## ON MODULES WITH SELF TOR VANISHING

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ABSTRACT. The long-standing Auslander and Reiten Conjecture states that a finitely generated module over a finite-dimensional algebra is projective if certain Ext-groups vanish. Several authors, including Avramov, Buchweitz, Iyengar, Jorgensen, Nasseh, Sather-Wagstaff, and Şega, have studied a possible counterpart of the conjecture, or question, for commutative rings in terms of vanishing of Tor. This has led to the notion of Tor-persistent rings. Our main result shows that the class of Tor-persistent local rings is closed under a number of standard procedures in ring theory.

#### 1. Introduction

Inspired by work of Şega [22, para. preceding Thm. 2.6], Avramov, Iyengar, Nasseh, and Sather-Wagstaff raise in [6]<sup>1</sup>, the question of whether every commutative noetherian ring is Tor-persistent. A commutative ring A is said to be Tor-persistent if every finitely generated A-module M with  $Tor_i^A(M,M) = 0$  for all  $i \gg 0$ , that is,  $Tor^A(M,M)$  is bounded, has finite projective dimension. We refer to [6] and the precursor [5] (by the same authors) for a history/background of this question. The mentioned works also contain information about several interesting classes of rings which are known to be Tor-persistent. This includes Gorenstein rings with an exact zero divisor whose radical to the fourth power is zero [22, Thm. 2], complete intersection rings [15, Cor. (1.2)] (see also [3, Thm. IV] and [14, Thm. 1.9]) and Golod rings [16, Thm. 3.1].

In [6, Prop. 1.6] it is shown that a commutative noetherian ring A is Tor-persistent if and only if the localization  $A_{\mathfrak{m}}$  is so for every maximal ideal  $\mathfrak{m} \subset A$ ; hence it suffices to study the question mentioned above for commutative noetherian *local* rings. Throughout this paper,  $(R, \mathfrak{m}, k)$  denotes such a ring. Our main result is the following:

## **1.1 Theorem.** The following conditions are equivalent:

- (i) R is Tor-persistent.
- (ii)  $\widehat{R}$  is Tor-persistent.
- (iii)  $R[X_1,...,X_n]$  is Tor-persistent.
- (iv)  $R[X_1,...,X_n]_{(\mathfrak{m},X_1,...,X_n)}$  is Tor-persistent.

While some papers in the literature approach the question raised in [6] by finding specific conditions that imply Tor-persistence, we show that Tor-persistence is a property preserved by standard procedures in local algebra. Our work is motivated by [10] where a result similar to Theorem 1.1 is proved for the so-called Auslander's condition. However, our arguments are somewhat different since the techniques used in *loc. cit.* do not work in our setting; see Remark 2.3 and [10, Cor. (2.2)].

It should be noticed that there is some overlap between this paper and [6]. For example, the equivalence  $(i) \Leftrightarrow (ii)$  in Theorem 1.1 is contained in [6, Prop. 1.5], and our Proposition 2.2 is akin to [6, Prop. 3.8]. However, the two papers have been written completely independently, indeed, [6] were only made available to us after we completed this work. Subsequently, we rewrote our introduction and adopted the terminology "Tor-persistent" coined in [6].

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<sup>&</sup>lt;sup>1</sup>Note that this work is announced under the different title *Vanishing of endohomology over local rings* in [5].

This short paper is organized as follows. In Section 2 we prove Theorem 1.1 and show how to construct new examples of Tor-persistent rings (Example 2.7). We also give a way to obtain certain kinds of regular sequences in power series rings (Lemma 2.6), which might be of independent interest. In Section 3 we consider another property for rings, called (TG); it is a slightly weaker property than Tor-persistence and it is related to the Gorenstein dimension. For this property we prove a result similar to Theorem 1.1 (see Theorem 3.2), and show that some results from Section 2 can be strengthened in this new setting.

# 2. Main results

**2.1 Lemma.** Let  $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, \ell)$  be a local homomorphism of commutative noetherian local rings. If S is Tor-persistent and has finite flat dimension over R, then R is Torpersistent.

*Proof.* Assume S is Tor-persistent and let M be a finitely generated R-module such that  $\operatorname{Tor}_i^R(M,M)=0$  for all  $i\gg 0$ . We have  $\operatorname{Tor}_i^R(M,S)=0$  for each i>d, where d is the flat dimension of S over R. Replacing M by a sufficiently high syzygy we can (by dimension shifting) assume that  $\operatorname{Tor}_i^R(M,M)=0$  and  $\operatorname{Tor}_i^R(M,S)=0$  for every i>0. In this case there is an isomorphism  $M\otimes_R^R S\cong M\otimes_R S$  in the derived category over S. This yields:

$$(M \otimes_R^{\mathbf{L}} M) \otimes_R^{\mathbf{L}} S \cong (M \otimes_R^{\mathbf{L}} S) \otimes_S^{\mathbf{L}} (M \otimes_R^{\mathbf{L}} S) \cong (M \otimes_R S) \otimes_S^{\mathbf{L}} (M \otimes_R S).$$

As the complex  $M \otimes_R^L M$  is homologically bounded (its homology is even concentrated in degree zero) and since S has finite flat dimension over R, the left-hand side is homologically bounded, and hence so is the right-hand side. That is,  $\operatorname{Tor}_i^S(M \otimes_R S, M \otimes_R S) = 0$  for all  $i \gg 0$ . As S is Tor-persistent, it follows that  $M \otimes_R S \cong M \otimes_R^L S$  has finite projective dimension over S. It follows from [4, (1.5.3)] that  $\operatorname{pd}_R(M)$  is finite.

**2.2 Proposition.** Let  $(R, \mathfrak{m}, k)$  be a commutative noetherian local ring and let  $\underline{x} = x_1, \dots, x_n$  be an R-regular sequence. If  $R/(\underline{x})$  is Tor-persistent, then R is Tor-persistent. The converse is true if  $x_i \notin \mathfrak{m}^2 + (x_1, \dots, x_{i-1})$  holds for every  $i = 1, \dots, n$ .

*Proof.* The first statement is a special case of Lemma 2.1. We now prove the (partial) converse. By assumption,  $\bar{x}_i$  is a non zero-divisor on  $R/(x_1,\ldots,x_{i-1})$ , which has the maximal ideal  $\bar{\mathfrak{m}} = \mathfrak{m}/(x_1,\ldots,x_{i-1})$ . Since  $x_i \notin \mathfrak{m}^2 + (x_1,\ldots,x_{i-1})$  we have  $\bar{x}_i \notin \bar{\mathfrak{m}}^2$ , so by induction it suffices to consider the case where n=1.

Let R be Tor-persistent and let  $x \in \mathfrak{m} \setminus \mathfrak{m}^2$  be a non zero-divisor on R. To see that R/(x) is Tor-persistent, let N be a finitely generated R/(x)-module with  $\operatorname{Tor}_i^{R/(x)}(N,N) = 0$  for all  $i \gg 0$ . By [21, 11.65] (see also [13, Lem. 2.1]) there is a long exact sequence,

$$\cdots \longrightarrow \operatorname{Tor}_{i-1}^{R/(x)}(N,N) \longrightarrow \operatorname{Tor}_{i}^{R}(N,N) \longrightarrow \operatorname{Tor}_{i}^{R/(x)}(N,N) \longrightarrow \cdots.$$

Therefore  $\operatorname{Tor}_{i}^{R}(N,N)=0$  for all  $i\gg 0$ . Since R is Tor-persistent, we get that  $\operatorname{pd}_{R}(N)$  is finite. As  $x\notin \mathfrak{m}^{2}$ , it follows that  $\operatorname{pd}_{R/(x)}(N)$  is finite; see e.g. [2, Prop. 3.3.5(1)].

- **2.3 Remark.** It would be interesting to know if the last assertion in Proposition 2.2 holds without the assumption  $x_i \notin \mathfrak{m}^2 + (x_1, \dots, x_{i-1})$ , i.e. if Tor-persistence is preserved when passing to the quotient by an ideal generated by *any* regular sequence; cf. Proposition 3.1.
- **2.4 Remark.** The sequence  $X_1, ..., X_n$  is regular on  $R[X_1, ..., X_n]$  and  $X_i$  does not belong to  $(\mathfrak{m}, X_1, ..., X_n)^2 + (X_1, ..., X_{i-1})$ . It follows from Proposition 2.2 that R is Tor-persistent if and only if  $R[X_1, ..., X_n]$  is Tor-persistent.

Proposition 2.2 can be used to construct new examples of Tor-persistent rings from known examples; see Example 2.7. However, to do so it is useful to have a concrete way of constructing regular sequences with the property mentioned in 2.2. In Lemma 2.6 below we give one such construction.

If A is a commutative ring and a is an element in A, then it can happen, perhaps surprisingly, that X - a is a zero-divisor on A[X]; see [12, p. 146] for an example. However, as is well-known, if A is noetherian, then the situation is much nicer:

- **2.5.** Let *A* be a commutative noetherian ring and consider an elemement  $f = f(X_1, ..., X_n)$  in  $A[X_1, ..., X_n]$ . It follows from [11, Thm. 5] that if f has some coefficient which is a unit in A, then f is a non zero-divisor on  $A[X_1, ..., X_n]$ .
- **2.6 Lemma.** Let  $(R, \mathfrak{m}, k)$  be a commutative noetherian local ring. Consider the power series ring  $S = R[X_1, ..., X_n]$  and write  $\mathfrak{n} = (\mathfrak{m}, X_1, ..., X_n)$  for its unique maximal ideal. Let  $0 = m_0 < m_1 < \cdots < m_{t-1} < m_t = n$  be integers and let  $f_1, ..., f_t \in \mathfrak{n}$  be elements such that, for every i = 1, ..., t, the following conditions hold:
  - (a)  $f_i \in R[X_1,\ldots,X_{m_i}] \subseteq S$ .
  - (b) The element  $\frac{\partial f_i}{\partial X_i}(0,...,0) \in R$  is a unit for some  $m_{i-1} < j$ .

Then  $f_1, ..., f_t$  is a regular sequence on  $R[X_1, ..., X_n]$  with  $f_i \notin \mathfrak{n}^2 + (f_1, ..., f_{i-1})$  for all i.

*Proof.* First note that condition (b) implies:

The power series  $f_i(0,...,0,X_{m_{i-1}+1},...,X_n)$  has a coefficient which is a unit in R. (2.1) Indeed, if  $m_{i-1} < j$ , then  $\frac{\partial f_i}{\partial X_i}(0,...,0)$  is a coefficient in  $f_i(0,...,0,X_{m_{i-1}+1},...,X_n)$ .

Next we show that  $f_1, \ldots, f_t$  is a regular sequence. With i = 1 condition (2.1) says that  $f_1(X_1, \ldots, X_n)$  has a coefficient which is a unit in R, and so  $f_1$  is a non zero-divisor on S by 2.5. Next we show that  $f_{i+1}$  is a non zero-divisor on  $S/(f_1, \ldots, f_i)$  where  $i \ge 1$ . Write

$$f_{i+1} = \sum_{\nu_{m_i+1},\dots,\nu_n} h_{\nu_{m_i+1},\dots,\nu_n} X_{m_i+1}^{\nu_{m_i+1}} \cdots X_n^{\nu_n} \in S \cong R[X_1,\dots,X_{m_i}][X_{m_i+1},\dots,X_n]$$
(2.2)

with  $h_* \in R[X_1, ..., X_{m_i}]$ . As  $f_1, ..., f_i \in R[X_1, ..., X_{m_i}]$  by (a) there is an isomorphism:

$$S/(f_1,...,f_i) \cong (R[X_1,...,X_{m_i}]/(f_1,...,f_i))[X_{m_i+1},...,X_n].$$
 (2.3)

In particular, the image  $\bar{f}_{i+1}$  of  $f_{i+1}$  in  $S/(f_1,...,f_i)$  can be identified with the element

$$\bar{f}_{i+1} = \sum_{\nu_{m_i+1},\dots,\nu_n} \tilde{h}_{\nu_{m_i+1},\dots,\nu_n} X_{m_i+1}^{\nu_{m_i+1}} \cdots X_n^{\nu_n}$$

in the right-hand side of (2.3), where  $\tilde{h}_*$  is the image of  $h_*$  in  $R[X_1,\ldots,X_{m_i}]/(f_1,\ldots,f_i)$ . Hence, to show that  $\bar{f}_{i+1}$  is a non zero-divisor, it suffices by 2.5 to argue that one of the coefficients  $\tilde{h}_*$  is a unit. By (2.1) we know that  $f_{i+1}(0,\ldots,0,X_{m_i+1},\ldots,X_n)$  has a coefficient which is a unit in R, and by (2.2) this means that one of the elements  $h_{v_{m_i+1},\ldots,v_n}(0,\ldots,0) \in R$  is a unit. Consequently  $h_{v_{m_i+1},\ldots,v_n} = h_{v_{m_i+1},\ldots,v_n}(X_1,\ldots,X_{m_i})$  will be a unit in  $R[X_1,\ldots,X_{m_i}]$ , so its image  $\tilde{h}_{v_{m_i+1},\ldots,v_n}$  is also a unit, as desired.

Next we show that  $f_i \notin \mathfrak{n}^2 + (f_1, \dots, f_{i-1})$  holds for all i. Suppose for contradiction that:

$$f_i = \sum_{v} p_v q_v + \sum_{w=1}^{i-1} g_w f_w$$
, where  $p_v, q_v \in \mathfrak{n}$  and  $g_w \in S$ .

By assumption (b) we have that  $\frac{\partial f_i}{\partial X_j}(0,\ldots,0) \in R$  is a unit for some  $m_{i-1} < j$ . It follows from the identity above that:

$$\frac{\partial f_i}{\partial X_j}(\underline{0}) = \sum_{\nu} \left( \frac{\partial p_{\nu}}{\partial X_j}(\underline{0}) q_{\nu}(\underline{0}) + p_{\nu}(\underline{0}) \frac{\partial q_{\nu}}{\partial X_j}(\underline{0}) \right) + \sum_{w=1}^{i-1} \left( \frac{\partial g_w}{\partial X_j}(\underline{0}) f_w(\underline{0}) + g_w(\underline{0}) \frac{\partial f_w}{\partial X_j}(\underline{0}) \right).$$

As already mentioned, the left-hand side is a unit, and this contradicts that the right-hand side belongs to m. Indeed, we have  $p_{\nu}(\underline{0}), q_{\nu}(\underline{0}), f_{w}(\underline{0}) \in \mathfrak{m}$  as  $p_{\nu}, q_{\nu}, f_{w} \in \mathfrak{n}$ . Furthermore,  $f_{1}, \ldots, f_{i-1}$  only depend on the variables  $X_{1}, \ldots, X_{m_{i-1}}$  by (a), so every  $\frac{\partial f_{w}}{\partial X_{i}}$  is zero.

**2.7 Example.** In R[U, V, W] the following (more or less arbitrarily chosen) sequence, corresponding to t = 2 and  $m_1 = 2$ , satisfies the assumptions of Lemma 2.6:

$$f_1 = a + U^3 + UV + V$$
 and  $f_2 = b + UV^2 + W + W^2$   $(a, b \in \mathfrak{m})$ 

Indeed, (a) is clear and (b) holds since  $\frac{\partial f_1}{\partial V}(0,0,0) = 1 = \frac{\partial f_2}{\partial W}(0,0,0)$ . So Proposition 2.2 implies that if R is Tor-persistent, then so is  $A = R[U,V,W]/(f_1,f_2)$ .

Note that the fiber product ring

$$R = k[X]/(X^4) \times_k k[Y]/(Y^3) \cong k[X,Y]/(X^4,Y^3,XY)$$

is artinian, not Gorenstein, and by [18, Thm. 1.1] it is Tor-persistent. Hence the following ring (where we have chosen  $a = Y^2$  and  $b = X^2$ ) is Tor-persistent as well:

$$A = k[X, Y, U, V, W]/(X^4, Y^3, XY, Y^2 + U^3 + UV + V, X^2 + UV^2 + W + W^2)$$
.

*Proof of Theorem 1.1.* The equivalence  $(i) \Leftrightarrow (iii)$  is noted in Remark 2.4. Let  $a_1, \ldots, a_n$  be a set of elements that generate m. We have  $\widehat{R} \cong R[X_1, \ldots, X_n]/(X_1 - a_1, \ldots, X_n - a_n)$  by [17, Thm. 8.12]. The sequence  $f_i = X_i - a_i$  clearly satisfies the assumptions in Lemma 2.6, so the equivalence  $(i) \Leftrightarrow (ii)$  follows. Note that  $R[X_1, \ldots, X_n]_{(\mathfrak{m}, X_1, \ldots, X_n]}$  and  $R[X_1, \ldots, X_n]$  have isomorphic completions (both are isomorphic to  $\widehat{R}[X_1, \ldots, X_n]$ ), so the equivalence  $(iii) \Leftrightarrow (iv)$  follows from the already established equivalence between (i) and (ii).

## 3. Connections with the Gorenstein dimension

In this section, we give a few remarks and observations pertaining Aulander's G-dimension [1] and self Tor vanishing. For a commutative noetherian local ring  $(R, \mathfrak{m}, k)$ , we consider the following property (which R may, or may not, have):

(TG) Every finitely generated *R*-module *M* satisfying  $\operatorname{Tor}_i^R(M,M) = 0$  for all  $i \gg 0$  has finite G-dimension, that is,  $\operatorname{G-dim}_R(M) < \infty$ .

Every Tor-persistent ring has the property (TG), see [9, Prop. (1.2.10)], and the converse holds if the maximal ideal m is decomposable; see [20, Thm. 5.5].

Testing finiteness of the G-dimension via the vanishing of Tor, in some form, is an idea pursued in a number of papers. For example, in [7, Thm. 3.11] it was proved that a finitely generated module M over a commutative noetherian ring R has finite G-dimension if and only if the stable homology  $\widehat{\operatorname{Tor}}_{i}^{R}(M,R)$  vanishes for every  $i \in \mathbb{Z}$ . Furthermore, finitely generated modules testing finiteness of the G-dimension via the vanishing of absolute homology, i.e. Tor, were also examined in [8].

For the property (TG) we have the following stronger version of Proposition 2.2.

**3.1 Proposition.** Let  $(R, \mathfrak{m}, k)$  be a commutative noetherian local ring and let  $\underline{x} = x_1, \dots, x_n$  be an R-regular sequence. Then R has the property (TG) if and only if  $R/(\underline{x})$  has it.

*Proof.* For the "if" part we proceed as in the proof of Lemma 2.1 with  $S = R/(\underline{x})$ . Note that having replaced M with a sufficiently high syzygy, the sequence  $\underline{x}$  becomes regular on M (this is standard but see also [19, Lem. 5.1]). From the finiteness of G-dim $_{R/(\underline{x})}(M/(\underline{x})M)$  we infer the finiteness of G-dim $_{R}(M)$  from [9, Cor. (1.4.6)]. For the "only if" part proceed as in the proof of Proposition 2.2. From the finiteness of G-dim $_{R}(N)$  one always gets finiteness of G-dim $_{R/(x)}(N)$  (the assumption  $x \notin \mathfrak{m}^2$  is not needed) by [9, Thm. p. 39].  $\square$ 

Now the arguments in the proof of Theorem 1.1 applies and give the following.

- **3.2 Theorem.** Let  $(R, \mathfrak{m}, k)$  be a commutative noetherian local ring. The following conditions are equivalent:
  - (i) R has the property (TG).
  - (ii)  $\widehat{R}$  has the property (TG).
- (iii)  $R[X_1,...,X_n]$  has the property (TG).
- (iv)  $R[X_1,...,X_n]_{(\mathfrak{m},X_1,...,X_n)}$  has the property (TG).

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