Regularization of some integral equations of the first kind in potential theory

Angelica Malaspina

Department of Mathematics, Computer Science and Economics
University of Basilicata, Potenza, Italy
angelica.malaspina@unibas.it

The real world is complex
an international symposium in honor of Christian Berg
Copenhagen, August 27, 2015
Regularization of some integral equations of the first kind in potential theory

Angelica Malaspina

Department of Mathematics, Computer Science and Economics
University of Basilicata, Potenza, Italy
angelica.malaspina@unibas.it

The real world is complex
an international symposium in honor of Christian Berg
Copenaghien, August 27, 2015
A. Cialdea, University of Basilicata, Italy
V. Leonessa, University of Basilicata, Italy
E. Dolce, University of Basilicata, Italy
BIEs $\Leftrightarrow$ REDUCTION

\[
\begin{align*}
\text{Direct method} & \quad \text{Green’s representation formula} \\
\text{Indirect method} & \quad \text{layer ansatz}
\end{align*}
\]
Dirichlet Problem for Laplace Equation in $\mathbb{R}^n$, $n \geq 2$

\[ \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \Sigma \end{cases} \]

\[ \Delta = \sum_{h=1}^{n} \frac{\partial^2}{\partial x_h^2} \]

$\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded domain

$\Sigma = \partial \Omega$ Lyapunov hypersurface: $\Sigma \in C^{1,\lambda}$, $0 < \lambda \leq 1$

$f$ assigned function on $\Sigma$
simple layer potential \[ u(x) = \frac{1}{2\pi} \int_{\Sigma} \varphi(y) \ln |x - y| d\sigma_y \]

BIE 1\(^{st}\) kind: \[ \frac{1}{2\pi} \int_{\Sigma} \varphi(y) \ln |x - y| ds_y = f(x), \quad x \in \Sigma \]

\[ f \in C^{1,\lambda}(\Sigma), \ 0 < \lambda \leq 1 \]

\[ \frac{1}{2\pi} \int_{\Sigma} \varphi(y) \frac{\partial}{\partial s_x} \ln |x - y| ds_y = \frac{df}{ds}(x), \quad x \in \Sigma. \quad (1) \]

If \( B \) and \( B' \) are two Banach spaces and \( S : B \to B' \) is a continuous linear operator, we say that \( S \) can be reduced on the left if \( \exists \) a continuous linear operator \( S' : B' \to B \) s. t. \( S'S = I + K \), where \( I \) stands for the identity operator of \( B \) and \( K : B \to B \) is compact. Analogously, one can define an operator \( S \) reducible on the right.
Dirichlet problem for Laplace equation in $\mathbb{R}^2$

Simple layer potential approach


**simple layer potential**

$$u(x) = \frac{1}{2\pi} \int_{\Sigma} \varphi(y) \ln|x-y| \, d\sigma_y$$

**BIE 1\textsuperscript{st} kind:**

$$\frac{1}{2\pi} \int_{\Sigma} \varphi(y) \ln|x-y| \, ds_y = f(x), \quad x \in \Sigma$$

$$f \in C^{1,\lambda}(\Sigma), \; 0 < \lambda \leq 1$$

$$\frac{1}{2\pi} \int_{\Sigma} \varphi(y) \frac{\partial}{\partial s_x} \ln|x-y| \, ds_y = \frac{df}{ds}(x), \quad x \in \Sigma. \quad (1)$$

---

If $B$ and $B'$ are two Banach spaces and $S : B \to B'$ is a continuous linear operator, we say that $S$ can be **reduced on the left** if $\exists$ a continuous linear operator $S' : B' \to B$ s. t. $S'S = I + K$, where $I$ stands for the identity operator of $B$ and $K : B \to B$ is compact.

Analogously, one can define an operator $S$ reducible on the right.
Indirect Method
Dirichlet Problem for Laplace Equation in $\mathbb{R}^2$: Muskhelishvili’s approach

Muskheilishvili’s approach

$\uparrow \quad \downarrow$

$n = 2$

$\downarrow \quad \uparrow$

theory of holomorphic derivative
functions of with respect to
1 complex variable the arc length

A. Malaspina, Univ. Basilicata
Dirichlet Problem for Laplace equation in $\mathbb{R}^n$, $n \geq 3$

Simple layer potential approach

Cialdea, 1988

$\Omega \subset \mathbb{R}^n$ bounded domain; $\mathbb{R}^n - \overline{\Omega}$ is connected, $\Sigma \in C^{1,\lambda}$

\[
\begin{cases}
\quad \Delta u = 0 \quad \text{in } \Omega \\
\quad u = f \quad \text{on } \Sigma, \; f \in W^{1,p}(\Sigma), \; 1 < p < +\infty
\end{cases}
\]

Simple layer potential

\[
u(x) = \int_{\Sigma} \varphi(y)s(x,y)\,d\sigma_y
\]

$\varphi \in L^p(\Sigma)$

$n = 2 \begin{cases}
\quad \text{theory of holomorphic functions of} \\
\quad 1 \text{ complex variable}
\end{cases} \begin{cases}
\quad \text{theory of differential forms}
\end{cases}
\quad n > 2$

$n = 2 \begin{cases}
\quad \text{derivative with respect to the arc length}
\end{cases} \begin{cases}
\quad \text{exterior differential}
\end{cases} \quad n > 2$
Neumann Problem for Laplace equation

\( \Omega \) simply connected domain

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = f & \text{on } \Sigma, \ f \in L^p(\Sigma), \ \int_\Sigma f \, d\sigma = 0
\end{cases}
\]

double layer pot.: \( u(x) = \int_\Sigma \psi(y) \frac{\partial}{\partial \nu_y} s(x, y) \, d\sigma_y, \ \psi \in W^{1,p}(\Sigma) \)

\[
\frac{\partial}{\partial \nu_x} \left( \int_\Sigma \psi(y) \frac{\partial}{\partial \nu_y} s(x, y) \, d\sigma_y \right) \, d\sigma_x = d_x \int_\Sigma d\psi(y) \wedge s_{n-2}(x, y)
\]

\[
s_{n-2}(x, y) = \sum_{j_1 < \ldots < j_{n-2}} s(x, y) \, dx^{j_1} \ldots dx^{j_{n-2}} \, dy^{j_1} \ldots dy^{j_{n-2}}
\]

\[
d_x \int_\Sigma d\psi(y) \wedge s_{n-2}(x, y) = f \, d\sigma
\]
We note that the method of Cialdea does not require the use of pseudo-differential operators nor the use of hypersingular integrals, differently from other methods, see e.g. Chapter 4 in


Generalization to other PDEs in a simply connected domain


Generalization to other PDEs in a simply connected domain


Recent generalizations to other PDEs in a simply connected domain


Recent generalizations to other PDEs in a simply connected domain


Recent generalizations to other PDEs in a simply connected domain


Generalization in multiply connected domains

Ω multiply connected domain of $\mathbb{R}^n$, $n \geq 2$

$$\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \overline{\Omega}_j$$

$\overline{\Omega}_j \subset \Omega_0$, $\overline{\Omega}_j \cap \overline{\Omega}_k = \emptyset$, $j \neq k$.


Lamè system

\[ \Omega \subset \mathbb{R}^n, \ n \geq 2, \text{ bounded domain (open connected set)} \]

\[ u(x) = (u_1(x), \ldots, u_n(x)) \]

\[ \Delta u + k \nabla \text{div } u = 0 \]

\[ k > 1 - 2/n \]

\[ \Delta u = \sum_{h=1}^{n} \frac{\partial^2 u_i}{\partial x^2_h}, \quad \nabla \cdot = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right), \quad \text{div } u = \sum_{h=1}^{n} \frac{\partial u_h}{\partial x_h}. \]
\[\Omega \text{ multiply connected domain of } \mathbb{R}^n, \ n \geq 2\]

\[\Sigma = \partial \Omega \in C^{1,\lambda}, \ 0 < \lambda \leq 1.\]

\[u(x) = (u_1(x), \ldots, u_n(x))\]

\[\begin{cases} 
\Delta u + k \nabla \text{div } u = 0 & \text{in } \Omega, \\
u = f & \text{on } \Sigma,
\end{cases}\]

\[k > 1 - 2/n\]

\[f \in [W^{1,p}(\Sigma)]^n, \ 1 < p < +\infty\]
simple layer potential  \( u(x) = \int_\Sigma \Gamma(x, y) \varphi(y) \, d\sigma_y \),  \( \varphi \in \left[ L^p(\Sigma) \right]^n \)

\[
\Gamma = \left( \Gamma_{ij}(x, y) \right)_{n \times n}
\]

\[
\Gamma_{ij}(x, y) = \begin{cases} 
\frac{1}{2\pi} \left( -\frac{k + 2}{2(k + 1)} \delta_{ij} \ln |x - y| + \frac{k}{2(k + 1)} \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} \right) & n = 2 \\
\frac{1}{\omega_n} \left( -\frac{k + 2}{2(k + 1)} \delta_{ij} |x - y|^{2-n} + \frac{k}{2(k + 1)} \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^n} \right) & n \geq 3
\end{cases}
\]

\( \omega_n \) being the hypersurface measure of the unit sphere in \( \mathbb{R}^n \).
BIS 1\textsuperscript{st} kind: \[ \int_\Sigma \Gamma(x, y) \varphi(y) d\sigma_y = f(x) \]

Singular integral system: \[ \int_\Sigma d_x[\Gamma(x, y)] \varphi(y) d\sigma_y = df(x) \quad (3) \]

\[ R : [L^p(\Sigma)]^n \rightarrow [L^1_1(\Sigma)]^n \]

\[ R\varphi = \int_\Sigma d_x[\Gamma(x, y)] \varphi(y) d\sigma_y \]

\[ R\varphi = df \]

\[ \varphi = (\varphi_1, \ldots, \varphi_n) \in [L^p(\Sigma)]^n, \quad df = (df_1, \ldots, df_n) \in [L^1_1(\Sigma)]^n \]
Let us define $R'_\xi : [L^p_1(\Sigma)]^n \rightarrow [L^p(\Sigma)]^n$ to be the singular integral operator

$$R'_i(\psi)(x) = (k - \xi)K_{jj}^\xi(\psi)(x)\nu_i(x) + \nu_j(x)K_{ij}^\xi(\psi)(x) + \xi\nu_j(x)K_{ji}^\xi(\psi)(x),$$

where $\psi = (\psi_1, \ldots, \psi_n) \in [L^p_1(\Sigma)]^n$, $\xi \in \mathbb{R}$

$$K_{jj}^\xi(\psi)(x) = \Theta_s(\psi_j)(x) -$$

$$\frac{1}{(n - 2)!}\delta_{hj}^{123\ldots n} \int_\Sigma \partial_{x_s} K_{hj}^\xi(x, y) \wedge \psi_i(y) \wedge dy^j \ldots dy^n$$

$$K_{hj}^\xi(x, y) = \left[ \frac{k(\xi + 1)(2 - n)}{2(k + 1)} \frac{(y_h - x_h)(y_j - x_j)}{|y - x|^2} + \frac{k - (2 + k)\xi}{2(k + 1)}\delta_{hj} \right] s(x, y)$$

$$\Theta_h(\psi_j)(x) = * \left( \int_\Sigma d_x [s_{n-2}(x, y)] \wedge \psi_j(y) \wedge dx^h \right)$$

$\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ denotes the outwards unit normal vector at a point $x$ on $\Sigma$. 
Then

\[ R'^\xi R \varphi = \frac{1}{4} \varphi + (T^\xi)^2 \varphi \] (5)

where

\[ T^\xi \varphi(x) = \int_{\Sigma} L^\xi_x [\Gamma(x, y)] \varphi(y) \, d\sigma_y \] (6)

and

\[ L^\xi_{i,x} [\Gamma^j(x, y)] = \]

\[ -\frac{1}{\omega_n} \left\{ \left[ \frac{2 + (1 - \xi)k}{2(1 + k)} \delta_{ij} + \frac{nk(\xi + 1)}{2(k + 1)} \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} \right] \frac{(x_p - y_p)\nu_p(x)}{|x - y|^n} \right. \]

\[ + \frac{k - (2 + k)\xi}{2(k + 1)} \left[ \frac{(x_j - y_j)\nu_i(x) - (x_i - y_i)\nu_j(x)}{|x - y|^n} \right] \right\}. \]

\( \omega_n \) hypersurface measure of the unit sphere in \( \mathbb{R}^n \), \( \Gamma^j(\cdot, \cdot) \) \( j \)-th column of \( \Gamma \), \( \forall j \).
1. If $\xi = k/(2 + k)$, $L_{i,y}^{k/(2+k)}[\Gamma^j(x,y)] = \mathcal{O}(|x - y|^{1-n+\lambda})$.
2. If $\xi \neq k/(2 + k)$, the kernels $L_{i,y}^{\xi}[\Gamma^j(x,y)]$ have a strong singularity on $\Sigma$.

**Reduction Theorem**

The operator

$$R : [L^p(\Sigma)]^n \rightarrow [L_1^p(\Sigma)]^n$$

$$R\varphi = \int_\Sigma d_x[\Gamma(x,y)]\varphi(y) \, d\sigma_y$$

can be reduced on the left. A reducing operator is given by $R' := R'^{\xi}$, with $\xi = k/(2 + k)$. 
Existence Theorem

Given $\omega \in [L^p_1(\Sigma)]^n$, there exists a solution of the singular integral system

$$\int_{\Sigma} d_x[\Gamma(x, y)] \varphi(y) d\sigma_y = \omega(x), \quad \varphi \in [L^p(\Sigma)]^n, \ x \in \Sigma$$

if, and only if,

$$\int_{\Sigma} \gamma \wedge \omega_i = 0, \quad i = 1, \ldots, n \tag{7}$$

for every $\gamma \in L^q_{n-2}(\Sigma)$ ($q = p/(p - 1)$) s.t. $\gamma$ is a weakly closed $(n - 2)$-form.

If $\omega_i = df_i \Rightarrow$ conditions (7) are satisfied and then $\exists$ a solution of $R\varphi = df$. 

A. Malaspina, Univ. Basilicata
Representation Theorem

For any \( f \in [W^{1,p}(\Sigma)]^n \) there exists a unique solution of the following BVP

\[
\begin{align*}
\Delta v + k \nabla \text{div } v &= 0 \quad \text{in } \Omega, \\
v &= f \quad \text{on } \Sigma.
\end{align*}
\]

It is given by

\[
v(x) = \int_\Sigma \Gamma(x, y) \varphi(y) \, d\sigma_y
\]

where the density \( \varphi \in [L^p(\Sigma)]^n \) solves the singular integral system

\[
R\varphi \equiv \int_\Sigma d_x[\Gamma(x, y)]\varphi(y) \, d\sigma_y = df.
\]
Remark

\[ R\varphi = df \]

\[ R' R\varphi = \frac{1}{4} \varphi + K^2 \varphi \]

is not an equivalent reduction

equivalent reduction: \( \mathcal{N}(R') = \{0\} \)

\[ R\varphi = \psi \iff R' R\varphi = R' \psi \]

\( R' \) still provides a kind of equivalence.

\[ \mathcal{N}(R'R) = \mathcal{N}(R) \]

If \( \psi \) is s. t. \( \exists \) a solution of \( R\varphi = \psi \), then

\[ R\varphi = \psi \iff R'R\varphi = R' \psi. \]
Traction problem for Lamè system

\( \Omega \) multiply connected domain of \( \mathbb{R}^n \)

\[
\begin{cases}
\Delta w + k \nabla \text{div} \ w = 0 & \text{in } \Omega, \\
Lw = f & \text{on } \Sigma,
\end{cases}
\tag{8}
\]

\( f \in [L^p(\Sigma)]^n \) s.t.

\[
\int_\Sigma f \cdot \alpha \ d\sigma = 0, \quad \forall \ \alpha \in \mathcal{R}
\]

stress operator: \( L_i w = (k - 1)(\text{div} \ w) \nu_i + \nu_j \partial_j w_i + \nu_j \partial_i w_j \quad i = 1, \ldots, n, \)

double layer pot. \( w(x) = \int_\Sigma L_y[\Gamma(x, y)] \psi(y) d\sigma_y, \ \psi \in [W^{1,p}(\Sigma)]^n \)
Traction problem for Lamè system

\( \Omega \) multiply connected domain of \( \mathbb{R}^n \)

\[
\begin{cases}
\Delta w + k \nabla \text{div} \ w = 0 & \text{in } \Omega, \\
Lw = f & \text{on } \Sigma
\end{cases}
\quad (8)
\]

\( f \in [L^p(\Sigma)]^n \) s.t.

\[
\int_{\Sigma} f \cdot \alpha \, d\sigma = 0, \quad \forall \alpha \in \mathcal{R}
\]

stress operator: \( L_i w = (k - 1)(\text{div} \ w) \nu_i + \nu_j \partial_j w_i + \nu_j \partial_i w_j \quad i = 1, \ldots, n, \)

double layer pot. \( w(x) = \int_{\Sigma} L_y[\Gamma(x, y)] \psi(y) \, d\sigma_y, \quad \psi \in [W^{1,p}(\Sigma)]^n \)
Theorem

Given \( f \in [L^p(\Sigma)]^n \), \( 1 < p < +\infty \), the traction problem admits a solution (determined up to an additive rigid displacement) \( \iff \)

\[
\int_{\Sigma_j} f \cdot \alpha \, d\sigma_x = 0, \quad \forall \alpha \in \mathcal{R}, \ j = 1, \ldots, m.
\]

Moreover, double layer potential is a solution of (8) if, and only if, its density \( \psi \) is given by

\[
\psi(x) = \int_{\Sigma} \Gamma(x, y) \phi(y) \, d\sigma_y, \quad x \in \Sigma,
\]

where \( \phi \) is a solution of the singular integral system

\[
\frac{1}{4} \phi + T^2 \phi = f,
\]

and where \( T \) is defined by operator (6) \( T^\xi \) with \( \xi = 1 \).
Theorem

Given \( f \in [L^p(\Sigma)]^n \) satisfying \( \int_\Sigma f \cdot \alpha \, d\sigma = 0, \ \forall \ \alpha \in \mathcal{R} \), the traction problem

\[
\begin{align*}
\Delta w + k \nabla \text{div} \ w &= 0 \quad \text{in } \Omega, \\
Lw &= f \quad \text{on } \Sigma,
\end{align*}
\]

admits a solution uniquely determined up to an additive rigid displacement

\[
w(x) = \int_\Sigma L_y [\Gamma(x, y)] \psi(y) \, d\sigma_y +
\sum_{j=1}^m \sum_{h=1}^{n(n+1)/2} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} f(t) v_h(t) \, d\sigma_t \int_{\Sigma_j} \Gamma(x, y) v_h(y) \, d\sigma_y,
\]

where \( \{v_h(x) : h = 1, \ldots, n(n+1)/2\} \) is an orthonormal basis of the space of all rigid displacements \( \mathcal{R} \).

A. Malaspina, Univ. Basilicata
Remarks on the plane for Laplace equation

If $\Omega \subset \mathbb{R}^n \ (n \geq 3)$ is a bounded simply connected domain

$$\begin{cases} 
\Delta u = 0 & \text{in } \Omega \\
u = f & \text{on } \Sigma, \ f \in W^{1,p}(\Sigma)
\end{cases}$$

is solvable by means of

$$u(x) = \int_{\Sigma} \varphi(y) s(x,y)(x,y) \, d\sigma_y, \quad \varphi \in L^p(\Sigma).$$

This is not always true for $n = 2$. In fact, there are some boundaries for which it is not possible to represent the solution of the Dirichlet problem by means of a simple layer potential. In particular it is not possible to represent the constant function.

If this occurs in the simply connected domain $\Omega \subset \mathbb{R}^2$ we say that its boundary $\Sigma$ is exceptional.

The simplest example of such a domain is given by the unit disc $D$, where

$$\int_{\partial D} \ln |x - y| \, ds_y = 0, \quad |x| \leq 1.$$
Remarks on the plane
for the Lamè system

Proposition

Let $C_r$ be the circle of radius $r$ centered at the origin. If $r = \exp(k/(2(k + 2)))$, $C_r$ is exceptional for the operator $\Delta + k\nabla \text{div}$, i.e. in $B_r$ (the ball of radius $r$ centered at the origin), for such a value of $r$, we cannot represent any smooth solution of the system $\Delta u + k\nabla \text{div} u = 0$ by means of a simple layer potential.
We have showed that also in some multiply connected domains one cannot represent any constant vectors by a simple layer potential and that this happens if, and only if, the exterior boundary $\Sigma_0$ (considered as the boundary of the simply connected domain $\Omega_0$) is exceptional.

**Theorem**

Let $\Omega \subset \mathbb{R}^2$ be an $(m + 1)$-connected domain. The following conditions are equivalent:

(i) there exists an Hölder continuous vector function $\varphi \neq 0$ such that

$$\int_{\Sigma} \Gamma(x, y) \varphi(y) \, ds_y = 0, \quad \forall \, x \in \Sigma;$$

(ii) there exists a constant vector which cannot be represented in $\Omega$ by a simple layer potential (i.e., there exists $c \in \mathbb{R}^2$ such that $c \notin S^p$);

(iii) $\Sigma_0$ is exceptional;
(iv) Let $\varphi_1, \ldots, \varphi_{2m+2}$ be linearly independent functions of

$$
P=\left\{ \varphi \in [L^p(\Sigma)]^2 : \int_\Sigma \varphi_j(y) \frac{\partial}{\partial s_x} \Gamma_{ij}(x, y)\, ds_y = 0, \text{a.e. } x \in \Sigma, i = 1, 2 \right\}
$$

and let $c_{jk} = (\alpha_{jk}, \beta_{jk}) \in \mathbb{R}^2$ be given by

$$
\int_\Sigma \Gamma(x, y) \varphi_j(y)\, ds_y = c_{jk},
$$

$x \in \Sigma_k, j = 1, \ldots, 2m+2, k = 0, 1, \ldots, m$. Then

$$
\det C = 0,
$$

where

$$
C = \begin{pmatrix}
\alpha_{1,0} & \ldots & \alpha_{2m+2,0} \\
\ldots & \ldots & \ldots \\
\alpha_{1,m} & \ldots & \alpha_{2m+2,m} \\
\beta_{1,0} & \ldots & \beta_{2m+2,0} \\
\ldots & \ldots & \ldots \\
\beta_{1,m} & \ldots & \beta_{2m+2,m}
\end{pmatrix}.
$$
THANK YOU FOR YOUR ATTENTION!
\[
L^\xi_i u = (k - \xi)(\text{div } u) \nu_i + \nu_j \partial_j u_i + \xi \nu_j \partial_i u_j,
\]

\(\xi \in \mathbb{R}\)

If \(\xi = 1\), \(L^1\) is the classical stress operator

If \(\xi = k/(2 + k)\), \(L^{k/(2+k)}\) is pseudostress operator
\[ s(x, y) = \begin{cases} 
\frac{1}{2\pi} \ln |x - y| & n = 2, \\
\frac{1}{(2 - n)\omega_n} |x - y|^{2-n} & n \geq 3
\end{cases} \]

\(\omega_n\) hypersurface measure of the unit sphere in \(\mathbb{R}^n\)