ON MODULES WITH SELF TOR VANISHING

OLGUR CELIKBAS AND HENRIK HOLM

ABSTRACT. The long-standing Auslander and Reiten Conjecture states that a finitely generated module M over a finite-dimensional algebra A with $\mathrm{Ext}^i_A(M,M) = \mathrm{Ext}^i_A(M,A) = 0$ for all $i > 0$ must be projective. Inspired by work of Avramov, Buchweitz, Jorgensen, Şega, and others, we consider in this paper a possible counterpart of the conjecture for commutative local rings, in terms of vanishing of Tor. Our main result shows that the class of local rings that does satisfy this counterpart is closed under standard procedures in ring theory.

1. INTRODUCTION

Throughout R denotes a commutative noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k. Our purpose is to establish a result, and also to give some observations and examples, which might be useful in further studies of the following open problem:

1.1 Question. If M is a finitely generated R-module with $\mathrm{Tor}_i^R(M,M) = 0$ for all $i \gg 0$, then must M have finite projective dimension?

This question, if true for all rings R, may be considered as the "homology counterpart" of the celebrated Auslander and Reiten Conjecture [2] for commutative rings. Question 1.1 was previously studied by several people, and various partial results were recently obtained for special cases; see, for example, [6, 18, 19, 20]. It is likely that a complete solution to the question above will yield a new go and perspective on homological commutative algebra.

Question 1.1 appears implicitly in Şega's paper [22, paragraph preceding Thm. 2.6], and in Thm. 2 in *loc. cit.* a positive answer is given in the case where R is Gorenstein, admits an exact zero divisor, and $\mathfrak{m}^4 = 0$. Other classes of rings over which Question 1.1 is known to be true include complete intersection rings [15, Cor. (1.2)] (see also [4, Thm. IV] and [14, Thm. 1.9]) and Golod rings [16, Thm. 3.1].

To further the study of the question we consider the following condition on R:

(TP) Every finitely generated R-module M satisfying $\mathrm{Tor}_i^R(M,M) = 0$ for all $i \gg 0$ has finite projective dimension, that is, $\mathrm{pd}_R(M) < \infty$.

To the best of our knowledge, our approach to Question 1.1 is different from that of the papers in the literature: instead of determining specific conditions that give an affirmative answer to Question 1.1, we show that the property (TP) is preserved by standard procedures in local algebra. Our main result can be stated as follows:

1.2 Theorem. *The following conditions are equivalent:*

(*i*) R *satisfies* (TP).

(*ii*) $\widehat{R}$ *satisfies* (TP).

(*iii*) $R[\![X_1,\ldots,X_n]\!]$ *satisfies* (TP).

(*iv*) $R[X_1,\ldots,X_n]_{(\mathfrak{m},X_1,\ldots,X_n)}$ *satisfies* (TP).

2010 *Mathematics Subject Classification.* 13D05, 13D07.

Key words and phrases. G-dimension; projective dimension; vanishing of Tor.

One can use Theorem 1.2 (in conjunction with Lemma 2.6) to construct new examples of rings satisfying (TP); see Example 2.7. We point out that our work is motivated by [10] where a result, similar to the one above, is proved for the so-called Auslander's condition. However, our arguments are quite different since the techniques used in *loc. cit.* do not work in our setting; see Remark 2.3 and [10, Cor. (2.2)].

In Section 2 we prove Theorem 1.2 and show how to construct examples of local rings satisfying (TP). Furthermore, we give a way to obtain certain kinds of regular sequences in power series rings, which might be of independent interest. In Section 3 we consider a slightly weaker version of the condition (TP), which we call (TG); this condition is related to the Gorenstein dimension. We prove a result similar to Theorem 1.2, and show that some results from Section 2 can be strengthened in this new setting.

2. Main results

2.1 Lemma. *Let $(R,\mathfrak{m},k) \to (S,\mathfrak{n},\ell)$ be a local homomorphism of commutative noetherian local rings. If S satisfies* (TP) *and has finite flat dimension over R, then R satisfies* (TP).

Proof. Assume S satisfies (TP) and let M be a finitely generated R-module such that $\operatorname{Tor}_i^R(M,M)=0$ for all $i \gg 0$. We have $\operatorname{Tor}_i^R(M,S)=0$ for each $i>d$, where d is the flat dimension of S over R. Replacing M by a sufficiently high syzygy we can (by dimension shifting) assume that $\operatorname{Tor}_i^R(M,M)=0$ and $\operatorname{Tor}_i^R(M,S)=0$ for every $i>0$. In this case there is an isomorphism $M \otimes_R^{\mathbf{L}} S \cong M \otimes_R S$ in the derived category over S. This yields:

$$(M \otimes_R^{\mathbf{L}} M) \otimes_R^{\mathbf{L}} S \cong (M \otimes_R^{\mathbf{L}} S) \otimes_S^{\mathbf{L}} (M \otimes_R^{\mathbf{L}} S) \cong (M \otimes_R S) \otimes_S^{\mathbf{L}} (M \otimes_R S)\,.$$

As the complex $M \otimes_R^{\mathbf{L}} M$ is homologically bounded (its homology is even concentrated in degree zero) and since S has finite flat dimension over R, the left-hand side is homologically bounded, and hence so is the right-hand side. That is, $\operatorname{Tor}_i^S(M \otimes_R S, M \otimes_R S)=0$ for all $i \gg 0$. As S satisfies (TP), it follows that $M \otimes_R S \cong M \otimes_R^{\mathbf{L}} S$ has finite projective dimension over S. It follows from [5, (1.5.3)] that $\operatorname{pd}_R(M)$ is finite. □

2.2 Proposition. *Let $(R,\mathfrak{m},k)$ be a commutative noetherian local ring and let $\underline{x}=x_1,\dots,x_n$ be an R-regular sequence. If $R/(\underline{x})$ satisfies* (TP)*, then R satisfies* (TP)*. The converse is true if $x_i \notin \mathfrak{m}^2+(x_1,\dots,x_{i-1})$ holds for every $i=1,\dots,n$.*

Proof. The first statement is a special case of Lemma 2.1. We now prove the (partial) converse. By assumption, $\bar{x}_i$ is a non zero-divisor on $R/(x_1,\dots,x_{i-1})$, which has the maximal ideal $\bar{\mathfrak{m}}=\mathfrak{m}/(x_1,\dots,x_{i-1})$. Since $x_i \notin \mathfrak{m}^2+(x_1,\dots,x_{i-1})$ we have $\bar{x}_i \notin \bar{\mathfrak{m}}^2$, so by induction it suffices to consider the case where $n=1$.

Assume that R satisfies (TP) and let $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ be a non zero-divisor on R. To see that $R/(x)$ satisfies (TP), let N be a finitely generated $R/(x)$-module with $\operatorname{Tor}_i^{R/(x)}(N,N)=0$ for all $i \gg 0$. By [21, 11.65] (see also [13, Lem. 2.1]) there is a long exact sequence,

$$\cdots \longrightarrow \operatorname{Tor}_{i-1}^{R/(x)}(N,N) \longrightarrow \operatorname{Tor}_i^R(N,N) \longrightarrow \operatorname{Tor}_i^{R/(x)}(N,N) \longrightarrow \cdots\,.$$

Therefore $\operatorname{Tor}_i^R(N,N)=0$ for all $i \gg 0$. Since R satisfies (TP), we get that $\operatorname{pd}_R(N)$ is finite. As $x \notin \mathfrak{m}^2$, it follows that $\operatorname{pd}_{R/(x)}(N)$ is finite; see, for example, [3, Prop. 3.3.5(1)]. □

2.3 Remark. It would be interesting to know if the last assertion in Proposition 2.2 holds without the assumption $x_i \notin \mathfrak{m}^2+(x_1,\dots,x_{i-1})$, i.e. if the property (TP) is preserved when passing to the quotient by an ideal generated by *any* regular sequence; cf. Proposition 3.1.

2.4 Remark. The sequence $X_1,\dots,X_n$ is regular on $R[\![X_1,\dots,X_n]\!]$ and X_i does not belong to $(\mathfrak{m},X_1,\dots,X_n)^2+(X_1,\dots,X_{i-1})$. It follows from Proposition 2.2 that R safisfies (TP) if and only if $R[\![X_1,\dots,X_n]\!]$ safisfies (TP).

Proposition 2.2 can be used to construct new examples of rings satisfying (TP) from known examples; see Example 2.7. However, to do so it is useful to have a concrete way of constructing regular sequences with the property mentioned in 2.2. In Lemma 2.6 below we give one such construction.

If A is a commutative ring and a is an element in A, then it can happen, perhaps surprisingly, that $X-a$ is a zero-divisor on $A[\![X]\!]$; see [12, p. 146] for an example. However, as is well-known, if A is noetherian, then the situation is much nicer:

2.5. Let A be a commutative noetherian ring and consider an elemement $f=f(X_1,\dots,X_n)$ in $A[\![X_1,\dots,X_n]\!]$. It follows from [11, Thm. 5] that if f has *some* coefficient which is a unit in A, then f is a non zero-divisor on $A[\![X_1,\dots,X_n]\!]$.

2.6 Lemma. *Let $(R,\mathfrak{m},k)$ be a commutative noetherian local ring. Consider the power series ring $S=R[\![X_1,\dots,X_n]\!]$ and write $\mathfrak{n}=(\mathfrak{m},X_1,\dots,X_n)$ for its unique maximal ideal. Let $0=m_0<m_1<\cdots<m_{t-1}<m_t=n$ be integers and let $f_1,\dots,f_t\in\mathfrak{n}$ be elements such that, for every $i=1,\dots,t$, the following conditions hold:*

(a) $f_i\in R[\![X_1,\dots,X_{m_i}]\!]\subseteq S$.

(b) *The element $\frac{\partial f_i}{\partial X_j}(0,\dots,0)\in R$ is a unit for some $m_{i-1}<j$.*

Then $f_1,\dots,f_t$ is a regular sequence on $R[\![X_1,\dots,X_n]\!]$ with $f_i\notin\mathfrak{n}^2+(f_1,\dots,f_{i-1})$ for all i.

Proof. First note that condition (b) implies:

$$\text{The power series } f_i(0,\dots,0,X_{m_{i-1}+1},\dots,X_n) \text{ has a coefficient which is a unit in } R. \quad (2.1)$$

Indeed, if $m_{i-1}<j$, then $\frac{\partial f_i}{\partial X_j}(0,\dots,0)$ *is* a coefficient in $f_i(0,\dots,0,X_{m_{i-1}+1},\dots,X_n)$.

Next we show that $f_1,\dots,f_t$ is a regular sequence. With $i=1$ condition (2.1) says that $f_1(X_1,\dots,X_n)$ has a coefficient which is a unit in R, and so f_1 is a non zero-divisor on S by 2.5. Next we show that f_{i+1} is a non zero-divisor on $S/(f_1,\dots,f_i)$ where $i\geqslant 1$. Write

$$f_{i+1} = \textstyle\sum_{v_{m_i+1},\dots,v_n} h_{v_{m_i+1},\dots,v_n} X_{m_i+1}^{v_{m_i+1}}\cdots X_n^{v_n} \in S\cong R[\![X_1,\dots,X_{m_i}]\!][\![X_{m_i+1},\dots,X_n]\!] \quad (2.2)$$

with $h_*\in R[\![X_1,\dots,X_{m_i}]\!]$. As $f_1,\dots,f_i\in R[\![X_1,\dots,X_{m_i}]\!]$ by (a) there is an isomorphism:

$$S/(f_1,\dots,f_i)\cong \big(R[\![X_1,\dots,X_{m_i}]\!]/(f_1,\dots,f_i)\big)[\![X_{m_i+1},\dots,X_n]\!]\,. \quad (2.3)$$

In particular, the image $\bar{f}_{i+1}$ of f_{i+1} in $S/(f_1,\dots,f_i)$ can be identified with the element

$$\bar{f}_{i+1} = \textstyle\sum_{v_{m_i+1},\dots,v_n} \tilde{h}_{v_{m_i+1},\dots,v_n} X_{m_i+1}^{v_{m_i+1}}\cdots X_n^{v_n}$$

in the right-hand side of (2.3), where $\tilde{h}_*$ is the image of h_* in $R[\![X_1,\dots,X_{m_i}]\!]/(f_1,\dots,f_i)$. Hence, to show that $\bar{f}_{i+1}$ is a non zero-divisor, it suffices by 2.5 to argue that one of the coefficients $\tilde{h}_*$ is a unit. By (2.1) we know that $f_{i+1}(0,\dots,0,X_{m_i+1},\dots,X_n)$ has a coefficient which is a unit in R, and by (2.2) this means that one of the elements $h_{v_{m_i+1},\dots,v_n}(0,\dots,0)\in R$ is a unit. Consequently $h_{v_{m_i+1},\dots,v_n}=h_{v_{m_i+1},\dots,v_n}(X_1,\dots,X_{m_i})$ will be a unit in $R[\![X_1,\dots,X_{m_i}]\!]$, so its image $\tilde{h}_{v_{m_i+1},\dots,v_n}$ is also a unit, as desired.

Next we show that $f_i\notin\mathfrak{n}^2+(f_1,\dots,f_{i-1})$ holds for all i. Suppose for contradiction that:

$$f_i = \textstyle\sum_v p_vq_v + \sum_{w=1}^{i-1} g_wf_w\,, \text{ where } p_v,q_v\in\mathfrak{n} \text{ and } g_w\in S.$$

By assumption (b) we have that $\frac{\partial f_i}{\partial X_j}(0,\dots,0) \in R$ is a unit for some $m_{i-1} < j$. It follows from the identity above that:

$$\tfrac{\partial f_i}{\partial X_j}(\underline{0}) \,=\, \textstyle\sum_v \left(\tfrac{\partial p_v}{\partial X_j}(\underline{0})\, q_v(\underline{0}) + p_v(\underline{0})\,\tfrac{\partial q_v}{\partial X_j}(\underline{0})\right) + \textstyle\sum_{w=1}^{i-1}\left(\tfrac{\partial g_w}{\partial X_j}(\underline{0})\, f_w(\underline{0}) + g_w(\underline{0})\,\tfrac{\partial f_w}{\partial X_j}(\underline{0})\right).$$

As already mentioned, the left-hand side is a unit, and this contradicts that the right-hand side belongs to $\mathfrak{m}$. Indeed, we have $p_v(\underline{0}), q_v(\underline{0}), f_w(\underline{0}) \in \mathfrak{m}$ as $p_v, q_v, f_w \in \mathfrak{n}$. Furthermore, $f_1,\dots,f_{i-1}$ only depend on the variables $X_1,\dots,X_{m_{i-1}}$ by (a), so every $\frac{\partial f_w}{\partial X_j}$ is zero. □

2.7 Example. In $R[\![U,V,W]\!]$ the following (more or less arbitrarily chosen) sequence, corresponding to $t=2$ and $m_1 = 2$, satisfies the assumptions of Lemma 2.6:

$$f_1 = a + U^3 + UV + V \quad \text{and} \quad f_2 = b + UV^2 + W + W^2 \qquad (a, b \in \mathfrak{m}).$$

Indeed, (a) is clear and (b) holds since $\frac{\partial f_1}{\partial V}(0,0,0) = 1 = \frac{\partial f_2}{\partial W}(0,0,0)$. So Proposition 2.2 implies that if R satisfies (TP), then so does $A = R[\![U,V,W]\!]/(f_1, f_2)$.

Note that the fiber product ring

$$R \,=\, k[\![X]\!]/(X^4) \times_k k[\![Y]\!]/(Y^3) \,\cong\, k[\![X,Y]\!]/(X^4,\, Y^3,\, XY)$$

is artinian, not Gorenstein, and by [18, Thm. 1.1] it satisfies (TP). Hence (TP) also holds for the following ring (where we have chosen $a = Y^2$ and $b = X^2$):

$$A = k[\![X,Y,U,V,W]\!]/(X^4,\, Y^3,\, XY,\, Y^2 + U^3 + UV + V,\; X^2 + UV^2 + W + W^2)\,. \qquad \square$$

Proof of Theorem 1.2. The equivalence $(i) \Leftrightarrow (iii)$ is noted in Remark 2.4. Let $a_1,\dots,a_n$ be a set of elements that generate $\mathfrak{m}$. We have $\widehat{R} \cong R[\![X_1,\dots,X_n]\!]/(X_1 - a_1,\dots,X_n - a_n)$ by [17, Thm. 8.12]. The sequence $f_i = X_i - a_i$ clearly satisfies the assumptions in Lemma 2.6, so the equivalence $(i) \Leftrightarrow (ii)$ follows. Note that $R[X_1,\dots,X_n]_{(\mathfrak{m},X_1,\dots,X_n)}$ and $R[\![X_1,\dots,X_n]\!]$ have isomorphic completions (both are isomorphic to $\widehat{R}[\![X_1,\dots,X_n]\!]$), so the equivalence $(iii) \Leftrightarrow (iv)$ follows from the already established equivalence between (i) and (ii). □

3. Connections with the Gorenstein dimension

In this section, we give a few remarks and observations pertaining Aulander's G-dimension [1] and self Tor vanishing. For a commutative noetherian local ring R, we consider the following variant of the condition (TP).

(TG) Every finitely generated R-module M satisfying $\operatorname{Tor}_i^R(M,M) = 0$ for all $i \gg 0$ has finite G-dimension, that is, $\operatorname{G-dim}_R(M) < \infty$.

In general, (TG) is weaker than (TP), see [9, Prop. (1.2.10)], but the two conditions are equivalent if the maximal ideal $\mathfrak{m}$ of R is decomposable; see [19, Thm. 5.5].

Testing finiteness of the G-dimension via the vanishing of Tor, in some form, is an idea pursued in a number of papers. For example, in [7, Thm. 3.11] it was proved that a finitely generated module M over a commutative noetherian ring R has finite G-dimension if and only if the stable homology $\widetilde{\operatorname{Tor}}_i^R(M,R)$ vanishes for every $i \in \mathbb{Z}$. Furthermore, finitely generated modules testing finiteness of the G-dimension via the vanishing of absolute homology, i.e. Tor, were also examined in [8].

For the property (TG) we have the following stronger version of Proposition 2.2.

3.1 Proposition. *Let $(R,\mathfrak{m},k)$ be a commutative noetherian local ring and let $\underline{x} = x_1,\dots,x_n$ be an R-regular sequence. Then R satisfies* (TG) *if and only if $R/(\underline{x})$ satisfies* (TG).

Proof. For the "if" part we proceed as in the proof of Lemma 2.1 with $S = R/(\underline{x})$. Note that having replaced M with a sufficiently high syzygy, the sequence $\underline{x}$ becomes regular on M (this is standard but see also [20, Lem. 5.1]). From the finiteness of $\text{G-dim}_{R/(\underline{x})}(M/(\underline{x})M)$ we infer the finiteness of $\text{G-dim}_R(M)$ from [9, Cor. (1.4.6)]. For the "only if" part proceed as in the proof of Proposition 2.2. From the finiteness of $\text{G-dim}_R(N)$ one always gets finiteness of $\text{G-dim}_{R/(x)}(N)$ (the assumption $x \notin \mathfrak{m}^2$ is not needed) by [9, Thm. p. 39]. □

Now the arguments in the proof of Theorem 1.2 applies and give the following.

3.2 Theorem. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. The following conditions are equivalent:*

- (i) *R satisfies* (TG).
- (ii) *$\widehat{R}$ satisfies* (TG).
- (iii) *$R[\![X_1, \ldots, X_n]\!]$ satisfies* (TG).
- (iv) *$R[X_1, \ldots, X_n]_{(\mathfrak{m}, X_1, \ldots, X_n)}$ satisfies* (TG). □

ACKNOWLEDGMENTS

Part of this work was completed when Holm visited West Virginia University in March 2018. He is grateful for the kind hospitality of the WVU Department of Mathematics.

REFERENCES

1. Maurice Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Secrétariat mathématique, Paris, 1967, Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des exposés de Maurice Auslander, par Marquerite Mangeney, Christian Peskine et Lucien Szpiro. École Normale Supérieure de Jeunes Filles. Available at http://www.numdam.org. MR0225844
2. Maurice Auslander and Idun Reiten, *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc. **52** (1975), 69–74. MR0389977
3. Luchezar L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1–118. MR1648664
4. Luchezar L. Avramov and Ragnar-Olaf Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. **142** (2000), no. 2, 285–318. MR1794064
5. Luchezar L. Avramov and Hans-Bjørn Foxby, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270. MR1455856
6. Luchezar L. Avramov, Srikanth B. Iyengar, Saeed Nasseh, and Sean Sather-Wagstaff, *Homology over trivial extensions of commutative dg algebras*, 2015, arXiv:1508.00748v1 [math.AC].
7. Olgur Celikbas, Lars Winther Christensen, Li Liang, and Greg Piepmeyer, *Stable homology over associative rings*, Trans. Amer. Math. Soc. **369** (2017), no. 11, 8061–8086. MR3695854
8. Olgur Celikbas and Sean Sather-Wagstaff, *Testing for the Gorenstein property*, Collect. Math. **67** (2016), no. 3, 555–568. MR3536062
9. Lars Winther Christensen, *Gorenstein dimensions*, Lecture Notes in Math., vol. 1747, Springer-Verlag, Berlin, 2000. MR1799866
10. Lars Winther Christensen and Henrik Holm, *Vanishing of cohomology over Cohen–Macaulay rings*, Manuscripta Math. **139** (2012), no. 3-4, 535–544. MR2974289
11. David E. Fields, *Zero divisors and nilpotent elements in power series rings*, Proc. Amer. Math. Soc. **27** (1971), 427–433. MR0271100
12. Robert Gilmer, Anne Grams, and Tom Parker, *Zero divisors in power series rings*, J. Reine Angew. Math. **278/279** (1975), 145–164. MR0387274
13. Craig Huneke and Roger Wiegand, *Tensor products of modules and the rigidity of* Tor, Math. Ann. **299** (1994), no. 3, 449–476. MR1282227
14. ———, *Tensor products of modules, rigidity and local cohomology*, Math. Scand. **81** (1997), no. 2, 161–183. MR1612887
15. David A. Jorgensen, *Tor and torsion on a complete intersection*, J. Algebra **195** (1997), no. 2, 526–537. MR1469637

16. ______, *A generalization of the Auslander-Buchsbaum formula*, J. Pure Appl. Algebra **144** (1999), no. 2, 145–155. MR1732626
17. Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Stud. Adv. Math., vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR1011461
18. Saeed Nasseh and Sean Sather-Wagstaff, *Vanishing of Ext and Tor over fiber products*, Proc. Amer. Math. Soc. **145** (2017), no. 11, 4661–4674. MR3691985
19. Saeed Nasseh and Ryo Takahashi, *Structure of irreducible homomorphisms to/from free modules*, to in appear in Algebr. Represent. Theory, 2016, arXiv:1606.02621v2 [math.AC].
20. ______, *Local rings with quasi-decomposable maximal ideal*, preprint, 2017, arXiv:1704.00719v1 [math.AC].
21. Joseph J. Rotman, *An introduction to homological algebra*, Pure and Applied Mathematics, vol. 85, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979. MR538169
22. Liana M. Şega, *Self-tests for freeness over commutative Artinian rings*, J. Pure Appl. Algebra **215** (2011), no. 6, 1263–1269. MR2769231

DEPARTMENT OF MATHEMATICS, WEST VIRGINIA UNIVERSITY, MORGANTOWN, WV 26506 U.S.A
E-mail address: olgur.celikbas@math.wvu.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITETSPARKEN 5, UNIVERSITY OF COPENHAGEN, 2100 COPENHAGEN Ø, DENMARK
E-mail address: holm@math.ku.dk
URL: http://www.math.ku.dk/~holm/