limits actually exist for every $T \in \mathcal{M}$ when $\mathcal{M}$ is a II_1–factor. There are examples of bounded operators on a Hilbert space which do *not* have this property (cf. Example 8.4).

Finally, in section 9, we show that for operators T which are decomposable in the sense of [LN, Section 1.2], our spectral subspaces $\mathcal{K}(T,F)$ for closed subsets F of $\mathbb{C}$ coincide with the subspaces in the spectral capacity of T. Moreover, we show that every II_1–factor contains a non–decomposable operator.

Throughout this paper we assume that $\mathcal{M}$ is a II_1–factor. However, our main results can easily be generalized to the case where $\mathcal{M}$ is a finite von Neumann algebra with a specified normal faithful trace τ. Indeed, such a von Neumann algebra $\mathcal{M}$ can be embedded into a II_1–factor $\mathcal{N}$ in such a way that the restriction of $\tau_{\mathcal{N}}$ to $\mathcal{M}$ agrees with τ (cf. [HW, Proof of Theorem 2.6]).

2 Some results on the Brown measure of a bounded operator

Consider a II_1-factor $\mathcal{M}$ with faithful tracial state τ. As was mentioned in the introduction, one can associate to every $T \in \mathcal{M}$ a probability measure μ_T on $\mathbb{C}$, the Brown measure of T.

2.1 Remark. (i) Note that if $T \in \mathcal{M}$, and if $A \in \mathcal{M}$ is invertible, then $\mu_T = \mu_{ATA^{-1}}$.

(ii) According to [Br], we have the following generalization of Weil's Theorem: For $T \in \mathcal{M}$ and $0 < p < \infty$,

$$\int_{\sigma(T)} |\lambda|^p \, \mathrm{d}\mu_T(\lambda) \leq \|T\|_p^p := \tau(|T|^p) \tag{2.1}$$

Fack and Kosaki (cf. [FK]) proved the generalized Hölder inequality

$$\|ST\|_r \leq \|S\|_p \|T\|_q, \qquad (S, T \in \mathcal{M}), \tag{2.2}$$

which holds for all $0 < p, q, r \leq \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. As a consequence of (2.1) and (2.2) we have (cf. [Br]):

$$\|T^n\|_{\frac{p}{n}} \leq \|T\|_p^n, \qquad (T \in \mathcal{M},\ p > 0). \tag{2.3}$$

The main result of this section is

2.2 Theorem. *Let $T \in \mathcal{M}$, and for $n \in \mathbb{N}$, let $\mu_n \in \mathrm{Prob}([0,\infty))$ denote the distribution of $(T^n)^* T^n$ w.r.t. τ, and let ν_n denote the push-forward measure of μ_n under the map $t \mapsto t^{\frac{1}{n}}$. Moreover, let ν denote the push-forward measure of μ_T under the map $z \mapsto |z|^2$, i.e. ν is determined by*

$$\nu([0, t^2]) = \mu_T(\overline{B(0,t)}), \qquad (t > 0).$$

Then $\nu_n \to \nu$ weakly in $\mathrm{Prob}([0,\infty))$.

We will obtain Theorem 2.2 as a consequence of

2.3 Theorem. *Let $T \in \mathcal{M}$. Then there is a sequence $(A_k)_{k=1}^\infty$ in $\mathcal{M}_{\mathrm{inv}}$, the set of invertible elements in $\mathcal{M}$, such that*

(i) $\|A_k T A_k^{-1}\| \leq \|T\|$ for all $k \in \mathbb{N}$,

(ii) $A_k T A_k^{-1}$ converges in $*$-distribution to a normal operator N in an ultrapower of $\mathcal{M}$, $\mathcal{M}^\omega$, with $\mu_N = \mu_T$.

Before proving Theorem 2.3 we state and prove some of its additional consequences:

2.4 Corollary. *For every $T \in \mathcal{M}$ and every $p > 0$,*

$$\int_{\mathbb{C}} |\lambda|^p \, \mathrm{d}\mu_T(\lambda) = \inf_{A \in \mathcal{M}_{\mathrm{inv}}} \|ATA^{-1}\|_p^p. \tag{2.4}$$

Proof. According to Remark 2.1 (i) and (2.1),

$$\int_{\mathbb{C}} |\lambda|^p \, \mathrm{d}\mu_T(\lambda) \leq \inf_{A \in \mathcal{M}_{\mathrm{inv}}} \|ATA^{-1}\|_p^p.$$

To see that $\geq$ holds in (2.4), take $(A_k)_{k=1}^\infty$ as in Theorem 2.3. Then $\mu_{(A_k T A_k^{-1})^* A_k T A_k^{-1}} \to \mu_{N^*N}$ in moments, and since all of the measures $(\mu_{(A_k T A_k^{-1})^* A_k T A_k^{-1}})_{k=1}^\infty$ are supported on

$[0, \|T\|^2]$, this implies weak convergence. In particular,

$$\begin{aligned}
\|A_kTA_k^{-1}\|_p^p &= \int_0^\infty t^{\frac{p}{2}}\,\mathrm{d}\mu_{(A_kTA_k^{-1})^*A_kTA_k^{-1}}(t)\\
&\to \int_0^\infty t^{\frac{p}{2}}\,\mathrm{d}\mu_{N^*N}(t)\\
&= \tau(|N|^p)\\
&= \int_{\mathbb{C}} |\lambda|^p\,\mathrm{d}\mu_N(\lambda)\\
&= \int_{\mathbb{C}} |\lambda|^p\,\mathrm{d}\mu_T(\lambda),
\end{aligned}$$

proving that $\geq$ holds in (2.4). ■

2.5 Theorem. *For every $T \in \mathcal{M}$ and every $p > 0$,*

$$\int_{\mathbb{C}} |\lambda|^p\,\mathrm{d}\mu_T(\lambda) = \lim_{n\to\infty} \|T^n\|_{\frac{p}{n}}^{\frac{p}{n}}. \tag{2.5}$$

Proof. According to [Br], μ_{T^n} is the push-forward measure of μ_T under the map $z \mapsto z^n$. Hence by (2.1), for all $n \in \mathbb{N}$,

$$\int_{\mathbb{C}} |\lambda|^p\,\mathrm{d}\mu_T(\lambda) = \int_{\mathbb{C}} |\lambda|^{\frac{p}{n}}\,\mathrm{d}\mu_{T^n}(\lambda) \leq \|T^n\|_{\frac{p}{n}}^{\frac{p}{n}}. \tag{2.6}$$

Let $\varepsilon > 0$. It follows from (2.4) that we may choose $A \in \mathcal{M}_{\text{inv}}$, such that

$$\|ATA^{-1}\|_p^p \leq \int_{\mathbb{C}} |\lambda|^p\,\mathrm{d}\mu_T(\lambda) + \varepsilon.$$

With $S = ATA^{-1}$ we then have (cf. (2.3)) that

$$\begin{aligned}
\|S^n\|_{\frac{p}{n}}^{\frac{p}{n}} &\leq \|S\|_p^p\\
&\leq \int_{\mathbb{C}} |\lambda|^p\,\mathrm{d}\mu_T(\lambda) + \varepsilon.
\end{aligned}$$

Since $T^n = A^{-1}S^nA$, the generalized Hölder inequality (2.2) implies that

$$\|T^n\|_{\frac{p}{n}} \leq \|A^{-1}\|\|A\|\|S^n\|_{\frac{p}{n}}.$$

Thus,

$$\|T^n\|_{\frac{p}{n}}^{\frac{p}{n}} \leq \|A^{-1}\|^{\frac{p}{n}}\|A\|^{\frac{p}{n}}\Big(\int_{\mathbb{C}} |\lambda|^p\,\mathrm{d}\mu_T(\lambda) + \varepsilon\Big),$$

and it follows that

$$\limsup_{n\to\infty} \|T^n\|_{\frac{p}{n}}^{\frac{p}{n}} \leq \int_{\mathbb{C}} |\lambda|^p\,\mathrm{d}\mu_T(\lambda) + \varepsilon. \tag{2.7}$$

Combining (2.6) and (2.7) we find that

$$\int_{\mathbb{C}} |\lambda|^p \,\mathrm{d}\mu_T(\lambda) \le \liminf_{n\to\infty} \|T^n\|_{\frac{p}{n}}^{\frac{p}{n}} \le \limsup_{n\to\infty} \|T^n\|_{\frac{p}{n}}^{\frac{p}{n}} \le \int_{\mathbb{C}} |\lambda|^p \,\mathrm{d}\mu_T(\lambda),$$

and the theorem follows. ■

2.6 Corollary. *For $T \in \mathcal{M}$ define $r'(T)$, the modified spectral radius of T, by*

$$r'(T) := \max\{|\lambda| \mid \lambda \in \operatorname{supp}(\mu_T)\}.$$

Then

$$r'(T) = \lim_{p\to\infty} \Big(\lim_{n\to\infty} \|T^n\|_{\frac{p}{n}}^{\frac{1}{n}} \Big). \tag{2.8}$$

Proof. $r'(T)$ is the essential supremum (w.r.t. μ_T) of the map $\lambda \mapsto |\lambda|$. Hence,

$$r'(T) = \lim_{p\to\infty} \Big(\int_{\mathbb{C}} |\lambda|^p \,\mathrm{d}\mu_T(\lambda) \Big)^{\frac{1}{p}}.$$

Now apply Theorem 2.5. ■

Proof of Theorem 2.3. If $A : [0,\infty) \to \mathcal{M}_{\mathrm{inv}}$ is a differentiable map, define $X : [0,\infty) \to \mathcal{M}$ by

$$X(t) = A(t)TA(t)^{-1}, \qquad (t > 0).$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t)^{-1} = -A(t)^{-1}A'(t)A(t)^{-1}.$$

Hence, X is differentiable with

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} X(t) &= A'(t)TA(t)^{-1} + A(t)T\frac{\mathrm{d}}{\mathrm{d}t}A(t)^{-1} \\ &= [A'(t)A(t)^{-1}, X(t)]. \end{aligned}$$

We will choose $A(t)$ to be the solution of the differential equation

$$\begin{cases} A'(t) = B(t)A(t) \quad , \quad t \ge 0 \\ A(0) = \mathbf{1} \end{cases} \tag{2.9}$$

for a suitable function $B : [0,\infty) \to \mathcal{M}_{\mathrm{sa}}$, chosen in such a way that the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = [B(t), X(t)]$$

implies that $t \mapsto \|X(t)\|_p^p$ is decreasing for all $p \in \mathbb{N}$. At first we consider the case $p = 2$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|X(t)\|_2^2 &= \frac{\mathrm{d}}{\mathrm{d}t} \tau(X(t)^*X(t)) \\ &= \tau([B(t), X(t)]^*X(t) + X(t)^*[B(t), X(t)]) \\ &= 2\tau\big(B(t)(X(t)X(t)^* - X(t)^*X(t))\big). \end{aligned}$$

Hence, if $B(t) = [X(t)^*, X(t)]$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|X(t)\|_2^2 = -2\|[X(t), X(t)^*]\|_2^2 \le 0.$$

Therefore, in the following we will try to solve the differential equation

$$\begin{cases} X'(t) = [[X(t)^*, X(t)], X(t)] \quad , \quad t \ge 0 \\ X(0) = T \end{cases} \tag{2.10}$$

and then solve

$$\begin{cases} A'(t) = [X(t)^*, X(t)]A(t) \quad , \quad t \ge 0 \\ A(0) = \mathbf{1}. \end{cases} \tag{2.11}$$

We are going to apply [L, Chapter 14, Theorem 3.1] to (2.10). Assuming that $T \neq 0$, set $\mathcal{U} = B(0, 2\|T\|) \subseteq \mathcal{M}$, and let

$$f(X) = [[X^*, X], X], \qquad X \in \mathcal{M}.$$

For fixed $X \in \mathcal{M}$, $f'(X)$ is a bounded operator on $\mathcal{M}$ given by

$$f'(X) : H \mapsto [[H^*, X], X] + [[X^*, H], X] + [[X^*, X], H].$$

For $X \in \mathcal{U}$,

$$\|f'(X)\| \le 12(2\|T\|)^2 = 48\|T\|^2.$$

Hence, $K = 48\|T\|^2$ is a Lipschitz constant for $f|_{\mathcal{U}}$. Also, $\|f(X)\|$ is bounded on $\mathcal{U}$ by

$$L = 4(2\|T\|)^3 = 32\|T\|^3.$$

With $a = \frac{1}{3}\|T\|$, $\overline{B_{2a}(T)} \subseteq \mathcal{U}$. Set

$$b_0 = \min\left\{\frac{1}{K}, \frac{a}{L}\right\} = \min\left\{\frac{1}{48\|T\|^2}, \frac{1}{96\|T\|^2}\right\}.$$

Then by [L, Chapter 14, Theorem 3.1], for all $b \in (0, b_0)$ there is a unique solution to

$$\begin{cases} X'(t) = f(X(t)) \\ X(0) = T \end{cases}$$

defined on the interval $(-b, b)$. In particular, such a solution exists and is unique on $\left[0, \frac{c}{\|T\|^2}\right]$, where $c = \frac{1}{100}$. Next, with $B = [X^*, X]$ we get that for every $p \in \mathbb{N}$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|X(t)\|_{2p}^{2p} &= \frac{\mathrm{d}}{\mathrm{d}t}\tau((X(t)^*X(t))^p) \\
&= p\,\tau\Big(\frac{\mathrm{d}}{\mathrm{d}t}\big(X(t)^*X(t)\big)(X(t)^*X(t))^{p-1}\Big) \\
&= p\,\tau([2X(t)^*B(t)X(t) - B(t)X(t)^*X(t) - X(t)^*X(t)B(t)](X(t)^*X(t))^{p-1}) \\
&= 2p\,\tau(B(t)[(X(t)X(t)^*)^p - (X(t)^*X(t))^p]) \\
&= 2p\,\tau([X(t)^*X(t) - X(t)X(t)^*][(X(t)X(t)^*)^p - (X(t)^*X(t))^p]).
\end{aligned}$$

For fixed $t \in \left[0, \frac{c}{\|T\|^2}\right]$ there is a unique compactly supported probability measure μ_t on $[0,\infty) \times [0,\infty)$, such that for all $f, g \in C([0,\infty))$,

$$\tau(f(X(t)^*X(t))g(X(t)X(t)^*)) = \int_{[0,\infty[\times[0,\infty[} f(u)g(v)\,\mathrm{d}\mu_t(u,v)$$

(cf. [Co, Proposition 1.1]). In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|X(t)\|_{2p}^{2p} = 2p\int_{[0,\infty[\times[0,\infty[} (u-v)(v^p-u^p)\,\mathrm{d}\mu_t(u,v) \le 0, \tag{2.12}$$

and it follows that $t \mapsto \|X(t)\|_p$ is decreasing for all $p \in \mathbb{N}$. Thus, also $t \mapsto \|X(t)\|_\infty = \lim_{p\to\infty}\|X(t)\|_p$ is decreasing. We can therefore extend the solution to (2.10) to the interval $\left[\frac{c}{\|T\|^2}, \frac{2c}{\|T\|^2}\right]$, and repeating the argument, we find that (2.10) has a solution defined on all of $[0,\infty[$ and satisfying $\|X(t)\| \le \|T\|$, $t \ge 0$. Then with $B(t) = [X(t)^*, X(t)]$ we have

$$\|B(t)\| \le 2\|T\|^2, \qquad t \ge 0. \tag{2.13}$$

In order to solve (2.11), we apply the method of proof of [L, Chap. 14, Theorem 3.1] and define $(A_n(t))_{n=0}^\infty$ recursively by

$$\begin{aligned} A_0(t) &= \mathbf{1}, \qquad t \ge 0, \\ A_n(t) &= \mathbf{1} + \int_0^t B(t')A_{n-1}(t')\,\mathrm{d}t', \qquad t \ge 0. \end{aligned}$$

By induction on n, we find that

$$\|A_n(t) - A_{n-1}(t)\| \le \frac{(2\|T\|^2 t)^n}{n!}, \qquad n \in \mathbb{N}.$$

Hence,

$$1 + \sum_{n=1}^{\infty}\|A_n(t) - A_{n-1}(t)\| \le e^{2\|T\|^2 t} < \infty,$$

and it follows that the limit

$$A(t) = \lim_{n\to\infty} A_n(t)$$

exists for all $t \ge 0$. Moreover, the convergence is uniform on compact subsets of $[0,\infty)$. Therefore,

$$A(t) = \mathbf{1} + \int_0^t B(t')A(t')\,\mathrm{d}t', \qquad t \ge 0,$$

showing that $A(t)$ is a C^1 solution to (2.11). By a similar argument, the problem

$$\begin{cases} C'(t) = -C(t)B(t) & , \quad t \ge 0, \\ C(0) = \mathbf{1} \end{cases} \tag{2.14}$$

has a C^1 solution defined on $[0,\infty)$. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(C(t)A(t)\big) = C'(t)A(t) + C(t)A'(t) = 0,$$

which implies that $C(t)A(t) = \mathbf{1}$, $t \geq 0$. Hence, $A(t)$ is invertible with inverse $C(t)$ for all $t \geq 0$. Now, observe that both $X(t)$ and $A(t)TA(t)^{-1}$ are C^1 solutions to the problem

$$\begin{cases} Y'(t) = [B(t), Y(t)] \quad , \quad t \geq 0, \\ Y(0) = T. \end{cases} \tag{2.15}$$

Then by [L, Chap. 14, Theorem 3.3], $X(t) = A(t)TA(t)^{-1}$ for all $t \geq 0$. That is, $A(t)$, $B(t)$ and $X(t)$ have all the properties we were asking for in the beginning of this proof.

Note that the map $t^p \mapsto p \cdot t^p$, $t \in [0, \infty)$, extends to a map from $C^1([0,\infty))$ into $C([0,\infty))$ given by

$$\phi(t) \mapsto t \cdot \phi'(t).$$

Let $\phi \in C^1([0, \infty[)$. Choosing a sequence of polynomials $(p_n)_{n=1}^{\infty}$, such that $p_n \to \phi$ and $p_n' \to \phi'$ uniformly on $[0, \|T\|^2]$, we get from (2.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau(\phi(X(t)^*X(t))) = -2\int_{[0,\infty[\times[0,\infty[} (u-v)(u\phi'(u) - v\phi'(v))\,\mathrm{d}\mu_t(u,v). \tag{2.16}$$

We shall need this identity later on.

Next we prove that as $t \to \infty$, $X(t)$ converges in $*$-moments to a normal operator N. Recall that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|X(t)\|_2^2 = -2\,\|[X(t)^*, X(t)]\|_2^2. \tag{2.17}$$

Moreover,

$$X'(t) = [[X(t)^*, X(t)], X(t)] = X(t)^*X(t)^2 - 2X(t)X(t)^*X(t) + X(t)^2X(t)^*. \tag{2.18}$$

Since $\tau(Y[Z,W]) = \tau(Z[W,Y]) = \tau(W[Y,Z])$, we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|[X(t)^*, X(t)]\|_2^2 &= 2\,\tau\Big([X(t)^*, X(t)] \cdot \frac{\mathrm{d}}{\mathrm{d}t}[X(t)^*, X(t)]\Big) \\ &= 2\,\tau\Big([X(t)^*, X(t)]\Big([X'(t)^*, X(t)] + [X(t)^*, X'(t)]\Big)\Big) \\ &= 2\,\tau(X'(t)^*[X(t), [X(t)^*, X(t)]]) + 2\,\tau(X'(t)[[X(t)^*, X(t)], X(t)^*]) \\ &= -2\,\tau(X'(t)^*[[X(t)^*, X(t)], X(t)]) - 2\,\tau(X'(t)[X(t)^*, [X(t)^*, X(t)]]) \\ &= -4\|X'(t)\|_2^2. \end{aligned} \tag{2.19}$$

Hence, $t \mapsto \|[X(t)^*, X(t)]\|_2^2$ is decreasing, and it follows that for all $t > 0$,

$$\|[X(t)^*, X(t)]\|_2^2 \leq \frac{1}{t}\int_0^t \|[X(u)^*, X(u)]\|_2^2\,\mathrm{d}u,$$

so by (2.17),

$$\|[X(t)^*, X(t)]\|_2^2 \leq \frac{1}{2t}\Big(\|X(0)\|_2^2 - \|X(t)\|_2^2\Big) \leq \frac{1}{2t}\|T\|_2^2.$$

This shows that

$$\lim_{t\to\infty} \|[X(t)^*, X(t)]\|_2^2 = 0. \tag{2.20}$$

According to (2.12), $t \mapsto \|X(t)\|_{2p}^{2p}$ is decreasing for every $p \in \mathbb{N}$. Therefore,

$$\lim_{t\to\infty} \tau((X(t)^*X(t))^p)$$

exists for every $p \in \mathbb{N}$. Combining this with (2.20) and the fact that $\|X(t)\|_\infty \le \|T\|$ for every $t \ge 0$, we get that the trace of any monomial in $X(t)$ and $X(t)^*$ with the same number of $X(t)$'s and $X(t)^*$'s also converges as $t \to \infty$. To prove that this holds for any monomial, it is, because of (2.20), sufficient to show that for all $m, n \in \mathbb{N}_0$, the limit

$$\lim_{t\to\infty} \tau((X(t)^*)^m X(t)^n) \tag{2.21}$$

exists. To see this, note that for every $\lambda \in \mathbb{C}$, $X(t) - \lambda\mathbf{1}$ satisfies the same differential equation as $X(t)$, because

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(X(t) - \lambda\mathbf{1}) &= [[X(t)^*, X(t)], X(t)] \\ &= [[X(t)^* - \overline{\lambda}\mathbf{1}, X(t) - \lambda\mathbf{1}], X(t) - \lambda\mathbf{1}].\end{aligned}$$

Hence, according to the above, the limit

$$\lim_{t\to\infty} \tau((X(t)^* - \overline{\lambda}\mathbf{1})^n (X(t) - \lambda\mathbf{1})^n) \tag{2.22}$$

exists for all $n \in \mathbb{N}$. Expanding $\tau((X(t)^* - \overline{\lambda}\mathbf{1})^n (X(t) - \lambda\mathbf{1})^n)$ in powers of λ and $\overline{\lambda}$, it is readily seen that the limit (2.21) exists. Hence, $(X(t))_{t\ge0}$ converges in $*$-moments to an operator N (N may be realized in the ultrapower $\mathcal{M}^\omega$ of $\mathcal{M}$). According to (2.20), N is normal.

In order to show that $\mu_N = \mu_T$, it suffices to show that

$$\Delta(T - \lambda\mathbf{1}) = \Delta(N - \lambda\mathbf{1}), \qquad (\lambda \in \mathbb{C}). \tag{2.23}$$

At first we consider the case $\lambda = 0$. Since $X(t) = A(t)TA(t)^{-1}$, $\Delta(T) = \Delta(X(t))$ for all $t \ge 0$. Hence,

$$\begin{aligned}\log\Delta(T) &= \log\Delta(X(t)) \\ &= \inf_{\varepsilon>0}\Big\{\frac{1}{2}\tau(\log(X(t)^*X(t) + \varepsilon\mathbf{1}))\Big\}.\end{aligned} \tag{2.24}$$

Applying now (2.16) to $\phi(u) = \log(u + \varepsilon)$, $u \ge 0$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau(\log(X(t)^*X(t)+\varepsilon\mathbf{1})) = -2\int_0^\infty\int_0^\infty (u-v)\Big(\frac{u}{u+\varepsilon} - \frac{v}{v+\varepsilon}\Big)\,\mathrm{d}\mu_t(u)\mathrm{d}\mu_t(v) \le 0. \tag{2.25}$$

Now, according to (2.24),

$$\begin{aligned}\log\Delta(T) &= \inf_{t>0}\inf_{\varepsilon>0}\Big\{\frac{1}{2}\tau(\log(X(t)^*X(t) + \varepsilon\mathbf{1}))\Big\} \\ &= \inf_{\varepsilon>0}\inf_{t>0}\Big\{\frac{1}{2}\tau(\log(X(t)^*X(t) + \varepsilon\mathbf{1}))\Big\} \\ &= \inf_{\varepsilon>0}\lim_{t\to\infty}\Big\{\frac{1}{2}\tau(\log(X(t)^*X(t) + \varepsilon\mathbf{1}))\Big\},\end{aligned}$$

where the last identity follows from (2.25). But $X(t)^*X(t)$ converges in moments to N^*N, and since $\sup_{t>0}\|X(t)^*X(t)\| < \infty$, it follows from the Weierstrass approximation theorem that

$$\forall\, \phi \in C([0,\infty[): \quad \lim_{t\to\infty} \tau(\phi(X(t)^*X(t))) = \tau_\omega(\phi(N^*N)), \tag{2.26}$$

where τ_ω denotes the tracial state on $\mathcal{M}^\omega$. Hence,

$$\log\Delta(T) = \inf_{\varepsilon>0}\left\{\frac{1}{2}\tau(\log(N^*N+\varepsilon\mathbf{1}))\right\} = \log\Delta(N).$$

The same arguments apply to $X(t)-\lambda\mathbf{1}$, and we obtain that

$$\log\Delta(T-\lambda\mathbf{1}) = \log\Delta(N-\lambda\mathbf{1}), \qquad (\lambda\in\mathbb{C}).$$

Hence, $\mu_T=\mu_N$. Theorem 2.3 now follows by taking $A_k = A(k)$, $k\in\mathbb{N}$. ■

Proof of Theorem 2.2. Let $p\in[0,\infty)$. Then, according to Theorem 2.5,

$$\begin{aligned}
\int_0^\infty t^{\frac{p}{2}}\,\mathrm{d}\nu(t) &= \int_{\mathbb{C}} |\lambda|^p\,\mathrm{d}\mu_T(\lambda)\\
&= \lim_{n\to\infty} \|T^n\|_{\frac{p}{n}}^{\frac{p}{n}}\\
&= \lim_{n\to\infty} \tau(((T^n)^*T^n)^{\frac{p}{2n}})\\
&= \lim_{n\to\infty} \int_0^\infty t^{\frac{p}{2n}}\,\mathrm{d}\mu_n(t)\\
&= \lim_{n\to\infty} \int_0^\infty t^{\frac{p}{2}}\,\mathrm{d}\nu_n(t).
\end{aligned}$$

In particular, $\nu_n\to\nu$ in moments. Since $\|(T^n)^*T^n\|\le\|T\|^{2n}$ and $\mathrm{supp}(\mu_T)\subseteq\overline{B(0,\|T\|)}$, ν and the ν_n are all supported on $[0,\|T\|^2]$. Hence, $\nu_n\to\nu$ weakly. ■

2.7 Corollary. *Let $T\in\mathcal{M}$. Then $\mu_T=\delta_0$ if and only if $((T^n)^*T^n)^{\frac{1}{n}}$ tends to 0 in the strong operator topology as $n\to\infty$.*

Proof. If $\mu_T=\delta_0$, then with the same notation as in Theorem 2.2, $\nu_n\to\delta_0$ weakly as $n\to\infty$. In particular,

$$\lim_{n\to\infty}\int_0^{\|T\|^2} t^4\,\mathrm{d}\nu_n(t) = 0,$$

i.e.

$$\lim_{n\to\infty}\tau([(T^n)^*T^n]^{\frac{4}{n}}) = 0.$$

It follows that for every vector ξ in the dense subspace $\mathcal{M}$ of $L^2(\mathcal{M},\tau)$,

$$\begin{aligned}
\|[(T^n)^*T^n]^{\frac{1}{n}}\xi\|_2^2 &= \tau([(T^n)^*T^n]^{\frac{2}{n}}\xi\xi^*)\\
&\le \|[(T^n)^*T^n]^{\frac{2}{n}}\|_2\|\xi\xi^*\|_2\\
&\to 0,
\end{aligned}$$

and hence, $((T^n)^*T^n)^{\frac{1}{n}}$ tends to 0 in the strong operator topology as $n \to \infty$. On the other hand, if $((T^n)^*T^n)^{\frac{1}{n}} \to 0$ strongly, then $((T^n)^*T^n)^{\frac{p}{n}} \to 0$ strongly for every $p \in \mathbb{N}$. Hence, with $\xi = \mathbf{1} \in \mathcal{M} \subseteq L^2(\mathcal{M}, \tau)$,

$$\tau([(T^n)^*T^n]^{\frac{2p}{n}}) = \|[(T^n)^*T^n]^{\frac{p}{n}}\xi\|_2^2 \to 0$$

as $n \to \infty$ for every $p \in \mathbb{N}$, i.e.

$$\lim_{n\to\infty} \int_0^{\|T\|^2} t^p \, \mathrm{d}\nu_n(t) = 0, \qquad (p \in \mathbb{N}).$$

It follows then from the Weierstrass approximation theorem that $\nu_n \to \delta_0$ weakly as $n \to \infty$. That is, $\nu = \delta_0$, and therefore $\mu_T = \delta_0$ as well. ∎

3 Constructing certain hyperinvariant subspaces

Consider a II_1-factor $\mathcal{M}$ acting on the Hilbert space $\mathcal{H}$ and with faithful, tracial state τ.

3.1 Definition. (i) For $T \in \mathcal{M}$ and $r > 0$, let $E(T, r)$ denote the set of $\xi \in \mathcal{H}$, for which there exists a sequence $(\xi_n)_{n=1}^\infty$ in $\mathcal{H}$, such that

$$\lim_{n\to\infty} \|\xi_n - \xi\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|T^n\xi_n\|^{\frac{1}{n}} \le r. \tag{3.1}$$

(ii) For $T \in \mathcal{M}$ and $r > 0$, let $F(T, r)$ denote the set of $\eta \in \mathcal{H}$, for which there exists a sequence $(\eta_n)_{n=1}^\infty$ in $\mathcal{H}$, such that

$$\lim_{n\to\infty} \|T^n\eta_n - \eta\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|\eta_n\|^{\frac{1}{n}} \le \frac{1}{r}. \tag{3.2}$$

3.2 Lemma. *For $T \in \mathcal{M}$ and $r > 0$ one has:*

(a) $E(T, r)$ and $F(T, r)$ are closed subspaces of $\mathcal{H}$.

(b) $r \mapsto E(T, r)$ is increasing, and $E(T, r) = \bigcap_{s>r} E(T, s)$.
$r \mapsto F(T, r)$ is decreasing, and $F(T, r) = \bigcap_{0<s<r} F(T, s)$.

(c) $E(T, r)$ and $F(T, r)$ are hyperinvariant for T, i.e. for every $S \in \{T\}'$, $S(E(T, r)) \subseteq E(T, r)$, and $S(F(T, r)) \subseteq F(T, r)$.

(d) The projections $P(T, r)$ and $Q(T, r)$ onto $E(T, r)$ and $F(T, r)$, respectively, belong to $W^*(T)$, and they are independent of the particular representation of $\mathcal{M}$ on a Hilbert space.

Proof. (a) It is easily seen that $E(T,r)$ and $F(T,r)$ are subspaces of $\mathcal{H}$. To see that $E(T,r)$ is closed, suppose $(\xi^{(k)})_{k=1}^{\infty}$ is a sequence in $E(T,r)$ converging to some $\xi \in \mathcal{H}$. We prove that ξ belongs to $E(T,r)$ as well. For each $k \in \mathbb{N}$ there is a sequence $(\xi_n^{(k)})_{n=1}^{\infty}$ in $\mathcal{H}$, such that

$$\lim_{n\to\infty} \|\xi_n^{(k)} - \xi^{(k)}\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|T^n \xi_n^{(k)}\|^{\frac{1}{n}} \leq r.$$

Choose a (strictly) increasing sequence of positive integers $(n_k)_{k=1}^{\infty}$ such that for all $k \in \mathbb{N}$,

$$\|\xi_n^{(k)} - \xi^{(k)}\| < \frac{1}{k}, \qquad (n \geq n_k), \tag{3.3}$$

and

$$\|T^n \xi_n^{(k)}\|^{\frac{1}{n}} < r + \frac{1}{k}, (n \geq n_k). \tag{3.4}$$

Then define a sequence $(\xi_n)_{n=1}^{\infty}$ in $\mathcal{H}$ by

$$\xi_n = \begin{cases} 0 & , \quad n < n_k \text{ for all } k \\ \xi_n^{(k)} & , \quad n_k \leq n < n_{k+1} \end{cases} \tag{3.5}$$

Then

$$\|\xi_n - \xi^k\| \leq \frac{1}{k}, \quad (n_k \leq n < n_{k+1}),$$

and

$$\|T^n \xi_n\|^{\frac{1}{n}} \leq r + \frac{1}{k}, \quad (n_k \leq n < n_{k+1}).$$

It follows that

$$\xi = \lim_{k\to\infty} \xi^{(k)} = \lim_{n\to\infty} \xi_n,$$

and

$$\limsup_{n\to\infty} \|T^n \xi_n\|^{\frac{1}{n}} \leq r.$$

Thus, $\xi \in E(T,r)$. Modifying the above arguments a bit, one easily sees that $F(T,r)$ is closed as well.

(b) Clearly, $r \mapsto E(T,r)$ is increasing, and therefore

$$E(T,r) \subseteq \bigcap_{s>r} E(T,s).$$

To see that the reverse inclusion holds, let $\xi \in \bigcap_{s>r} E(T,s)$. For each $k \in \mathbb{N}$ take a sequence $(\xi_n^{(k)})_{n=1}^{\infty}$ in $\mathcal{H}$, such that

$$\lim_{n\to\infty} \|\xi_n^{(k)} - \xi\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|T^n \xi_n^{(k)}\|^{\frac{1}{n}} < r + \frac{1}{k}.$$

As in the above, choose a (strictly) increasing sequence of positive integers $(n_k)_{k=1}^{\infty}$ such that for all $k \in \mathbb{N}$, (3.3) and (3.4) hold. Let ξ_n be given by (3.5), and as in the proof of (a), note that $(\xi_n)_{n=1}^{\infty}$ satisfies (3.1), i.e. $\xi \in E(T,r)$.

The second statement in (b) is proved in a similar way.

(c) This follows immediately from the definition of $E(T,r)$ and $F(T,r)$ and the fact that when $ST = TS$, then $ST^n = T^nS$.

(d) Since $E(T,r)$ and $F(T,r)$ are hyperinvariant for T, $P(T,r)$ and $Q(T,r)$ belong to $W^*(T) = \{T, T^*\}''$. The independence of the representation of $\mathcal{M}$ follows exactly as in [**?**, Lemma 2.3 and 2.4]. ■

3.3 Remark. If $T \in \mathcal{M}$ is invertible, then $E(T,r) = F(T^{-1}, \frac{1}{r})$. Indeed "$\subseteq$" follows by choosing $(\xi_n)_{n=1}^\infty$ as in Definition 3.1 (i) and setting $\eta_n = T^n\xi_n$. Then (3.2) holds for $\eta = \xi$ and with T replaced by T^{-1} and r replaced by $\frac{1}{r}$. The reverse inclusion follows by choosing $(\eta_n)_{n=1}^\infty$ as in Definition 3.1 (ii) (with T replaced by T^{-1} and r replaced by $\frac{1}{r}$) and setting $\xi_n = T^{-n}\eta_n$.

3.4 Lemma. *Let $T \in \mathcal{M}$, and let $r > 0$. Then*

(i) $\mathrm{supp}(\mu_T) \subseteq \overline{B(0,r)}$ *iff* $E(T,r) = \mathcal{H}$.

(ii) $\mathrm{supp}(\mu_T) \subseteq \mathbb{C} \setminus B(0,r)$ *iff* $F(T,r) = \mathcal{H}$.

Proof. According to Lemma 3.2 (d) we may assume that $\mathcal{M}$ acts on the Hilbert space $\mathcal{H} = L^2(\mathcal{M}, \tau)$. For $a \in \mathcal{M}$ we let $[a]$ denote the corresponding element in $\mathcal{H}$.

Throughout the proof we let $\mu_n = \mu_{(T^n)^*T^n} = \mu_{T^n(T^n)^*}$, and we let ν_n denote the push-forward measure of μ_n under the map $t \mapsto t^{\frac{1}{n}}$. Then, according to Theorem 2.2, $\nu_n \to \nu$ weakly in $\mathrm{Prob}([0,\infty))$, where ν is determined by

$$\nu([0,t^2]) = \mu_T(\overline{B(0,t)}), \qquad (t > 0).$$

(i) Suppose that $\mathrm{supp}(\mu_T) \subseteq \overline{B(0,r)}$. Consider a fixed $s > r$. Since $\mathrm{supp}(\mu_T) \subseteq \overline{B(0,r)}$, ν is supported on $[0,r^2]$,

$$\mu_n(]s^{2n},\infty)) = \mu_n(]s^2,\infty)) \to 0 \text{ as } n \to \infty.$$

Define a sequence of projections in $\mathcal{M}$, $(E_n)_{n=1}^\infty$, by

$$E_n = 1_{[0,s^{2n}]}((T^n)^*T^n).$$

Note that for all $a \in \mathcal{M}$,

$$\begin{aligned} \|(\mathbf{1} - E_n)[a]\|_2^2 &= \tau(a^*(\mathbf{1} - E_n)a) \\ &\leq \|a\|^2\tau(\mathbf{1} - E_n) \\ &= \|a\|^2\mu_n((s^{2n},\infty)) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $\{[a] \mid a \in \mathcal{M}\}$ is dense in $\mathcal{H}$, this shows that $E_n \to \mathbf{1}$ in the strong operator topology, i.e. for all $\xi \in \mathcal{H}$,

$$\lim_{n\to\infty} \|E_n\xi - \xi\| = 0.$$

Moreover, since $E_n(T^n)^*T^nE_n \leq s^{2n}\mathbf{1}$,

$$(\|T^nE_n\xi\|)^{\frac{1}{n}} \leq (\|T^nE_n\|\|\xi\|)^{\frac{1}{n}} \leq s\|\xi\|^{\frac{1}{n}}$$

for all $\xi \in \mathcal{H}$, and hence $E(T,s) = \mathcal{H}$. Since $s > r$ was arbitrary, Lemma 3.2 (b) now implies that $E(T,r) = \mathcal{H}$.

Next assume that $E(T,r) = \mathcal{H}$. Let $\xi = [\mathbf{1}] \in \mathcal{H}$, and take a sequence $(\xi_n)_{n=1}^\infty$ in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \|\xi_n - \xi\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|T^n\xi_n\|^{\frac{1}{n}} \leq r.$$

Since $\|\xi_n\| \to \|\xi\| = 1$ as $n \to \infty$, we may as well assume that $\|\xi_n\| = 1$ for all $n \in \mathbb{N}$. Let $\mu_n' \in \text{Prob}([0,\infty))$ denote the distribution of $(T^n)^*T^n$ w.r.t. the vector state $x \mapsto (x\xi_n, \xi_n)$. Let $s > r$, and take $s_1 \in (r,s)$. Then $\|T^n\xi_n\| \leq s_1^n$ eventually as $n \to \infty$. Hence for all large n,

$$\int_0^\infty t\,\mathrm{d}\mu_n'(t) = ((T^n)^*T^n\xi_n, \xi_n) \leq s_1^{2n},$$

and therefore

$$\mu_n'(]s^{2n},\infty[) \leq s^{-2n}\int_{s^{2n}}^\infty t\,\mathrm{d}\mu_n'(t) \leq \left(\frac{s_1}{s}\right)^{2n}.$$

It follows that with $E_n = 1_{[0,s^{2n}]}((T^n)^*T^n)$ as before,

$$\|E_n\xi_n - \xi_n\|^2 \leq \left(\frac{s_1}{s}\right)^{2n}$$

for all large $n \in \mathbb{N}$, and hence

$$\lim_{n\to\infty} \|E_n\xi_n - \xi_n\| = 0.$$

Since $\xi_n \to \xi$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \|E_n\xi_n - \xi\| = 0$$

as well. That is,

$$\tau(\mathbf{1} - E_n) = \|(\mathbf{1} - E_n)\xi\|^2 \to 0 \text{ as } n \to \infty,$$

which is equivalent to saying that

$$\mu_n(]s^{2n},\infty)) \to 0 \text{ as } n \to \infty.$$

We now have that for every $\phi \in C_c^+(\mathbb{R})$ with $\text{supp}(\phi) \subseteq (s^2,\infty)$,

$$\begin{aligned}\int_0^\infty \phi(t)\,\mathrm{d}\nu(t) &= \lim_{n\to\infty}\int_0^\infty \phi(t)\,\mathrm{d}\nu_n(t)\\ &\leq \|\phi\|_\infty \limsup_{n\to\infty} \nu_n(]s^2,\infty[)\\ &= 0.\end{aligned}$$

Hence, $\mathrm{supp}(\nu) \subseteq [0, s^2]$, i.e. $\mathrm{supp}(\mu_T) \subseteq \overline{B(0,s)}$ for all $s > r$, and we conclude that $\mathrm{supp}(\mu_T) \subseteq \overline{B(0,r)}$.

(ii) At first assume that $\mathrm{supp}(\mu_T) \subseteq \mathbb{C} \setminus B(0,r)$. Let $0 < s < r$, and define $F_n \in \mathcal{M}$ by

$$F_n = 1_{[s^{2n},\infty)}(T^n(T^n)^*).$$

By an argument similar to the one given in the proof of (i), we find that for all $\xi \in \mathcal{H}$,

$$\|F_n\xi - \xi\| \to 0 \text{ as } n \to \infty.$$

Now, consider a fixed $\xi \in \mathcal{H}$, and with

$$g_n(t) = \begin{cases} 0 & , \quad 0 \leq t < s^{2n} \\ \frac{1}{t} & , \quad t \geq s^{2n} \end{cases}$$

let

$$\eta_n = (T^*)^n g_n(T^n(T^n)^*)\xi.$$

Then, since $t\, g_n(t) = 1_{[s^{2n},\infty[}(t)$,

$$T^n\eta_n = F_n\xi.$$

Thus,

$$\|T^n\eta_n - \xi\| \to 0 \text{ as} n \to \infty.$$

Moreover, since $t\, g_n(t)^2 = g_n(t) \leq \frac{1}{s^{2n}}$,

$$\|\eta_n\|^2 = (g_n(T^n(T^n)^*\xi, \xi) \leq \frac{1}{s^{2n}}\|\xi\|.$$

Thus,

$$\limsup_{n\to\infty} \|\eta_n\|^{\frac{1}{n}} \leq \frac{1}{s},$$

and this shows that $F(T,s) = \mathcal{H}$. It now follows from Lemma 3.2 (b) that $F(T,r) = \mathcal{H}$.

On the other hand, suppose that $F(T,r) = \mathcal{H}$, and let $\xi = [\mathbf{1}]$. Choose a sequence $(\eta_n)_{n=1}^\infty$ in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \|T^n\eta_n - \xi\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|\eta_n\|^{\frac{1}{n}} \leq \frac{1}{r}.$$

Since $\|\xi\| = 1$, we may assume that $\|T^n\eta_n\| = 1$ for all $n \in \mathbb{N}$.

Put $\xi_n = T^n\eta_n$, and let $\mu'_n \in \mathrm{Prob}([0,\infty))$ denote the distribution of $T^n(T^n)^*$ w.r.t. the vector state $x \mapsto (x\xi_n, \xi_n)$. Since for every $\varepsilon > 0$,

$$\|(T^n(T^n)^* + \varepsilon\mathbf{1})^{-\frac{1}{2}}T^n\| \leq 1,$$

we have that

$$\begin{aligned} \int_0^\infty \frac{1}{t}\,\mathrm{d}\mu'_n(t) &= \sup_{\varepsilon>0} \int_0^\infty \frac{1}{t+\varepsilon}\,\mathrm{d}\mu'_n(t) \\ &= \sup_{\varepsilon>0} \|(T^n(T^n)^* + \varepsilon\mathbf{1})^{-\frac{1}{2}}\xi_n\|^2 \\ &= \sup_{\varepsilon>0} \|(T^n(T^n)^* + \varepsilon\mathbf{1})^{-\frac{1}{2}}T^n\eta_n\|^2 \\ &\leq \|\eta_n\|^2. \end{aligned}$$

Now, let $s \in (0, r)$, and choose $s_1 \in (s, r)$. Then $\|\eta_n\|^{\frac{1}{n}} \leq \frac{1}{s_1}$ eventually as $n \to \infty$, and hence

$$\int_0^\infty \frac{1}{t}\, \mathrm{d}\mu_n'(t) \leq \frac{1}{s_1},$$

eventually as $n \to \infty$. With $F_n = 1_{[s^{2n},\infty)}(T^n(T^n)^*)$ as above we then have that

$$\begin{aligned} \|F_n\xi_n - \xi_n\|^2 &= \mu_n'([0, s^{2n})) \\ &\leq \int_0^{2^n} \frac{s^{2n}}{t} \mathrm{d}\mu_n'(t) \\ &\leq \left(\frac{s}{s_1}\right)^{2n} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $\xi_n \to \xi$, it follows that

$$\|\mathbf{1} - F_n\|_2^2 = \|\xi - F_n\xi\|^2 \to 0 \text{ as } n \to \infty,$$

i.e.

$$\lim_{n\to\infty} \nu_n([0, s)) = \lim_{n\to\infty} \mu_n([0, s^{2n})) = 0.$$

Since $\nu_n \to \nu$ weakly, it follows that $\mathrm{supp}(\nu) \subseteq [s^2, \infty)$, and hence $\mathrm{supp}(\mu_T) \subseteq \mathbb{C} \setminus B(0, s)$ for all $s \in (0, r)$. We thus conclude that $\mathrm{supp}(\mu_T) \subseteq \mathbb{C} \setminus B(0, r)$. ■

3.5 Corollary. *Let $T \in \mathcal{M}$, let $\lambda \in \mathbb{C}$, and let $r > 0$. Then*

(a) If E is any closed, T-invariant subspace affiliated with $\mathcal{M}$, such that $\mathrm{supp}(\mu_{T|_E}) \subseteq \overline{B(\lambda, r)}$*, then $E \subseteq E(T - \lambda\mathbf{1}, r)$.*

(b) If F is any closed, T-invariant subspace affiliated with $\mathcal{M}$, such that $\mathrm{supp}(\mu_{T|_F}) \subseteq \mathbb{C} \setminus B(\lambda, r)$*, then $F \subseteq F(T - \lambda\mathbf{1}, r)$.*

Proof. (a) Suppose that E is a closed, T-invariant subspace affiliated with $\mathcal{M}$, such that $\mathrm{supp}(\mu_{T|_E}) \subseteq \overline{B(\lambda, r)}$. Then E is invariant for $T - \lambda\mathbf{1}$ as well, and

$$\mathrm{supp}(\mu_{T-\lambda\mathbf{1}|_E}) \subseteq \overline{B(0, r)}.$$

According to Lemma 3.4, this implies that

$$E(T - \lambda\mathbf{1}|_E, r) = E.$$

It is easily seen that $E(T - \lambda\mathbf{1}|_E, r) \subseteq E(T - \lambda\mathbf{1}, r)$, and hence,

$$E \subseteq E(T - \lambda\mathbf{1}, r).$$

(b) follows in a similar way, so we leave out the proof of it. ■

3.6 Lemma. *For $T \in \mathcal{M}$ and $s > r > 0$ one has that $E(T, r) \perp F(T^*, s)$.*

Proof. Let $\xi \in E(T,r)$, and $\eta \in F(T^*,s)$. Take sequences $(\xi_n)_{n=1}^\infty$ and $(\eta_n)_{n=1}^\infty$ in $\mathcal{H}$, such that $\xi_n \to \xi$, $(T^*)^n\eta_n \to \eta$,

$$\limsup_{n\to\infty} \|T^n\xi_n\| \le r,$$

and

$$\limsup_{n\to\infty} \|\eta_n\| \le \frac{1}{s}.$$

Choose positive numbers r' and s' such that $s > s' > r' > r$. Eventually as $n \to \infty$, $\|T^n\xi_n\| \le (r')^n$ and $\|\eta_n\| \le \frac{1}{(s')^n}$, whence

$$\begin{aligned} |(\xi,\eta)| &= \lim_{n\to\infty} |(\xi_n,(T^*)^n\eta_n)| \\ &= \lim_{n\to\infty} |(T^n\xi_n,\eta_n)| \\ &\le \limsup_{n\to\infty} \|T^n\xi_n\|\|\eta_n\| \\ &\le \limsup_{n\to\infty} \left(\frac{r'}{s'}\right)^n \\ &= 0. \end{aligned}$$

That is, $\xi \perp \eta$. ∎

4 Some results from free probability theory

In order to proceed, we will need a number of results which are based on free probability theory. Recall from [VDN] that a C^*–probability space is a pair $(\mathcal{A},\varphi)$ where $\mathcal{A}$ is a unital C^*–algebra and φ is a state on $\mathcal{A}$. We say that $(\mathcal{A},\varphi)$ is a W^*–probability space if $\mathcal{A}$ is a von Neumann algebra and if φ is a normal state on $\mathcal{A}$.

Let $\mathcal{H}$ be a Hilbert space. The full Foch–space over $\mathcal{H}$, $\mathcal{T}(\mathcal{H})$, is the Hilbert space

$$\mathcal{T}(\mathcal{H}) = \mathbb{C}\Omega \oplus \Big(\bigoplus_{n=1}^\infty \mathcal{H}^{\otimes n}\Big),$$

where Ω, the vacuum vector, is a unit vector in the one–dimensional Hilbert space $\mathbb{C}\Omega$. Ω induces the vector state ω on $B(\mathcal{T}(\mathcal{H}))$.

To a given orthonormal set $(e_i)_{i\in I}$ in $\mathcal{H}$ we associate the creation operators $(\ell_i)_{i\in I}$ in $B(\mathcal{T}(\mathcal{H}))$ given by

$$\ell_i\Omega = e_i \qquad \text{and} \qquad \ell_i\xi = e_i \otimes \xi, \quad \xi \in \mathcal{H}^{\otimes n}.$$

Then $\ell_i^*\ell_j = \delta_{ij}\mathbf{1}$ and

$$\omega(\ell_i^m(\ell_i^*)^n) = \begin{cases} 1, & (m,n) = (0,0), \\ 0, & (m,n) \in \mathbb{N}_0^2 \setminus \{(0,0)\}. \end{cases} \tag{4.1}$$

Moreover, the $*$–algebras $\mathcal{A}_i = \mathrm{alg}(\ell_i,\ell_i^*)$, $i \in I$, are free in $B(\mathcal{T}(\mathcal{H}))$ w.r.t. ω (cf. [VDN]).

A family of creation operators $(\ell_i)_{i\in I}$ obtained from an orthonormal set $(e_i)_{i\in I}$ as above is called a *free family of creation operators*.

4.1 Lemma. *Let $(\mathcal{A},\varphi)$ be a C^*–probability space, and let $(s_i)_{i\in I}$ be operators in $\mathcal{A}$ which satisfy*

(i) $s_i^* s_j = \delta_{ij}\mathbf{1}$, $i,j\in I$,

(ii) $\varphi(s_i s_i^*) = 0$, $i\in I$.

Then $(s_i)_{i\in I}$ has the same $*$–distribution as a free family of creation operators $(\ell_i)_{i\in I}$.

Proof. Let $\mathcal{B}_i = \mathrm{alg}(s_i, s_i^*) \subseteq \mathcal{A}$, $i\in I$. Since $s_i^* s_i = \mathbf{1}$,

$$\mathcal{B}_i = \mathrm{span}\{s_i^m (s_i^*)^n \mid (m,n)\in\mathbb{N}_0^2\}. \tag{4.2}$$

By (ii) and the Schwarz inequality

$$|\varphi(ab)| \le \varphi(aa^*)^{\frac12}\varphi(b^*b)^{\frac12}, \qquad a,b\in\mathcal{A},$$

we have that

$$\varphi(s_i^m (s_i^*)^n) = \begin{cases} 1, & (m,n) = (0,0),\\ 0, & (m,n)\in\mathbb{N}_0^2\setminus\{(0,0)\}.\end{cases} \tag{4.3}$$

Let $(\ell_i)_{i\in I}$ be the set of creation operators coming from an orthonormal set $(e_i)_{i\in I}$ as described above. Then by (4.1), ℓ_i and s_i have the same $*$–distribution for all $i\in I$. Hence, in order to show that the families $(\ell_i)_{i\in I}$ and $(s_i)_{i\in I}$ have the same $*$–distributions, it suffices to show that $(\mathcal{B}_i)_{i\in I}$ are free subalgebras in $(\mathcal{A},\varphi)$.

Put

$$\mathcal{B}_i^0 = \{b\in\mathcal{B}_i \mid \varphi(b) = 0\}.$$

It follows from (4.2) and (4.3) that

$$\mathcal{B}_i^0 = \mathrm{span}\{s_i (s_i^*)^n \mid (m,n)\in\mathbb{N}_0^2\setminus\{(0,0)\}\}.$$

Thus, in order to prove that $(\mathcal{B}_i)_{i\in I}$ are free, it suffices to show that

$$\varphi(s_{i_1}^{m_1}(s_{i_1}^*)^{n_1} s_{i_2}^{m_2}(s_{i_2}^*)^{n_2}\cdots s_{i_k}^{m_k}(s_{i_k}^*)^{n_k}) = 0 \tag{4.4}$$

for all $k\in\mathbb{N}$, all $(m_1,n_1),\dots(m_k,n_k)\in\mathbb{N}_0^2\setminus\{(0,0)\}$ and all $i_1,\dots,i_k\in I$ such that $i_1\ne i_2\ne\cdots\ne i_{k-1}\ne i_k$.

Given $k\in\mathbb{N}$, $(m_1,n_1),\dots(m_k,n_k)\in\mathbb{N}_0^2\setminus\{(0,0)\}$ and $i_1,\dots,i_k\in I$ such that $i_1\ne i_2\ne\cdots\ne i_{k-1}\ne i_k$, assume that

$$\varphi(s_{i_1}^{m_1}(s_{i_1}^*)^{n_1} s_{i_2}^{m_2}(s_{i_2}^*)^{n_2}\cdots s_{i_k}^{m_k}(s_{i_k}^*)^{n_k}) \ne 0.$$

By (ii) and the Schwarz inequality,

$$\varphi(as_i^*) = 0, \qquad a\in\mathcal{A},\ i\in I. \tag{4.5}$$

Hence $n_k = 0$, which implies that $m_k\ne 0$. By (i), $s_{i_{k-1}}^* s_{i_k} = 0$, and therefore n_{k-1} must be zero. Then $m_{k-1}\ne 0$, and continuing this way we find that $n_1 = n_2 = \cdots = n_k = 0$, $m_1, m_2,\dots,m_k\ge 1$ and

$$\varphi(s_{i_1}^{m_1} s_{i_2}^{m_2}\cdots s_{i_k}^{m_k}) \ne 0.$$

However, by (4.5), $\varphi(s_i a) = \overline{\varphi(a^* s_i^*)} = 0$ for $i \in I$ and $a \in \mathcal{A}$. Thus, since $m_1 \geq 1$, $\varphi(s_{i_1}^{m_1} s_{i_2}^{m_2} \cdots s_{i_k}^{m_k}) = 0$ and we have reached a contradiction. This proves (4.4) and completes the proof of the lemma. ■

Next we will apply the following result due to Shlyakhtenko:

4.2 Lemma. *[S, Corollary 2.5] Let $(\mathcal{A}, \varphi)$ be a C^*–probability space, $\mathcal{B} \subseteq \mathcal{A}$ a C^*–subalgebra of $\mathcal{A}$ with $\mathbf{1} \in \mathcal{B}$, and let $\ell_1, \ldots, \ell_n \in \mathcal{A}$. Suppose the following three conditions hold:*

(1) $\ell_1, \ldots, \ell_n$ have the same $*$–distribution w.r.t. φ as a free family of n creation operators.

(2) For all $p, q \geq 0$ with $p + q > 0$, for all $b_1, \ldots, b_p, b'_1, \ldots, b'_{q+1} \in \mathcal{B}$ and for all $1 \leq i_1, \ldots, i_p, j_1, \ldots, j_q \leq n$,

$$\varphi(b_1 \ell_{i_1} b_2 \cdots \ell_{i_p} b'_1 \ell_{j_1}^* b'_2 \cdots \ell_{j_q}^* b'_{q+1}) = 0.$$

(3) The GNS representation of $C^*(\mathcal{B}, \ell_1, \ldots, \ell_n)$ associated with φ is faithful.

Then the following two conditions are equivalent:

(a) $\mathcal{B}$ and $C^*(\ell_1, \ldots, \ell_n)$ are free in $(\mathcal{A}, \varphi)$

(b) $\ell_i^* b \ell_j = \delta_{ij} \varphi(b) \mathbf{1}$ for $b \in \mathcal{B}$, $1 \leq i, j \leq n$

Combining Lemma 4.1 and Lemma 4.2, we get:

4.3 Lemma. *Let $(\mathcal{A}, \varphi)$ be a C^*–probability space, $\mathcal{B} \subseteq \mathcal{A}$ a C^*–subalgebra of $\mathcal{A}$ with $\mathbf{1} \in \mathcal{B}$, and let $\ell_1, \ldots, \ell_n \in \mathcal{A}$. Suppose the GNS representation of $C^*(\mathcal{B}, \ell_1, \ldots, \ell_n)$ associated to φ is faithful.*

(i) If the following two conditions hold:

(a1) $\ell_1, \ldots, \ell_n$ have the same $*$–distribution w.r.t. φ as a free family of n creation operators

(a2) $\mathcal{B}$ and $C^*(\ell_1, \ldots, \ell_n)$ are free in $(\mathcal{A}, \varphi)$

Then the following two conditions also hold:

(b1) $\ell_i^* b \ell_j = \delta_{ij} \varphi(b) \mathbf{1}$ for $b \in \mathcal{B}$, $1 \leq i, j \leq n$

(b2) $\varphi(b^* \ell_i \ell_i^* b) = 0$ for $b \in \mathcal{B}$, $1 \leq i \leq n$

(ii) Conversely, (b1) together with (b2) implies (a1) and (a2).

Proof. (i) Suppose (a1) and (a2) hold. Then (1) and (a) in Lemma 4.2 are fulfilled. Moreover, by assumption, (3) in Lemma 4.2 also holds. We prove that (b2) holds: Let $\mathcal{C} = C^*(\ell_1, \ldots, \ell_n)$ and let

$$\mathcal{B}^0 = \{b \in \mathcal{B} \,|\, \varphi(b) = 0\} \quad \text{and} \quad \mathcal{C}^0 = \{c \in \mathcal{C} \,|\, \varphi(c) = 0\}.$$

For $b \in \mathcal{B}$, let $b^0 = b - \varphi(b)\mathbf{1}$. Then for $1 \leq i \leq n$,

$$\begin{aligned} \varphi(b^*\ell_i\ell_i^*b) &= \varphi(((b^*)^0 + \varphi(b^*)\mathbf{1})\ell_i\ell_i^*(b^0 + \varphi(b)\mathbf{1})) \\ &= \varphi((b^*)^0\ell_i\ell_i^*b^0) + \overline{\varphi(b)}\varphi(\ell_i\ell_i^*b^0) + \varphi(b)\varphi((b^*)^0\ell_i\ell_i^*) + |\varphi(b)|^2\varphi(\ell_i\ell_i^*). \end{aligned}$$

According to (a1), $\varphi(\ell_i\ell_i^*) = 0$, that is $\ell_i\ell_i^* \in \mathcal{C}^0$. Moreover, $b^0, (b^*)^0 \in \mathcal{B}^0$ so by the freeness assumption (a2),

$$\varphi((b^*)^0\ell_i\ell_i^*b^0) = \varphi(\ell_i\ell_i^*b^0) = \varphi((b^*)^0\ell_i\ell_i^*) = \varphi(\ell_i\ell_i^*) = 0.$$

This proves (b2). We will now show that (2) of Lemma 4.2 holds: By the Schwarz inequality and by (b2),

$$\varphi(b\ell_i a) = 0, \qquad b \in \mathcal{B},\, a \in \mathcal{A}, \tag{4.6}$$

because

$$|\varphi(b\ell_i a)| \leq \varphi(b\ell_i(b\ell_i)^*)^{\frac{1}{2}}\varphi(a^*a)^{\frac{1}{2}} = 0.$$

Moreover, by complex conjugation of (4.6), it follows that

$$\varphi(a\ell_i^*b) = 0, \qquad b \in \mathcal{B},\, a \in \mathcal{A}. \tag{4.7}$$

This proves that (2) (in addition to (1), (3) and (a)) of Lemma 4.2 holds, and thus (b1)=(b) holds. This proves (i).

(ii) Assume now that (b1) and (b2) hold. Then (b) of Lemma 4.2 holds. Moreover, by assumption, condition (3) of Lemma 4.2 holds. From the proof of (i) we know that (b2) implies (4.6) and (4.7), and the latter two imply (2) of Lemma 4.2.

By (b1) and (b2), $\ell_1, \ldots, \ell_n$ satisfy the conditions in Lemma 4.1. Hence, $(\ell_1, \ldots, \ell_n)$ has the same $*$–distribution as a set of n free creation operators. This proves (a1) as well as (1) in Lemma 4.2. Thus, (1), (2), (3) and (b) of Lemma 4.2 hold, and then by Lemma 4.2, (a2)=(a) holds. This completes the proof of (ii). ■

Recall from [VDN] that when $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ are two W^*–probability spaces, one can define a *reduced* free product,

$$(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) * (\mathcal{A}_2, \varphi_2),$$

which is a von Neumann algebra $\mathcal{A}$ realized on the free product of Hilbert spaces with distinguished unit vector,

$$(\mathcal{H}, \xi) = (\mathcal{H}_{\varphi_1}, \xi_{\varphi_1}) * (\mathcal{H}_{\varphi_2}, \xi_{\varphi_2}),$$

where $\mathcal{H}_{\varphi_i}$ with distinguished unit vector ξ_{φ_i} is the GNS Hilbert space in the representation π_{φ_i} of $\mathcal{A}_i$. φ is the vector state on $\mathcal{A}$ induced by the unit vector $\xi \in \mathcal{H}$ which is cyclic for $\mathcal{A}$. In particular, the GNS representation π_φ of $\mathcal{A}$ is one–to–one and can be identified with the inclusion map $\mathcal{A} \hookrightarrow B(\mathcal{H})$. If π_{φ_1} and π_{φ_2} are one–to–one, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are naturally embedded in $\mathcal{A}$ as a free pair of subalgebras w.r.t. φ, and $\varphi_i = \varphi|_{\mathcal{A}_i}$, $i = 1, 2$.

Let $(\mathcal{A}, \varphi)$ be a C^*–probability space. Following the notation of [V1] (or [VDN]), we say that $s_1, \dots, s_n$ in $(\mathcal{A}, \varphi)$ form a *semicircular family* in $(\mathcal{A}, \varphi)$ if $s_1, \dots, s_n$ are free self–adjoint elements with

$$\varphi(s_i^p) = \frac{1}{2\pi} \int_{-2}^{2} t^p \sqrt{4 - t^2}\, \mathrm{d}t, \qquad p \in \mathbb{N}.$$

Elements $x_1, \dots, x_n$ in $(\mathcal{A}, \varphi)$ are said to form a *circular family* if there is a semicircular family $(s_1, \dots, s_{2n})$ in $(\mathcal{A}, \varphi)$ such that

$$x_j = \frac{s_j + \mathrm{i} s_{n+j}^*}{\sqrt{2}}, \qquad 1 \le j \le n.$$

4.4 Proposition. *Let $(\mathcal{N}, \tau)$ be a finite von Neumann algebra with a faithful, normal, tracial state τ, and let $\mathcal{M} \subseteq \mathcal{N}$ be a von Neumann subalgebra of $\mathcal{N}$. Let $n \in \mathbb{N}$, and let $u = (u_{ij})_{i,j=1}^n \in \mathcal{U}(M_n(\mathcal{M}))$. Moreover, suppose that $(x_1, \dots, x_n)$ is a circular family in $\mathcal{N}$ which is $*$-free from $\mathcal{M}$. Then with $y_1, \dots, y_n \in \mathcal{N}$ given by*

$$y_i = \sum_{j=1}^{n} u_{ij} x_j, \qquad (1 \le i \le n), \tag{4.8}$$

$(y_1, \dots, y_n)$ is a circular family which is $*$-free from $\mathcal{M}$.

Proof. We may without loss of generality assume that

$$\mathcal{N} = \mathcal{M} * W^*(x_1, \dots, x_n).$$

By [V1] (or [VDN]), $W^*(x_1, \dots, x_n) \cong L(\mathbb{F}_{2n})$, the von Neumann algebra of the free group on $2n$ generators. Moreover, we can realize the circular family as follows:

Take an orthonormal set $e_1, e_1', \dots, e_n, e_n'$ in a Hilbert space $\mathcal{H}$, and let $\ell_i = \ell(e_i)$ and $\ell_i' = \ell(e_i')$ be the corresponding creation operators on $\mathcal{T}(\mathcal{H})$. Moreover, let ω be the state on $B(\mathcal{T}(\mathcal{H}))$ given by the vacuum vector Ω. Then

$$x_i = \ell_i + (\ell_i')^*, \qquad 1 \le i \le n,$$

is a circular family in $(B(\mathcal{T}(\mathcal{H})), \omega)$ and the restriction ω' of ω to $W^*(x_1, \dots, x_n)$ is a faithful, normal, tracial state. Furthermore, Ω is cyclic for $W^*(x_1, \dots, x_n)$ and we can therefore identify the GNS representation $\pi_{\omega'}$ with the inclusion map $W^*(x_1, \dots, x_n) \hookrightarrow B(\mathcal{T}(\mathcal{H}))$. The GNS vector $\xi_{\omega'}$ then corresponds to Ω. Hence, the reduced free products

$$(\mathcal{N}, \tau) = (\mathcal{M}, \tau|_{\mathcal{M}}) * (W^*(x_1, \dots, x_n), \omega')$$

and

$$(\tilde{\mathcal{N}}, \varphi) = (\mathcal{M}, \tau|_{\mathcal{M}}) * (B(\mathcal{T}(\mathcal{H})), \omega)$$

are both realized on the free product of Hilbert spaces,

$$(\mathcal{H},\xi)=(L^2(\mathcal{M},\tau|_{\mathcal{M}}),\mathbf{1})*(\mathcal{T}(\mathcal{H}),\Omega).$$

We may therefore regard $\mathcal{N}$ as a subalgebra of $\tilde{\mathcal{N}}$ such that $\varphi|_{\mathcal{N}}=\tau$. We will now apply Lemma 4.3 (i) to $\mathcal{A}=\tilde{\mathcal{N}}$, $\mathcal{B}=\mathcal{M}$ and the family of creation operators $(\ell_1,\dots,\ell_n,\ell'_1,\dots,\ell'_n)$. Since ξ is cyclic for $\tilde{\mathcal{N}}=W^*(\mathcal{M},\ell_1,\dots,\ell_n,\ell'_1,\dots,\ell'_n)$, the GNS representation of $\tilde{\mathcal{N}}$ associated to φ is one–to–one, which implies that the GNS representation of the restriction of φ to $C^*(\mathcal{M},\ell_1,\dots,\ell_n,\ell'_1,\dots,\ell'_n)$ is also one–to–one. Hence, the assumptions of Lemma 4.3 and (a1) and (a2) in Lemma 4.3 are fulfilled. It then follows that (b1) and (b2) hold, that is

(i) $\forall x\in\mathcal{M}\ \forall i,j\in\{1,\dots,n\}$:

$$\begin{aligned}\ell_i^*x\ell_j &= \delta_{ij}\cdot\tau(x)\mathbf{1},\\ (\ell'_i)^*x\ell'_j &= \delta_{ij}\cdot\tau(x)\mathbf{1},\\ \ell_i^*x\ell'_j &= (\ell'_i)^*x\ell_j=0.\end{aligned}$$

(ii) $\forall x\in\mathcal{M}\ \forall i\in\{1,\dots,n\}$:

$$\varphi(x^*\ell_i\ell_i^*x)=\varphi(x^*\ell'_i(\ell'_i)^*x)=0.$$

Put

$$s_i=\sum_{j=1}^n u_{ij}\ell_j,\qquad(1\le i\le n)\tag{4.9}$$

and

$$s'_i=\sum_{j=1}^n \ell'_j u_{ij}^*,\qquad(1\le i\le n).\tag{4.10}$$

Then

$$y_i=s_i+(s'_i)^*,\qquad(1\le i\le n).\tag{4.11}$$

We will now apply Lemma 4.3 (ii) to $\mathcal{A}=\tilde{\mathcal{N}}$, $\mathcal{B}=\mathcal{M}$ and the family of operators $(s_1,\dots,s_n,s'_1,\dots,s'_n)$. Since $u=(u_{ij})_{i,j=1}^n$ is unitary, (4.9) and (4.10) imply that

$$\begin{aligned}\ell_j &= \sum_{i=1}^n u_{ij}^*s_i, && 1\le j\le n,\\ \ell'_j &= \sum_{i=1}^n s'_i u_{ij}^*, && 1\le j\le n.\end{aligned}$$

Hence,

$$C^*(\mathcal{M},s_1,\dots,s_n,s'_1,\dots,s'_n)=C^*(\mathcal{M},\ell_1,\dots,\ell_n,\ell'_1,\dots,\ell'_n),$$

and the conditions of Lemma 4.3 are fulfilled by $s_1,\dots,s_n,s'_1,\dots,s'_n$ as well. We will now make sure that (b1) and (b2) in Lemma 4.3 are fulfilled, i.e. that (i') $\forall x \in \mathcal{M}\ \forall i,j \in \{1,\ldots,n\}$:

$$\begin{aligned} s_i^* x s_j &= \delta_{ij}\cdot\tau(x)\mathbf{1}, \\ (s_i')^* x s_j' &= \delta_{ij}\cdot\tau(x)\mathbf{1}, \\ s_i^* x s_j' &= (s_i')^* x s_j = 0. \end{aligned}$$

(ii') $\forall x \in \mathcal{M}\ \forall i \in \{1,\ldots,n\}$:

$$\varphi(x^* s_i s_i^* x) = \varphi(x^* s_i'(s_i')^* x) = 0.$$

If (i') and (ii') hold, then according to Lemma 4.3 (ii), $(s_1,\ldots,s_n,s_1',\ldots,s_n')$ has the same $*$–distribution w.r.t. φ as free creation operators, and $\mathcal{M}$ and $C^*(s_1,\ldots,s_n,s_1',\ldots,s_n')$ are free in $(\tilde{\mathcal{N}},\varphi)$. Hence, $y_i = s_i + (s_i')^*$, $1\le i\le n$, is a circular family which is $*$–free from $\mathcal{M}$.

To prove (i') and (ii'), let $x\in\mathcal{M}$, and let $1\le i,j\le n$. Then (i) implies that

$$\begin{aligned} s_i^* x s_j &= \sum_{p,q=1}^n \ell_p^* u_{ip}^* x u_{jq} \ell_q \\ &= \sum_{p,q=1}^n \delta_{pq}\varphi(u_{ip}^* x u_{jq})\mathbf{1} \\ &= \sum_{p=1}^n \varphi(x u_{jp}(u^*)_{pi})\mathbf{1} \\ &= \varphi(x(uu^*)_{ji})\mathbf{1} \\ &= \delta_{ij}\cdot\tau(x)\mathbf{1}, \end{aligned}$$

and similarly,

$$(s_i')^* x s_j' = \delta_{ij}\cdot\tau(x)\mathbf{1}.$$

Moreover, according to (i),

$$s_i^* x s_j' = \sum_{p,q=1}^n \ell_p^* u_{ip}^* x \ell_q' u_{jq}^* = \sum_{p,q=1}^n 0\cdot u_{jq}^* = 0,$$

and hence

$$(s_i')^* x s_j = (s_j^* x^* s_i')^* = 0.$$

This proves (i'). Next we prove that (ii') holds, making use of (i) and (ii):

Let $x\in\mathcal{M}$. Then

$$\varphi(x^* s_i s_i^* x) = \sum_{p,q=1}^n \varphi(x^* u_{ip}\ell_p \ell_q^* u_{iq}^* x), \tag{4.12}$$

and

$$\varphi(x^* s_i'(s_i')^* x) = \sum_{p,q=1}^n \varphi(x^* \ell_p' u_{ip}^* (\ell_q')^* x). \tag{4.13}$$

Since (ii) holds, we get, using the Schwarz inequality, that

$$\varphi(y^*\ell_i a) = \varphi(y^*\ell_i' a) = 0 \tag{4.14}$$

for $y \in \mathcal{M}$, $a \in \tilde{\mathcal{N}}$ and $1 \le i \le n$. Applying this to $y = x^* u_{ip}$ and $a = \ell_p^* u_{iq}^* x$, we get, using (4.12), that

$$\varphi(x^* s_i s_i^* x) = 0.$$

Similarly, applying (4.14) to $y = x$ and $a = u_{ip}^* u_{iq} (\ell_q')^* x$, we get from (4.13) that

$$\varphi(x^* s_i' (s_i')^* x) = 0.$$

This proves (ii'). ■

Let $\mathcal{M}$ be a II_1–factor, and let $T \in \mathcal{M}$. We regard $\mathcal{M}$ as a subfactor of $\mathcal{N} = \mathcal{M} * L(\mathbb{F}_4)$ with tracial state τ, and we choose a circular system $\{x, y\}$ that generates $L(\mathbb{F}_4)$ and which therefore is free from $\mathcal{M}$. Then by [HS, Theorem 5.2], the unbounded operator $z = xy^{-1}$ is in $L^p(\mathcal{N}, \tau)$, $0 < p < 1$. Thus, for all $a \in (0, \infty)$, $T + az \in L^p(\mathcal{N}, \tau)$, $0 < p < 1$, and therefore $T + az$ has a well–defined Fuglede–Kadison determinant and a well–defined Brown measure (cf. [HS, Section 2]).

4.5 Proposition. *Let $T \in \mathcal{M}$, and let $z = xy^{-1}$ as above. Then for each $a > 0$,*

$$\Delta(T + az) = \Delta(T^*T + a^2\mathbf{1})^{\frac{1}{2}}, \qquad (a > 0). \tag{4.15}$$

Proof. Take a unitary $u \in \mathcal{M}$ such that $T = u|T|$, and for fixed $a > 0$ put

$$w = (T^*T + a^2\mathbf{1})^{-\frac{1}{2}}(|T|y + au^*x).$$

Then

$$T + az = u(T^*T + a^2\mathbf{1})^{\frac{1}{2}} w y^{-1},$$

and hence

$$\begin{aligned} \Delta(T + az) &= \Delta(u)\Delta((T^*T + a^2\mathbf{1})^{\frac{1}{2}})\Delta(w)\Delta(y^{-1}) \\ &= \Delta(T^*T + a^2\mathbf{1})^{\frac{1}{2}}\Delta(w)\Delta(y)^{-1}. \end{aligned}$$

It follows that if w is circular, then $\Delta(T + az) = \Delta(T^*T + a^2\mathbf{1})^{\frac{1}{2}}$. To see that w is circular, define $v \in \mathcal{U}(M_2(\mathcal{M}))$ by

$$v = \begin{pmatrix} (T^*T + a^2\mathbf{1})^{-\frac{1}{2}}|T| & a(T^*T + a^2\mathbf{1})^{-\frac{1}{2}}u^* \\ a(T^*T + a^2\mathbf{1})^{-\frac{1}{2}} & -(T^*T + a^2\mathbf{1})^{-\frac{1}{2}}|T|u^* \end{pmatrix},$$

and define $x', y' \in \mathcal{M}$ by

$$\begin{pmatrix} y' \\ x' \end{pmatrix} = v \begin{pmatrix} y \\ x \end{pmatrix}.$$

Then, according to Proposition 4.4, $\{x', y'\}$ is a circular system (which is $*$-free from $\mathcal{M}$). Since $w = y'$, w is then circular, and hence $\Delta(w) = \Delta(y)$. ■

4.6 Corollary. *The Brown measure of* $T + az$ *is given by*

$$\mathrm{d}\mu_{T+az}(\lambda) = \frac{a^2}{\pi}\tau\big((T(\lambda)^*T(\lambda) + a^2\mathbf{1})^{-1}(T(\lambda)T(\lambda)^* + a^2\mathbf{1})^{-1}\big)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2, \tag{4.16}$$

where $\lambda_1 = \mathrm{Re}\lambda$, $\lambda_2 = \mathrm{Im}\lambda$, *and* $T(\lambda) = T - \lambda\mathbf{1}$.

Proof. By [HS, Definition 2.13] and Proposition 4.5,

$$\begin{aligned}\mathrm{d}\mu_{T+az}(\lambda) &= \frac{1}{2\pi}\nabla^2\big(\log\Delta(T + az - \lambda\mathbf{1})\big)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2\\ &= \frac{1}{2\pi}\nabla^2\big(\tfrac{1}{2}\log\Delta\big((T-\lambda\mathbf{1})^*(T-\lambda\mathbf{1}) + a^2\mathbf{1}\big)\big)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2,\end{aligned}$$

where the Laplacian $\nabla^2 = \frac{\partial^2}{\partial\lambda_1^2} + \frac{\partial^2}{\partial\lambda_2^2}$ is taken in the distribution sense. By the proof of [HS, Lemma 2.8],

$$L_a(\lambda) := \tfrac{1}{2}\log\Delta\big((T-\lambda\mathbf{1})^*(T-\lambda\mathbf{1}) + a^2\mathbf{1}\big)$$

is a C^2–function in (λ_1, λ_2) and its Laplacian is given by

$$(\nabla^2 L_a)(\lambda) = 2a^2\tau\big((T(\lambda)^*T(\lambda) + a^2)^{-1}(T(\lambda)T(\lambda)^* + a^2)^{-1}\big)$$

(replace (a, b, ε) by $(T, \mathbf{1}, a^2)$ in the function g_ε in the proof of [HS, Lemma 2.8]). This proves (4.16). ■

4.7 Remark. In [HS, Theorem 5.2] it was shown that the Brown measure of z is given by

$$\mathrm{d}\mu_z(\lambda) = \frac{1}{\pi(1+|\lambda|^2)}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2.$$

Note that this can also be obtained as a special case of (4.16) with $T = 0$ and $a = 1$.

4.8 Corollary. *Let* T *and* $T + az$ *be as in Proposition 4.5. Then*

$$\mu_{T+az} \overset{w^*}{\to} \mu_T \quad \text{as} \quad a \to 0+\,.$$

Proof. It suffices to show that for all $\varphi \in C_c^\infty(\mathbb{C})$,

$$\int_{\mathbb{C}} \varphi\,\mathrm{d}\mu_{T+az} \to \int_{\mathbb{C}} \varphi\,\mathrm{d}\mu_T \quad \text{as} \quad a \to 0+\,. \tag{4.17}$$

As in the proof of Corollary 4.6, put

$$\begin{aligned}L_a(\lambda) &= \log\Delta(T + az - \lambda\mathbf{1})\\ &= \frac{1}{2}\log\Delta\big((T-\lambda\mathbf{1})^*(T-\lambda\mathbf{1}) + a^2\mathbf{1}\big),\end{aligned} \tag{4.18}$$

and let

$$L(\lambda) = \log\Delta(T - \lambda\mathbf{1}) = \frac{1}{2}\log\Delta((T-\lambda\mathbf{1})^*(T-\lambda\mathbf{1})). \tag{4.19}$$

Then

$$\mathrm{d}\mu_{T+az}(\lambda) = \frac{1}{2\pi}\nabla^2 L_a(\lambda)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2,$$

and

$$\mathrm{d}\mu_T(\lambda) = \frac{1}{2\pi}\nabla^2 L(\lambda)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2.$$

Hence, for all $\varphi \in C_c^\infty(\mathbb{C})$,

$$\int_{\mathbb{C}} \varphi\,\mathrm{d}\mu_{T+az} = \frac{1}{2\pi}\int_{\mathbb{C}} (\nabla^2\varphi)(\lambda)L_a(\lambda)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2, \tag{4.20}$$

and

$$\int_{\mathbb{C}} \varphi\,\mathrm{d}\mu_T = \frac{1}{2\pi}\int_{\mathbb{C}} (\nabla^2\varphi)(\lambda)L(\lambda)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2. \tag{4.21}$$

By (4.18) and (4.19), $L_a(\lambda) \searrow L(\lambda)$ as $a \searrow 0$. Since L and L_a are subharmonic and hence locally integrable, (4.17) then follows from (4.20), (4.21) and Lebesgue's dominated convergence theorem. ∎

5 A certain Lipschitz condition

Consider a von Neumann algebra $\mathcal{N}$ equipped with a faithful normal tracial state τ, and assume that $\mathcal{N}$ contains the von Neumann algebra $\mathcal{M}$ as a sub-algebra and a circular system $\{x, y\}$ which is $*$-free from $\mathcal{M}$. We let $z = xy^{-1} \in L^p(\mathcal{N}, \tau)$, $(0 < p < 1)$. By [HS, Theorem 5.2 and Theorem 5.4],

$$z, z^{-1} \in L^p(\mathcal{N}, \tau) \quad \text{for all} \quad p \in (0, 1). \tag{5.1}$$

Moreover, if $p \in (0, \frac{2}{3})$ and $\lambda \in \mathbb{C}$, then $z^2, z^{-2}, (z^2 - \lambda\mathbf{1})^{-1} \in L^p(\mathcal{N}, \tau)$, and

$$\|(z^2 - \lambda\mathbf{1})^{-1}\|_p \leq \|z^{-2}\|_p = \|z^2\|_p < \infty. \tag{5.2}$$

As an application of (5.1), (5.2), and Proposition 4.4, in this section we prove:

5.1 Theorem. *For every $T \in \mathcal{M}$ and every $p \in (0, 1)$, $T + z$ has an inverse $(T + z)^{-1} \in L^p(\mathcal{N}, \tau)$. Moreover, for each $p \in (0, \frac{2}{3})$ there is a constant $C_p^{(1)} > 0$ such that*

$$\forall S, T \in \mathcal{M}: \quad \|(S + z)^{-1} - (T + z)^{-1}\|_p \leq C_p^{(1)}\|S - T\|. \tag{5.3}$$

Theorem 5.1 is a consequence of the following three results, which we prove later on.

5.2 Lemma. *Let $S, T \in \mathcal{M}$. Then*

$$\|(\mathbf{1} + S^*S)^{-\frac{1}{2}}S^* - (\mathbf{1} + T^*T)^{-\frac{1}{2}}T^*\| \;\leq\; \tfrac{5}{4}\,\|S - T\|, \tag{5.4}$$

and

$$\|(\mathbf{1} + S^*S)^{-\frac{1}{2}} - (\mathbf{1} + T^*T)^{-\frac{1}{2}}\| \;\leq\; \tfrac{2}{\pi}\,\|S - T\|. \tag{5.5}$$

5.3 Proposition. *For $T \in \mathcal{M}$ define $u(T) \in \mathcal{U}(M_2(\mathcal{M}))$ by*

$$u(T) = \begin{pmatrix} (\mathbf{1}+T^*T)^{-\frac{1}{2}}T^* & -(\mathbf{1}+T^*T)^{-\frac{1}{2}} \\ (\mathbf{1}+TT^*)^{-\frac{1}{2}} & (\mathbf{1}+TT^*)^{-\frac{1}{2}}T \end{pmatrix}. \tag{5.6}$$

Then for all $S, T \in \mathcal{M}$,

$$\|u(S) - u(T)\| \leq 2\|S - T\|. \tag{5.7}$$

According to Proposition 4.4, if $u = (u_{ij})_{i,j=1}^2 \in \mathcal{U}(M_2(\mathcal{M}))$, then $u_{11}x + u_{12}y$ and $u_{21}x + u_{22}y$ are $*$-free circular elements in $\mathcal{N}$, and they are $*$-free from $\mathcal{M}$. In particular, we may define $g_u(z) \in L^p(\mathcal{N}, \tau)$, $(0 < p < 1)$, by

$$g_u(z) = (u_{11}x + u_{12}y)(u_{21}x + u_{22}y)^{-1} = (u_{11}z + u_{12})(u_{21}z + u_{22})^{-1}. \tag{5.8}$$

5.4 Proposition. *(i) Let $p \in (0,1)$. Then for every $u \in \mathcal{U}(M_2(\mathcal{M}))$,*

$$\|g_u(z)\|_p = \|z\|_p < \infty. \tag{5.9}$$

(ii) Let $p \in (0, \frac{2}{3})$. Then there is a constant $C_p^{(2)} > 0$ such that for all $u, v \in \mathcal{U}(M_2(\mathcal{M}))$,

$$\|g_u(z) - g_v(z)\|_p \leq C_p^{(2)}\|u - v\|. \tag{5.10}$$

Proof of Theorem 5.1. Let $T \in \mathcal{M}$, and let $u(T) \in \mathcal{U}(M_2(\mathcal{M}))$ be given by (5.6). Moreover define (a circular system) $\{v, w\} \subseteq \mathcal{N}$ by

$$\begin{pmatrix} v \\ w \end{pmatrix} = u(T)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (\mathbf{1}+T^*T)^{-\frac{1}{2}}(T^*x - y) \\ (\mathbf{1}+TT^*)^{-\frac{1}{2}}(x + Ty) \end{pmatrix}. \tag{5.11}$$

Then

$$\begin{aligned} T + z &= (Ty + x)y^{-1} \\ &= (\mathbf{1}+TT^*)^{\frac{1}{2}}wy^{-1}, \end{aligned}$$

so for each $p \in (0,1)$, $T + z$ has an inverse $(T+z)^{-1} \in L^p(\mathcal{N}, \tau)$ given by

$$(T+z)^{-1} = yw^{-1}(\mathbf{1}+TT^*)^{-\frac{1}{2}}. \tag{5.12}$$

Moreover,

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= u(T)^*\begin{pmatrix} v \\ w \end{pmatrix} \\ &= \begin{pmatrix} T(\mathbf{1}+T^*T)^{-\frac{1}{2}} & (\mathbf{1}+TT^*)^{-\frac{1}{2}} \\ -(\mathbf{1}+T^*T)^{-\frac{1}{2}} & T^*(\mathbf{1}+TT^*)^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} \\ &= \begin{pmatrix} (\mathbf{1}+TT^*)^{-\frac{1}{2}}T & (\mathbf{1}+TT^*)^{-\frac{1}{2}} \\ -(\mathbf{1}+T^*T)^{-\frac{1}{2}} & (\mathbf{1}+T^*T)^{-\frac{1}{2}}T^* \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}. \end{aligned}$$

In particular,

$$y = (\mathbf{1} + T^*T)^{-\frac{1}{2}}(T^*w - v), \tag{5.13}$$

and hence by (5.12),

$$\begin{aligned}(T+z)^{-1} &= yw^{-1}(\mathbf{1}+TT^*)^{-\frac{1}{2}} \\ &= (\mathbf{1}+T^*T)^{-\frac{1}{2}}(T^*w-v)w^{-1}(\mathbf{1}+TT^*)^{-\frac{1}{2}} \\ &= T^*(\mathbf{1}+TT^*)^{-1} - (\mathbf{1}+T^*T)^{-\frac{1}{2}}vw^{-1}(\mathbf{1}+TT^*)^{-\frac{1}{2}} \\ &= T^*(\mathbf{1}+TT^*)^{-1} - (\mathbf{1}+T^*T)^{-\frac{1}{2}}g_{u(T)}(z)(\mathbf{1}+TT^*)^{-\frac{1}{2}}.\end{aligned}$$

Recall (cf. [FK]) that if $0 < p < 1$, then for $a, b \in L^p(\mathcal{M})$,

$$\|a+b\|_p^p \leq \|a\|_p^p + \|b\|_p^p .$$

It follows that for $S, T \in \mathcal{M}$ and $0 < p < \frac{2}{3}$,

$$\|(S+z)^{-1} - (T+z)^{-1}\|_p^p \leq \|A\|_p^p + \|B\|_p^p, \tag{5.14}$$

where

$$A = S^*(\mathbf{1}+SS^*)^{-1} - T^*(\mathbf{1}+TT^*)^{-1}, \tag{5.15}$$

and

$$B = (\mathbf{1}+S^*S)^{-\frac{1}{2}}g_{u(S)}(z)(\mathbf{1}+SS^*)^{-\frac{1}{2}} - (\mathbf{1}+T^*T)^{-\frac{1}{2}}g_{u(T)}(z)(\mathbf{1}+TT^*)^{-\frac{1}{2}}. \tag{5.16}$$

According to (5.4),

$$\|A\|_p \leq \|A\| = \|(\mathbf{1}+SS^*)^{-\frac{1}{2}}S - (\mathbf{1}+TT^*)^{-\frac{1}{2}}T\| \leq \tfrac{5}{4}\,\|S^* - T^*\| = \tfrac{5}{4}\,\|S-T\|.$$

To get an estimate of $\|B\|_p$, we write B as a sum, $B = B_1 + B_2 + B_3$, where B_1, B_2 and B_3 are given by

$$\begin{aligned}B_1 &= [(\mathbf{1}+S^*S)^{-\frac{1}{2}} - (\mathbf{1}+T^*T)^{-\frac{1}{2}}]g_{u(S)}(z)(\mathbf{1}+SS^*)^{-\frac{1}{2}}, \\ B_2 &= (\mathbf{1}+T^*T)^{-\frac{1}{2}}(g_{u(S)}(z) - g_{u(T)}(z))(\mathbf{1}+SS^*)^{-\frac{1}{2}}, \\ B_3 &= (\mathbf{1}+T^*T)^{-\frac{1}{2}}g_{u(T)}(z)[(\mathbf{1}+SS^*)^{-\frac{1}{2}} - (\mathbf{1}+TT^*)^{-\frac{1}{2}}].\end{aligned}$$

According to (5.5),

$$\|(\mathbf{1}+S^*S)^{-\frac{1}{2}} - (\mathbf{1}+T^*T)^{-\frac{1}{2}}\| \leq \tfrac{2}{\pi}\,\|S-T\| \leq \|S-T\|,$$

and

$$\|(\mathbf{1}+SS^*)^{-\frac{1}{2}} - (\mathbf{1}+TT^*)^{-\frac{1}{2}}\| \leq \tfrac{2}{\pi}\,\|S^*-T^*\| \leq \|S-T\|.$$

Moreover, by Proposition 5.4 and Proposition 5.3,

$$\|g_{u(S)}(z) - g_{u(T)}(z)\|_p \leq C_p^{(2)}\|u(S) - u(T)\| \leq 2C_p^{(2)}\|S-T\|,$$

and

$$\|g_{u(S)}(z)\|_p = \|z\|_p.$$

It follows that

$$\|B_1\|_p \leq \|z\|_p \|S-T\|, \tag{5.17}$$

$$\|B_2\|_p \leq 2C_p^{(2)} \|S-T\| \tag{5.18}$$

and

$$\|B_3\|_p \leq \|z\|_p \|S-T\|. \tag{5.19}$$

Altogether we have shown that

$$\begin{array}{rcl} \|A\|_p^p + \|B\|_p^p & \leq & \|A\|_p^p + \|B_1\|_p^p + \|B_2\|_p^p + \|B_3\|_p^p \\ & \leq & \Big((\frac{5}{4})^p + 2\|z\|_p^p + (2C_p^{(2)})^p \Big) \|S-T\|^p, \end{array}$$

so by (5.14), (5.3) holds with $C_p^{(1)} = \Big((\frac{5}{4})^p + 2\|z\|_p^p + 2^p C_p^{(2)p} \Big)^{\frac{1}{p}}$. ■

Proof of Lemma 5.2. For all $x > 0$ we have:

$$x^{-\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \frac{a^{-\frac{1}{2}}}{x+a} \, \mathrm{d}a.$$

Therefore every $R \in \mathcal{M}$ satisfies

$$(\mathbf{1} + R^*R)^{-\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty (R^*R + (a+1)\mathbf{1})^{-1} a^{-\frac{1}{2}} \, \mathrm{d}a,$$

(as a Banach space valued integral).

For $a \geq 1$ define

$$\begin{array}{rcl} A(a) & = & (a\mathbf{1} + S^*S)^{-1}S^* - (a\mathbf{1} + T^*T)^{-1}T^*, \\ B(a) & = & (a\mathbf{1} + S^*S)^{-1} - (a\mathbf{1} + T^*T)^{-1}, \\ C(a) & = & (a\mathbf{1} + SS^*)^{-1} - (a\mathbf{1} + TT^*)^{-1}. \end{array}$$

Then

$$\begin{array}{rcl} A(a) & = & (a\mathbf{1} + S^*S)^{-1}S^* - T^*(a\mathbf{1} + TT^*)^{-1} \\ & = & (a\mathbf{1} + S^*S)^{-1}[S^*(a\mathbf{1} + TT^*) - (a\mathbf{1} + S^*S)T^*](a\mathbf{1} + TT^*)^{-1} \\ & = & a(a\mathbf{1} + S^*S)^{-1}(S^* - T^*)(a\mathbf{1} + TT^*)^{-1} + \\ & & \qquad (a\mathbf{1} + S^*S)^{-1}S^*(T-S)T^*(a\mathbf{1} + TT^*)^{-1}. \end{array}$$

Now, for every $R \in \mathcal{M}$,

$$\|(a\mathbf{1} + R^*R)^{-1}\| \leq \tfrac{1}{a},$$

and

$$\begin{aligned}
\|(a\mathbf{1}+R^*R)^{-1}R^*\| &= \|(a\mathbf{1}+R^*R)^{-1}R^*R(a\mathbf{1}+R^*R)^{-1}\|^{\frac{1}{2}} \\
&\leq \Big(\sup_{t\geq 0}\frac{t}{(a+t)^2}\Big)^{\frac{1}{2}} \qquad (5.20)\\
&= \frac{1}{2\sqrt{a}}. \qquad (5.21)
\end{aligned}$$

Hence

$$\|A(a)\| \leq \tfrac{1}{a}\,\|S^*-T^*\| + \tfrac{1}{4a}\,\|T-S\| = \tfrac{5a}{4}\,\|S-T\|. \qquad (5.22)$$

It follows that

$$\begin{aligned}
\|(\mathbf{1}+S^*S)^{-\frac{1}{2}}S^* - (\mathbf{1}+T^*T)^{-\frac{1}{2}}T^*\| &\leq \frac{1}{\pi}\int_0^\infty \|A(a+1)\|a^{-\frac{1}{2}}\,\mathrm{d}a \\
&\leq \tfrac{5}{4\pi}\,\|S-T\|\textstyle\int_0^\infty (a+1)a^{-\frac{1}{2}}\,\mathrm{d}a \\
&= \tfrac{5}{4}\,\|S-T\|, \qquad (5.23)
\end{aligned}$$

which is (5.4).

Also, since

$$\begin{aligned}
B(a) &= (a\mathbf{1}+S^*S)^{-1}(T^*T-S^*S)(a\mathbf{1}+T^*T)^{-1} \\
&= (a\mathbf{1}+S^*S)^{-1}(T^*-S^*)T(a\mathbf{1}+T^*T)^{-1} \\
&\qquad +(a\mathbf{1}+S^*S)^{-1}S^*(T-S)(a\mathbf{1}+T^*T)^{-1},
\end{aligned}$$

where

$$\begin{aligned}
\|(a\mathbf{1}+S^*S)^{-1}(T^*-S^*)T(a\mathbf{1}+T^*T)^{-1}\| &\leq \tfrac{1}{a}\|T-S\|\|T(a\mathbf{1}+T^*T)^{-1}\| \\
&= \tfrac{1}{a}\|T-S\|\|(a\mathbf{1}+T^*T)^{-1}T^*\| \\
&\leq \tfrac{1}{a}\tfrac{1}{2\sqrt{a}}\|T-S\|,
\end{aligned}$$

and

$$\begin{aligned}
\|(a\mathbf{1}+S^*S)^{-1}S^*(T-S)(a\mathbf{1}+T^*T)^{-1}\| &\leq \tfrac{1}{a}\|(a\mathbf{1}+S^*S)^{-1}S^*\|\|T-S\| \\
&\leq \tfrac{1}{a}\tfrac{1}{2\sqrt{a}}\|T-S\|,
\end{aligned}$$

we find that

$$\|B(a)\| \leq \tfrac{1}{a\sqrt{a}}\,\|S-T\|, \qquad (5.24)$$

whence

$$\begin{aligned}
\|(\mathbf{1}+S^*S)^{-\frac{1}{2}} - (\mathbf{1}+T^*T)^{-\frac{1}{2}}\| &\leq \frac{1}{\pi}\int_0^\infty \|B(a+1)\|a^{-\frac{1}{2}}\,\mathrm{d}a \\
&\leq \|S-T\|\,\frac{1}{\pi}\int_0^\infty (a+1)^{-\frac{3}{2}}a^{-\frac{1}{2}}\,\mathrm{d}a \\
&= \tfrac{2}{\pi}\,\|S-T\|, \qquad (5.25)
\end{aligned}$$

and this is (5.5). ■

Proof of Proposition 5.3. With

$$\begin{aligned} A &= (\mathbf{1}+S^*S)^{-\frac{1}{2}}S^* - (\mathbf{1}+T^*T)^{-\frac{1}{2}}T^*, \\ B &= (\mathbf{1}+S^*S)^{-\frac{1}{2}} - (\mathbf{1}+T^*T)^{-\frac{1}{2}}, \\ C &= (\mathbf{1}+SS^*)^{-\frac{1}{2}} - (\mathbf{1}+TT^*)^{-\frac{1}{2}} \end{aligned}$$

one has that

$$u(S) - u(T) = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix},$$

and hence by Lemma 5.2,

$$\begin{aligned} \|u(S)-u(T)\| &\leq \left\| \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\| \\ &= \|A\| + \max\{\|B\|, \|C\|\} \\ &\leq \tfrac{5}{4}\|S-T\| + \tfrac{2}{\pi}\|S-T\| \\ &\leq 2\|S-T\|. \end{aligned}$$

■

Next we want to prove Proposition 5.4. To this end we need a series of lemmas which we state and prove in the following.

5.5 Lemma. *Let $u \in \mathcal{U}(M_2(\mathcal{M}))$ with $\|u-\mathbf{1}\| < 1$. Then there exist $w_1, w_2 \in \mathcal{U}(\mathcal{M})$ and $\ell \in \mathcal{M}$ such that*

$$u = wv, \tag{5.26}$$

where

$$w = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}, \tag{5.27}$$

and

$$v = \begin{pmatrix} (\mathbf{1}-\ell\ell^*)^{\frac{1}{2}} & l \\ -\ell^* & (\mathbf{1}-\ell^*\ell)^{\frac{1}{2}} \end{pmatrix}. \tag{5.28}$$

w_1, w_2 and ℓ are uniquely determined by (5.26), (5.27) and (5.28). Moreover, $\|\ell\| < 1$, $\|v-\mathbf{1}\| \leq 2\|u-\mathbf{1}\|$ and $\|w_i-\mathbf{1}\| \leq 3\|u-\mathbf{1}\|$, $i=1,2$.

Proof. $u = (u_{ij})_{i,j=1}^2$ with $u_{ij} \in \mathcal{M}$, where $\|u_{ii}-\mathbf{1}\| < 1$, $i=1,2$. Thus, u_{11} and u_{22} are invertible and have unique polar decompositions $u_{ii} = w_i|u_{ii}|$, where $w_i \in \mathcal{U}(\mathcal{M})$, $i=1,2$. Put $\ell = w_1^*u_{12}$, $m = w_2^*u_{21}$ and

$$v = \begin{pmatrix} w_1^* & 0 \\ 0 & w_2^* \end{pmatrix} u = \begin{pmatrix} |u_{11}| & \ell \\ m & |u_{22}| \end{pmatrix}.$$

Since v is unitary, $|u_{11}|^2 + \ell\ell^* = \mathbf{1}$, $\ell\ell^* + |u_{22}|^2 = \mathbf{1}$ and $m|u_{11}| + |u_{22}|\ell^* = 0$. Hence

$$|u_{11}| = (\mathbf{1}-\ell\ell^*)^{\frac{1}{2}}, \tag{5.29}$$

$$|u_{22}| = (\mathbf{1} - \ell^*\ell)^{\frac{1}{2}} \tag{5.30}$$

and

$$m = -|u_{22}|\ell^*|u_{11}|^{-1} = -\ell^*. \tag{5.31}$$

Thus, v is given by (5.28). Moreover, due to the uniqueness of the polar decompositions of u_{11} and u_{22}, w_1, w_2 and ℓ are uniquely determined by (5.26), (5.27) and (5.28).

Since $\|u - \mathbf{1}\| < 1$, we find that

$$\|\ell\| = \|u_{12}\| \leq \|u - \mathbf{1}\| < 1. \tag{5.32}$$

Also, when $t \in [0, 1]$, then $0 \leq 1 - \sqrt{t} \leq 1 - t$, and hence

$$\|\mathbf{1} - (\mathbf{1} - \ell\ell^*)^{\frac{1}{2}}\| \leq \|\mathbf{1} - (\mathbf{1} - \ell\ell^*)\| = \|\ell\|^2,$$

and

$$\|\mathbf{1} - (\mathbf{1} - \ell^*\ell)^{\frac{1}{2}}\| \leq \|\mathbf{1} - (\mathbf{1} - \ell^*\ell)\| = \|\ell\|^2.$$

Since

$$\mathbf{1} - v = \begin{pmatrix} \mathbf{1} - (\mathbf{1} - \ell\ell^*)^{\frac{1}{2}} & 0 \\ 0 & \mathbf{1} - (\mathbf{1} - \ell^*\ell)^{\frac{1}{2}} \end{pmatrix} + \begin{pmatrix} 0 & -l \\ \ell^* & 0 \end{pmatrix},$$

we may use the estimates obtained above to see that

$$\begin{aligned} \|v - \mathbf{1}\| &\leq \max\{\|\mathbf{1} - (\mathbf{1} - \ell\ell^*)^{\frac{1}{2}}\|, \|\mathbf{1} - (\mathbf{1} - \ell^*\ell)^{\frac{1}{2}}\|\} + \|\ell\| \\ &\leq \|\ell\|^2 + \|\ell\| \\ &\leq 2\|\ell\| \\ &\leq 2\|u - \mathbf{1}\|. \end{aligned} \tag{5.33}$$

Finally, w given by (5.27) satisfies

$$\begin{aligned} \|w - \mathbf{1}\| &= \|uv^* - \mathbf{1}\| \\ &= \|(u - v)v^*\| \\ &= \|u - v\| \\ &\leq \|u - \mathbf{1}\| + \|v - \mathbf{1}\| \\ &\leq 3\|u - \mathbf{1}\|, \end{aligned}$$

and hence $\|w_i - \mathbf{1}\| \leq 3\|u - \mathbf{1}\|$, $i = 1, 2$. ■

5.6 Lemma. *Let $\ell \in \mathcal{M}$ with $\|\ell\| < 1$. Then there exists $w_3 \in \mathcal{U}(\mathcal{M})$ such that*

$$\begin{pmatrix} (\mathbf{1} - \ell\ell^*)^{\frac{1}{2}} & l \\ -\ell^* & (\mathbf{1} - \ell^*\ell)^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} w_3 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} (\mathbf{1} - |\ell|^2)^{\frac{1}{2}} & |\ell| \\ -|\ell| & (\mathbf{1} - |\ell|^2)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} w_3^* & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{5.34}$$

Proof. Since $\mathcal{M}$ is finite, we may take a unitary $w_3 \in \mathcal{U}(\mathcal{M})$ such that $\ell = w_3|\ell|$. Let $m_1 = w_3^*(\mathbf{1} - \ell\ell^*)^{\frac{1}{2}} w_3$ and $m_2 = (\mathbf{1} - \ell^*\ell)^{\frac{1}{2}}$. Then

$$\begin{pmatrix} w_3^* & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} (\mathbf{1} - \ell\ell^*)^{\frac{1}{2}} & l \\ -\ell^* & (\mathbf{1} - \ell^*\ell)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} w_3 & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} m_1 & |\ell| \\ -|\ell| & m_2 \end{pmatrix}.$$

In particular, $\begin{pmatrix} m_1 & |\ell| \\ -|\ell| & m_2 \end{pmatrix}$ is unitary, and hence $m_1^2 + |\ell|^2 = \mathbf{1}$, so $m_1 = (\mathbf{1} - |\ell|^2)^{\frac{1}{2}}$. This proves (5.34). ■

5.7 Lemma. *Let $m \in \mathcal{M}_+$ with $\|m\| < 1$. Then there exist unitaries $w_4, w_5 \in \mathcal{M}$ and $\theta \in \mathbb{R}$ such that*

$$\begin{aligned} v' &:= \begin{pmatrix} (\mathbf{1} - m^2)^{\frac{1}{2}} & m \\ -m & (\mathbf{1} - m^2)^{\frac{1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} w_4^* & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \cos\theta\mathbf{1} & \sin\theta w_5 \\ -\sin\theta w_5^* & \cos\theta\mathbf{1} \end{pmatrix} \begin{pmatrix} \cos\theta\mathbf{1} & \sin\theta w_5^* \\ -\sin\theta w_5 & \cos\theta\mathbf{1} \end{pmatrix} \begin{pmatrix} w_4 & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \end{aligned} \tag{5.35}$$

$0 \leq \theta \leq \|v' - \mathbf{1}\|$ *and* $\|w_4 - \mathbf{1}\| \leq 2\|v' - \mathbf{1}\|$.

Proof. If $m = 0$, then $v' = \mathbf{1}$, and (5.35) holds with $\theta = 0$, $w_4 = w_5$. Now, assume that $m \neq 0$. Then we may choose $\theta \in (0, \frac{\pi}{4})$ such that $\sin(2\theta) = \|m\|$. Let

$$m' = \frac{1}{\sin(2\theta)} m = \frac{1}{2\cos\theta\sin\theta} m. \tag{5.36}$$

Then $m' = (m')^*$, $\|m'\| = 1$, and with

$$w_5 = m' + \mathrm{i}\sqrt{\mathbf{1} - (m')^2} \in \mathcal{U}(\mathcal{M}) \tag{5.37}$$

one has that

$$m' = \tfrac{1}{2}(w_5 + w_5^*).$$

Let

$$v'' = \begin{pmatrix} \cos\theta\mathbf{1} & \sin\theta w_5 \\ -\sin\theta w_5^* & \cos\theta\mathbf{1} \end{pmatrix} \begin{pmatrix} \cos\theta\mathbf{1} & \sin\theta w_5^* \\ -\sin\theta w_5 & \cos\theta\mathbf{1} \end{pmatrix} = \begin{pmatrix} k & m \\ -m & k^* \end{pmatrix},$$

where

$$k = \cos^2\theta\mathbf{1} - \sin^2\theta w_5^2. \tag{5.38}$$

Since v'' is unitary, $kk^* = k^*k = \mathbf{1} - m^2$. Then let

$$k = w_4(\mathbf{1} - m^2)^{\frac{1}{2}} \tag{5.39}$$

be the polar decomposition of k. $\sin^2\theta < \cos^2\theta$, when $0 < \theta < \frac{\pi}{4}$, so k is invertible, and w_4 must be unitary. Moreover, w_4 belongs to the abelian von Neumann algebra $W^*(k)$, and therefore it commutes with $m = (\mathbf{1} - k^*k)^{\frac{1}{2}}$. It follows that

$$\begin{pmatrix} w_4^* & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} k & m \\ -m & k^* \end{pmatrix} \begin{pmatrix} w_4 & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} (\mathbf{1} - m^2)^{\frac{1}{2}} & m \\ -m & (\mathbf{1} - m^2)^{\frac{1}{2}} \end{pmatrix}, \tag{5.40}$$

that is, (5.35) holds. Also, since $2\theta \in (0, \frac{\pi}{2})$, $\|m\| = \sin(2\theta) \geq \frac{2}{\pi}\, 2\theta$, and we get that

$$\theta \leq \tfrac{\pi}{4}\, \|m\| \leq \|m\| \leq \|v' - \mathbf{1}\|. \tag{5.41}$$

w_5^2 is unitary, and hence by (5.38) and the spectral mapping theorem, the spectrum of k must be contained in the boundary of the closed ball $\overline{B(\cos^2\theta, \sin^2\theta)} \subseteq \mathbb{C}$. As $\cos^2\theta > \sin^2\theta$, this ball is contained in $\{z \in \mathbb{C} \,|\, \mathrm{Re} z > 0\}$, so there is a continuous determination of $\arg z$ in $\sigma(k)$ taking values in $(-\frac{\pi}{2}, \frac{\pi}{2})$. In particular,

$$w_4 = \mathrm{e}^{\mathrm{i}\arg(k)} \tag{5.42}$$

and we get that

$$\begin{aligned} \|w_4 - \mathbf{1}\| &\leq \|\arg(k)\| \\ &\leq \max\{|\arg(z)| \,|\, z \in \partial B(\cos^2\theta, \sin^2\theta)\} \\ &= \arcsin(\tan^2\theta). \end{aligned} \tag{5.43}$$

Making use of the estimates

$$0 \leq \tan t \leq \frac{4}{\pi} t, \qquad (0 \leq t \leq \frac{\pi}{4}),$$

and

$$0 \leq \arcsin t \leq \frac{\pi}{2} t, \qquad (0 \leq t \leq 1),$$

we finally find that

$$\begin{aligned} \|w_4 - \mathbf{1}\| &\leq \arcsin(\tan^2\theta) \\ &\leq \arcsin(\tfrac{16}{\pi^2}\theta^2) \\ &\leq \tfrac{\pi}{2}\tfrac{16}{\pi^2}\theta^2 \\ &\leq 2\theta \\ &\leq 2\|v' - \mathbf{1}\|. \qquad \blacksquare \end{aligned}$$

5.8 Definition. For $p \in (0,1)$ define $d_p : \mathcal{U}(M_2(\mathcal{M})) \times \mathcal{U}(M_2(\mathcal{M})) \to [0, \infty[$ by

$$d_p(u, v) = \|g_u(z) - g_v(z)\|_p^p, \qquad (u, v \in \mathcal{U}(M_2(\mathcal{M})). \tag{5.44}$$

Clearly, for all $u, v, w \in \mathcal{U}(M_2(\mathcal{M}))$, we have:

$$d_p(u, w) \leq d_p(u, v) + d_p(v, w). \tag{5.45}$$

5.9 Proposition. *Let $0 < p < \frac{2}{3}$. Then*

(i) d_p is right-invariant on $\mathcal{U}(M_2(\mathcal{M}))$.

(ii) d_p *is left-invariant w.r.t. multiplication by diagonal unitaries.*

(iii) For all $w_1, w_2 \in \mathcal{U}(\mathcal{M})$ *one has that* $w = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$ *satisfies*

$$d_p(w, \mathbf{1}) \leq \big(\|w_1 - \mathbf{1}\|^p + \|w_2 - \mathbf{1}\|^p\big)\|z\|_p^p. \tag{5.46}$$

Proof. (i) Let $u, v, w \in \mathcal{U}(M_2(\mathcal{M}))$. Then with

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = w \begin{pmatrix} x \\ y \end{pmatrix}$$

and $z' = x'(y')^{-1}$ we have that

$$g_{uw}(z) - g_{vw}(z) = g_u(z') - g_v(z').$$

According to Proposition 4.4, $\{x', y'\}$ is a circular system which is $*$-free from $\mathcal{M}$, so

$$\begin{aligned} d_p(uw, vw) &= \|g_{uw}(z) - g_{vw}(z)\|_p \\ &= \|g_u(z') - g_v(z')\|_p \\ &= \|g_u(z) - g_v(z)\|_p \\ &= d_p(u, v). \end{aligned}$$

(ii) Let $u = (u_{ij})_{i,j=1}^2, v = (v_{ij})_{i,j=1}^2 \in \mathcal{U}(M_2(\mathcal{M}))$, and let $w = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} \in \mathcal{U}(M_2(\mathcal{M}))$. Then $w_1, w_2 \in \mathcal{U}(\mathcal{M})$, and hence

$$\begin{aligned} d_p(wu, wv) &= \|g_{wu}(z) - g_{wv}(z)\|_p^p \\ &= \|w_1(u_{11}x + u_{12}y)(u_{21}x + u_{22}y)^{-1}w_2^* - w_1(v_{11}x + v_{12}y)(v_{21}x + v_{22}y)^{-1}w_2^*\|_p^p \\ &= \|(u_{11}x + u_{12}y)(u_{21}x + u_{22}y)^{-1} - (v_{11}x + v_{12}y)(v_{21}x + v_{22}y)^{-1}\|_p^p \\ &= \|g_u(z) - g_v(z)\|_p^p \\ &= d_p(u, v). \end{aligned}$$

(iii) Let $w_1, w_2 \in \mathcal{U}(\mathcal{M})$, and let $w = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$. Then

$$\begin{aligned} d_p(w, \mathbf{1}) &= \|w_1 x (w_2 y)^{-1} - xy^{-1}\|_p^p \\ &= \|w_1 xy^{-1} - xy^{-1}w_2\|_p^p \\ &\leq \|(w_1 - \mathbf{1})xy^{-1}\|_p^p + \|xy^{-1}(\mathbf{1} - w_2)\|_p^p \\ &\leq (\|(w_1 - \mathbf{1})\|^p + \|(w_2 - \mathbf{1})\|^p)\|z\|_p^p. \qquad \blacksquare \end{aligned}$$

5.10 Corollary. *Let* $n \in \mathbb{N}$*, let* $u, u_1, \ldots, u_n \in \mathcal{U}(M_2(\mathcal{M}))$*, and let* $w \in \mathcal{U}(M_2(\mathcal{M}))$ *be diagonal. Then* *(iv)* $d_p(u_1 \cdots u_n, \mathbf{1}) \leq \sum_{i=1}^n d_p(u_i, \mathbf{1})$,

(v) $d_p(wuw^*, \mathbf{1}) = d_p(u, \mathbf{1})$.

Proof. (iv) follows from (5.45) and from Proposition 5.9 (i). Indeed, if $u_1, u_2 \in \mathcal{U}(M_2(\mathcal{M}))$, then

$$\begin{aligned} d_p(u_1 u_2, \mathbf{1}) &= d_p(u_1, u_2^*) \\ &\leq d_p(u_1, \mathbf{1}) + d_p(\mathbf{1}, u_2^*) \\ &= d_p(u_1, \mathbf{1}) + d_p(u_2, \mathbf{1}). \end{aligned}$$

Then (iv) follows for general $n \in \mathbb{N}$ by induction.

(v) is a consequence of Proposition 5.9 (i) and (ii). ■

Proof of Proposition 5.4.(i) Let $u = (u_{ij})_{i,j=1}^2 \in \mathcal{U}(M_2(\mathcal{M}))$. Then, as already mentioned, $u_{11}x + u_{12}y$ and $u_{21}x + u_{22}y$ are $*$-free circular elements. In particular, $g_u(z) = (u_{11}x + u_{12}y)(u_{21}x + u_{22}y)^{-1}$ has the same $*$-distribution as z (in the sense of [HS, Definition 3.2]), whence $\|g_u(z)\|_p = \|z\|_p < \infty$ for $p \in (0, 1)$.

(ii) Now assume that $p \in (0, \frac{2}{3})$. At first we consider $u \in \mathcal{U}(M_2(\mathcal{M}))$ with $\|u - \mathbf{1}\| < 1$. Take $w = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$ and v as in Lemma 5.5. Then, according to Corollary 5.10 (iv),

$$d_p(u, \mathbf{1}) \leq d_p(w, \mathbf{1}) + d_p(v, \mathbf{1}), \tag{5.47}$$

where

$$\begin{aligned} d_p(w, \mathbf{1}) &\leq \big(\|w_1 - \mathbf{1}\|^p + \|w_2 - \mathbf{1}\|^p\big)\|z\|_p^p \\ &\leq 2 \cdot 3^p \|u - \mathbf{1}\|^p \|z\|_p^p \end{aligned} \tag{5.48}$$

(cf. Proposition 5.9 and Lemma 5.5).

As in Lemma 5.6, take $m = |\ell| \in \mathcal{M}_+$ an $w_3 \in \mathcal{U}(\mathcal{M})$ such that

$$v = \begin{pmatrix} w_3 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} (\mathbf{1} - m^2)^{\frac{1}{2}} & m \\ -m & (\mathbf{1} - m^2)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} w_3^* & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

and put

$$v' = \begin{pmatrix} (\mathbf{1} - m^2)^{\frac{1}{2}} & m \\ -m & (\mathbf{1} - m^2)^{\frac{1}{2}} \end{pmatrix}. \tag{5.49}$$

Then

$$\|v - \mathbf{1}\| = \|v' - \mathbf{1}\| \leq 2\|u - \mathbf{1}\|, \tag{5.50}$$

and because of Corollary 5.10 (v), $d_p(v', \mathbf{1}) = d_p(v, \mathbf{1})$, so

$$d_p(u, \mathbf{1}) \leq 2 \cdot 3^p \|u - \mathbf{1}\|^p \|z\|_p^p + d_p(v', \mathbf{1}). \tag{5.51}$$

As in Lemma 5.7 we may take $w_4, w_5 \in \mathcal{U}(\mathcal{M})$ and $\theta \in \mathbb{R}$ such that $0 \le \theta \le \|v' - \mathbf{1}\|$, $\|w_4 - \mathbf{1}\| \le 2\|v' - \mathbf{1}\|$ and

$$v' = \begin{pmatrix} w_4^* & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \cos\theta\mathbf{1} & \sin\theta w_5 \\ -\sin\theta w_5^* & \cos\theta\mathbf{1} \end{pmatrix} \begin{pmatrix} \cos\theta\mathbf{1} & \sin\theta w_5^* \\ -\sin\theta w_5 & \cos\theta\mathbf{1} \end{pmatrix} \begin{pmatrix} w_4 & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{5.52}$$

Then, according to Proposition 5.9 (iii) and Corollary 5.10 (iv), with

$$v'' = \begin{pmatrix} \cos\theta\mathbf{1} & \sin\theta w_5 \\ -\sin\theta w_5^* & \cos\theta\mathbf{1} \end{pmatrix} \begin{pmatrix} \cos\theta\mathbf{1} & \sin\theta w_5^* \\ -\sin\theta w_5 & \cos\theta\mathbf{1} \end{pmatrix} \tag{5.53}$$

we have that

$$\begin{aligned} d_p(v', \mathbf{1}) &\le 2\|w_4 - \mathbf{1}\|_p^p \|z\|_p^p + d_p(v'', \mathbf{1}) \\ &\le 2 \cdot 2^p \|v' - \mathbf{1}\|^p \|z\|_p^p + d_p(v'', \mathbf{1}) \\ &\le 2 \cdot 4^p \|u - \mathbf{1}\|^p \|z\|_p^p + d_p(v'', \mathbf{1}). \end{aligned} \tag{5.54}$$

Altogether we have shown that

$$d_p(u, \mathbf{1}) \le 2 \cdot (3^p + 4^p)\|u - \mathbf{1}\|^p \|z\|_p^p + d_p(v'', \mathbf{1}). \tag{5.55}$$

Since

$$\begin{pmatrix} \cos\theta\mathbf{1} & \sin\theta w_5 \\ -\sin\theta w_5^* & \cos\theta\mathbf{1} \end{pmatrix} = \begin{pmatrix} w_5 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} w_5^* & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

and

$$\begin{pmatrix} \cos\theta\mathbf{1} & \sin\theta w_5^* \\ -\sin\theta w_5 & \cos\theta\mathbf{1} \end{pmatrix} = \begin{pmatrix} w_5^* & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} w_5 & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

Corollary 5.10 implies that with

$$u(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{5.56}$$

one has that

$$d_p(v'', \mathbf{1}) \le 2 d_p(u(\theta), \mathbf{1}). \tag{5.57}$$

Thus, when $\|u - \mathbf{1}\| < 1$,

$$d_p(u, \mathbf{1}) \le 4^{p+1}\|u - \mathbf{1}\|^p \|z\|_p^p + 2 d_p(u(\theta), \mathbf{1}), \tag{5.58}$$

where $0 \le \theta \le \max\{2\|u - \mathbf{1}\|, \frac{\pi}{4}\}$.

If $\|u - \mathbf{1}\| \ge 1$, then

$$\begin{aligned} d_p(u, \mathbf{1}) &= \|g_u(z) - z\|_p^p \\ &\le \|g_u(z)\|_p^p + \|z\|_p^p \\ &= 2\|z\|_p^p \\ &\le 4^{p+1}\|u - \mathbf{1}\|^p \|z\|_p^p, \end{aligned}$$

so (5.58) still holds in this case.

To get an estimate of $d_p(u(\theta), \mathbf{1})$, where $0 \leq \theta \leq \max\{2\|u - \mathbf{1}\|, \frac{\pi}{4}\}$, let $\phi = \frac{\theta}{2}$. Then $u(\theta) = u(\phi)^2$, so by the right-invariance of d_p and the inequality

$$1 + \tan^2 \phi \leq 2, \qquad 0 \leq \phi \leq \frac{\pi}{8},$$

we get that

$$\begin{aligned} d_p(u(\theta), \mathbf{1}) &= d_p(u(\phi), u(\phi)^{-1}) \\ &= d_p\Big(u(\phi)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, u(\phi)^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big) \\ &= \Big\|\frac{\sin\phi\, z + \cos\phi}{\cos\phi\, z - \sin\phi} - \frac{-\sin\phi\, z + \cos\phi}{\cos\phi\, z + \sin\phi}\Big\|_p^p \\ &= \Big\|\frac{2\tan\phi\,(z^2 + \mathbf{1})}{z^2 - \tan^2\phi}\Big\|_p^p \\ &= \Big\|2\tan\phi + \frac{2\tan\phi\,(1 + \tan^2\phi)}{z^2 - \tan\phi}\Big\|_p^p \\ &\leq (2\tan\phi)^p + (4\tan\phi)^p\Big\|\frac{1}{z^2 - \tan\phi}\Big\|_p^p. \end{aligned}$$

Hence, by (5.2) and the fact that

$$\tan\phi \leq 2\phi = \theta, \qquad 0 \leq \phi \leq \frac{\pi}{8},$$

$$\begin{aligned} d_p(u(\theta), \mathbf{1}) &\leq (2\tan\phi)^p + (4\tan\pi)^p\|z^2\|_p^p \\ &\leq (2\theta)^p + (4\theta)^p\|z^2\|_p^p \\ &\leq 2\theta^p\big(1 + 2\|z^2\|_p^p\big) \\ &\leq 4 \cdot 2^p\|u - 1\|^p\big(1 + 2\|z^2\|_p^p\big). \end{aligned}$$

Finally, by insertion of the above estimate into (5.58), we find that with

$$C_p^{(2)} = \big(4^{p+1}\|z\|_p^p + 2^{p+2}(1 + 2\|z^2\|_p^p)\big)^{\frac{1}{p}}, \tag{5.59}$$

one has that

$$d_p(u, \mathbf{1})^{\frac{1}{p}} \leq C_p\|u - \mathbf{1}\|. \tag{5.60}$$

Then for all $u, v \in \mathcal{U}(M_2(\mathcal{M}))$,

$$\begin{aligned} \|g_u(z) - g_v(z)\|_p &= d_p(u, v)^{\frac{1}{p}} \\ &= d_p(uv^{-1}, \mathbf{1})^{\frac{1}{p}} \\ &\leq C_p^{(2)}\|uv^{-1} - \mathbf{1}\| \\ &= C_p^{(2)}\|u - v\|, \end{aligned}$$

and this is (5.10). ■

6 Integration in $L^p(\mathcal{M},\tau)$ for $0<p<1$

In this section we consider a II_1-factor $\mathcal{M}$ with faithful tracial state τ and a fixed $p\in(0,1)$. In this case

$$\|A\|_p=\big(\tau(|A|^p)\big)^{\frac{1}{p}},\qquad A\in\mathcal{M},$$

does not define a norm on $\mathcal{M}$ but only a quasi–norm satisfying

$$\|A+B\|_p^p\le\|A\|_p^p+\|B\|_p^p,\qquad A,B\in\mathcal{M} \tag{6.1}$$

(cf. [FK]). $L^p(\mathcal{M},\tau)$ is complete w.r.t. the metric

$$d_p(A,B)=\|A-B\|_p^p,\qquad A,B\in\mathcal{M}.$$

Although $L^p(\mathcal{M},\tau)$ is not a locally convex vector space, one can define a Riemann type integral of a vector valued function

$$f:[a,b]\to L^p(\mathcal{M},\tau),$$

provided f fulfills the condition that for some $\alpha>\frac{1-p}{p}$ and for some positive constant C,

$$\forall\,x,y\in[a,b]:\quad \|f(x)-f(y)\|_p\le C|x-y|^\alpha. \tag{6.2}$$

This is a special case of an integral introduced by Turpin and Waelbroeck in 1968–71 (cf. [TuWa], [Wa]) and further developed by Kalton in 1985 (cf. [Ka, Section 3]). We shall only need the properties of the Turpin–Waelbroeck integral stated in Definition 6.1 and Theorem 6.2 below. For the convenience of the reader we have enclosed a self–contained proof of Theorem 6.2 in the appendix (section 10).

Suppose (6.2) is fulfilled. Then for each $n\in\mathbb{N}_0$ define $S_n\in L^p(\mathcal{M},\tau)$ by

$$S_n=\frac{b-a}{2^n}\sum_{k=1}^{2^n}f\Big(a+k\frac{b-a}{2^n}\Big),\qquad (n\in\mathbb{N}_0). \tag{6.3}$$

Then for every $n\ge2$,

$$\begin{aligned}S_n-S_{n-1}&=\frac{b-a}{2^n}\sum_{k=1}^{2^n}(-1)^{k-1}f\Big(a+k\frac{b-a}{2^n}\Big)\\&=\frac{b-a}{2^n}\sum_{j=1}^{2^{n-1}}T_{n,j},\end{aligned}$$

where

$$T_{n,j}=f\Big(a+(2j-1)\frac{b-a}{2^n}\Big)-f\Big(a+2j\frac{b-a}{2^n}\Big).$$

Hence, by (6.2)

$$\|T_{n,j}\|_p\le C\left(\frac{b-a}{2^n}\right)^\alpha,\qquad 1\le j\le 2^{n-1},$$

and then by (6.1),

$$\|S_n - S_{n-1}\|_p^p \le \frac{2^{n-1}}{2^{np}} C^p \Big(\frac{b-a}{2^n}\Big)^{p\alpha+p} = \frac{C^p(b-a)^p}{2\cdot 2^{n(p+p\alpha+1)}}. \tag{6.4}$$

Since $\alpha > \frac{1-p}{p}$, $p + p\alpha - 1 > 0$, and hence

$$\sum_{n=2}^{\infty} \|S_n - S_{n-1}\|_p^p < \infty.$$

It follows that $(S_n)_{n=1}^{\infty}$ is a Cauchy sequence in $L^p(\mathcal{M},\tau)$ w.r.t. the metric d_p, and therefore $\lim_{n\to\infty} S_n$ exists in $L^p(\mathcal{M},\tau)$.

6.1 Definition. For $f : \mathbb{R} \to L^p(\mathcal{M},\tau)$ as above we define $\int_a^b f(x)\,\mathrm{d}x \in L^p(\mathcal{M},\tau)$ by

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{n\to\infty} S_n. \tag{6.5}$$

6.2 Theorem. *Suppose $f : [a,b] \to L^p(\mathcal{M},\tau)$ satisfies (6.2) for some $\alpha > \frac{1}{p} - 1$. Let $I = (a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b)$ be an arbitrary partition of $[a,b]$, and for arbitrary $t_i \in [x_{i-1}, x_i]$ define*

$$M = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}). \tag{6.6}$$

Then with $\delta(I) = \max_{1\le i\le n}(x_i - x_{i-1})$,

$$\Big\| M - \int_a^b f(x)\,\mathrm{d}x \Big\|_p^p \le \frac{C^p(b-a)\delta(I)^{p+\alpha p-1}}{p+p\alpha-1}. \tag{6.7}$$

It follows that if $(I_n)_{n=1}^{\infty}$ is a sequence of partitions of $[a,b]$, such that the fineness $\delta(I_n)$ of the n'th partition I_n tends to zero as n tends to infinity, and if for each $n \in \mathbb{N}$ we associate to I_n a finite sum $M_n \in L^p(\mathcal{M},\tau)$ as in (6.6), then $\|M_n - \int_a^b f(x)\,\mathrm{d}x\|_p \to 0$ as $n \to \infty$.

Proof. See section 10.

By [HS, Section 2], the definition of the Fuglede–Kadison determinant $\Delta(T)$ and the Brown measure μ_T can be extended to all unbounded operators in $\mathcal{M}^\Delta$, where

$$\mathcal{M}^\Delta = \{T \in \tilde{\mathcal{M}} \mid \int_0^\infty \log^+ t\,\mathrm{d}\mu_{|T|}(t) < \infty\}.$$

Note that $L^p(\mathcal{M},\tau) \subseteq \mathcal{M}^\Delta$ for all $p \in (0,\infty)$.

6.3 Definition. For $T \in \mathcal{M}^\Delta$ define $r'(T)$, the *modified spectral radius* of T, by

$$r'(T) = \sup\{|z| \mid z \in \operatorname{supp}(\mu_T)\}. \tag{6.8}$$

6.4 Proposition. *For each $T \in \mathcal{M}^\Delta$ and each $p \in (0,\infty)$ we have that*

$$r'(T) \le \limsup_{m\to\infty} \|T^m\|_p^{\frac{1}{m}}. \tag{6.9}$$

Proof. $r'(T)$ is the essential supremum of $|\lambda|$ w.r.t. μ_T. Hence by application of [HS, Proposition 2.15] and [HS, Theorem 2.19],

$$\begin{aligned}
r'(T) &= \lim_{q\to\infty} \Big(\int_{\mathbb{C}} |z|^q \,\mathrm{d}\mu_T(z)\Big)^{\frac{1}{q}} \\
&= \lim_{m\to\infty} \Big(\int_{\mathbb{C}} |z|^{pm} \,\mathrm{d}\mu_T(z)\Big)^{\frac{1}{pm}} \\
&= \lim_{m\to\infty} \Big(\int_{\mathbb{C}} |z^m|^{p} \,\mathrm{d}\mu_T(z)\Big)^{\frac{1}{pm}} \\
&= \lim_{m\to\infty} \Big(\int_{\mathbb{C}} |z|^{p} \,\mathrm{d}\mu_{T^m}(z)\Big)^{\frac{1}{pm}} \\
&\le \limsup_{m\to\infty} \Big(\|T^m\|_p^p\Big)^{\frac{1}{pm}} \\
&= \limsup_{m\to\infty} \|T^m\|_p^{\frac{1}{m}}. \qquad \blacksquare
\end{aligned}$$

6.5 Proposition. *Let $T \in \mathcal{M}^\Delta$ and let $P \in \mathcal{M}$ be a non–trivial projection such that $PTP = TP$. Then*

$$\Delta(T) = \Delta_{P\mathcal{M}P}(PTP)^{\tau(P)}\, \Delta_{P^\perp\mathcal{M}P^\perp}(P^\perp TP^\perp)^{1-\tau(P)}, \tag{6.10}$$

and

$$\mu_T = \tau(P)\mu_{PTP} + (1-\tau(P))\mu_{P^\perp TP^\perp}, \tag{6.11}$$

where the Brown measures μ_{PTP} and $\mu_{P^\perp TP^\perp}$ are computed relative to the II_1–factors $P\mathcal{M}P$ and $P^\perp\mathcal{M}P^\perp$, respectively.

Proof. Equation (6.10) was proven in [HS, Proposition 2.24]. Since $PTP = TP$, we have that for all $\lambda \in \mathbb{C}$, $P(T-\lambda\mathbf{1})P = (T-\lambda\mathbf{1})P$, and thus by (6.10), we have for all $\lambda \in \mathbb{C}$,

$$\begin{aligned}
&\log\Delta(T-\lambda\mathbf{1}) \\
&\quad = \tau(P)\log\Delta_{P\mathcal{M}P}(P(T-\lambda\mathbf{1})P) + (1-\tau(P))\log\Delta_{P^\perp\mathcal{M}P^\perp}(P^\perp(T-\lambda\mathbf{1})P^\perp).
\end{aligned} \tag{6.12}$$

Equation (6.11) now follows by taking the Laplacian (in the Schwartz distribution sense) on both sides of (6.12) (cf. [HS, Definition 2.13]). $\blacksquare$

6.6 Theorem. *Suppose $T \in \mathcal{M}^{\Delta}$ and that T has empty point spectrum.*[2] *Moreover, assume that for some $p \in (\frac{1}{2}, 1)$ there exist $\alpha \in (\frac{1}{p} - 1, 1]$ and a positive constant C such that*

(i) $(T - \lambda\mathbf{1})^{-1} \in L^p(\mathcal{M}, \tau)$ *for all* $\lambda \in \mathbb{C}$,

(ii) $\|(T - \lambda\mathbf{1})^{-1} - (T - \mu\mathbf{1})^{-1}\|_p \leq C|\lambda - \mu|^{\alpha}$ *for all* $\lambda, \mu \in \mathbb{C}$.

Then there is a T-invariant subspace $\mathcal{K}$ affiliated with $\mathcal{M}$ such that when $P \in \mathcal{M}$ denotes the projection onto $\mathcal{K}$, and when we write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

according to the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$, then $T_{11} \in L^p(P\mathcal{M}P)$, $T_{22}^{-1} \in L^p(P^{\perp}\mathcal{M}P^{\perp})$, $\operatorname{supp}(\mu_{T_{11}}) \subseteq \overline{B(0,1)}$ *and* $\operatorname{supp}(\mu_{T_{22}}) \subseteq \{z \in \mathbb{C} \,|\, |z| \geq 1\}$.

Proof. Define $f : \mathbb{R} \to L^p(\mathcal{M}, \tau)$ by

$$f(t) = (\mathrm{e}^{\mathrm{i}t}\mathbf{1} - T)^{-1}\mathrm{e}^{\mathrm{i}t}, \qquad (t \in \mathbb{R}).$$

Then f is Hölder continuous with exponent α. Indeed,

$$\begin{aligned} \|f(t) - f(s)\|_p^p &= \|(\mathrm{e}^{\mathrm{i}t}\mathbf{1} - T)^{-1}\mathrm{e}^{\mathrm{i}t} - (\mathrm{e}^{\mathrm{i}s}\mathbf{1} - T)^{-1}\mathrm{e}^{\mathrm{i}s}\|_p^p \\ &\leq \|((\mathrm{e}^{\mathrm{i}t}\mathbf{1} - T)^{-1} - (\mathrm{e}^{\mathrm{i}s}\mathbf{1} - T)^{-1})\mathrm{e}^{\mathrm{i}t}\|_p^p + \|(\mathrm{e}^{\mathrm{i}s}\mathbf{1} - T)^{-1}(\mathrm{e}^{\mathrm{i}s} - \mathrm{e}^{\mathrm{i}t})\|_p^p \\ &\leq C^p|\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{\mathrm{i}s}|^{\alpha p} + (C')^p|s - t|^p \\ &\leq C^p|s - t|^{\alpha p} + (C')^p|s - t|^p, \end{aligned} \tag{6.13}$$

where

$$C' = \max\{\|(\lambda\mathbf{1} - T)^{-1}\|_p \,|\, |\lambda| = 1\} < \infty.$$

It follows that f is Hölder continuous with exponent $\min\{\alpha, 1\} = \alpha$ and that we may define $E \in L^p(\mathcal{M}, \tau)$ by

$$E = \frac{1}{2\pi\mathrm{i}} \int_{\partial B(0,1)} (\lambda\mathbf{1} - T)^{-1} \,\mathrm{d}\lambda := \frac{1}{2\pi} \int_0^{2\pi} f(t) \,\mathrm{d}t. \tag{6.14}$$

We are going to prove that the range projection P of E has the properties stated in Theorem 6.6.

At first we prove that $E^2 = E$. To see this, note that with

$$\begin{aligned} s_k^{(n)} &= \frac{2\pi k}{n}, \qquad (1 \leq k \leq n), \\ t_k^{(n)} &= \frac{2\pi(k - \frac{1}{2})}{n}, \qquad (1 \leq k \leq n) \end{aligned}$$

[2] The *point spectrum* of T is the set of eigenvalues of T.

$$E_n = \frac{1}{n}\sum_{k=1}^{n}(\mathrm{e}^{\mathrm{i}s_k^{(n)}}\mathbf{1} - T)^{-1}\mathrm{e}^{\mathrm{i}s_k^{(n)}} \tag{6.15}$$

and

$$F_n = \frac{1}{n}\sum_{k=1}^{n}(\mathrm{e}^{\mathrm{i}t_k^{(n)}}\mathbf{1} - T)^{-1}\mathrm{e}^{\mathrm{i}t_k^{(n)}} \tag{6.16}$$

one has that

$$\lim_{n\to\infty}\|E - E_n\|_p = \lim_{n\to\infty}\|E - F_n\|_p = 0.$$

Moreover,

$$\begin{aligned} E_nF_n &= \frac{1}{n^2}\sum_{k,l_1}^{n}\mathrm{e}^{\mathrm{i}s_k^{(n)}}\mathrm{e}^{\mathrm{i}t_l^{(n)}}(\mathrm{e}^{\mathrm{i}s_k^{(n)}}\mathbf{1} - T)^{-1}(\mathrm{e}^{\mathrm{i}t_l^{(n)}}\mathbf{1} - T)^{-1} \\ &= \frac{1}{n^2}\sum_{k,l_1}^{n}\frac{\mathrm{e}^{\mathrm{i}s_k^{(n)}}\mathrm{e}^{\mathrm{i}t_l^{(n)}}}{\mathrm{e}^{\mathrm{i}t_l^{(n)}} - \mathrm{e}^{\mathrm{i}s_k^{(n)}}}\Big((\mathrm{e}^{\mathrm{i}s_k^{(n)}}\mathbf{1} - T)^{-1} - (\mathrm{e}^{\mathrm{i}t_l^{(n)}}\mathbf{1} - T)^{-1}\Big) \\ &= \frac{1}{n}\sum_{k=1}^{n}a_k^{(n)}\mathrm{e}^{\mathrm{i}s_k^{(n)}}(\mathrm{e}^{\mathrm{i}s_k^{(n)}}\mathbf{1} - T)^{-1} + \frac{1}{n}\sum_{l=1}^{n}b_l^{(n)}\mathrm{e}^{\mathrm{i}t_l^{(n)}}(\mathrm{e}^{\mathrm{i}t_l^{(n)}}\mathbf{1} - T)^{-1}, \end{aligned}$$

where

$$a_k^{(n)} = \frac{1}{n}\sum_{l=1}^{n}\frac{\mathrm{e}^{\mathrm{i}t_l^{(n)}}}{\mathrm{e}^{\mathrm{i}t_l^{(n)}} - \mathrm{e}^{\mathrm{i}s_k^{(n)}}} = \frac{1}{n}\sum_{l=1}^{n}\frac{1}{1 - \mathrm{e}^{-\mathrm{i}t_l^{(n)}}\mathrm{e}^{\mathrm{i}s_k^{(n)}}},$$

and

$$b_l^{(n)} = \frac{1}{n}\sum_{k=1}^{n}\frac{\mathrm{e}^{\mathrm{i}s_k^{(n)}}}{\mathrm{e}^{\mathrm{i}s_k^{(n)}} - \mathrm{e}^{\mathrm{i}t_l^{(n)}}} = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1 - \mathrm{e}^{-\mathrm{i}s_k^{(n)}}\mathrm{e}^{\mathrm{i}t_l^{(n)}}}.$$

It follows that

$$a_k^{(n)} = b_l^{(n)} = \frac{1}{n}\sum_{m=1}^{n}\frac{1}{1 - \theta_m},$$

where $\theta_m = \mathrm{e}^{\mathrm{i}\frac{(2m-1)\pi}{n}}$, $m = 1, \dots, n$, are the roots of the polynomial

$$p(z) = z^n + 1, \qquad (z \in \mathbb{C}).$$

Consequently,

$$p(z) = \prod_{m=1}^{n}(z - \theta_m),$$

and

$$\frac{p'(z)}{p(z)} = \sum_{m=1}^{n}\frac{1}{z - \theta_m}.$$

This implies that

$$a_k^{(n)} = b_l^{(n)} = \frac{1}{n}\frac{p'(1)}{p(1)} = \frac{1}{2},$$

and we have thus shown that

$$E_n F_n = \tfrac{1}{2}(E_n + F_n). \tag{6.17}$$

Now, $E_n F_n = \frac{1}{2}(E_n + F_n) \to E$ w.r.t. $\|\cdot\|_p$ and hence w.r.t. $\|\cdot\|_{\frac{p}{2}}$. Moreover,

$$\begin{aligned} \|E_n F_n - E^2\|_{\frac{p}{2}}^{\frac{p}{2}} &\leq \|(E_n - E)F_n\|_{\frac{p}{2}}^{\frac{p}{2}} + \|E(F_n - E)\|_{\frac{p}{2}}^{\frac{p}{2}} \\ &\leq \big(\|E_n - E\|_p \|\|F_n\|_p\big)^{\frac{p}{2}} + \big(\|E\|_p \|\|F_n - E\|_p\big)^{\frac{p}{2}} \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

and it follows that $E = E^2$.

Clearly, E_n and T commute for every $n \in \mathbb{N}$. Since the map $(a, b) \mapsto ab$ is continuous w.r.t the measure topology on products of sets which are bounded in measure, this implies that

$$ET = TE \text{ in } \tilde{\mathcal{M}}. \tag{6.18}$$

In particular, $\ker(E)$ and $R(E)$ (the range of E) are T-invariant.

Let P denote the projection onto $\overline{R(E)}$. Then $P \in \mathcal{M}$, and PTP agrees with TP on $R(E) + P(\mathcal{H})^{\perp}$ which is a dense subset of $\mathcal{H}$. Hence $PTP = TP$ in $\tilde{\mathcal{M}}$, i.e. $\mathcal{K} = \overline{R(E)}$ is T-invariant.

Next take a fixed $m \in \mathbb{N}$. As above we find that the map

$$t \mapsto (\mathrm{e}^{\mathrm{i}t}\mathbf{1} - T)^{-1}\mathrm{e}^{\mathrm{i}(m+1)t}$$

is Hölder continuous with exponent α, and thus we may define $G \in L^p(\mathcal{M}, \tau)$ by

$$G = \frac{1}{2\pi\mathrm{i}}\int_{\partial B(0,1)} (\lambda\mathbf{1} - T)^{-1}\lambda^m \, \mathrm{d}\lambda. \tag{6.19}$$

We know that $G = \lim_{n\to\infty} G_n$ in p-norm, where

$$G_n = \frac{1}{n}\sum_{k=1}^{n} (\mathrm{e}^{\mathrm{i}s_k^{(n)}}\mathbf{1} - T)^{-1}\mathrm{e}^{\mathrm{i}(m+1)s_k^{(n)}},$$

with $s_k^{(n)} = \frac{2\pi k}{n}$. With E_n as in (6.15) we want to prove that for fixed $n > m$,

$$G_n = T^m E_n, \tag{6.20}$$

i.e. that

$$\frac{1}{n}\sum_{k=1}^{n} (\mathrm{e}^{\mathrm{i}s_k^{(n)}}\mathbf{1} - T)^{-1}\mathrm{e}^{\mathrm{i}(m+1)s_k^{(n)}} = \frac{1}{n}\sum_{k=1}^{n} (\mathrm{e}^{\mathrm{i}s_k^{(n)}}\mathbf{1} - T)^{-1}T^m\mathrm{e}^{\mathrm{i}s_k^{(n)}}. \tag{6.21}$$

Note that (6.21) holds if

$$\frac{1}{n}\sum_{k=1}^{n}(\mathrm{e}^{\mathrm{i}s_k^{(n)}}-z)^{-1}\mathrm{e}^{\mathrm{i}(m+1)s_k^{(n)}}=\frac{1}{n}\sum_{k=1}^{n}(\mathrm{e}^{\mathrm{i}s_k^{(n)}}-z)^{-1}z^m\mathrm{e}^{\mathrm{i}s_k^{(n)}} \tag{6.22}$$

as rational functions of $z\in\mathbb{C}$.

Now, with

$$q(z)=z^n-1, \qquad (z\in\mathbb{C}),$$

$\mathrm{e}^{\mathrm{i}s_k^{(n)}}$, $k=1,\dots,n$ are the roots of q, and therefore

$$\mathrm{Res}\Big(\frac{1}{q};\mathrm{e}^{\mathrm{i}s_k^{(n)}}\Big)=\frac{1}{q'(\mathrm{e}^{\mathrm{i}s_k^{(n)}})}=\frac{1}{n\cdot\mathrm{e}^{\mathrm{i}(n-1)s_k^{(n)}}}=\frac{\mathrm{e}^{\mathrm{i}s_k^{(n)}}}{n}.$$

It follows that

$$\frac{1}{q(z)}=\frac{1}{n}\sum_{k=1}^{n}(z-\mathrm{e}^{\mathrm{i}s_k^{(n)}})^{-1}\mathrm{e}^{\mathrm{i}s_k^{(n)}}. \tag{6.23}$$

Also, $\lim_{|z|\to\infty}\frac{z^m}{q(z)}=0$ and

$$\mathrm{Res}\Big(\frac{z^m}{q(z)};\mathrm{e}^{\mathrm{i}s_k^{(n)}}\Big)=\frac{\mathrm{e}^{\mathrm{i}ms_k^{(n)}}}{q'(\mathrm{e}^{\mathrm{i}s_k^{(n)}})}=\frac{1}{n}\mathrm{e}^{\mathrm{i}(m+1)s_k^{(n)}}.$$

Hence

$$\frac{z^m}{q(z)}=\frac{1}{n}\sum_{k=1}^{n}(z-\mathrm{e}^{\mathrm{i}s_k^{(n)}})^{-1}\mathrm{e}^{\mathrm{i}(m+1)s_k^{(n)}}. \tag{6.24}$$

Comparing (6.23) with (6.24) we find that (6.22) holds when $n>m$, that is $G_n=T^mE_n$. Take the limit as $n\to\infty$ on both sides of (6.20) (w.r.t. the measure topology) and conclude that

$$G=T^mE. \tag{6.25}$$

(6.25) enables us to make an estimate of $\|T^mE\|_p$: As in (6.13) one can show that

$$\begin{aligned}\|(\mathrm{e}^{\mathrm{i}s}\mathbf{1}-T)^{-1}\mathrm{e}^{\mathrm{i}(m+1)s}-(\mathrm{e}^{\mathrm{i}t}\mathbf{1}-T)^{-1}\mathrm{e}^{\mathrm{i}(m+1)t}\|_p^p &\le C^p|\mathrm{e}^{\mathrm{i}s}-\mathrm{e}^{\mathrm{i}t}|^{\alpha p}+(C')^p|\mathrm{e}^{\mathrm{i}(m+1)s}-\mathrm{e}^{\mathrm{i}(m+1)t}|^p\\ &\le C^p|s-t|^{\alpha p}+(C')^p|\mathrm{e}^{\mathrm{i}(m+1)s}-\mathrm{e}^{\mathrm{i}(m+1)t}|^p,\end{aligned}$$

and since $\alpha\le 1$ and $|\mathrm{e}^{\mathrm{i}(m+1)s}-\mathrm{e}^{\mathrm{i}(m+1)t}|\le 2$,

$$\begin{aligned}|\mathrm{e}^{\mathrm{i}(m+1)s}-\mathrm{e}^{\mathrm{i}(m+1)t}| &\le 2\Big|\frac{\mathrm{e}^{\mathrm{i}(m+1)s}-\mathrm{e}^{\mathrm{i}(m+1)t}}{2}\Big|^\alpha\\ &= 2^{1-\alpha}|\mathrm{e}^{\mathrm{i}(m+1)s}-\mathrm{e}^{\mathrm{i}(m+1)t}|^\alpha\\ &\le 2^{1-\alpha}(m+1)^\alpha|s-t|^\alpha\\ &\le 2^{1-\alpha}(m+1)|s-t|^\alpha.\end{aligned}$$

It follows that $t \mapsto (\mathrm{e}^{\mathrm{i}t}\mathbf{1}-T)^{-1}\mathrm{e}^{\mathrm{i}(m+1)t}$ is Hölder continuous with exponent α and constant

$$\big(C^p + (C')^p(m+1)^p 2^{p(1-\alpha)}\big)^{\frac{1}{p}}.$$

Since $0 < p < 1$,

$$\Big(\frac{x+y}{2}\Big) \geq \frac{1}{2}(x^p + y^p),$$

whence

$$x^p + y^p \leq 2^{1-p}(x+y)^p \leq 2^p(x+y)^p,$$

because $p \geq \frac{1}{2}$. Consequently, $t \mapsto (\mathrm{e}^{\mathrm{i}t}\mathbf{1}-T)^{-1}\mathrm{e}^{\mathrm{i}(m+1)t}$ is Hölder continuous with exponent α and constant

$$C'' = 2(C + C'(m+1)2^{1-\alpha}). \tag{6.26}$$

Then according to Lemma 10.4,

$$\|G\|_p^p \leq \frac{(C'')^p(2\pi)^{\alpha p}}{\alpha p + p - 1} + \|(\mathbf{1}-T)^{-1}\|_p^p, \tag{6.27}$$

and combining (6.26) with (6.27) we find that there exist positive constants k_1, k_2 (which are independent of m) such that

$$\|T^m E\|_p \leq k_1 + m k_2, \qquad (m \in \mathbb{N}).$$

In particular,

$$\lim_{m\to\infty} \Big(\tau(P)^{-\frac{1}{p}}\|T^m E\|_p\Big)^{\frac{1}{m}} \leq 1.$$

Since $\mathcal{K}$ is T-invariant, with $T_{11} = T_{|\mathcal{K}}$, we have that $T_{11}^m = T_{|\mathcal{K}}^m$. Moreover, TE and $T_{|\mathcal{K}}$ agree on $R(E)$, and hence they must agree as operators in $(P\mathcal{M}P)\tilde{}$. It follows that

$$T_{11}^m = T_{|\mathcal{K}}^m = T^m E_{|\mathcal{K}},$$

and then, by Proposition 6.4,

$$\begin{aligned} r'(T_{11}) &\leq \lim_{m\to\infty} \|T_{11}^m\|_p^{\frac{1}{m}} \\ &= \lim_{m\to\infty} \Big(\tau(P)^{-\frac{1}{p}}\|T^m E\|_p\Big)^{\frac{1}{m}} \\ &\leq 1. \end{aligned}$$

Thus $\mathrm{supp}(\mu_{T_{11}}) \subseteq \overline{B(0,1)}$.

Finally, to prove that $\mathrm{supp}(\mu_{T_{22}}) \subseteq \{z \in \mathbb{C} \,|\, |z| \geq 1\}$, define for fixed $m \in \mathbb{N}$ $H \in L^p(\mathcal{M},\tau)$ by

$$H = -\frac{1}{2\pi\mathrm{i}}\int_{\partial B(0,1)} (\lambda\mathbf{1} - T)^{-1}\lambda^{-m}\,\mathrm{d}\lambda \tag{6.28}$$

As above one may prove that the right hand side makes sense as a Riemann integral in $L^p(\mathcal{M},\tau)$. Moreover, $H=\lim_{n\to\infty} H_n$ in p-norm, where

$$H_n = -\frac{1}{n}\sum_{k=1}^{n}(\mathrm{e}^{\mathrm{i}s_k^{(n)}}\mathbf{1}-T)^{-1}\mathrm{e}^{\mathrm{i}(1-m)s_k^{(n)}},$$

with $s_k^{(n)}=\frac{2\pi k}{n}$. We are going to prove that

$$H_n = T^{-m}(1-E_n), \qquad (n>m), \tag{6.29}$$

and as in the above, (6.29) holds, as soon as

$$-\frac{1}{n}\sum_{k=1}^{n}(\mathrm{e}^{\mathrm{i}s_k^{(n)}}-z)^{-1}\mathrm{e}^{\mathrm{i}(1-m)s_k^{(n)}} = z^{-m}\Big(1-\frac{1}{n}\sum_{k=1}^{n}(\mathrm{e}^{\mathrm{i}s_k^{(n)}}\mathbf{1}-z)^{-1}\mathrm{e}^{\mathrm{i}s_k^{(n)}}\Big), \qquad (z\in\mathbb{C}). \tag{6.30}$$

According to (6.23),

$$z^{-m}\Big(1-\frac{1}{n}\sum_{k=1}^{n}(\mathrm{e}^{\mathrm{i}-s_k^{(n)}}\mathbf{1}-z)^{-1}\mathrm{e}^{\mathrm{i}s_k^{(n)}}\Big) = z^{-m}\Big(1+\frac{1}{q(z)}\Big) = \frac{z^{n-m}}{z^n-1}. \tag{6.31}$$

Since $\frac{z^{n-m}}{z^n-1}\to 0$ as $|z|\to\infty$ and

$$\operatorname{Res}\Big(\frac{z^{n-m}}{z^n-1};\mathrm{e}^{\mathrm{i}s_k^{(n)}}\Big) = \frac{\mathrm{e}^{(n-m)\mathrm{i}s_k^{(n)}}}{q'(\mathrm{e}^{\mathrm{i}s_k^{(n)}})} = \frac{1}{n}\mathrm{e}^{\mathrm{i}(1-m)s_k^{(n)}},$$

$\frac{z^{n-m}}{z^n-1}$ has the following partial fraction decomposition:

$$\frac{z^{n-m}}{z^n-1} = \frac{1}{n}\sum_{k=1}^{n}\mathrm{e}^{\mathrm{i}(1-m)s_k^{(n)}}\frac{1}{z-\mathrm{e}^{\mathrm{i}s_k^{(n)}}} = -\frac{1}{n}\sum_{k=1}^{n}\mathrm{e}^{\mathrm{i}(1-m)s_k^{(n)}}\frac{1}{\mathrm{e}^{\mathrm{i}s_k^{(n)}}-z} \tag{6.32}$$

Combining (6.31) with (6.32) we find that (6.30) holds, and hence $H_n=T^{-m}(1-E_n)$ when $n>m$. Taking the limit as $n\to\infty$ on both sides of (6.29) (w.r.t. the measure topology) we arrive at the identity

$$H = T^{-m}(\mathbf{1}-E). \tag{6.33}$$

As in the above this implies that there are constants $k_3,k_4>0$ such that

$$\|T^{-m}(\mathbf{1}-E)\|_p \le k_3+k_4 m, \qquad (m\in\mathbb{N}),$$

whence

$$\|(T^*)^{-m}(\mathbf{1}-E^*)\|_p = \|(\mathbf{1}-E)T^{-m}\|_p = \|T^{-m}(\mathbf{1}-E)\|_p \le k_3+k_4 m,$$

because T and E commute.

Clearly, $(\mathbf{1}-E^*)^2 = \mathbf{1}-E^*$, and $(T^*)^{-1}$ commutes with $\mathbf{1}-E^*$. As in the above this implies that $\overline{R(\mathbf{1}-E^*)}$ is $(T^*)^{-1}$-invariant and that

$$\|(T^*)^{-m}Q\|_p \leq k_3 + k_4 m, \tag{6.34}$$

where $Q \in \mathcal{M}$ denotes the projection onto $\overline{R(\mathbf{1}-E^*)}$. Now

$$\overline{R(\mathbf{1}-E^*)}^{\perp} = \overline{\ker(\mathbf{1}-E)} = \overline{\{x \in \mathcal{D}(E) \,|\, Ex = x\}} = \overline{R(E)} = \mathcal{K},$$

and hence

$$(T^*)^{-m}Q = Q(T^*)^{-m}Q = (Q(T^*)^{-1}Q)^m = (T_{22}^*)^{-m}. \tag{6.35}$$

As above it follows from (6.34) and (6.35) that

$$r'(T_{22}^{-1}) \leq \lim_{m\to\infty} \|T_{22}^{-m}\|_p^{\frac{1}{m}} = \lim_{m\to\infty} \|(T_{22}^*)^{-m}\|_p^{\frac{1}{m}} \leq 1,$$

i.e. $\mathrm{supp}(\mu_{T_{22}^{-1}}) \subseteq \{z \in \mathbb{C} \,|\, |z| \leq 1\}$. Since $T \in \mathcal{M}^{\Delta}$, it follows that $T_{22} \in (P^{\perp}\mathcal{M}P^{\perp})^{\Delta}$, and thus, by [HS, Proposition 2.16], $\mathrm{supp}(\mu_{T_{22}}) \subseteq \{z \in \mathbb{C} \,|\, |z| \geq 1\}$. ■

7 The invariant subspace problem relative to a II_1-factor

The purpose of the present section is to combine the results of the previous sections to give a proof of our main theorem:

7.1 Theorem. *For every $T \in \mathcal{M}$ and every Borel set $B \subseteq \mathbb{C}$ there is a largest closed T-invariant subspace $\mathcal{K} = \mathcal{K}_T(B)$ affiliated with $\mathcal{M}$, such that the Brown measure of $T|_{\mathcal{K}}$, $\mu_{T|_{\mathcal{K}}}$, is concentrated on B.*[3] *Moreover, $\mathcal{K}$ is hyperinvariant*[4] *for T, and if $P = P_T(B) \in \mathcal{M}$ denotes the projection onto $\mathcal{K}$, then*

(i) $\tau(P) = \mu_T(B)$,

(ii) the Brown measure of $P^{\perp}TP^{\perp}$, considered as an element of $P^{\perp}\mathcal{M}P^{\perp}$, is concentrated on $\mathbb{C} \setminus B$.

7.2 Corollary. *Let $T \in \mathcal{M}$, and suppose that μ_T is not concentrated on a singleton. Then there is a non-trivial subspace affiliated with $\mathcal{M}$ which is hyperinvariant for T.*

Proof. Take a Borel set $B \subseteq \mathbb{C}$ such that $\mu_T(B) > 0$ and $\mu_T(B^c) > 0$. Then with $P = P_T(B) \in \mathcal{M}$ as in Theorem 7.1, one has that P is hyperinvariant for T, and $\tau(P) = \mu_T(B) \in (0,1)$. Since τ is faithful, P must be non-trivial. ■

[3] If $\mathcal{K} = \{0\}$, then we define $\mu_{T|_{\mathcal{K}}} := 0$. If $\mathcal{K} \neq \{0\}$, then $\mu_{T|_{\mathcal{K}}}$ is computed relative to the II_1-factor $P\mathcal{M}P$, where $P \in \mathcal{M}$ denotes the projection onto $\mathcal{K}$.

[4] $\mathcal{K}$ is said to be *hyperinvariant* for T if it is invariant under every operator $S \in \{T\}'$.

First case: Spectral subspaces associated with the set $B(0,1)$).

Consider a fixed operator $T \in \mathcal{M}$. As in sections 4 and 5, we can choose a circular system $\{x,y\}$ in $L(\mathbb{F}_4)$ and embed $\mathcal{M}$ and $L(\mathbb{F}_4)$ into the free product $\mathcal{N} = \mathcal{M} * L(\mathbb{F}_4)$. This way $\{x,y\}$ becomes a circular system in the II_1–factor $\mathcal{N}$ such that $\mathcal{M}$ is free from $\{x,y\}$. We will denote the trace on $\mathcal{N}$ by τ as well. For each $n \in \mathbb{N}$ we define an operator $T_n \in L^p(\mathcal{N},\tau)$, $(0<p<1)$, by

$$T_n = T + \tfrac{1}{n}xy^{-1}. \tag{7.1}$$

Note that $\|T - T_n\|_p \to 0$ as $n \to \infty$. According to Theorem 6.6 and Theorem 5.1 we have:

7.3 Theorem. *For each $n \in \mathbb{N}$ there exists a projection $P_n \in \mathcal{N}$ such that*

(i) $P_nT_nP_n = T_nP_n$,

(ii) $\mu_{P_nT_nP_n}$ *(computed relative to $P_n\mathcal{N}P_n$) is concentrated on* $\overline{B(0,1)}$,

(iii) $\mu_{P_n^\perp T_n P_n^\perp}$ *(computed relative to $P_n^\perp\mathcal{N}P_n^\perp$) is concentrated on* $\mathbb{C}\setminus B(0,1)$.

Proof. We will show that T_n satisfies the assumptions of Theorem 6.6 with $\alpha = 1$ and $p \in (\frac{1}{2},\frac{2}{3})$. Clearly, $T_n \in \mathcal{N}^\Delta$, and by Theorem 5.1 (i),

$$(T_n - \lambda\mathbf{1})^{-1} = n(nT - n\lambda\mathbf{1} + z)^{-1} \in L^p(\mathcal{N},\tau)$$

for $p \in (0,1)$, $\lambda \in \mathbb{C}$. Moreover, by Theorem 5.1 (ii),

$$\begin{aligned}\|(T_n-\lambda\mathbf{1})^{-1} - (T_n-\mu\mathbf{1})^{-1}\|_p &= n\|(nT-n\lambda\mathbf{1}+z)^{-1} - (nT-n\mu\mathbf{1}+z)^{-1}\|_p \\ &\le n^2C_p^{(1)}|\lambda-\mu|\end{aligned}$$

for $p \in (0,\frac{2}{3})$ and $\lambda,\mu \in \mathbb{C}$. Thus, Theorem 7.3 follows from Theorem 6.6. ∎

7.4 Proposition. *Let $n \in \mathbb{N}$, and let P_n be as in Theorem 7.3. Then*

$$0 < \tau(P_n) < 1, \tag{7.2}$$

$$\mu_{T_n}(\partial B(0,1)) = 0, \tag{7.3}$$

$$\mu_{T_n}(B(0,1)) = \tau(P_n). \tag{7.4}$$

Moreover, for every Borel set $A \in \mathbb{B}(\mathbb{C})$,

$$\mu_{T_n}(A \cap B(0,1)) = \tau(P_n)\mu_{P_nT_nP_n}(A), \tag{7.5}$$

$$\mu_{T_n}(A \setminus B(0,1)) = \tau(P_n^\perp)\mu_{P_n^\perp T_nP_n^\perp}(A), \tag{7.6}$$

where the Brown measures of $P_nT_nP_n$ and $P_n^\perp T_nP_n^\perp$ are computed relative to $P_n\mathcal{N}P_n$ and $P_n^\perp\mathcal{N}P_n^\perp$, respectively.

Proof. According to Corollary 4.6, μ_{T_n} has a density φ_n w.r.t. Lebesgue measure on $\mathbb{C}$, and $\varphi_n(z) > 0$ for all $z \in \mathbb{C}$. Therefore $\operatorname{supp}(\mu_{T_n}) = \mathbb{C}$. Moreover,

$$\mu_{T_n} = \tau(P_n)\mu_{P_nT_nP_n} + \tau(P_n^\perp)\mu_{P_n^\perp T_nP_n^\perp} \tag{7.7}$$

(cf. Proposition 6.5). It then follows from (ii) and (iii) of Theorem 7.3 that (7.2) holds. Moreover, since $\partial B(0,1)$ is a nullset w.r.t. Lebesgue measure on $\mathbb{C}$, (7.3) holds. Then by (ii) of Theorem 7.3,

$$\begin{aligned}
\tau(P_n) &= \tau(P_n)\mu_{P_nT_nP_n}(B(0,1)) \\
&= \mu_{T_n}(B(0,1)) - \tau(P_n^\perp)\mu_{P_n^\perp T_nP_n^\perp}(B(0,1)) \\
&\overset{\text{Theorem 7.3 (iii)}}{=} \mu_{T_n}(B(0,1)).
\end{aligned}$$

(7.5) and (7.6) now follow from (7.7) and the fact that $\mu_{P_nT_nP_n}$ and $\mu_{P_n^\perp T_nP_n^\perp}$ are concentrated on $B(0,1)$ and $\mathbb{C}\setminus B(0,1)$, respectively. ∎

Now, take a free ultrafilter ω on $\mathbb{N}$, and let

$$\begin{aligned}
\mathcal{J}_\omega &= \{(x_n)_{n=1}^\infty \in \ell^\infty(\mathcal{N}) \mid \lim_\omega \|x_n\|_2 = 0\}, \\
\mathcal{N}^\omega &= \ell^\infty(\mathcal{N})/\mathcal{J}_\omega.
\end{aligned}$$

Moreover, let $\rho : \ell^\infty(\mathcal{N}) \to \mathcal{N}^\omega$ denote the quotient mapping, let $\tilde{T} = \rho((T)_{n=1}^\infty)$ denote the copy of T in $\mathcal{N}^\omega$, and define a projection $P \in \mathcal{N}^\omega$ by

$$P = \rho((P_n)_{n=1}^\infty).$$

Recall that the ultrapower $\mathcal{N}^\omega$ is a II$_1$-factor equipped with a faithful tracial state τ_ω given by

$$\tau_\omega(\rho(x)) = \lim_{n\to\omega} \tau(x_n), \qquad x = (x_n)_{n=1}^\infty \in \ell^\infty(\mathcal{N}).$$

For $0 < p < \infty$,

$$\|\rho(x)\|_p = \lim_{n\to\omega} \|x_n\|_p, \qquad x = (x_n)_{n=1}^\infty \in \ell^\infty(\mathcal{N}),$$

where the p–norm on the left–hand side is computed w.r.t. the trace τ_ω.

7.5 Proposition. *With P and $\tilde{T}$ as defined above, $P\tilde{T}P = \tilde{T}P$.*

Proof. For all $p \in (0,1)$,

$$\begin{aligned}
\|P\tilde{T}P - \tilde{T}P\|_p^p &= \lim_{n\to\omega} \|P_nTP_n - TP_n\|_p^p \\
&= \lim_{n\to\omega} \|(T-T_n)P_n + T_nP_n - P_nT_nP_n - P_n(T-T_n)P_n\|_p^p,
\end{aligned}$$

where

$$\begin{aligned}
\|(T-T_n)P_n + T_nP_n - P_nT_nP_n - P_n(T-T_n)P_n\|_p^p &\le 2\|T-T_n\|_p^p \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Since ω is a free ultrafilter, any convergent sequence converges along ω as well. Hence,

$$\|P\tilde{T}P - \tilde{T}P\|_p = \lim_{n\to\omega} \|P_nTP_n - TP_n\|_p = 0, \qquad 0 < p < 1,$$

and therefore $\tilde{T}P = P\tilde{T}P$. ■

7.6 Lemma. *Let $\mu, \mu_1, \mu_2, \ldots \in \mathrm{Prob}(\mathbb{C})$, and suppose that $\mu_n \to \mu$ weakly as $n \to \infty$. Then*

(i) For every open set $\mathcal{U} \subseteq \mathbb{C}$,

$$\mu(\mathcal{U}) \leq \liminf_{n\to\infty} \mu_n(\mathcal{U}).$$

(ii) For every closed set $F \subseteq \mathbb{C}$,

$$\mu(F) \geq \limsup_{n\to\infty} \mu_n(F).$$

(iii) For every Borel set $B \subseteq \mathbb{C}$ with $\mu(\partial B) = 0$,

$$\mu(B) = \lim_{n\to\infty} \mu_n(B).$$

Proof. This is standard. In order to prove (i), write $1_{\mathcal{U}}$ as the pointwise limit of an increasing sequence of non-negative functions $\phi_n \in C_c(\mathbb{C})$. To prove (ii), use (i) and the fact that $\mu(F) = 1 - \mu(F^c)$, where F^c is open. (iii) follows from (i) and (ii) with $\mathcal{U} = \mathrm{int}B$ (the interior of B) and $F = \overline{B}$. Note that $\mu(F) = \mu(\mathcal{U}) = \mu(B)$. ■

7.7 Proposition. *If $\mu_T(\partial B(0,1)) = 0$, then the sequence $(\tau(P_n))_{n=1}^{\infty}$ converges as $n \to \infty$. Moreover,*

$$\mu_T(B(0,1)) = \lim_{n\to\infty} \tau(P_n) = \tau_\omega(P). \tag{7.8}$$

Proof. According to Corollary 4.6, μ_{T_n}converges weakly to μ_T as n tends to infinity. Then by Proposition 7.4 and Lemma 7.6,

$$\begin{aligned} \mu_T(B(0,1)) &= \lim_{n\to\infty} \mu_{T_n}(B(0,1)) \\ &= \lim_{n\to\infty} \tau(P_n), \end{aligned}$$

and since ω is a free ultrafilter on $\mathbb{N}$, $\tau(P_n)$ converges along ω to $\mu_T(B(0,1))$ as well. Thus,

$$\tau_\omega(P) = \lim_{n\to\omega} \tau(P_n) = \mu_T(B(0,1)). \qquad ■$$

7.8 Lemma. *(i) Consider a (classical) probability space $(X, \mathcal{E}, \mu)$. For every $f \in \bigcup_{p>0} L^p(X, \mathcal{E}, \mu)$ one has that*

$$\exp\left\{ \int_X \log|f| \, \mathrm{d}\mu \right\} = \inf_{p>0} \|f\|_p = \lim_{p\to 0+} \|f\|_p,$$

where $\exp(-\infty) := 0$.

(ii) For every $S \in \bigcup_{p>0} L^p(\mathcal{N}, \tau)$,

$$\Delta(S) = \inf_{p>0} \|S\|_p = \lim_{p\to 0+} \|S\|_p.$$

Proof. (i) follows from [Ru, Section 3, exercise 5(d)]. (ii) now follows by application of (i) to $(\mathbb{R}, \mathbb{B}, \mu_{|S|})$ and $f = \mathrm{id}_{\mathbb{R}}$. ■

7.9 Lemma. *Suppose $\mu_T(\partial B(0,1)) = 0$ and that $\tau_\omega(P) \in (0,1)$. Then for every complex number λ,*

$$\lim_{n\to\omega} \Delta_{P_n \mathcal{N} P_n}(P_n T_n P_n - \lambda P_n) \leq \Delta_{P\mathcal{N}^\omega P}(P\tilde{T}P - \lambda P), \tag{7.9}$$

and

$$\lim_{n\to\omega} \Delta_{P_n^\perp \mathcal{N} P_n^\perp}(P_n^\perp T_n P_n^\perp - \lambda P_n^\perp) \leq \Delta_{P^\perp \mathcal{N}^\omega P^\perp}(P^\perp \tilde{T} P^\perp - \lambda P^\perp). \tag{7.10}$$

Proof. We consider the case $\lambda = 0$ only. The general case can be taken care of in the same way. According to Lemma 7.8

$$\Delta_{P_n \mathcal{N} P_n}(P_n T_n P_n) = \inf_{0<p<1} \|P_n T_n P_n\|_{L^p(P_n \mathcal{N} P_n)},$$

where

$$\|P_n T_n P_n\|^p_{L^p(P_n \mathcal{N} P_n)} = \frac{1}{\tau(P_n)} \|P_n T_n P_n\|_p^p.$$

It follows that

$$\Delta_{P_n \mathcal{N} P_n}(P_n T_n P_n) = \inf_{0<p<1} (\tau(P_n)^{-\frac{1}{p}} \|P_n T_n P_n\|_p), \tag{7.11}$$

and by the same argument,

$$\Delta_{P\mathcal{N}^\omega P}(P\tilde{T}P) = \inf_{0<p<1} (\tau_\omega(P)^{-\frac{1}{p}} \|P\tilde{T}P\|_p), \tag{7.12}$$

where

$$\tau_\omega(P)^{-\frac{1}{p}} = \lim_{n\to\omega} \tau(P_n)^{-\frac{1}{p}}, \tag{7.13}$$

and

$$\|P\tilde{T}P\|_p = \lim_{n\to\omega} \|P_n T P_n\|_p, \qquad 0 < p < 1.$$

Now,

$$\|P_n T P_n - P_n T_n P_n\|_p \leq \|T - T_n\|_p \to 0 \text{ as } n \to \omega,$$

and hence

$$\|P\tilde{T}P\|_p = \lim_{n\to\omega} \|P_n T P_n\|_p = \lim_{n\to\omega} \|P_n T_n P_n\|_p. \tag{7.14}$$

Combining now (7.12), (7.13) and (7.14), we obtain:

$$\Delta_{P\mathcal{N}^\omega P}(P\tilde{T}P) = \inf_{0<p<1} (\lim_{n\to\omega} \tau(P_n)^{-\frac{1}{p}} \|P_n T_n P_n\|_p).$$

According to (7.11), for every $p \in (0,1)$,

$$\lim_{n\to\omega} (\tau(P_n)^{-\frac{1}{p}} \|P_n T_n P_n\|_p) \geq \lim_{n\to\omega} \Delta_{P_n \mathcal{N} P_n}(P_n T_n P_n),$$

and it follows that

$$\Delta_{P\mathcal{N}^\omega P}(P\tilde{T}P) \geq \lim_{n\to\omega} \Delta_{P_n\mathcal{N}P_n}(P_nT_nP_n).$$

By similar arguments one can prove that (7.10) holds. ■

7.10 Lemma. *Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be sequences in $\mathbb{R}\cup\{-\infty\}$ and let $a,b\in\mathbb{R}$.*

(i) If

$$\begin{aligned} \limsup_{n\to\infty} a_n &\leq a, \\ \limsup_{n\to\infty} b_n &\leq b, \\ \liminf_{n\to\infty}(a_n+b_n) &\geq a+b, \end{aligned}$$

then

$$\lim_{n\to\infty} a_n = a \qquad \text{and} \qquad \lim_{n\to\infty} b_n = b.$$

(ii) (i) holds for convergence along a free ultrafilter ω on $\mathbb{N}$ as well.

Proof. (i) Let $\varepsilon > 0$. Then, eventually as $n\to\infty$, $a_n \leq a+\varepsilon$, $b_n \leq b+\varepsilon$, and $a_n+b_n \geq a+b-\varepsilon$. Hence,

$$a_n = (a_n+b_n) - b_n \geq a - 2\varepsilon$$

and

$$b_n = (a_n+b_n) - a_n \geq b - 2\varepsilon,$$

eventually as $n\to\infty$. It follows that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. (ii) follows in a similar way. ■

7.11 Proposition. *Suppose $\mu_T(\partial B(0,1)) = 0$ and $\tau_\omega(P)\in(0,1)$. Let $\lambda\in\mathbb{C}$ with $\Delta(T-\lambda\mathbf{1})>0$. Then*

$$\lim_{n\to\omega} \Delta_{P_n\mathcal{N}P_n}(P_nT_nP_n - \lambda P_n) = \Delta_{P\mathcal{N}^\omega P}(P\tilde{T}P - \lambda P), \tag{7.15}$$

and

$$\lim_{n\to\omega} \Delta_{P_n^\perp\mathcal{N}P_n^\perp}(P_n^\perp T_nP_n^\perp - \lambda P_n^\perp) = \Delta_{P^\perp\mathcal{N}^\omega P^\perp}(P^\perp\tilde{T}P^\perp - \lambda P^\perp). \tag{7.16}$$

Proof. We will apply Lemma 7.10 with

$$\begin{aligned} a_n &= \tau(P_n)\log\Delta_{P_n\mathcal{N}P_n}(P_nT_nP_n - \lambda P_n), \\ b_n &= \tau(P_n^\perp)\log\Delta_{P_n^\perp\mathcal{N}P_n^\perp}(P_n^\perp T_nP_n^\perp - \lambda P_n^\perp), \\ a &= \tau_\omega(P)\log\Delta_{P\mathcal{N}^\omega P}(P\tilde{T}P - \lambda P), \\ b &= \tau_\omega(P^\perp)\log\Delta_{P^\perp\mathcal{N}^\omega P^\perp}(P^\perp\tilde{T}P^\perp - \lambda P^\perp). \end{aligned}$$

Since $\tau(P_n)$ converges to $\tau_\omega(P)$ as $n \to \omega$, we get from Lemma 7.9 that

$$\lim_{n\to\omega} a_n \leq a \qquad \text{and} \qquad \lim_{n\to\omega} b_n \leq b.$$

Moreover, by Proposition 6.5,

$$a_n + b_n = \log \Delta(T_n - \lambda \mathbf{1}) \qquad \text{and} \qquad a + b = \log \Delta(T - \lambda \mathbf{1}).$$

Hence, by the proof of Corollary 4.8,

$$\lim_{n\to\omega}(a_n + b_n) = a + b.$$

$a, b \in \mathbb{R}$ because $\Delta(T - \lambda \mathbf{1}) > 0$. Then by Lemma 7.10 (ii),

$$\lim_{n\to\omega} a_n = a \qquad \text{and} \qquad \lim_{n\to\omega} b_n = b.$$

Since $\lim_{n\to\omega} \tau(P_n) = \tau_\omega(P) > 0$ and $\lim_{n\to\omega} \tau(P_n^\perp) = \tau_\omega(P^\perp) > 0$, this completes the proof. ∎

We will now prove that under the assumptions $\mu_T(\partial B(0,1)) = 0$ and $0 < \tau_\omega(P) < 1$,

$$\lim_{n\to\omega} \mu_{P_n T_n P_n} = \mu_{P\tilde{T}P}$$

and

$$\lim_{n\to\omega} \mu_{P_n^\perp T_n P_n^\perp} = \mu_{P^\perp \tilde{T} P^\perp}$$

(weak convergence in Prop($\mathbb{C}$)).

7.12 Lemma. *Suppose $\mu_T(\partial B(0,1)) = 0$ and $0 < \tau_\omega(P) < 1$. Define maps $\rho, \sigma : \mathbb{B}(\mathbb{C}) \to [0, \infty[$ by*

$$\rho(A) = \frac{1}{\tau_\omega(P)} \mu_T(A \cap B(0,1)), \tag{7.17}$$

$$\sigma(A) = \frac{1}{\tau_\omega(P^\perp)} \mu_T(A \setminus B(0,1)). \tag{7.18}$$

Then $\rho, \sigma \in$ Prob($\mathbb{C}$), and

$$\lim_{n\to\infty} \mu_{P_n T_n P_n} = \rho \qquad \text{and} \qquad \lim_{n\to\infty} \mu_{P_n^\perp T_n P_n^\perp} = \sigma$$

(weak convergence in Prob($\mathbb{C}$)*).*

Proof. $\rho, \sigma \in$ Prob($\mathbb{C}$) because $\tau_\omega(P) = \mu_T(B(0,1))$ and $\tau_\omega(P^\perp) = \mu_T(\mathbb{C} \setminus B(0,1))$ (cf. Proposition 7.7). Put

$$\rho_n = \mu_{P_n T_n P_n} \qquad \text{and} \qquad \sigma_n = \mu_{P_n^\perp T_n P_n^\perp}.$$

Then by (7.5) and (7.6),

$$\rho_n(A) = \frac{1}{\tau(P_n)} \mu_{T_n}(A \cap B(0,1)) \qquad \text{and} \qquad \rho_n(A) = \frac{1}{\tau(P_n^\perp)} \mu_{T_n}(A \setminus B(0,1)).$$

Hence, for every continuous function $\varphi : \mathbb{C} \to [0,1]$,

$$\int_{\mathbb{C}} \varphi \,\mathrm{d}\rho_n = \frac{1}{\tau(P_n)} \int_{B(0,1)} \varphi \,\mathrm{d}\mu_{T_n},$$

and by the definition of ρ,

$$\int_{\mathbb{C}} \varphi \,\mathrm{d}\rho = \frac{1}{\tau_\omega(P)} \int_{B(0,1)} \varphi \,\mathrm{d}\mu_T.$$

Choose $f_k \in C_c(\mathbb{C})$, $0 \le f_k \le 1$, such that $f_k \nearrow 1_{B(0,1)}$ as $k \to \infty$. Since $\mu_{T_n} \to \mu_T$ weakly as $n \to \infty$,

$$\begin{aligned}\liminf_{n\to\infty} \int_{\mathbb{C}} \varphi \,\mathrm{d}\rho_n &\ge \lim_{n\to\infty} \Big(\frac{1}{\tau(P_n)} \int_{\mathbb{C}} \varphi \cdot f_k \,\mathrm{d}\mu_{T_n} \Big) \\ &= \frac{1}{\tau_\omega(P)} \int_{\mathbb{C}} \varphi \cdot f_k \,\mathrm{d}\mu_T, \qquad k \in \mathbb{N}.\end{aligned}$$

Letting $k \to \infty$, we get that

$$\liminf_{n\to\infty} \int_{\mathbb{C}} \varphi \,\mathrm{d}\rho_n \ge \frac{1}{\tau_\omega(P)} \int_{\mathbb{C}} \varphi \,\mathrm{d}\mu_T = \int_{\mathbb{C}} \varphi \,\mathrm{d}\rho. \tag{7.19}$$

The same argument applied to the function $1 - \varphi$ gives

$$\liminf_{n\to\infty} \int_{\mathbb{C}} (1-\varphi) \,\mathrm{d}\rho_n \ge \int_{\mathbb{C}} (1-\varphi) \,\mathrm{d}\rho,$$

and hence

$$\limsup_{n\to\infty} \int_{\mathbb{C}} \varphi \,\mathrm{d}\rho_n \le \int_{\mathbb{C}} \varphi \,\mathrm{d}\rho. \tag{7.20}$$

Combining (7.19) and (7.20), we find that for every continuous function $\varphi : \mathbb{C} \to [0,1]$,

$$\lim_{n\to\infty} \int_{\mathbb{C}} \varphi \,\mathrm{d}\rho_n = \int_{\mathbb{C}} \varphi \,\mathrm{d}\rho,$$

proving that $\rho_n \to \rho$ weakly as $n \to \infty$. Since $\mu_{T_n}(\partial B(0,1)) = \mu_T(\partial B(0,1)) = 0$, a similar proof gives that $\sigma_n \to \sigma$ weakly as $n \to \infty$: Simply replace f_k in the above by $g_k \in C_c(\mathbb{C})$ such that $0 \le g_k \le 1$ and $g_k \nearrow 1_{\mathbb{C}\setminus\overline{B(0,1)}}$ as $k \to \infty$. ■

7.13 Proposition. *Suppose $\mu_T(\partial B(0,1)) = 0$ and $0 < \tau_\omega(P) < 1$, and let ρ, σ be as in Lemma 7.12. Then for all $\lambda \in \mathbb{C}$,*

$$\limsup_{n\to\infty} \log \Delta_{P_n \mathcal{N} P_n}(P_n T_n P_n - \lambda P_n) \le \int_{\mathbb{C}} \log|z-\lambda| \,\mathrm{d}\rho(z), \tag{7.21}$$

and

$$\limsup_{n\to\infty} \log \Delta_{P_n^\perp \mathcal{N} P_n^\perp}(P_n^\perp T_n P_n^\perp - \lambda P_n^\perp) \le \int_{\mathbb{C}} \log|z-\lambda| \,\mathrm{d}\sigma(z). \tag{7.22}$$

Moreover, if $\Delta(T-\lambda\mathbf{1})>0$, *then*

$$\lim_{n\to\infty}\log\Delta_{P_n\mathcal{N}P_n}(P_nT_nP_n-\lambda P_n)=\int_{\mathbb{C}}\log|z-\lambda|\,\mathrm{d}\rho(z), \tag{7.23}$$

and

$$\lim_{n\to\infty}\log\Delta_{P_n^\perp\mathcal{N}P_n^\perp}(P_n^\perp T_nP_n^\perp-\lambda P_n^\perp)=\int_{\mathbb{C}}\log|z-\lambda|\,\mathrm{d}\sigma(z). \tag{7.24}$$

Proof. At first note that by (7.17) and (7.18), $\mathrm{supp}(\rho),\mathrm{supp}(\sigma)\subseteq\mathrm{supp}(\mu_T)\subseteq\sigma(T)$, whence $\mathrm{supp}(\rho)$ and $\mathrm{supp}(\sigma)$ are compact. Therefore, the right–hand sides of (7.21) and (7.22) are well–defined. Let $\rho_n=\mu_{P_nT_nP_n}$ and $\sigma_n=\mu_{P_n^\perp T_nP_n^\perp}$. Then by Lemma 7.12,

$$\lim_{n\to\infty}\rho_n=\rho \qquad\text{and}\qquad \lim_{n\to\infty}\sigma_n=\sigma \tag{7.25}$$

(weak convergence in $\mathrm{Prob}(\mathbb{C})$). Note that (7.21) and (7.22) are equivalent to

$$\limsup_{n\to\infty}\int_{\mathbb{C}}\log|z-\lambda|\,\mathrm{d}\rho_n(z)\le\int_{\mathbb{C}}\log|z-\lambda|\,\mathrm{d}\rho(z) \tag{7.26}$$

and

$$\limsup_{n\to\infty}\int_{\mathbb{C}}\log|z-\lambda|\,\mathrm{d}\sigma_n(z)\le\int_{\mathbb{C}}\log|z-\lambda|\,\mathrm{d}\sigma(z), \tag{7.27}$$

respectively. Let $p\in(0,1)$. By [HS, Theorem 2.19],

$$\begin{aligned}\int|z|^p\,\mathrm{d}\rho_n(z) &\le \|P_nT_nP_n\|^p_{L^p(P_n\mathcal{N}P_n)}\\ &= \frac{1}{\tau(P_n)}\|P_nT_nP_n\|_p^p\\ &\le \frac{1}{\tau(P_n)}\|T+\tfrac{1}{n}xy^{-1}\|_p^p\\ &\le \frac{1}{\tau(P_n)}\big(\|T\|_p^p+\|xy^{-1}\|_p^p\big),\end{aligned}$$

and since ρ has compact support,

$$\int|z|^p\,\mathrm{d}\rho(z)<\infty.$$

Now, $\tau(P_n)\to\tau_\omega(P)$ as $n\to\infty$, and we can therefore choose a constant C_p depending only on p such that

$$\int|z|^p\,\mathrm{d}\rho(z)<C_p \qquad\text{and}\qquad \int|z|^p\,\mathrm{d}\rho_n(z)<C_p, \qquad n\in\mathbb{N}.$$

Then since $|z-\lambda|^p\le|z|^p+|\lambda|^p$,

$$\int|z-\lambda|^p\,\mathrm{d}\rho(z)\le C_p(\lambda) \qquad\text{and}\qquad \int|z-\lambda|^p\,\mathrm{d}\rho_n(z)\le C_p(\lambda),$$

where $C_p(\lambda) := C_p + |\lambda|^p$. We will prove (7.26) and (7.27) in the case $\lambda = 0$ only. For general λ the proof is essentially the same. For $1 < R < \infty$ define

$$\varphi_R(z) = \begin{cases} 0 & |z| \le 1, \\ \log|z|, & 1 < |z| \le R, \\ \log R, & |z| > R. \end{cases}$$

Then

$$\log^+|z| = \varphi_R(z) + \log^+\Big(\frac{|z|}{R}\Big). \tag{7.28}$$

φ_R is continuous and bounded. Hence, (7.25) implies that

$$\lim_{n\to\infty} \int_{\mathbb{C}} \varphi_R(z)\,\mathrm{d}\rho_n(z) = \int_{\mathbb{C}} \varphi_R(z)\,\mathrm{d}\rho(z). \tag{7.29}$$

Moreover, for $0 < p < 1$,

$$\begin{aligned} \int_{\mathbb{C}} \log^+\Big(\frac{|z|}{R}\Big)\,\mathrm{d}\rho(z) &\le \frac{1}{p}\int_{\mathbb{C}} \Big(\frac{|z|}{R}\Big)^p \mathrm{d}\rho(z) \\ &\le \frac{R^{-p}C_p}{p}. \end{aligned}$$

Similarly,

$$\int_{\mathbb{C}} \log^+\Big(\frac{|z|}{R}\Big)\,\mathrm{d}\rho_n(z) \le \frac{R^{-p}C_p}{p}.$$

Then by (7.28) and (7.29),

$$\limsup_{n\to\infty}\Big|\int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\rho_n(z) - \int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\rho(z)\Big| \le \frac{2R^{-p}C_p}{p}$$

for all $R > 1$, implying that

$$\lim_{n\to\infty} \int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\rho_n(z) = \int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\rho(z). \tag{7.30}$$

Now choose a sequence $(f_k)_{k=1}^\infty$ of compactly supported continuous functions on $\mathbb{C}$ such that $f_k \ge 0$ and $f_k \nearrow \log^-|z|$ as $k \to \infty$. Then as in the proof of Lemma 7.12,

$$\begin{aligned} \liminf_{n\to\infty} \int_{\mathbb{C}} \log^-|z|\,\mathrm{d}\rho_n(z) &\ge \lim_{n\to\infty}\int_{\mathbb{C}} f_k\,\mathrm{d}\rho_n \\ &= \int_{\mathbb{C}} f_k\,\mathrm{d}\rho. \end{aligned}$$

Letting $k \to \infty$, we find that

$$\liminf_{n\to\infty} \int_{\mathbb{C}} \log^-|z|\,\mathrm{d}\rho_n(z) \ge \int_{\mathbb{C}} \log^-|z|\,\mathrm{d}\rho(z). \tag{7.31}$$

Since $\log|z| = \log^+|z| - \log^-|z|$, (7.26) (with $\lambda = 0$) now follows from (7.30) and (7.31). (7.27) follows from a similar proof. This finishes the proof of (7.21) and (7.22).

Now, assume that $\Delta(T-\lambda\mathbf{1})>0$. We will apply Lemma 7.10 with

$$\begin{aligned} a_n &= \tau(P_n)\log\Delta_{P_n\mathcal{N}P_n}(P_nT_nP_n-\lambda P_n),\\ b_n &= \tau(P_n^\perp)\log\Delta_{P_n^\perp\mathcal{N}P_n^\perp}(P_n^\perp T_nP_n^\perp-\lambda P_n^\perp),\\ a &= \tau_\omega(P)\int_{\mathbb{C}}\log|z-\lambda|\,\mathrm{d}\rho(z),\\ b &= \tau_\omega(P^\perp)\int_{\mathbb{C}}\log|z-\lambda|\,\mathrm{d}\sigma(z). \end{aligned}$$

Then by (7.21), (7.22) and Proposition 7.7,

$$\limsup_{n\to\infty} a_n \le a \qquad \text{and} \qquad \limsup_{n\to\infty} b_n \le b,$$

and by Proposition 6.5,

$$a_n+b_n=\log\Delta(T_n-\lambda\mathbf{1}).$$

Since $\mu_T=\tau_\omega(P)\rho+\tau_\omega(P^\perp)\sigma$, we also have

$$a+b=\int_{\mathbb{C}}\log|z-\lambda|\,\mathrm{d}\mu_T(z)=\log\Delta(T-\lambda\mathbf{1}).$$

Hence,

$$\lim_{n\to\infty}(a_n+b_n)=a+b>-\infty.$$

In particular, $a,b\in\mathbb{R}$. Then by Lemma 7.10,

$$\lim_{n\to\infty} a_n = a \qquad \text{and} \qquad \lim_{n\to\infty} b_n = b.$$

Since $\lim_{n\to\infty}\tau(P_n)=\tau_\omega(P)\in(0,1)$, (7.23) now follows. A similar argument shows that (7.24) holds. ■

7.14 Proposition. *If $\mu_T(\partial B(0,1))=0$ and $0<\tau_\omega(P)<1$, then w.r.t. weak convergence in* $\mathrm{Prob}(\mathbb{C})$,

$$\lim_{n\to\infty}\mu_{P_nT_nP_n}=\mu_{P\tilde{T}P}, \tag{7.32}$$

and

$$\lim_{n\to\infty}\mu_{P_n^\perp T_nP_n^\perp}=\mu_{P^\perp\tilde{T}P^\perp}. \tag{7.33}$$

Moreover,

$$\mathrm{supp}(\mu_{P\tilde{T}P})\subseteq\overline{B(0,1)}$$

and

$$\mathrm{supp}(\mu_{P^\perp\tilde{T}P^\perp})\subseteq\mathbb{C}\setminus B(0,1).$$

Proof. Combining (7.23), (7.24), and Proposition 7.11, we find that when $\lambda\in\mathbb{C}$ and $\Delta(T-\lambda\mathbf{1})>0$, then

$$\log\Delta_{P\mathcal{N}^\omega P}(P\tilde{T}P-\lambda P)=\int_{\mathbb{C}}\log|z-\lambda|\,\mathrm{d}\rho(z), \tag{7.34}$$

and

$$\log \Delta_{P^{\perp}\mathcal{N}^{\omega}P^{\perp}}(P^{\perp}\tilde{T}P^{\perp} - \lambda P^{\perp}) = \int_{\mathbb{C}} \log|z-\lambda|\,\mathrm{d}\sigma(z). \tag{7.35}$$

The map $\lambda \mapsto \log \Delta(T - \lambda\mathbf{1})$ is subharmonic, and hence $\Delta(T-\lambda\mathbf{1}) > 0$ for a.e. $\lambda \in \mathbb{C}$ w.r.t. Lebesgue measure. Both sides of (7.34) and (7.35) are subharmonic functions of λ, and since two subharmonic functions which agree almost everywhere are identical, (7.34) and (7.35) hold for all $\lambda \in \mathbb{C}$, showing that

$$\rho = \mu_{P\tilde{T}P} \qquad \text{and} \qquad \sigma = \mu_{P^{\perp}\tilde{T}P^{\perp}}.$$

Then (7.32) and (7.33) follow from Lemma 7.12. Moreover, by the definition of ρ and σ in Lemma 7.12, it is clear that

$$\operatorname{supp}(\rho) \subseteq \overline{B(0,1)} \qquad \text{and} \qquad \operatorname{supp}(\sigma) \subseteq \mathbb{C} \setminus B(0,1). \qquad \blacksquare$$

7.15 Proposition. *Let $T \in \mathcal{M}$, and suppose that for some $r > 0$, $\mu_T(\partial B(0,r)) = 0$. Then there is a $\tilde{T}$-invariant projection $P_r \in \mathcal{N}^{\omega}$, such that* $\operatorname{supp}(\mu_{P_r\tilde{T}P_r}) \subseteq \overline{B(0,r)}$, $\operatorname{supp}(\mu_{P_r^{\perp}\tilde{T}P_r^{\perp}}) \subseteq \mathbb{C} \setminus B(0,r)$, *and*

$$\tau_{\omega}(P_r) = \mu_T(\overline{B(0,r)}).$$

Proof. If $\mu_T(B(0,r)) = 1$, take $P_r = \mathbf{1}$ and define $\mu_{P_r^{\perp}\tilde{T}P_r^{\perp}}$ to be 0. If $\mu_T(B(0,r)) = 0$, take $P_r = 0$ and define $\mu_{P_r\tilde{T}P_r}$ to be 0. If $0 < \mu_T(B(0,r)) < 1$, then $0 < \mu_{\frac{1}{r}T}(B(0,1)) < 1$, and we can take $P_r \in \mathcal{N}^{\omega}$ to be the $\frac{1}{r}\tilde{T}$–invariant projection found in the above with

$$\operatorname{supp}(\mu_{P_r\frac{1}{r}\tilde{T}P_r}) \subseteq \overline{B(0,1)} \qquad \text{and} \qquad \operatorname{supp}(\mu_{P_r^{\perp}\frac{1}{r}\tilde{T}P_r^{\perp}}) \subseteq \mathbb{C} \setminus B(0,1)$$

and with

$$\tau_{\omega}(P_r) = \mu_{\frac{1}{r}T}(B(0,1)) = \mu_{\frac{1}{r}T}(\overline{B(0,1)}).$$

Clearly, P_r satisfies the conditions listed in Proposition 7.15. $\blacksquare$

The T-invariant subspaces $(E(T,r))_{r>0}$ and $(F(T,r))_{r>0}$.

Recall from Section 3 that for each $r > 0$ one can define T-hyperinvariant subspaces $E(T,r)$ and $F(T,r)$, for which the projections $P_{E(T,r)}$ and $P_{F(T,r)}$ are independent of the representation of $\mathcal{M}$ on a Hilbert space. In particular, we may regard $E(T,r)$ and $F(T,r)$ as subspaces of the Hilbert space $\mathcal{H} = L^2(\mathcal{N}^{\omega}, \tau_{\omega})$, on which $\mathcal{N}^{\omega}$ acts.

7.16 Lemma. *Let $T \in \mathcal{M}$ and let $r > 0$. If $\mu_T(\partial B(0,r)) = 0$, then*

(i) $\operatorname{supp}(\mu_{T|_{E(T,r)}}) \subseteq \overline{B(0,r)}$,

(ii) $\operatorname{supp}(\mu_{T|_{F(T,r)}}) \subseteq \mathbb{C} \setminus B(0,r)$,

(iii) $\tau(P_{E(T,r)}) = \mu_T(\overline{B(0,r)})$,

(iv) $\tau(P_{F(T,r)}) = \mu_T(\mathbb{C} \setminus B(0,r))$,

(v) $E(T,r) = F(T^*,r)^\perp$,

(vi) $F(T,r) = E(T^*,r)^\perp$.

Proof. In the proof of (i)–(vi) we will consider $\mathcal{M}$ as a von Neumann algebra acting on $L^2(\mathcal{N}^\omega, \tau_\omega)$ and thus identify $T \in \mathcal{M}$ with $\tilde{T} \in \mathcal{N}^\omega$. Let P_r denote the projection from Proposition 7.15. Then

$$\mathrm{supp}(\mu_{P_rTP_r}) \subseteq \overline{B(0,r)},$$

$$\mathrm{supp}(\mu_{P_r^\perp TP_r^\perp}) \subseteq \mathbb{C} \setminus B(0,r),$$

and

$$\tau_\omega(P_r) = \mu_T(B(0,r)) = \mu_T(\overline{B(0,r)}).$$

For $A \subseteq \mathbb{C}$, let $A^* = \{\overline{z} \,|\, z \in A\}$. Then

$$\mathrm{supp}(\mu_{P_r^\perp T^* P_r^\perp}) \subseteq (\mathbb{C} \setminus B(0,r))^* = \mathbb{C} \setminus B(0,r).$$

Since $P_r(\mathcal{H})$ is T–invariant and $P_r(\mathcal{H})^\perp$ is T^*–invariant, it follows from Corollary 3.5 that

$$P_r(\mathcal{H}) \subseteq E(T,r) \qquad \text{and} \qquad P_r(\mathcal{H})^\perp \subseteq F(T^*,r). \tag{7.36}$$

Hence,

$$\tau(P_{E(T,r)}) \geq \tau_\omega(P_r) = \mu_T(\overline{B(0,r)}), \tag{7.37}$$

$$\tau(P_{F(T^*,r)}) \geq 1 - \tau_\omega(P_r) = \mu_T(\mathbb{C} \setminus B(0,r)). \tag{7.38}$$

Now choose a sequence $(s_n)_{n=1}^\infty$ from $]r,\infty[$ such that $\mu_T(\partial B(0,s_n)) = 0$ for all $n \in \mathbb{N}$ and $s_n \searrow r$ as $n \to \infty$. Applying (7.38) to s_n, we find that

$$\tau(F(T^*,s_n)) \geq \mu_T(\mathbb{C} \setminus B(0,s_n)), \qquad n \in \mathbb{N}.$$

Moreover, $E(T,r) \perp F(T^*,s_n)$ (cf. Lemma 3.6). Hence,

$$\tau(P_{E(T,r)}) \leq 1 - \tau(P_{F(T^*,s_n)}) \leq \mu_T(\mathbb{C} \setminus B(0,s_n)).$$

Letting $n \to \infty$, we see that

$$\tau(P_{E(T,r)}) \leq \mu_T(\mathbb{C} \setminus B(0,r)).$$

Thus the inequality (7.37) is an equality. Next, choose a sequence $(t_n)_{n=1}^\infty$ from $(0,r)$ such that $t_n \nearrow r$ as $n \to \infty$ and with $\mu_T(\partial B(0,t_n)) = 0$ for all n. Arguing as above, we find that

$$\tau(P_{F(T^*,r)}) \leq 1 - \tau(P_{E(T,t_n)}) \leq \mu_T(\mathbb{C} \setminus \overline{B(0,t_n)}).$$

Letting $n \to \infty$, we obtain

$$\tau(P_{F(T^*,r)}) \leq \mu_T(\mathbb{C} \setminus B(0,r)).$$

Hence, (7.38) is an equality too. With (7.37) and (7.38) being equalities it follows from (7.36) that

$$P_r(\mathcal{H}) = E(T,r) \qquad \text{and} \qquad P_r(\mathcal{H})^\perp = F(T^*,r).$$

Altogether, we have proven (i), (ii), and (v) for T and (ii), (iv), and (vi) with T replaced by T^*. Since $\mu_{T^*}(\partial B(0,r)) = \mu_T(\partial B(0,r)) = 0$, and since T was arbitrary, this finishes the proof. ∎

7.17 Remark. If $T \in \mathcal{M}$, and if E_1 and E_2 are closed, T-invariant subspaces affiliated with $\mathcal{M}$ with $E_1 \subseteq E_2 \neq \{0\}$, then $\operatorname{supp}(\mu_{T|_{E_1}}) \subseteq \operatorname{supp}(\mu_{T|_{E_2}})$. Indeed, E_1 is invariant under $T|_{E_2}$, and if P_i denotes the projection onto E_i, then E_1 is affiliated with $P_2\mathcal{M}P_2$, whence

$$\begin{aligned}\mu_{T|_{E_2}} &= \tau_2(P_1)\cdot\mu_{(T|_{E_2})|_{E_1}} + \tau_2(P_2-P_1)\cdot\mu_{(P_2-P_1)T|_{E_2}(P_2-P_1)} \\ &= \tau_2(P_1)\cdot\mu_{T|_{E_1}} + \tau_2(P_2-P_1)\cdot\mu_{(P_2-P_1)T|_{E_2}(P_2-P_1)},\end{aligned}$$

where $\tau_2 = \frac{1}{\tau(P_2)}\tau$ denotes the trace on $P_2\mathcal{M}P_2$. It follows that $\operatorname{supp}(\mu_{T|_{E_1}}) \subseteq \operatorname{supp}(\mu_{T|_{E_2}})$.

7.18 Lemma. *For $T \in \mathcal{M}$ and for arbitrary $r > 0$, (i), (ii), (iii), and (iv) of Lemma 7.16 hold. Moreover,*

(v') $E(T,r) = \bigcap_{s>r} F(T^*,s)^\perp \supseteq F(T^*,r)^\perp$,

(vi') $F(T,r) = \bigcap_{s<r} E(T^*,s)^\perp \supseteq E(T^*,r)^\perp$.

Proof. Choose sequences $(s_n)_{n=1}^\infty$ from (r,∞) and $(t_n)_{n=1}^\infty$ from $(0,r)$ such that $s_n \searrow r$, $t_n \nearrow r$, and with $\mu_T(\partial B(0,s_n)) = \mu_T(\partial B(0,t_n)) = 0$ for all n. According to Remark 7.17 and Lemma 7.16 (i) and (ii),

$$\begin{aligned}\operatorname{supp}(\mu_{T|_{E(T,r)}}) &= \bigcap_{n=1}^\infty \operatorname{supp}(\mu_{T|_{E(T,s_n)}}) \\ &\subseteq \bigcap_{n=1}^\infty \overline{B(0,s_n)} \\ &= \overline{B(0,r)}.\end{aligned}$$

Similarly, with the aid of the sequence $(t_n)_{n=1}^\infty$ we find that

$$\operatorname{supp}(\mu_{T|_{F(T,r)}}) \subseteq \mathbb{C}\setminus B(0,r).$$

According to Lemma 7.16 (iii) and (iv),

$$\begin{aligned}\tau(P_{E(T,s_n)}) &= \mu_T(\overline{B(0,s_n)}), \\ \tau(P_{F(T,t_n)}) &= \mu_T(\mathbb{C}\setminus B(0,t_n)).\end{aligned}$$

Letting $n \to \infty$ and applying Lemma 3.2 (b) we then have

$$\begin{aligned} \tau(P_{E(T,r)}) &= \mu_T(\overline{B(0,r)}), \\ \tau(P_{F(T,r)}) &= \mu_T(\mathbb{C} \setminus B(0,0)). \end{aligned}$$

By application of Lemma 7.16 (v),

$$E(T,r) = \bigcap_{n=1}^{\infty} E(T,s_n) = \bigcap_{n=1}^{\infty} F(T^*,s_n)^{\perp} \supseteq \bigcap_{s>r} F(T^*,s)^{\perp},$$

and then by Lemma 3.6,

$$E(T,r) = \bigcap_{s>r} F(T^*,s)^{\perp}.$$

Since $F(T^*,s) \subseteq F(T^*,r)$, $s > r$ (cf. Lemma 3.2 (b)), this proves (v'). (vi') is proven in a similar way using the sequence $(t_n)_{n=1}^{\infty}$. ■

7.19 Corollary. *For every $T \in \mathcal{M}$, every $\lambda \in \mathbb{C}$ and every $r > 0$ one has:*

(i) $E := E(T - \lambda\mathbf{1}, r)$ is the largest, closed T-invariant subspace affiliated with $\mathcal{M}$, such that $\operatorname{supp}(\mu_{T|_E}) \subseteq \overline{B(\lambda,r)}$.

(ii) $F := F(T - \lambda\mathbf{1}, r)$ is the largest, closed T-invariant subspace affiliated with $\mathcal{M}$, such that $\operatorname{supp}(\mu_{T|_F}) \subseteq \mathbb{C} \setminus B(\lambda,r)$.

Proof. It suffices to consider the case $\lambda = 0$, because $\mu_{T-\lambda\mathbf{1}|_E}$ and $\mu_{T-\lambda\mathbf{1}|_F}$ are the push-forward measures of $\mu_{T|_E}$ and $\mu_{T|_F}$, respectively, under the map $z \mapsto z - \lambda$. Moreover, because of Corollary 3.5 we only have to prove that $\operatorname{supp}(\mu_{T|_E}) \subseteq \overline{B(0,r)}$ and $\operatorname{supp}(\mu_{T|_F}) \subseteq \mathbb{C} \setminus B(0,r)$, and these properties follow from Lemma 7.18. ■

Spectral subspaces corresponding to closed sets.

7.20 Proposition. *For every $T \in \mathcal{M}$ and every closed set $F \subseteq \mathbb{C}$ there is a largest, closed T-invariant subspace $\mathcal{K} = \mathcal{K}_T(F)$ affiliated with $\mathcal{M}$, such that* $\operatorname{supp}(\mu_{T|_{\mathcal{K}}}) \subseteq F$. *Moreover, $\mathcal{K}$ is hyperinvariant for T.*

Proof. We may write $\mathbb{C} \setminus F$ as a union of countably many open balls $(B(\lambda_k, r_k))_{k=1}^{\infty}$:

$$\mathbb{C} \setminus F = \bigcup_{k=1}^{\infty} B(\lambda_k, r_k).$$

With

$$\mathcal{K} := \bigcap_{k=1}^{\infty} F(T - \lambda_k \mathbf{1}, r_k), \tag{7.39}$$

$\mathcal{K}$ is hyperinvariant for T, and according to Remark 7.17,

$$\operatorname{supp}(\mu_{T|_{\mathcal{K}}}) \subseteq \mathbb{C} \setminus B(\lambda_k, r_k)$$

for every $k \in \mathbb{N}$. Hence, $\operatorname{supp}(\mu_{T|_{\mathcal{K}}}) \subseteq F$.

In order to prove that $\mathcal{K}$ is the largest closed subspace of $\mathcal{H}$ having this property, assume that $\mathcal{K}'$ is a closed T-invariant subspace affiliated with $\mathcal{M}$, such that $\operatorname{supp}(\mu_{T|_{\mathcal{K}'}}) \subseteq F$. Then, for every $k \in \mathbb{N}$, $\operatorname{supp}(\mu_{T|_{\mathcal{K}'}}) \subseteq \mathbb{C} \setminus B(\lambda_k, r_k)$. Therefore, by Corollary 3.5, $\mathcal{K}' \subseteq F(T - \lambda_k \mathbf{1}, r_k)$, and it follows that $\mathcal{K}' \subseteq \mathcal{K}$. ∎

7.21 Definition. For $T \in \mathcal{M}$ and F a closed subset of $\mathbb{C}$ we denote by $P_T(F) \in W^*(T)$ the projection onto the subspace $\mathcal{K}_T(F)$ found in Proposition 7.20.

The following proposition is a trivial consequence of Proposition 7.20 and Remark 7.17:

7.22 Proposition. *For every $T \in \mathcal{M}$ one has that*

(i) $P_T(\emptyset) = 0$ *and* $P_T(\mathbb{C}) = \mathbf{1}$

(ii) If F_1 and F_2 are closed subsets of $\mathbb{C}$ with $F_1 \subseteq F_2$, then $P_T(F_1) \leq P_T(F_2)$

(iii) If $(F_i)_{i \in I}$ is a family of closed subsets of $\mathbb{C}$, then $P_T\big(\bigcap_{i \in I} F_i\big) = \bigwedge_{i \in I} P_T(F_i)$

7.23 Lemma. *For every $T \in \mathcal{M}$ and every closed subset F of $\mathbb{C}$,*

$$\tau(P_T(F)) \leq \mu_T(F). \tag{7.40}$$

Proof. This is an easy consequence of the fact that

$$\mu_T = \tau(P_T(F)) \cdot \mu_{P_T(F) T P_T(F)} + \tau(P_T(F)^{\perp}) \cdot \mu_{P_T(F)^{\perp} T P_T(F)^{\perp}}$$

with $\mu_{P_T(F) T P_T(F)}(F) = 1$. ∎

7.24 Proposition. *Let $T \in \mathcal{M}$. Then for every closed subset F of $\mathbb{C}$ with $\mu_T(\partial F) = 0$,*

(i) $\tau(P_T(F)) = \mu_T(F)$,

(ii) $\mathcal{K}_T(F) = \mathcal{K}_{T^*}(\overline{\mathbb{C} \setminus F^*})^{\perp}$.

Proof. As in the proof of Proposition 7.20, write $\mathbb{C} \setminus F$ as a union of countably many open balls $(B(\lambda_k, r_k))_{k=1}^{\infty}$:

$$\mathbb{C} \setminus F = \bigcup_{k=1}^{\infty} B(\lambda_k, r_k).$$

Then $\mathcal{K}_T(F)$ is given by

$$\mathcal{K}_T(F) = \bigcap_{k=1}^{\infty} F(T - \lambda_k \mathbf{1}, r_k).$$

By Proposition 7.22 (ii),

$$\mathcal{K}_{T^*}(\overline{\mathbb{C}\setminus F^*}) \supseteq \mathcal{K}_{T^*}(\overline{B(\lambda_k, r_k)}) = E(T^* - \lambda_k \mathbf{1}, r_k),$$

and then by Lemma 7.16 (v'),

$$\mathcal{K}_{T^*}(\overline{\mathbb{C}\setminus F^*}) \supseteq F(T - \lambda_k \mathbf{1}, r_k)^{\perp}.$$

It follows that

$$\mathcal{K}_T(F) = \bigcap_{k=1}^{\infty} F(T - \lambda_k \mathbf{1}, r_k) \supseteq \mathcal{K}_{T^*}(\overline{\mathbb{C}\setminus F^*})^{\perp}. \tag{7.41}$$

Then for the corresponding projections we have:

$$\tau(P_T(F)) \geq 1 - \tau(P_{T^*}(\overline{\mathbb{C}\setminus F^*}). \tag{7.42}$$

According to Lemma 7.23,

$$\tau(P_T(F)) \leq \mu_T(F), \tag{7.43}$$

and since $\mu_T(\partial F) = 0$,

$$1 - \tau(P_{T^*}(\overline{\mathbb{C}\setminus F^*})) \geq 1 - \mu_{T^*}(\overline{\mathbb{C}\setminus F^*}) = 1 - \mu_T(\mathbb{C}\setminus F) = \mu_T(F). \tag{7.44}$$

We deduce from (7.42), (7.43), and (7.44) that

$$\tau(P_T(F)) = \mu_T(F) = \tau(\mathbf{1} - P_{T^*}(\overline{\mathbb{C}\setminus F^*})).$$

Then by (7.41),

$$\mathcal{K}_T(F) = \mathcal{K}_{T^*}(\overline{\mathbb{C}\setminus F^*})^{\perp}. \qquad \blacksquare$$

7.25 Proposition. *For every $T \in \mathcal{M}$ and every closed subset F of $\mathbb{C}$, $\tau(P_T(F)) = \mu_T(F)$, and hence $\mu_{P_T(F)^{\perp} T P_T(F)^{\perp}}$ is concentrated on $\mathbb{C}\setminus F$.*

Proof. For $t > 0$ define

$$F_t = \Big\{ z \in \mathbb{C} \,|\, \mathrm{dist}(z, F) \leq \frac{1}{t} \Big\}.$$

The map $t \mapsto \mu_T(F_t) \in [0,1]$ is decreasing, and therefore it has at most countably many points of discontinuity. This entails that $\mu_T(\partial F_t) = 0$ for all but countably many $t > 0$. Choose $t_1 > t_2 > t_3 > \ldots$, such that $t_n \searrow 0$ as $n \to \infty$ and $\mu_T(\partial F_{t_n}) = 0$ for all $n \in \mathbb{N}$. Since $F_{t_n} \searrow \bigcap_{m=1}^{\infty} F_{t_m} = F$, we have that $P_T(F_{t_n}) \searrow P_T(F)$, and hence

$$\begin{aligned} \tau(P_T(F)) &= \lim_{n\to\infty} \tau(P_T(F_{t_n})) \\ &= \lim_{n\to\infty} \mu_T(F_{t_n}) \\ &= \mu_T(F), \end{aligned}$$

as claimed. Finally, since

$$\mu_T = \tau(P_T(F)) \cdot \mu_{P_T(F) T P_T(F)} + \tau(P_T(F)^{\perp}) \cdot \mu_{P_T(F)^{\perp} T P_T(F)^{\perp}},$$

where $\mu_{P_T(F)TP_T(F)}(F) = 1$, we conclude that

$$\tau(P_T(F)^{\perp}) \cdot \mu_{P_T(F)^{\perp}TP_T(F)^{\perp}}(F) = 0.$$

Hence, if $P_T(F) \neq \mathbf{1}$, then $\mu_{P_T(F)^{\perp}TP_T(F)^{\perp}}(F) = 0$. If $P_T(F) = \mathbf{1}$, then, by definition, $\mu_{P_T(F)^{\perp}TP_T(F)^{\perp}} = 0$, and this measure is trivially concentrated on $\mathbb{C} \setminus F$. ■

7.26 Lemma. *Let $T \in \mathcal{M}$, and let $P \in \mathcal{M}$ be a T-invariant projection. Then for every closed subset F of $\mathbb{C}$,*

$$\mathcal{K}_{T|_{P(\mathcal{H})}}(F) = \mathcal{K}_T(F) \cap P(\mathcal{H}). \tag{7.45}$$

Proof. Let $Q \in P\mathcal{M}P$ denote the projection onto $\mathcal{K}_{T|_{P(\mathcal{H})}}(F)$, and let $R = P_T(F) \wedge P$. We will prove that $Q \leq R$ and $R \leq Q$.

Clearly, $Q \leq P$. In order to see that $Q \leq P_T(F)$, recall that $P_T(F)$ is the largest projection with the properties

(i) $P_T(F)TP_T(F) = TP_T(F)$,

(ii) $\mu_{P_T(F)TP_T(F)}$ (computed relative to $P_T(F)\mathcal{M}P_T(F)$) is concentrated on F.

Since

$$QTQ = QTPQ = TPQ = TQ, \tag{7.46}$$

and $\mu_{QTQ} = \mu_{QTPQ}$ (computed relative to $Q\mathcal{M}Q$) is concentrated on F, we get that $Q \leq P_T(F)$, and hence $Q \leq R$.

Similarly, to prove that $R \leq Q$, prove that

(i') $RTPR = TPR$, i.e. $RTR = TR$,

(ii') $\mu_{RTPR} = \mu_{RTR}$ (computed relative to $R\mathcal{M}R$) is concentrated on F.

Note that if $P_T(F) = 0$, then $R \leq Q$, so we may assume that $P_T(F) \neq 0$. (i') holds, because $R(\mathcal{H}) = P(\mathcal{H}) \cap P_T(F)(\mathcal{H})$ is T-invariant when $P(\mathcal{H})$ and $P_T(F)(\mathcal{H})$ are T-invariant. In order to prove (ii'), at first note that $R(\mathcal{H})$ is $TP_T(F)$-invariant. Hence

$$\mu_{TP_T(F)} = \tau_1(R) \cdot \mu_{RTR} + \tau_1(R^{\perp}) \cdot \mu_{R^{\perp}TR^{\perp}}, \tag{7.47}$$

where $\tau_1 = \frac{1}{\tau(P_T(F))} \cdot \tau|_{P_T(F)\mathcal{M}P_T(F)}$. It follows that

$$\tau_1(R) \cdot \mu_{RTR}(F^c) \leq \mu_{TP_T(F)}(F^c) = 0, \tag{7.48}$$

and thus, if $R \neq 0$, then $\mu_{RTR}(F^c) = 0$, and (ii') holds. If $R = 0$, then $R \leq Q$ is trivially fulfilled. ■

The general case.

Proof of Theorem 7.1. Define

$$P_T(B) := \bigvee_{K \text{ compact},\ K\subseteq B} P_T(K) \tag{7.49}$$

and

$$\mathcal{K}_T(B) = P_T(B)(\mathcal{H}). \tag{7.50}$$

Then $\mathcal{K}_T(B)$ is T-hyperinvariant. Since $\mu_{T|_{\mathcal{K}_T(B)}}$ is a regular measure (cf. [Fo, Theorem 7.8]),

$$\mu_{T|_{\mathcal{K}_T(B)}}(B) = \sup\{\mu_{T|_{\mathcal{K}_T(B)}}(K) \mid K \text{ compact},\ K \subseteq B\}.$$

For every compact set $K \subseteq B$, $\mathcal{K}_T(K) \subseteq \mathcal{K}_T(B)$ is $T|_{\mathcal{K}_T(B)}$-invariant, so with $P = P_T(B)$,

$$\mu_{T|_{\mathcal{K}_T(B)}} = \tau_{P\mathcal{M}P}(P_T(K)) \cdot \mu_{T|_{\mathcal{K}_T(K)}} + \tau_{P\mathcal{M}P}(P - P_T(K)) \cdot \mu_{(P-P_T(K))T(P-P_T(K))}.$$

Therefore, by Proposition 7.25,

$$\mu_{T|_{\mathcal{K}_T(B)}}(K) = \frac{1}{\tau(P)} \cdot \tau(P_T(K)) \cdot \mu_{T|_{\mathcal{K}_T(K)}}(K) + 0 = \frac{1}{\tau(P)} \cdot \tau(P_T(K)).$$

$(P_T(K))_{K \text{ compact},\ K\subseteq B}$ is an increasing net of projections with SO-limit $P_T(B)$. Therefore,

$$\mu_{T|_{\mathcal{K}_T(B)}}(B) = \sup\left\{\frac{1}{\tau(P)} \cdot \tau(P_T(K)) \mid K \text{ compact},\ K \subseteq B\right\} = 1.$$

This shows that $\mu_{T|_{\mathcal{K}_T(B)}}$ is concentrated on B. Moreover, by similar arguments,

$$\begin{aligned}
\mu_T(B) &= \sup\{\mu_T(K) \mid K \text{ compact},\ K \subseteq B\} \\
&= \sup\{\tau(P_T(K)) \mid K \text{ compact},\ K \subseteq B\} \\
&= \tau(P_T(B)),
\end{aligned}$$

proving that (i) of Theorem 7.1 holds. (ii) then follows as in the foregoing proof.

Finally, suppose that $Q \in \mathcal{M}$ is a T-invariant projection and that $\mu_{T|_{Q(\mathcal{H})}}$ is concentrated on B. Then by Lemma 7.26 and Proposition 7.25,

$$\begin{aligned}
\tau_{Q\mathcal{M}Q}(P \wedge Q) &= \sup\{\tau_{Q\mathcal{M}Q}(P_T(K) \wedge Q) \mid K \text{ compact},\ K \subseteq B\} \\
&= \sup\{\tau_{Q\mathcal{M}Q}(P_{T|_{Q(\mathcal{H})}}(K)) \mid K \text{ compact},\ K \subseteq B\} \\
&= \sup\{\mu_{T|_{Q(\mathcal{H})}}(K) \mid K \text{ compact},\ K \subseteq B\} \\
&= \mu_{T|_{Q(\mathcal{H})}}(B) \\
&= 1.
\end{aligned}$$

Hence, $P \wedge Q = Q$, and we get that $Q \leq P$. ∎

7.27 Corollary. *For every Borelset* $B \subseteq \mathbb{C}$,

$$P_{T^*}(\mathbb{C} \setminus B^*) = \mathbf{1} - P_T(B),$$

where $B^* = \{\overline{z} \mid z \in B\}$.

Proof. Let $\mathcal{K} = \mathcal{K}_T(B)$ and $P = P_T(B)$ be as in Theorem 7.1. Then by Theorem 7.1 (ii), $\mathcal{K}^\perp$ is a closed T^*–invariant subspace, and the Brown measure of $P^\perp T^* P^\perp = (P^\perp T P^\perp)^*$ is concentrated on B^*. Hence,

$$P_{T^*}(\mathbb{C} \setminus B^*) \geq P^\perp = P_T(B)^\perp. \tag{7.51}$$

But

$$\begin{aligned} \tau(P_{T^*}(\mathbb{C} \setminus B^*)) &= \mu_{T^*}(\mathbb{C} \setminus B^*) \\ &= \mu_T(\mathbb{C} \setminus B) \\ &= 1 - \mu_T(B) \\ &= \tau(P_T(B)^\perp). \end{aligned}$$

Hence, equality must hold in (7.51). ■

7.28 Corollary. *Let* $T \in \mathcal{M}$*, let* $B \subseteq \mathbb{C}$ *be a Borel set, and let* $\mathcal{K}$ *be a closed,* T*–invariant subspace of* $\mathcal{H}$ *which is affiliated with* $\mathcal{M}$*. Then the following two conditions are equivalent:*

(i) $\mathcal{K} = \mathcal{K}_T(B)$,

(ii) $\mu_{P_\mathcal{K} T P_\mathcal{K}}$ *is concentrated on* B *and* $\mu_{P_\mathcal{K}^\perp T P_\mathcal{K}^\perp}$ *is concentrated on* $\mathbb{C} \setminus B$.

Proof. That (i) implies (ii) is a consequence of Theorem 7.1. Now, suppose that (ii) holds. Then $\mathcal{K} \subseteq \mathcal{K}_T(B)$. Moreover, $\mathcal{K}^\perp$ is T^*–invariant, and $\mu_{P_\mathcal{K}^\perp T^* P_\mathcal{K}^\perp}$ is concentrated on $(\mathbb{C} \setminus B)^* = \mathbb{C} \setminus B^*$. Therefore, $\mathcal{K}^\perp \subseteq \mathcal{K}_{T^*}(\mathbb{C} \setminus B^*) = \mathcal{K}_T(B)^\perp$. Hence, $\mathcal{K} = \mathcal{K}_T(B)$. ■

8 Realizing $P_{E(T,r)}$ and $P_{F(T,r)}$ as spectral projections

Recall from Section 3 that for every $T \in \mathcal{M}$ and every $r > 0$ we defined T-hyperinvariant subspaces $E(T,r)$ and $F(T,r)$. The aim of the present section is to show that the corresponding projections, $P_{E(T,r)}$ and $P_{F(T,r)}$, have the following property:

8.1 Theorem. *For every* $T \in \mathcal{M}$ *we have:*

(a) There is a unique operator $A \in \mathcal{M}^+$*, such that for every* $r > 0$,

$$P_{E(T,r)} = 1_{[0,r]}(A). \tag{8.1}$$

Moreover,

$$A = \mathrm{SO} - \lim_{n \to \infty} ((T^*)^n T^n)^{\frac{1}{2n}}. \tag{8.2}$$

(b) There is a unique operator $B \in \mathcal{M}^+$, such that for every $r > 0$,

$$P_{F(T,r)} = 1_{[r,\infty)}(B). \tag{8.3}$$

Moreover,

$$B = \mathrm{SO} - \lim_{n\to\infty} (T^n(T^*)^n)^{\frac{1}{2n}}. \tag{8.4}$$

In the proof of Theorem 8.1 we shall need the following two lemmas. The first one of them is elementary, and we omit the proof of it.

8.2 Lemma. *Let $P, P_1, P_2, \ldots$ be projections in $B(\mathcal{H})$. Then the following are equivalent:*

(i) $P_n \to P$ in the strong operator topology,

(ii) $\|P_n\xi - \xi\| \to 0$ for every $\xi \in P(\mathcal{H})$ and $\|P_n^\perp\eta - \eta\| \to 0$ for every $\eta \in P(\mathcal{H})^\perp$.

8.3 Lemma. *Let $A, A_1, A_2, \ldots$ be operators from $B(\mathcal{H})^+$, and assume that*

$$M := \max\{\|A\|, \sup_{n\in\mathbb{N}} \|A_n\|\} < \infty.$$

If

$$1_{[0,r]}(A_n) \to 1_{[0,r]}(A) \tag{8.5}$$

in the strong operator topology for all but countably many $r \in [0, M]$, then $A_n \to A$ in the strong operator topology.

Proof. We prove this, approximating A_n and A with linear combinations of projections of the form $1_{[0,r]}(A_n)$ and $1_{[0,r]}(A)$, respectively. Choose $\alpha > 0$ such that (8.5) holds for all $r \in \alpha\mathbb{Q}_+ \cap [0, M]$. For each $k \in \mathbb{N}$ define

$$B_k = \frac{\alpha}{k} \sum_{k=1}^{[kM\alpha^{-1}]} 1_{(\frac{n\alpha}{k},\infty)}(A),$$

$$B_{n,k} = \frac{\alpha}{k} \sum_{k=1}^{[kM\alpha^{-1}]} 1_{(\frac{n\alpha}{k},\infty)}(A_n).$$

Then, by the Borel functional calculus for normal operators,

$$\|A - B_k\| \leq \frac{\alpha}{k},$$

and

$$\|A_n - B_{n,k}\| \leq \frac{\alpha}{k}.$$

By assumption, for fixed $k \in \mathbb{N}$, $B_k = \mathrm{SO} - \lim_{n\to\infty} B_{n,k}$, and it follows that $A_n \to A$ in the strong operator topology. ∎

Proof of Theorem 8.1. According to Lemma 3.2 (b), if we define $E(T,0) := \bigcap_{r>0} E(T,r)$, then $r \mapsto P_{E(T,r)}$ is increasing and SO-continuous from the right. Moreover, according to Lemma 3.4, $P_{E(T,r)} = \mathbf{1}$ for every $r \geq r'(T)$. [KR1, Theorem 5.2.4] then implies that there is one and only one operator $A \in \mathcal{M}^+$, such that (8.1) holds. Moreover, $\|A\| \leq r'(T) \leq \|T\|$.

Now, take $r > 0$ such that $\mu_T(\partial B(0,r)) = 0$. Then by Lemma 7.16 (v),

(i) $E(T,r) = F(T^*,r)^\perp$.

Moreover, we claim that

(ii) $\overline{\bigcup_{0<s<r} E(T,s)} = E(T,r)$,

(iii) $\overline{\bigcup_{r<t<\infty} F(T^*,t)} = F(T^*,r)$.

The one inclusion $\subseteq$ in (ii) is obvious. On the other hand, since $\mu_T(\partial B(0,r)) = 0$, we get by Lemma 7.16 (iii),

$$\begin{aligned}
\tau(P_{\overline{\bigcup_{0<s<r} E(T,s)}}) &= \lim_{s\to r-} \tau(P_{E(T,s)}) \\
&= \lim_{s\to r-} \mu_T(\overline{B(0,s)}) \\
&= \mu_T(B(0,r)) \\
&= \tau(P_{E(T,r)}).
\end{aligned}$$

Hence, "=" holds in (ii). Similarly, "$\subseteq$" in (iii) is obvious. On the other hand, by Lemma 7.16 (v),

$$\begin{aligned}
\tau(P_{\overline{\bigcup_{r<t<\infty} F(T^*,t)}}) &= \lim_{t\to r+} \tau(P_{F(T^*,t)}) \\
&= \lim_{t\to r+} \tau(\mathbf{1} - P_{E(T,t)}) \\
&= \lim_{t\to r+} \mu_T(\mathbb{C} \setminus B(0,t)) \\
&= \mu_T(\mathbb{C} \setminus B(0,r)) \\
&= \tau(\mathbf{1} - P_{E(T,r)}) \\
&= \tau(P_{F(T^*,r)}).
\end{aligned}$$

Thus, (iii) holds.

Now, let $\xi \in E(T,r)$ and $\eta \in E(T,r)^\perp = F(T^*,r)$ with $\|\xi\| = \|\eta\| = 1$. Let $\varepsilon > 0$. According to (ii) and (iii), we may take $s \in (0,r)$, $t \in (r,\infty)$, $\xi' \in E(T,s)$ and $\eta' \in F(T^*,t)$ with $\|\xi'\| = \|\eta'\| = 1$, $\|\xi - \xi'\| < \frac{\varepsilon}{2}$ and $\|\eta - \eta'\| < \frac{\varepsilon}{2}$. Next choose ξ'_n and η'_n in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \|\xi'_n - \xi'\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|T^n \xi'_n\|^{\frac{1}{n}} \leq s,$$

and

$$\lim_{n\to\infty} \|(T^*)^n \eta'_n - \eta'\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|\xi'_n\|^{\frac{1}{n}} \leq \frac{1}{t}.$$

We can, without loss of generality, assume that $\|\xi_n'\| = \|(T^*)^n\eta_n'\| = 1$ for all $n \in \mathbb{N}$. Then let ρ_n (respectively σ_n) denote the distribution of $(T^*)^nT^n$ w.r.t the vector state on $\mathcal{M}$ induced by ξ_n' (respectively $(T^*)^n\eta_n'$). Arguing as in the proof of Lemma 3.4, we get that

$$\rho_n((r^{2n}, \infty)) \to 0 \quad \text{as} \quad n \to \infty,$$

and

$$\sigma_n([0, r^{2n}]) \to 0 \quad \text{as} \quad n \to \infty.$$

Put $A_n = ((T^*)^nT^n)^{\frac{1}{2n}}$. Then

$$\|1_{[0,r]}(A_n)\xi_n' - \xi_n\|^2 = \rho_n((r^{2n}, \infty)) \to 0 \quad \text{as} \quad n \to \infty,$$

and

$$\|1_{(r,\infty)}(A_n)(T^*)^n\eta_n' - (T^*)^n\eta_n'\|^2 = \sigma_n([0, r^{2n}]) \to 0 \quad \text{as} \quad n \to \infty.$$

Since $\|\xi_n' - \xi'\| \to 0$ and $\|\eta - \eta'\| < \frac{\varepsilon}{2}$, we get that

$$\|1_{[0,r]}(A_n)\xi - \xi\| < \varepsilon,$$

eventually as $n \to \infty$, and similarly one argues that

$$\|1_{]r,\infty[}(A_n)\eta - \eta\| < \varepsilon,$$

eventually as $n \to \infty$. Thus, with $P_n = 1_{[0,r]}(A_n)$ and $P = P_{E(T,r)} = 1_{[0,r]}(A)$ we have shown that

$$\lim_{n\to\infty} \|P_n\xi - \xi\| = 0$$

for all unit vectors $\xi \in P(\mathcal{H})$ and hence for all $\xi \in P(\mathcal{H})$. Similarly,

$$\lim_{n\to\infty} \|(\mathbf{1} - P_n)\eta - \eta\| = 0$$

for all $\eta \in P(\mathcal{H})^\perp$. It then follows from Lemma 8.2 that

$$\text{SO} - \lim_{n\to\infty} 1_{[0,r]}(A_n) = 1_{[0,r]}(A)$$

for all $r > 0$ with $\mu_T(\partial B(0, r)) = 0$. Then by Lemma 8.3 (with $M = \|T\|$), $A_n \to A$ as $n \to \infty$ in strong operator topology. This proves (a).

(b) According to (a) applied to T^*, the limit

$$B = \text{SO} - \lim_{n\to\infty} ((T^n)^*T^n)^{\frac{1}{2n}}$$

exists and is uniquely determined by

$$1_{[0,r]}(B) = P_{E(T^*,r)}, \qquad (r > 0).$$

Moreover, for all but countably many $r > 0$,

$$1_{[r,\infty)}(B) = 1_{(r,\infty)}(B),$$

and

$$F(T,r)^{\perp} = E(T^*,r).$$

Hence, for these r's,

$$1_{[r,\infty)}(B) = 1 - 1_{[0,r]}(B) = 1 - P_{E(T^*,r)} = P_{F(T,r)}.$$

Since $r \mapsto 1_{[r,\infty)}(B)$ and $r \mapsto P_{F(T,r)}$ are both SO-continuous from the left, it follows that

$$1_{[r,\infty)}(B) = P_{F(T,r)}$$

for all $r > 0$.

Conversely, if $B' \in \mathcal{M}^+$ and $1_{[r,\infty)}(B') = P_{F(T,r)}$ for all $r > 0$, then the argument above may be reversed to show that

$$1_{[0,r]}(B') = P_{E(T^*,r)}, \qquad (r > 0),$$

and hence, by the uniqueness in (a), $B = B'$. ■

8.4 Example. According to Theorem 8.1 , for every $T \in \mathcal{M}$, the sequence $(((T^*)^n T^n)^{\frac{1}{2n}})_{n=1}^{\infty}$ converges in strong operator topology. The following example due to Voiculescu (personal communication, 2004) shows that this is not the case for all $T \in B(\mathcal{H})$ when $\dim\mathcal{H} = \infty$. Indeed, let $T \in B(l^2(\mathbb{N}))$ be the weighted shift given by

$$Te_n = c_n e_{n+1}\,, \quad n \in \mathbb{N}\,,$$

where $(e_n)_{n=1}^{\infty}$ is the standard basis for $l^2(\mathbb{N})$, and $(c_n)_{n=1}^{\infty}$ is given by

$$c_n = \begin{cases} 1 & \text{if } \; 2^k \leq n < 2^{k+1}\,, \quad k \; \text{ even} \\ 2 & \text{if } \; 2^k \leq n < 2^{k+1}\,, \quad k \; \text{ odd} \end{cases}$$

Then

$$(T^*)^n T^n e_1 = \left(\prod_{i=1}^{n} c_i \right)^2 e_1\,,$$

and thus

$$((T^*)^n T^n)^{\frac{1}{2n}} e_1 = \left(\prod_{i=1}^{n} c_i \right)^{\frac{1}{n}} e_1\,.$$

Since

$$\limsup_{n\to\infty} \left(\prod_{i=1}^{n} c_i \right)^{\frac{1}{n}} = 2^{\frac{2}{3}}\,,$$

and

$$\liminf_{n\to\infty} \left(\prod_{i=1}^{n} c_i \right)^{\frac{1}{n}} = 2^{\frac{1}{3}}\,,$$

it follows that $(((T^*)^n T^n)^{\frac{1}{2n}})_{n=1}^{\infty}$ is not SO-convergent in $B(l^2(\mathbb{N}))$.

9 Local spectral theory and decomposability

A bounded operator T on a Hilbert space $\mathcal{H}$ is said to be *decomposable* (cf. [LN, Definition 1.1.1]), if for every open cover $\mathbb{C} = U \cup V$ of the complex plane, there are T-invariant closed subspaces $\mathcal{H}'$ and $\mathcal{H}''$ of $\mathcal{H}$, such that the spectra of the restrictions of T satisfy $\sigma(T|_{\mathcal{H}'}) \subseteq U$ and $\sigma(T|_{\mathcal{H}''}) \subseteq V$, and such that $\mathcal{H} = \mathcal{H}' + \mathcal{H}''$.

Given $T \in B(\mathcal{H})$, a *spectral capacity for* T is a mapping E from the set of closed subsets of $\mathbb{C}$ into the set of all closed, T-invariant subspaces of $\mathcal{H}$, such that

(i) $E(\emptyset) = \{0\}$ and $E(\mathbb{C}) = \mathcal{H}$,

(ii) $E(\overline{U}_1) + \cdots + E(\overline{U}_n) = \mathcal{H}$ for every (finite) open cover $\{U_1, \ldots, U_n\}$ of $\mathbb{C}$,

(iii) $E\Big(\bigcap_{n=1}^{\infty} F_n\Big) = \bigcap_{n=1}^{\infty} E(F_n)$ for every countable family $(F_n)_{n=1}^{\infty}$ of closed subsets of $\mathbb{C}$,

(iv) $\sigma(T|_{E(F)}) \subseteq F$ for every closed subset F of $\mathbb{C}$ (with the convention that $\sigma(T|_{\{0\}}) = \emptyset$).

Finally, given $T \in B(\mathcal{H})$ and $\xi \in \mathcal{H}$, the *local resolvent set*, $\rho_T(\xi)$, of T at ξ is the union of all open subsets U of $\mathbb{C}$, for which there exist holomorphic vector-valued functions $f_U : U \to \mathcal{H}$, such that $(T - \lambda\mathbf{1})f_U(\lambda) = \xi$ for all $\lambda \in U$. Note that according to Neumann's lemma,

$$\{z \in \mathbb{C} \,|\, |z| > \|T\|\} \subseteq \rho_T(\xi),$$

and therefore, $\sigma_T(\xi) := \mathbb{C} \setminus \rho_T(\xi)$, the *local spectrum* of T at ξ, is compact. For any subset A of $\mathbb{C}$, the corresponding *local spectral subspace* of T is

$$\mathcal{H}_T(A) = \{\xi \in \mathcal{H} \,|\, \sigma_T(\xi) \subseteq A\}. \tag{9.1}$$

It is not hard to see that $\mathcal{H}_T(A)$ is T-hyperinvariant.

Now, the three definitions given above, i.e. that of decomposability, that of a spectral capacity and that of a local spectral subspace, are closely related, as the following theorem indicates:

9.1 Theorem. *[LN, Proposition 1.2.23] Let T be a bounded operator on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) T is decomposable,

(ii) T has a spectral capacity,

(iii) for every closed subset F of $\mathbb{C}$, $\mathcal{H}_T(F)$ is closed and

$$\sigma((\mathbf{1} - p_T(F))T|_{\mathcal{H}_T(F)^{\perp}}) \subseteq \overline{\sigma(T) \setminus F},$$

where $p_T(F)$ denotes the projection onto $\mathcal{H}_T(F)$.

Moreover, if T is decomposable, then the map $F \mapsto \mathcal{H}_T(F)$ is the unique spectral capacity for T.

Now, if the operator T appearing in our generalized version of Theorem 1.1 is decomposable, how is the local spectral subspace $\mathcal{H}_T(B)$ related to the T-invariant subspace $\mathcal{K}_T(B)$ for $B \in \mathbb{B}(\mathbb{C})$?

9.2 Proposition. *If T is a decomposable operator in the II_1-factor $\mathcal{M}$, then for every $B \in \mathbb{B}(\mathbb{C})$, $\mathcal{K}_T(B) = \overline{\mathcal{H}_T(B)}$.*

Proof. Since $\sigma_T(\xi)$ is compact for every $\xi \in \mathcal{H}$, we have that

$$\mathcal{H}_T(B) = \bigcup_{K \subseteq B,\ K \text{ compact}} \mathcal{H}_T(K).$$

Moreover, by definition

$$\mathcal{K}_T(B) = \overline{\bigcup_{K \subseteq B,\ K \text{ compact}} \mathcal{K}_T(K)}.$$

Hence, it suffices to prove that for every compact set $K \subseteq \mathbb{C}$, $\mathcal{K}_T(K) = \mathcal{H}_T(K)$. Since T is decomposable, Theorem 9.1 implies that for every closed subset F of $\mathbb{C}$,

$$\sigma(T|_{\mathcal{H}_T(F)}) \subseteq F,$$

and

$$\sigma((\mathbf{1} - Q_T(F))T|_{\mathcal{H}_T(F)^\perp}) \subseteq \overline{\sigma(T) \setminus F},$$

where $Q_T(F) \in W^*(T)$ denotes the projection onto $\mathcal{H}_T(F)$. In particular,

$$\text{supp}(\mu_{T|_{\mathcal{H}_T(F)}}) \subseteq F,$$

and

$$\text{supp}(\mu_{T|_{(\mathbf{1}-Q_T(F))T|_{\mathcal{H}_T(F)^\perp}}}) \subseteq \overline{\sigma(T) \setminus F},$$

It follows that $Q_T(F) \leq P_T(F)$, where $P_T(F) \in W^*(T)$ denotes the projection onto $\mathcal{K}_T(F)$. Now,

$$\begin{aligned} \tau(P_T(F)) &= \mu_T(F) \\ &= \tau(Q_T(F))\mu_{T|_{\mathcal{H}_T(F)}}(F) + \tau(Q_T(F)^\perp)\mu_{(\mathbf{1}-Q_T(F))T|_{\mathcal{H}_T(F)^\perp}}(F), \end{aligned}$$

and since $F \cap \overline{\sigma(T) \setminus F} \subseteq \partial F$, we get that

$$\tau(P_T(F)) \leq \tau(Q_T(F)) + \tau(Q_T(F)^\perp)\mu_{(\mathbf{1}-Q_T(F))T|_{\mathcal{H}_T(F)^\perp}}(\partial F).$$

Hence, if $\mu_T(\partial F) = 0$, then $\tau(P_T(F)) \leq \tau(Q_T(F))$, and it follows that $P_T(F) = Q_T(F)$.

For a general closed subset F of $\mathbb{C}$, define

$$F_t = \left\{ z \in \mathbb{C} \,\middle|\, \text{dist}(z, F) \leq \frac{1}{t} \right\}, \qquad (t > 0).$$

Then $F_t \searrow F$ as $t \nearrow \infty$. Take $0 < t_1 \leq t_2 \leq \ldots$, such that $t_n \to \infty$ and such that for all $n \in \mathbb{N}$, $\mu_T(\partial F_{t_n}) = 0$. Then, since $F \mapsto Q_T(F)$ is a spectral capacity for T, we have:

$$\begin{aligned} Q_T(F) &= \bigcap_{n=1}^{\infty} Q_T(F_{t_n}) \\ &= \bigcap_{n=1}^{\infty} P_T(F_{t_n}) \\ &= P_T(F). \qquad \blacksquare \end{aligned}$$

9.3 Corollary. *If $T \in \mathcal{M}$ is decomposable, then* $\mathrm{supp}(\mu_T) = \sigma(T)$.

Proof. We know from [Br] that $\mathrm{supp}(\mu_T) \subseteq \sigma(T)$ always holds. Now, let $T \in \mathcal{M}$ be decomposable, and let $F = \mathrm{supp}(\mu_T)$. Then $\mathcal{K}_T(F) = \mathcal{H}$. Hence, by Proposition 9.2 and Theorem 9.1,

$$\mathcal{H}_T(F) = \overline{\mathcal{H}_T(F)} = \mathcal{K}_T(F) = \mathcal{H}.$$

Therefore, by condition (iv) in the definition of a spectral capacity,

$$\sigma(T) = \sigma(T|_{\mathcal{H}_T(F)}) \subseteq F = \mathrm{supp}(\mu_T). \qquad \blacksquare$$

9.4 Corollary. *Every II_1–factor $\mathcal{M}$ contains a non–decomposable operator.*

Proof. According to [DH1, Example 6.6], $\mathcal{N} = \bigoplus_{k=2}^{\infty} B(\mathbb{C}^k)$ contains an operator T for which $\sigma(T) = \overline{\mathbb{D}}$ and $\mu_T = \delta_0$. Since $\mathcal{N}$ embeds into the hyperfinite II_1–factor $\mathcal{R}$, and since every II_1–factor contains $\mathcal{R}$ as a von Neumann subalgebra, $\mathcal{M}$ contains a copy of T. According to Corollary 9.3, T is not decomposable. $\blacksquare$

9.5 Remark. By [DH2], Voiculescu's circular operator is decomposable. More generally, every DT–operator is decomposable.

10 Appendix: Proof of Theorem 6.2

Let $0 < p < 1$ and consider a fixed map $f : [a, b] \to L^p(\mathcal{M}, \tau)$ which is Hölder continuous with exponent $\alpha > \frac{1-p}{p}$. That is, f satisfies (6.2) for some positive constant C. For every closed subinterval $[c, d]$ of $[a, b]$ we let $\int_c^d f(x)\,\mathrm{d}x$ be given by Definition 6.1.

10.1 Lemma. *Let $a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$ and $a = y_0 < y_1 < \cdots < y_{n-1} < y_n = b$ be partitions of the interval $[a, b]$, and define*

$$T_x = \sum_{i=1}^{m} f(x_i)(x_i - x_{i-1}),$$

$$T_y = \sum_{j=1}^{n} f(y_j)(y_j - y_{j-1}).$$

Then with $\delta_x = \max_{1 \le i \le m}(x_i - x_{i-1})$ *and* $\delta_y = \max_{1 \le j \le n}(y_j - y_{j-1})$ *one has that*

$$\|T_x - T_y\|_p^p \le C^p (m+n) \max\{\delta_x, \delta_y\}^{p+\alpha p}. \tag{10.1}$$

Proof. Let $a = z_0 < z_1 < \cdots < z_{r-1} < z_r = b$ be the paron of the interval $[a,b]$ containing all of the points $x_1, \ldots, x_{m-1}$ and $y_1, \ldots, y_{n-1}$. For $1 \le k \le r$, let $i(k) \in \{1, \ldots, m\}$ and $j(k) \in \{1, \ldots, n\}$ be those indices for which

$$[z_{k-1}, z_k] \subseteq [x_{i(k)-1}, x_{i(k)}]$$

and

$$[z_{k-1}, z_k] \subseteq [y_{j(k)-1}, y_{j(k)}].$$

Then

$$T_x = \sum_{k=1}^{r} f(x_{i(k)})(z_k - z_{k-1}), \tag{10.2}$$

and

$$T_y = \sum_{k=1}^{r} f(y_{j(k)})(z_k - z_{k-1}). \tag{10.3}$$

Since $z_k \in [x_{i(k)-1}, x_{i(k)}] \cap [y_{j(k)-1}, y_{j(k)}]$, both of the points $x_{i(k)}$ and $y_{j(k)}$ must belong to the interval $[z_k, z_k + \max\{\delta_x, \delta_y\}]$, and it follows that

$$\|f(x_{i(k)}) - f(y_{j(k)})\|_p \le C \max\{\delta_x, \delta_y\}^{\alpha}. \tag{10.4}$$

Combining (10.2), (10.3) and (10.4) we find that

$$\|T_x - T_y\|_p^p \le \sum_{k=1}^{r} (z_k - z_{k-1})^p C^p \max\{\delta_x, \delta_y\}^{\alpha p} \le (m+n) C^p \max\{\delta_x, \delta_y\}^{\alpha p + p}. \qquad \blacksquare$$

10.2 Lemma. *For every* $c \in (a,b)$,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x. \tag{10.5}$$

Proof. Let $(S_n)_{n=1}^{\infty}$, $(S_n^{(1)})_{n=1}^{\infty}$ and $(S_n^{(2)})_{n=1}^{\infty}$ be the sequences defining $\int_a^b f(x)\,\mathrm{d}x$, $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$, respectively (cf. Definition 6.1). That is, with

$$\begin{aligned} x_i &= a + i\frac{b-a}{2^n}, \qquad (i = 0, \ldots, 2^n), \\ y_j &= a + j\frac{c-a}{2^n}, \qquad (j = 0, \ldots, 2^n), \\ y_j &= c + (j - 2^n)\frac{b-c}{2^n}, \qquad (j = 2^n + 1, \ldots, 2 \cdot 2^n), \end{aligned}$$

one has:

$$\begin{aligned} S_n &= \sum_{i=1}^{2^n} f(x_i)(x_i - x_{i-1}), \\ S_n^{(1)} + S_n^{(2)} &= \sum_{j=1}^{2\cdot 2^n} f(y_j)(y_j - y_{j-1}). \end{aligned}$$

Then with $\delta_x = \max_{1\le i\le 2^n}(x_i - x_{i-1}) = \frac{b-a}{2^n}$ and $\delta_y = \max_{1\le j\le 2\cdot 2^n}(y_j - y_{j-1}) \le \frac{b-a}{2^n}$ we get from Lemma 10.1 that

$$\begin{aligned} \|S_n - (S_n^{(1)} + S_n^{(2)})\|_p^p &\le 3\cdot 2^n C^p\Big(\frac{b-a}{2^n}\Big)^{p+\alpha p} \\ &= 3C^p(b-a)^{p+\alpha p}2^{-n(p+\alpha p-1)}, \end{aligned}$$

and since $p+\alpha p - 1 > 0$, we conclude that $\|S_n - (S_n^{(1)} + S_n^{(2)})\|_p \to 0$ as $n\to\infty$. ■

10.3 Lemma.

$$\int_{-b}^{-a} f(-x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x. \tag{10.6}$$

Proof. $\int_{-b}^{-a} f(-x)\,\mathrm{d}x = \lim_{n\to\infty} S_n'$, where

$$\begin{aligned} S_n' &= \frac{b-a}{2^n}\sum_{k=1}^{2^n} f\Big(b - k\frac{b-a}{2^n}\Big) \\ &= \frac{b-a}{2^n}\sum_{k=1}^{2^n} f\Big(a + (2^n-k)\frac{b-a}{2^n}\Big) \\ &= \frac{b-a}{2^n}\sum_{l=0}^{2^n-1} f\Big(a + l\frac{b-a}{2^n}\Big), \qquad (n\in\mathbb{N}_0). \end{aligned}$$

Hence

$$\|S_n - S_n'\|_p = \frac{b-a}{2^n}\|f(b)-f(a)\|_p \to 0 \quad \text{as } n\to\infty,$$

and it follows that

$$\int_{-b}^{-a} f(-x)\,\mathrm{d}x = \lim_{n\to\infty} S_n' = \lim_{n\to\infty} S_n = \int_a^b f(x)\,\mathrm{d}x. \qquad ■$$

10.4 Lemma. *For every $c\in[a,b]$,*

$$\Big\|f(c) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\Big\|_p^p \le \frac{C^p(b-a)^{p+\alpha p}}{\alpha p + p - 1}. \tag{10.7}$$

Proof. At first we consider the case $c = b$. Taking (6.4) into account we obtain:

$$\begin{aligned}\left\|(b-a)f(b) - \int_a^b f(x)\,\mathrm{d}x\right\|_p^p &= \lim_{n\to\infty} \|S_0 - S_n\|_p^p \\ &\le \sum_{n=1}^{\infty} \|S_n - S_{n-1}\|_p^p \\ &\le \frac{C^p (b-a)^{p+\alpha p}}{2} \sum_{n=1}^{\infty} 2^{-n(p+p\alpha-1)},\end{aligned}$$

where

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{-n(p+p\alpha-1)} &= \frac{1}{2^{p+p\alpha-1}-1} \\ &= \frac{1}{\mathrm{e}^{\log 2(p+p\alpha-1)}-1} \\ &\le \frac{1}{\log 2(p+p\alpha-1)} \\ &\le \frac{2}{p+p\alpha-1}.\end{aligned}$$

Hence

$$\left\|(b-a)f(b) - \int_a^b f(x)\,\mathrm{d}x\right\|_p^p \le \frac{C^p (b-a)^{p+\alpha p}}{p+p\alpha-1},$$

and then by Lemma 10.3,

$$\left\|(b-a)f(a) - \int_a^b f(x)\,\mathrm{d}x\right\|_p^p = \left\|(b-a)f(-(-a)) - \int_{-b}^{-a} f(-x)\,\mathrm{d}x\right\|_p^p \le \frac{C^p (b-a)^{p+\alpha p}}{p+p\alpha-1}.$$

It follows now that for arbitrary $c \in (a,b)$,

$$\left\|(c-a)f(c) - \int_a^c f(x)\,\mathrm{d}x\right\|_p^p \le \frac{C^p (c-a)^{p+\alpha p}}{p+p\alpha-1} \tag{10.8}$$

and

$$\left\|(b-c)f(c) - \int_c^b f(x)\,\mathrm{d}x\right\|_p^p \le \frac{C^p (b-c)^{p+\alpha p}}{p+p\alpha-1}, \tag{10.9}$$

whence

$$\begin{aligned}\left\|(b-a)f(c) - \int_a^b f(x)\,\mathrm{d}x\right\|_p^p &= \left\|(b-c)f(c) + (c-a)f(c) - \int_a^c f(x)\,\mathrm{d}x - \int_c^b f(x)\,\mathrm{d}x\right\|_p^p \\ &\le \left\|(b-c)f(c) - \int_a^c f(x)\,\mathrm{d}x\right\|_p^p + \left\|(c-a)f(c) - \int_a^c f(x)\,\mathrm{d}x\right\|_p^p \\ &\le \frac{C^p (c-a)^{p+\alpha p}}{p+p\alpha-1} + \frac{C^p (b-c)^{p+\alpha p}}{p+p\alpha-1}.\end{aligned}$$

Since $p + \alpha p > 1$,

$$(c-a)^{p+\alpha p} + (b-c)^{p+\alpha p} \le (b-a)^{p+\alpha p},$$

and (10.7) follows. ∎

Proof of Theorem 6.2. According to Lemma 10.2,

$$\begin{aligned} \Big\| M - \int_a^b f(x)\,\mathrm{d}x \Big\|_p^p &= \Big\| \sum_{i=1}^n \Big[f(t_i)(x_i - x_{i-1}) - \int_{x_{i-1}}^{x_i} f(x)\,\mathrm{d}x \Big] \Big\|_p^p \\ &\le \sum_{i=1}^n \Big\| f(t_i)(x_i - x_{i-1}) - \int_{x_{i-1}}^{x_i} f(x)\,\mathrm{d}x \Big\|_p^p, \end{aligned}$$

and by Lemma 10.4,

$$\begin{aligned} \Big\| f(t_i)(x_i - x_{i-1}) - \int_{x_{i-1}}^{x_i} f(x)\,\mathrm{d}x \Big\|_p^p &\le \frac{C^p (x_i - x_{i-1})^{p+\alpha p}}{p + \alpha p - 1} \\ &= \frac{C^p (x_i - x_{i-1})(x_i - x_{i-1})^{p+\alpha p-1}}{p + \alpha p - 1} \\ &\le \frac{C^p (x_i - x_{i-1})\delta(I)^{p+\alpha p-1}}{p + \alpha p - 1}. \end{aligned}$$

Consequently,

$$\Big\| M - \int_a^b f(x)\,\mathrm{d}x \Big\|_p^p \le \frac{C^p}{p + \alpha p - 1} \sum_{i=1}^n (x_i - x_{i-1})\delta(I)^{p+\alpha p-1} = \frac{C^p (b-a)\delta(I)^{p+\alpha p-1}}{p + \alpha p - 1}. \qquad ∎$$

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