Hausdorff dimension of the Julia sets of some rational maps

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Singular perturbation and McMullen maps

Consider a singular perturbation of the unicritical polynomials

\[ f_\lambda(z) = z^q + \frac{\lambda}{z^p}, \text{ where } p \geq 2, \ q \geq 2, \ \lambda \in \mathbb{C} \setminus \{0\}. \]

\( f_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is called the **McMullen family**, since McMullen studied this family first in 1988.
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1. 0 and $\infty$ are critical points, $f_\lambda(0) = \infty = f_\lambda(\infty)$;
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3. \textbf{free} critical points: $\omega_j = \frac{p+q}{p} \sqrt{\frac{\lambda}{p}} e^{2\pi ij/(p+q)}$, where $0 \leq j < p+q;$
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4. The dynamics of $f_\lambda$ depends on the **one of** the free critical orbits.
Escape Trichotomy Theorem

Theorem (Devaney, Look and Uminsky, 2005)

Suppose that the free critical points of $f_\lambda$ are attracted by $\infty$. Then one and only one of the following three cases happens:

1. $f_\lambda(\omega_j) \in B_\lambda$ for some $j$, then $J_\lambda$ is a **Cantor set**;

2. $f_\lambda(\omega_j) \in T_\lambda \neq B_\lambda$ for some $j$, then $J_\lambda$ is a **Cantor circles**;

3. $f^{\circ k}_\lambda(\omega_j) \in T_\lambda \neq B_\lambda$ for some $j$ and $k \geq 2$, then $J_\lambda$ is a **Sierpinski carpet**.

Moreover, if the orbit of $\omega_j$ is bounded, then $J(f_\lambda)$ is connected.

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**Cantor circles**: Cantor set $\times$ unit circle

**Sierpinski carpet**: compact, connected, locally connected, has empty interior, and the complementary domains are bounded by pairwise disjoint simple closed curves
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Singular perturbation and dimension

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A connected Julia set

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Hausdorff dimension of the Julia sets

Copenhagen, Aug 14, 2018
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For hyperbolic cases:

\[ 0 < \dim_H < 2 \quad 1 < \dim_H < 2 \quad 1 < \dim_H < 2 \]

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Theorem (Y., Qiu-Ren-Y., Fu-Y., 2018)

We have

1. \( \dim_H(\text{Cantor Julia sets}) \) can take any value in \((0, 2]\);

Moreover, there exist Sierpinski carpet Julia sets with positive area, and Sierpinski carpet Julia sets with zero area but with Hausdorff dimension two.

Remark: these results are sharp.

Except: We don’t know the existence of Cantor Julia sets with positive area.
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Some facts:

1. Garber (1978) and Stallard (1994): $\dim_H(J(f)) > 0$, where $f$ is a non-constant, non-linear meromorphic function.

2. Bodart and Zinsmeister (1996): \( \lim_{c \to \infty} \dim_H(J(z^2 + c)) = 0. \)

3. Shishikura (1998): \( \sup_{c \in \mathbb{C} \setminus M} \dim_H(J(z^2 + c)) = 2. \)


5. $\dim_H$(hyperbolic Cantor Julia sets) can take any value in $(0, 2)$.

Theorem (Y., 2018)

There exist cubic polynomials whose Julia sets are Cantor sets having Hausdorff dimension two.

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Shishikura’s criterion

Theorem (Shishikura, 1998)

Suppose that a rational map $f_0$ of degree $d \geq 2$ has a **parabolic fixed point** $z_0$ with multiplier $e^{2\pi i p/q}$ ($p, q \in \mathbb{Z}$, $(p, q) = 1$) and that the immediate parabolic basin of $z_0$ contains only one critical point of $f_0$. Then for any $\varepsilon > 0$ and $b > 0$, there exist a neighborhood $\mathcal{N}$ of $f_0$ in the space of rational maps of degree $d$, a neighborhood $\mathcal{V}$ of $z_0$ in $\mathbb{C}$, positive integers $N_1$ and $N_2$ such that if $f \in \mathcal{N}$, and if $f$ has a fixed point in $\mathcal{V}$ with multiplier $e^{2\pi i \alpha}$, where

$$q \alpha = p \pm \frac{1}{a_1 \pm \frac{1}{a_2 + \beta}}$$

with integers $a_1 \geq N_1$, $a_2 \geq N_2$ and $\beta \in \mathbb{C}$, $0 \leq \text{Re} \beta < 1$, $|\text{Im} \beta| \leq b$, then

$$\dim_H(J(f)) > 2 - \varepsilon.$$
Branner-Hubbard’s characterization on cubic poly.

\[ P_{a,b}(z) = z^3 - 3a^2z + b \]

**Figure:** The space \( \mathcal{L}^+(\zeta) \) for some \( \zeta > 1 \). The set \( \mathcal{B}^+(\zeta) \subset \mathcal{L}^+(\zeta) \) has been drawn and zoomed in several times. The *copies of the Mandelbrot set* and some decorations of the *point components* of \( \mathcal{B}^+(\zeta) \) can be seen clearly.
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\[ \zeta \in \mathbb{C} \setminus \overline{D} \] is the Böttcher coordinate of the co-critical point \(-2a\)

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3. $\dim_H(\text{Cantor circle Julia sets})$ can take any value very close to 2.
4. Zdunik (1990): $\dim_H(\text{Julia set (non-circle or interval) with a non-degenerate continua}) > 1$.
5. Häggisnky-Pilgrim (2012): The lower bound $\alpha$ of $\dim_H(\text{Cantor circle Julia sets})$ is determined by the combinatorics.

$$\sum_{i=1}^{n} \left( \frac{1}{d_i} \right)^{\alpha} = 1,$$

where $\sum_{i=1}^{n} \frac{1}{d_i} < 1$, and $\alpha$ is the conformal dimension.
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2. Urbanski (1994), Yin (2000): \( \dim_H(\text{geom. finite Julia sets}) < 2. \)
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Theorem (Qiu-Y., 2018)

Let $\mathcal{H}$ be a hyperbolic component containing a rational map $f_0$ whose Julia set $J(f_0)$ is a Cantor set of circles. Then

$$\inf_{f \in \mathcal{H}} \dim_H(J(f)) = \dim_C(J(f_0)) \quad \text{and} \quad \sup_{f \in \mathcal{H}} \dim_H(J(f)) = 2.$$
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Theorem (Qiu-Ren-Y., 2018)

Let $f_\lambda(z) = z^q + \lambda/z^p$, where $1/p + 1/q < 1$. Then

$$\lim_{\lambda \to 0} \dim_H(J(f_\lambda)) = 1 + \alpha_{p,q},$$

where $\alpha_{p,q} = \alpha \in (0,1)$ is the unique positive root of $p^{-\alpha} + q^{-\alpha} = 1$.

Moreover, if $p = q \geq 3$ then

$$\left| \dim_H(J(f_\lambda)) - \left(1 + \frac{\log 2}{\log p}\right) \right| \leq \frac{2^{p+1} \log(2p)}{\log^2 p} |\lambda|^{1 - \frac{2}{p}} + O(|\lambda|^{2(1 - \frac{2}{p})}).$$
The proof idea

Ingredients in the proof:

1. Decompose the dynamics of $f_\lambda$ to an IFS;
2. Koebe’s distortion theorem on the estimation of contraction factors;
3. Falconer’s criterion on the Hausdorff dimension of the attractor of the IFS;
4. Put the calculation on the logarithm plane.
Sierpinski carpet Julia sets

Some known results:

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2. Barański-Wardal (2015): For $f_{\lambda,p}(z) = z^p + \lambda/z^p$ with $p \geq 2$, there exists $\lambda = \lambda(p)$ such that $J(f_{\lambda,p})$ is a Sierpinski carpet, and $\lim_{p \to \infty} \dim_H(J(f_{\lambda,p})) = 1$. 
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1. Zdunik (1990): \( \dim_H(\text{Sierpinski carpet Julia set}) > 1 \).

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4. Qiu-Wang-Yin (2012): There exists renormalizable parameters \( \lambda \)'s such that the Julia sets of \( f_{\lambda}(z) = z^p + \lambda/z^p \) with \( p \geq 3 \) are Sierpinski carpets.
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Theorem (Fu-Y., 2018)

Let \( \mathcal{H} \) be a Sierpinski carpet hyperbolic component (actually holds for the hyperbolic Julia sets with a simply connected attracting basin). Then

\[
\sup_{f \in \mathcal{H}} \dim_H(J(f)) = 2.
\]
Theorem (Y.-Yin, 2018, a refinement of Shishikura’s result)

There exist non-renormalizable quadratic polynomials whose periodic points are all repelling and whose Julia sets have Hausdorff dimension two. Moreover, such parameters are dense on the boundary of the Mandelbrot set.
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There exist Sierpinski carpet Julia sets with zero area but with Hausdorff dimension two.
One may consider the Lebesgue area and the Hausdorff dimension of some special Julia sets (or subsets):

1. degenerated Sierpinski carpets;

Conjecture

For each hyperbolic component $H$ in the space of rational maps with degree at least two, 

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One may consider the Lebesgue area and the Hausdorff dimension of some special Julia sets (or subsets):

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1. degenerated Sierpinski carpets;
2. Boundaries of Siegel disks (McMullen, Buff-Chéritat, Graczyk-Jones, Cheraghi-DeZotti, …)
3. hedgehogs, Cremer Julia sets (with Cheraghi-DeZotti, work in progress)

Conjecture

For each hyperbolic component \( \mathcal{H} \) in the space of rational maps with degree at least two,

\[
\sup_{f \in \mathcal{H}} \dim_H(J(f)) = 2.
\]

Thank you for your attention!