Approximation of semigroups generated by differential operators associated with Markov operators

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joint work with
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*On differential operators associated with Markov operators*,
Given

- a convex compact subset $K$ of $\mathbb{R}^d$ ($d \geq 1$) with non-empty interior
- a Markov operator $T$ on $C(K)$ (i.e., a positive linear operator $T$ on $C(K)$ such that $T(1) = 1$, $1$ being the constant function of value 1)

it is possible to associate with $T$ an elliptic second-order differential operator $W_T$, defined by setting, for every $u \in C^2(K)$,

$$W_T(u) := \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \quad (1)$$

where, for each $i, j = 1, \ldots, d$ and $x \in K$,

$$\alpha_{ij}(x) := T(pr_i pr_j)(x) - (pr_i pr_j)(x), \quad (2)$$

$pr_i$ being the $i$-th coordinate function (i.e., $pr_i(x) = x_i$ for every $x \in K$).
Differential operators associated with Markov operators

\[ W_T(u) := \frac{1}{2} \sum_{i,j=1}^{d} ( T(pr_i pr_j) - pr_i pr_j ) \frac{\partial^2 u}{\partial x_i \partial x_j} \]  

Difficulties

- The boundary \( \partial K \) of \( K \) is generally non-smooth, due to the presence of possible sides and corners.
- \( W_T \) degenerates on the set

\[ \partial_T K := \{ x \in K \mid T(f)(x) = f(x) \text{ for every } f \in \mathcal{C}(K) \} \]

(4)

of all interpolation points for \( T \) which contains the set \( \partial_e K \) of the extreme points of \( K \) if, in addition,

\[ T(h) = h \quad \text{for every } h \in \{ pr_1, \ldots, pr_d \}. \]

(5)
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**Interests**

Operators of the form (3) are of concern in the study of several diffusion problems arising in biology, financial mathematics and other fields.

**Our main aim**

Proving that, under suitable hypotheses on \( T \), the operator \((W_T, \mathcal{C}^2(K))\) is closable and its closure generates a Markov semigroup \((T(t))_{t \geq 0}\) on \( \mathcal{C}(K) \).

\[ W_T \quad \longrightarrow \quad (B_n)_{n \geq 1} \]

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\[ W_T \quad \rightarrow \quad \text{via asymptotic formula} \quad (B_n)_{n \geq 1} \]
Trotter-Schnabl-type theorem

Let \((L_n)_{n\geq 1}\) be a sequence of linear contractions \(a\) on a Banach space \((E, \| \cdot \|)\) over \(\mathbb{R}\) or \(\mathbb{C}\) and let \((\rho(n))_{n\geq 1}\) be a sequence of positive real numbers such that \(\lim_{n\to \infty} \rho(n) = 0\). Let \((A_0, D_0)\) be a linear operator defined on a subspace \(D_0\) of \(E\) and assume that

(i) there exists a family \((E_i)_{i \in I}\) of finite dimensional subspaces of \(D_0\) which are invariant under \(L_n\) and whose union \(\bigcup_{i \in I} E_i\) is dense in \(E\).

(ii) \(\lim_{n \to \infty} \frac{L_n(u) - u}{\rho(n)} = A_0(u)\) for every \(u \in D_0\).

Then \((A_0, D_0)\) is closable and its closure \((A, D(A))\) is the generator of a contractive \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on \(E\).

\(^a\|L_n\| \leq 1\) for every \(n \geq 1\).
Moreover, if \( t \geq 0 \) and if \((k(n))_{n \geq 1}\) is a sequence of positive integers satisfying \( \lim_{n \to \infty} k(n) \rho(n) = t \), then, for every \( f \in E \),

\[
T(t)(f) = \lim_{n \to \infty} L_n^{k(n)}(f).
\]

Furthermore, \( \bigcup_{i \in I} E_i \) is a core \(^a\) for \((A, D(A))\).

\(^a\)A core for a linear operator \( A : D(A) \to C(K) \) is a linear subspace of \( D(A) \) which is dense in \( D(A) \) with respect to the graph norm \( \|u\|_A := \|A(u)\|_\infty + \|u\|_\infty \) \((u \in D(A))\).
Moreover, if \( t \geq 0 \) and if \( (k(n))_{n \geq 1} \) is a sequence of positive integers satisfying \( \lim_{n \to \infty} k(n) \rho(n) = t \), then, for every \( f \in E \),

\[
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For us

- \((E, \| \cdot \|) = (\mathcal{C}(K), \| \cdot \|_\infty)\)
- \((A_0, D_0) = (W_T, \mathcal{C}^2(K))\)
- \(L_n = B_n\)
It is well-known that for every $x \in K$ there exists a (unique) probability Borel measure $\tilde{\mu}_x^T$ on $K$ such that, for every $f \in C(K)$,

$$T(f)(x) = \int_K f \, d\tilde{\mu}_x^T. \quad (7)$$

Then, for every $n \geq 1$, we define the $n$-th Bernstein-Schnabl operator $B_n$ associated with $T$ by setting, for every $f \in C(K)$ and $x \in K$,

$$B_n(f)(x) := \int_K \cdots \int_K f \left( \frac{x_1 + \cdots + x_n}{n} \right) \, d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n). \quad (8)$$
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Properties of $B_n$’s

- $B_n$ is a positive linear operator from $C(K)$ into $C(K)$.
- $B_n(1) = 1 \implies \|B_n\| = 1$ for every $n \geq 1$.
- $B_1 = T$.
- If $K = [0, 1]$ and $T = T_1$ is the canonical projection, i.e.

$$T_1(f)(x) = xf(1) + (1 - x)f(0) \quad (f \in C([0, 1]), x \in [0, 1]), \quad (9)$$

then $B_n$’s turn into the classical Bernstein operators on $[0, 1]$. 
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Properties of $B_n$’s

Assume that

$$T(h) = h \quad \text{for every } h \in \{pr_1, \ldots, pr_d\}. \quad (10)$$

**Theorem 1**

For every $i, j = 1, \ldots, d$ and $n \geq 1$,

$$B_n(pr_i) = pr_i \quad \text{and} \quad B_n(pr_i pr_j) = \frac{1}{n} T(pr_i pr_j) + \frac{n-1}{n} pr_i pr_j. \quad (11)$$

Moreover, for every $f \in \mathcal{C}(K)$,

$$\lim_{n \to \infty} B_n(f) = f \quad \text{uniformly on } K. \quad (12)$$

Finally, for every $n \geq 1$ and $f \in \mathcal{C}(K)$,

$$B_n(f) = f \quad \text{on } \partial_T K. \quad (13)$$
Properties of $B_n$’s

For every $m \geq 1$ denote by $P_m(K)$ the (restriction to $K$ of all) polynomials of degree at most $m$. Moreover, we set

$$P_{\infty}(K) := \bigcup_{m \geq 1} P_m(K).$$  \hfill (14)

Observe that $P_{\infty}$ is a subalgebra of $\mathcal{C}(K)$ and it is dense in $\mathcal{C}(K)$.

**Theorem 2**

If

$$T(P_m(K)) \subset P_m(K) \quad \text{for every } m \geq 1,$$  \hfill (15)

then

$$B_n(P_m(K)) \subset P_m(K)$$  \hfill (16)

for every $n, m \geq 1$.

In the Trotter-Schnabl-type theorem:

(i) $(E_i)_{i \in I}$ is $(P_m(K))_{m \geq 1}$
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In the Trotter-Schnabl-type theorem:

(i) $(E_i)_{i \in I}$ is $(P_m(K))_{m \geq 1}$
Asymptotic formula for $B_n$'s

**Theorem 3**

For every $u \in C^2(K)$,

$$\lim_{n \to \infty} n(B_n(u) - u) = W_T(u) \quad \text{uniformly on } K.$$  

(17)

In the Trotter-Schnabl-type theorem:

(ii) $L_n = B_n$ and $\rho(n) = 1/n$
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Theorem 3

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(ii) $L_n = B_n$ and $\rho(n) = 1/n$
Theorem 4

Let $K$ be a convex compact subset of $\mathbb{R}^d$, $d \geq 1$, having non-empty interior and consider a Markov operator $T$ on $C(K)$ preserving coordinate functions. Furthermore, assume that

$$T(P_m(K)) \subset P_m(K) \quad \text{for every } m \geq 2. \quad (18)$$

Then the operator $(W_T, C^2(K))$ is closable and its closure $(A_T, D(A_T))$ generates a Markov semigroup $(T(t))_{t \geq 0}$ on $C(K)$ such that, if $t \geq 0$ and $(k(n))_{n \geq 1}$ is a sequence of positive integers satisfying $\lim_{n \to \infty} k(n)/n = t$,

$$T(t)(f) = \lim_{n \to \infty} B_n^{k(n)}(f) \quad \text{uniformly on } K \quad (19)$$

for every $f \in C(K)$.

Moreover, $P_\infty(K)$, and hence $C^2(K)$ as well, is a core for $(A_T, D(A_T))$. 
Finally, for every $t \geq 0$ and $m \geq 1$,

$$T(t)(P_m(K)) \subset P_m(K)$$  \hspace{1cm} (20)

and, if $t \geq 0$ and $f \in \mathcal{C}(K)$,

$$T(t)(f) = f \quad \text{on} \; \partial_T K.$$  \hspace{1cm} (21)
Consider the abstract Cauchy problem

\[
\begin{cases}
\frac{\partial u}{\partial t}(x, t) = A_T(u(\cdot, t))(x) & x \in K, \ t \geq 0, \\
u(x, 0) = u_0(x) & u_0 \in D(A_T), \ x \in K.
\end{cases}
\]  
(22)

Since \((A_T, D(A_T))\) generates the Markov semigroup \((T(t))_{t \geq 0}\), (22) admits a unique solution \(u : K \times [0, +\infty[ \rightarrow \mathbb{R}\) given by

\[
u(x, t) = T(t)(u_0)(x) \quad (x \in K, t \geq 0).
\]  
(23)

Hence, by Theorem 4,

\[
u(x, t) = T(t)(u_0)(x) = \lim_{n \to \infty} B_n^{k(n)}(u_0)(x),
\]  
(24)

where \((k(n))_{n \geq 1}\) is a sequence of positive integers satisfying \(\lim_{n \to \infty} k(n)/n = t\), and the limit is uniform with respect to \(x \in K\).
Recall that

\[ A_T = W_T \quad \text{on } C^2(K). \quad (25) \]

Therefore, if \( u_0 \in P_m(K) (m \geq 1) \) then \( u(x, t) \) is the unique solution to the Cauchy problem

\[
\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x, t) & x \in K, \ t \geq 0, \\
u(x, 0) = u_0(x) & x \in K
\end{cases}
\]

and

\[ u(\cdot, t) \in P_m(K) \quad \text{for every } t \geq 0. \quad (27) \]
In

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*On differential operators associated with Markov operators*,

- \((B_n)_{n\geq 1}\) preserves Hölder continuous functions \(\implies\) the same holds for 
  \((T(t))_{t\geq 0}\)

Further properties both for \((B_n)_{n\geq 1}\) and \((T(t))_{t\geq 0}\) have been investigated in

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*Markov Operators, Positive Semigroups and Approximation Processes*

Examples of $W_T$ satisfying Theorem 4

$T(h) = h$ for every $h \in \{pr_1, \ldots, pr_d\}$ \hspace{1cm} (Hp1)

$T(P_m(K)) \subset P_m(K)$ for every $m \geq 2$ \hspace{1cm} (Hp2)
Examples of $W_T$ satisfying Theorem 4

1. Consider the $d$-dimensional simplex

$$K_d := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i \geq 0 \text{ for every } i = 1, \ldots, d \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\}$$

(28)

and the projection $T_d$ on $K_d$ defined by

$$T_d(f)(x) := \left( 1 - \sum_{i=1}^{d} x_i \right) f(v_0) + \sum_{i=1}^{d} x_i f(v_i)$$

(29)

($f \in \mathcal{C}(K_d)$, $x = (x_1, \ldots, x_d) \in K_d$), where

$$v_0 := (0, \ldots, 0), \quad v_1 := (1, 0, \ldots, 0), \ldots, \quad v_d := (0, \ldots, 0, 1)$$

(30)

are the vertices of the simplex.

$T_d$ is a Markov operator that satisfies (Hp1) and (Hp2) (note that $T_d(\mathcal{C}(K_d)) \subset P_1(K_d)$).
Examples of $W_T$ satisfying Theorem 4

Then Theorem 4 applies to the differential operator associated with $T_d$ given by

$$W_{T_d}(u)(x) = \frac{1}{2} \sum_{i,j=1}^{d} x_i (\delta_{ij} - x_j) \frac{\partial^2 u}{\partial x_i \partial x_j}(x)$$

$$= \frac{1}{2} \sum_{i=1}^{d} x_i (1 - x_i) \frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1 \leq i < j \leq d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x)$$

(31)

$(u \in C^2(K_d), \ x = (x_1, \ldots, x_d) \in K_d); \ \delta_{ij}$ stands for the Kronecker symbol.

The operator $W_{T_d}$ falls into the class of the so called Fleming-Viot operators.

The coefficients of $W_{T_d}$ vanish on the vertices of the simplex.
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The operator $W_{T_d}$ falls into the class of the so called Fleming-Viot operators.

The coefficients of $W_{T_d}$ vanish on the vertices of the simplex.
2. Let $S : C(K_d) \rightarrow C(K_d)$ be the Markov operator defined by

$$
S(f)(x) := \begin{cases} 
1 - \frac{x_1}{d} & f(0, x_2, \ldots, x_d) \\
1 - \sum_{i=2}^{d} x_i & f(1 - \sum_{i=2}^{d} x_i, x_2, \ldots, x_d) \quad \text{if } \sum_{i=2}^{d} x_i \neq 1, 
\end{cases}
$$

if $\sum_{i=2}^{d} x_i = 1$

$$
(\sum_{i=2}^{d} x_i \neq 1, \quad f \in C(K_d),\ x = (x_1, \ldots, x_d) \in K_d).
$$
Examples of $W_T$ satisfying Theorem 4

One has

$$S(pr_1 pr_j) = \begin{cases} 
(1 - \sum_{i=2}^{d} pr_i) pr_1 & \text{if } j = 1, \\
pr_1 pr_j & \text{if } 1 < j \leq d 
\end{cases}$$

(33)

and $S(pr_i pr_j) = pr_i pr_j$ for every $1 < i \leq j \leq d$.

Then $S$ verifies (Hp1) and (Hp2) since, if $m_1, \ldots, m_d$ are positive integers,

$$S \left( pr_1^{m_1} \cdots pr_d^{m_d} \right) = \begin{cases} 
pr_2^{m_2} \cdots pr_d^{m_d} & \text{if } m_1 = 0, \\
\left( 1 - \sum_{i=2}^{d} pr_i \right)^{m_1-1} pr_1 \ pr_2^{m_2} \cdots pr_d^{m_d} & \text{if } m_1 \geq 1.
\end{cases}$$

(34)
Examples of $W_T$ satisfying Theorem 4

Therefore Theorem 4 applies to the differential operator associated with $S$ given by

$$ W_S(u)(x) = \frac{1}{2} x_1 \left( 1 - \sum_{i=1}^{d} x_i \right) \frac{\partial^2 u}{\partial x_1^2}(x) $$

$$ (35) $$

$$(u \in C^2(K_d), \ x = (x_1, \ldots, x_d) \in K_d).$$

Note that $W_S$ degenerates on the faces $\{x = (x_1, \ldots, x_d) \in K_d \mid x_1 = 0\}$ and $\left\{ x = (x_1, \ldots, x_d) \in K_d \mid \sum_{i=1}^{d} x_i = 1 \right\}$. 
Examples of $W_T$ satisfying Theorem 4

3. Consider the convex combination of the above operators, that is the Markov operator $V := \frac{T_d + S}{2}$. Then $V$ satisfies (Hp1) and (Hp2) and Theorem 4 applies to the differential operator

$$W_V(u)(x) = \frac{1}{4} \left( \left( 2x_1(1 - x_1) - x_1 \sum_{i=2}^{d} x_i \right) \frac{\partial^2 u}{\partial x_1^2}(x) \right.$$

$$+ \sum_{i=2}^{d} x_i(1 - x_i) \frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1 \leq i < j \leq d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \right)$$

$$\left( u \in C^2(K_d), \quad x = (x_1, \ldots, x_d) \in K_d \right).$$

Observe that $W_V$ degenerates on the vertices of $K_d$ as well.
Examples of $W_T$ satisfying Theorem 4

4. Let $Q_d := [0, 1]^d$, $d \geq 1$, and for every $i = 1, \ldots, d$ consider a Markov operator $U_i$ on $C([0, 1])$ satisfying (Hp1) and (Hp2).

If $U := \bigotimes_{i=1}^d U_i$ is the tensor product of the family $(U_i)_{1 \leq i \leq d}$, then $U$ is a Markov operator on $C(Q_d)$ satisfying (Hp1) and (Hp2) because

$$U (pr_1^{m_1} \cdots pr_d^{m_d}) = (U_1 (e_1^{m_1}) \circ pr_1) \cdots (U_d (e_1^{m_d}) \circ pr_d)$$

for every positive integers $m_1, \ldots, m_d$.

Therefore, Theorem 4 applies to the differential operator

$$W_U(u)(x) = \frac{1}{2} \sum_{i=1}^d \alpha_i(x) \frac{\partial^2 u}{\partial x_i^2}(x), \quad (37)$$

($u \in C^2(Q_d)$, $x = (x_1, \ldots, x_d) \in Q_d$), where

$$\alpha_i(x) := U_i(e_2)(x_i) - x_i^2 \quad (1 \leq i \leq d). \quad (38)$$
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On differential operators associated with Markov operators,

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Thank you for your attention