CONSTRUCTION OF TOTALLY REFLEXIVE MODULES
FROM AN EXACT PAIR OF ZERO DIVISORS

HENRIK HOLM

Abstract. Let $A$ be a local ring which admits an exact pair $x, y$ of zero divisors as defined by Henriques and Šega. Assuming that this pair is regular and that there exists a regular element on the $A$-module $A/(x, y)$, we explicitly construct an infinite family of non-isomorphic indecomposable totally reflexive $A$-modules. In this setting, our construction provides an answer to a question raised by Christensen, Piepmeyer, Striuli, and Takahashi. Furthermore, we compute the module of homomorphisms between any two given modules from the infinite family mentioned above.

1. Introduction

Throughout this paper, $A$ is a commutative noetherian local ring.

As indicated by the title, this paper is concerned with explicit constructions of totally reflexive $A$-modules, as defined by Auslander [1] in 1967 (the terminology “totally reflexive” was introduced in 2002 by Avramov and Martsinkovsky [4]).

Totally reflexive $A$-modules always exist, indeed, $A$ itself is totally reflexive. However, in general, there need not exist non-free totally reflexive $A$-modules. If $A$ is Gorenstein then the totally reflexive $A$-modules are exactly the maximal Cohen-Macaulay modules, and their representation theory is a classical field of study.

A main result of Christensen, Piepmeyer, Striuli, and Takahashi [6, Theorem B] asserts that if $A$ is not Gorenstein, then existence of one non-free totally reflexive $A$-module implies the existence of infinitely many non-isomorphic indecomposable totally reflexive $A$-modules. Unfortunately, the known proof of this interesting result is not constructive, which is why the authors of [6] raise the following.

Question ([6 (4.8)]). Assume that $A$ is not Gorenstein, and that there exists a non-free totally reflexive $A$-module. Are there constructions that produce infinite families of non-isomorphic indecomposable totally reflexive $A$-modules?

Under suitable assumptions, we give in this paper exactly such a construction. However, before we reveal the details of our construction, we will mention a couple of related results from the literature.

Assume that $A$ is complete or has an uncountable residue field. Assume furthermore that there exist a prime ideal $p$ in $A$ with grade $p > 0$ and $\dim A/p > 1$, and a totally reflexive $A$-module $M$ such that $M_p$ is not $A_p$-free. Then Takahashi [11] proves the existence of uncountably many non-isomorphic indecomposable totally reflexive $A$-modules. The proof is not constructive, but in [11, Example 4.3]
one does find an example \((A = \mathbb{Q}[x, y, z]/(x^2))\), where \(\mathbb{Q}\) is a complete local domain which is not a field) where uncountably many non-isomorphic indecomposable totally reflexive \(A\)-modules are actually constructed.

Assume that \(A\) has an embedded deformation of codimension \(c \geq 2\), i.e., there is a local ring \((\mathbb{Q}, q, k)\) and a \(Q\)-regular sequence \(x\) in \(q\) of length \(c\) such that \(A = \mathbb{Q}/(x)\). Avramov, Gasharov, and Peeva \([2]\) constructs a non-free totally reflexive \(A\)-module \(G\), from which infinitely many non-isomorphic indecomposable totally reflexive \(A\)-modules can be constructed by using results of Avramov and Iyengar \([3]\):

Consider the graded ring \(R = k[\chi_1, \ldots, \chi_c]\) where \(\chi_i\) has degree 2. From \([3, 4.1]\) and \([3\) proof of 7.4(1)] it follows that given a closed subset \(X \subseteq \text{Spec} R\) there is a module \(M_X \in \text{Thick}_A(G \oplus A) \subseteq D(A)\) with cohomological support \(\text{Supp}^*(M_X, k)\) equal to \(X\). As \(M_X \in \text{Thick}_A(G \oplus A)\) some syzygy \(G_X\) of \(M_X\) is totally reflexive, and by \([3, 4.2]\) one also has \(\text{Supp}^*(G_X, k) = X\). Clearly, if \(X \neq Y\) then \(G_X\) and \(G_Y\) are not isomorphic. Furthermore, if \(X\) is irreducible, it follows by \([3\) (3.6.2)] that 
\[
\text{Supp}^*(G_X, k) = X\]
for some indecomposable summand \(G_X\) of \(G_X\).

In the author’s opinion, the construction of \(M_X\) given in \([3\) proof of 4.1)] is not as explicit and accessible as one could hope for. Under suitable assumptions, this paper offers a construction of infinitely many non-isomorphic indecomposable totally reflexive \(A\)-modules, which is different from the ones mentioned above. Our construction has the advantage of being fairly elementary; it applies to large classes of examples, cf. Example \([2, 4]\) and Lemma \([2, 6]\); and it makes explicit computations of relevant \(\text{Hom}\)-modules possible.

Given an exact pair of zero divisors \(x, y\) in \(A\), as defined by Henriques and Sega \([9]\), it is easily seen that \(A/(x)\) and \(A/(y)\) are non-free totally reflexive \(A\)-modules. Thus, if \(A\) is not Gorenstein, one should by the main result in \([6]\) be able to construct infinitely many non-isomorphic indecomposable totally reflexive \(A\)-modules. In this paper, we define for each \(a\) in \(A\) two totally reflexive \(A\)-modules \(G_a\) and \(H_a\) as cokernels of certain \(2 \times 2\) matrices with entries in \(A\). A special case of our main Theorem \((6.1)\) reads as follows.

**Theorem.** Let \(x, y \in A\) be a regular exact pair of zero divisors, and let \(a \in A\). If \(a\) is regular on the \(A\)-module \(A/(x,y)\) then

\[
G_{a, a^2, a^3, a^4, \ldots, H_{a, a^2, a^3, H_{a, a^2, a^3}, \ldots}
\]

is a (double) infinite family of non-isomorphic, indecomposable, non-free, totally reflexive \(A\)-modules. Furthermore, one has the following identities,

\[
\text{Hom}_A(H_{a^n}, G_{a^n}) \cong \text{Hom}_A(H_{a^n}, G_{a^{m+n}}) \cong G_{a^{m+n}},
\]

\[
\text{Hom}_A(G_{a^n}, H_{a^m}) \cong \text{Hom}_A(G_{a^n}, H_{a^n}) \cong H_{a^{m+n}},
\]

\[
\text{Hom}_A(G_{a^n}, G_{a^n}) \cong \text{Hom}_A(H_{a^n}, H_{a^m}) \cong \begin{cases} 
H_{a^{m-n}} & \text{for } m > n \\
A & \text{for } m = n \\
G_{a^{m-n}} & \text{for } m < n
\end{cases}.
\]

The paper is organized as follows: In Section \([2]\) we introduce exact pairs of zero divisors and discuss regularity. In Section \([3]\) the modules \(G_a\) and \(H_a\) are defined, and in Sections \([4]\) and \([5]\) we study the indecomposability and the isomorphism classes of these modules. Section \([6]\) simply contains our main result. The final Section \([7]\) is the technical part of this paper. In this section, on which several results in Sections \([4]\), \([5]\) and \([6]\) are based, we demonstrate how to compute the module of homomorphisms between various combinations of \(G_a\) and \(H_b\).
2. Exact pairs of zero divisors and regular elements

In this section, we consider an exact pair of zero divisors as defined by Henriques and Šega [9]. We introduce a notion of regularity for such a pair and give examples.

(2.1) Definition. Two non-units \(x, y \in A\) are called an exact pair of zero divisors if \(\text{Ann}_A(x) = (y)\) and \(\text{Ann}_A(y) = (x)\).

Let \(M\) be a finitely generated \(A\)-module, and let \(a \in A\) be an element. Recall that \(a\) is weakly regular on \(M\) if multiplication by \(a\) on \(M\) is a monomorphism. If, in addition, the element \(a\) is not a unit, then \(a\) is called regular on \(M\).

A (weakly) regular element on the \(A\)-module \(M = A\) is simply referred to as a (weakly) regular element in the ring \(A\).

(2.2) Lemma. Let \(x, y \in A\) be an exact pair of zero divisors. Then the following conditions are equivalent:

(i) \(x\) is regular on \(A/(y)\).
(ii) \(y\) is regular on \(A/(x)\).
(iii) \((x) \cap (y) = 0\).

Proof. By symmetry, it suffices to prove (i) \(\Leftrightarrow\) (iii).

First assume (i), and let \(z\) be in \((x) \cap (y)\). As \(z \in (x)\) we have \(z = ax\) for some \(a\). As \(z \in (y)\) it follows that \(x[a]_y = [z]_y = [0]_y\) in \(A/(y)\), and since multiplication by \(x\) on \(A/(y)\) is a monomorphism, we conclude that \([a]_y = [0]_y\), that is, \(a \in (y)\). Consequently, \(z = ax = 0\) since, in particular, \(yx = 0\).

Next assume (iii). We must argue that multiplication by \(x\) on \(A/(y)\) is a monomorphism. Thus, \([a]_y\) in \(A/(y)\) satisfy \(x[a]_y = [0]_y\), that is, \(ax \in (y)\). Then \(ax \in (x) \cap (y) = 0\), and hence \(a \in \text{Ann}_A(x) = (y)\). Thus \([a]_y = [0]_y\) in \(A/(y)\). \(\square\)

(2.3) Definition. An exact pair of zero divisors \(x, y \in A\) that satisfies the equivalent conditions in Lemma (2.2) is called a regular exact pair of zero divisors.

(2.4) Example. Let \(Q\) be a commutative noetherian local ring, and let \(f, g\) be regular elements in \(Q\). Set \(A = Q/(fg)\) and let \(x, y \in A\) denote the cosets of \(f, g\) with respect to the ideal \((fg)\). Then \(x, y\) is an exact pair of zero divisors in \(A\); and this pair is regular if and only if \((f) \cap (g) = (fg)\) in \(Q\).

In particular, if \(f\) and \(g\) are non-zero and non-units in a noetherian local UFD \(Q\), then \(x, y\) is an exact pair of zero divisors in \(A\); and this pair is regular if and only if \(f\) and \(g\) are relatively prime.

Note that local UFDs come in all shapes and sizes (Gorenstein, non-Gorenstein etc.), in fact, by Heitmann [8], Theorem 8 every complete local ring of depth \(> 1\), in which no integer is a zero divisor, is the completion of a local UFD.

(2.5) Remark. Let \(a\) be a proper ideal in \(A\), and let \(M\) be a finitely generated module over \(A/a\). Then \(\text{depth}_A M = \text{depth}_{A/a} M\) as a sequence \(x_1, \ldots, x_n\) of non-units in \(A\) is regular on \(M\) if and only if the corresponding sequence \([x_1]_a, \ldots, [x_n]_a\) in \(A/a\) is regular on \(M\). In particular, the depth of the \(A\)-module \(A/a\) equals the depth of the ring \(A/a\).

Several results in this paper refer to a weakly regular element \(a\) on the \(A\)-module \(A/(x, y)\). For example, \(a\) could be a unit in \(A\). However, only in the case where \(a\) is
regular on the $A$-module $A/(x, y)$ do we get interesting applications of our results. For the existence of regular elements on $A/(x, y)$, we give the following.

(2.6) Lemma. Let $Q$ be a commutative noetherian local ring, and let $f, g$ be regular elements in $Q$ such that $(f) \cap (g) = (fg)$. Consider the regular exact pair $x, y$ of zero divisors in $A = Q/(fg)$ constructed in Example (2.4).

If depth $Q > 2$ then there exists regular elements on the $A$-module $A/(x, y)$.

Proof. We must prove that the $A$-module $A/(x, y)$ has depth $> 0$, equivalently, that the ring $A/(x, y)$ has depth $> 0$, cf. Remark (2.5). Since there is a ring isomorphism $A/(x, y) \cong Q/(f, g)$, we are required to show that $Q/(f, g)$ has depth $> 0$.

To finish the proof it suffices to argue that $f, g$ is a $Q$-regular sequence, because then depth $Q/(f, g) = \text{depth } Q - 2$ by Bruns and Herzog [5, Proposition 1.2.10(d)].

However, it is given that $f$ is regular on $Q$, and that $g$ is regular on $Q/(f)$ follows from the assumption $(f) \cap (g) = (fg)$ and the regularity of $g$ on $Q$. \qed

3. Two families of totally reflexive modules

Assuming that the ring $A$ admits an exact pair of zero divisors, we introduce in this section two families $(G_a)_{a \in A}$ and $(H_a)_{a \in A}$ of totally reflexive $A$-modules.

We begin with the following definition due to Auslander [1] §3.2.2. Although Auslander himself did not use the terminology “totally reflexive module”, this usage was introduced by Avramov and Martsinkovsky [4] §2 and has grown to be standard in several later papers on the subject.

(3.1) Definition. A finitely generated $A$-module $G$ is totally reflexive if it satisfies the following three conditions:

1. $\text{Ext}^i_A(G, A) = 0$ for all $i > 0$.
2. $\text{Ext}^i_A(\text{Hom}_A(G, A), A) = 0$ for all $i > 0$.
3. The biduality map $G \rightarrow \text{Hom}_A(\text{Hom}_A(G, A), A)$ is an isomorphism.

(3.2) Definition. Let $x, y \in A$ be an exact pair of zero divisors, and let $a \in A$. We define two $2 \times 2$ matrices,

$$
\gamma_a = \begin{pmatrix} x & a \\ 0 & y \end{pmatrix} \quad \text{and} \quad \eta_a = \begin{pmatrix} y & -a \\ 0 & x \end{pmatrix}.
$$

Furthermore, considering $\gamma_a$ and $\eta_a$ as $A$-linear maps $A^2 \rightarrow A^2$ acting on column vectors by multiplication from the left, we define two finitely generated $A$-modules,

$$
G_a = \text{Coker } \gamma_a \quad \text{and} \quad H_a = \text{Coker } \eta_a.
$$

(3.3) Observation. Let $x, y \in A$ be an exact pair of zero divisors, let $a$ be any element in $A$, and let $u$ be a unit in $A$. Then the commutative diagrams,

$$
\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \gamma_a & \eta_a \\ \eta_a & \gamma_a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \cong \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \cong \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}
$$

show that $G_{ua} \cong G_a$ and $H_{ua} \cong H_a$. In particular, $G_{-a} \cong G_a$ and $H_{-a} \cong H_a$. 

**Lemma.** Let \( x, y \in A \) be an exact pair of zero divisors, and let \( a \in A \). Then there is an exact complex of free \( A \)-modules given by

\[
F = \cdots \longrightarrow A^2 \xrightarrow{\gamma_a} A^2 \xrightarrow{\eta_a} A^2 \xrightarrow{\gamma_a} A^2 \xrightarrow{\eta_a} \cdots .
\]

Furthermore, there is an isomorphism of complexes \( \text{Hom}_A(F, A) \cong F \),

\[
\cdots \xrightarrow{\eta_a^t} A^2 \xrightarrow{\gamma_a^t} A^2 \xrightarrow{\eta_a^t} A^2 \xrightarrow{\gamma_a^t} \cdots \cong \phi \cong \phi \cong \phi.
\]

where \((-)^t\) is transposition of matrices, and \( \phi \) is the isomorphism given by

\[
\phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

In particular, the complex \( \text{Hom}_A(F, A) \) is exact.

**Proof.** As \( \gamma_a \eta_a = \eta_a \gamma_a = 0 \) we conclude that \( F \) is a complex. To show that \( F \) is exact, we must argue that \( \text{Ker} \gamma_a \subseteq \text{Im} \eta_a \) and \( \text{Ker} \eta_a \subseteq \text{Im} \gamma_a \). We only prove the first inclusion, as the other one is proved analogously. To this end, assume that

\[
\gamma_a \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1 x + b_2 a \\ b_2 y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

As \( b_2 y = 0 \) we get that \( b_2 = c_2 x \) for some \( c_2 \) in \( A \). Hence \( (b_1 + c_2 a)x = b_1 x + b_2 a = 0 \), so there exists \( c_1 \) in \( A \) with \( b_1 + c_2 a = c_1 y \). Consequently,

\[
\eta_a \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 y - c_2 a \\ c_2 x \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},
\]

as desired. The last assertion of the lemma is straightforward to check. \( \square \)

**Proposition.** Let \( x, y \in A \) be an exact pair of zero divisors, and let \( a \in A \). Then the \( A \)-modules \( G_a \) and \( H_a \) are totally reflexive. Furthermore, \( G_a \) and \( H_a \) are each others dual, that is, \( G_a \cong \text{Hom}_A(H_a, A) \) and \( H_a \cong \text{Hom}_A(G_a, A) \).

**Proof.** By Lemma \((3.4)\), the exact complex \( F \) consists of finitely generated free \( A \)-modules, and furthermore \( \text{Hom}_A(F, A) \) is exact. Since \( G_a \) and \( H_a \) are cokernels of differentials in \( F \), it follows that they are totally reflexive, cf. \([1]\) prop. 8 in §3.2.2.

The duality is immediate from the isomorphism \( F \cong \text{Hom}_A(F, A) \). \( \square \)

The proof of the following result uses the derived category \( \text{D}(A) \), and the right derived \( \text{Hom} \)-functor \( \text{RHom}_A(-, -) \). We refer to Weibel \([12]\) Chapter 10] for details.

**Corollary.** Let \( x, y \in A \) be an exact pair of zero divisors, and let \( a, b \in A \). Then there are the following isomorphisms of \( A \)-modules:

(a) \( \text{Ext}_A^i(H_b, G_a) \cong \text{Ext}_A^i(H_a, G_b) \) for every \( i \in \mathbb{Z} \).

(b) \( \text{Ext}_A^i(G_b, H_a) \cong \text{Ext}_A^i(G_a, H_b) \) for every \( i \in \mathbb{Z} \).

(c) \( \text{Ext}_A^i(G_a, G_b) \cong \text{Ext}_A^i(H_b, H_a) \) for every \( i \in \mathbb{Z} \).
Proof. We only prove part (a), as the proofs of (b) and (c) are similar.

As $H_a$ is totally reflexive, $\text{Ext}_A^i(H_a, A) = 0$ for all $i > 0$. By Proposition (3.5) we also have $G_a \cong \text{Hom}_A(H_a, A)$, and therefore $G_a \cong \text{RHom}_A(H_a, A)$ in the derived category $\mathcal{D}(A)$. Consequently, one has the following isomorphisms in $\mathcal{D}(A)$,

$$
\text{RHom}_A(H_b, G_a) \cong \text{RHom}_A(H_b, \text{RHom}_A(H_a, A)) \\
\cong \text{RHom}_A(H_a, \text{RHom}_A(H_b, A)) \\
\cong \text{RHom}_A(H_a, G_b),
$$

where the second isomorphism is the so-called swap-isomorphism. The desired result is obtained by taking homology on the displayed isomorphisms. \hfill \Box

(3.7) Remark. Corollary (3.6)(c) implies that $\text{Hom}_A(G_a, G_a)$ and $\text{Hom}_A(H_a, H_a)$ are isomorphic as $A$-modules. One should be careful to conclude from this fact alone that $\text{End}_A(G_a)$ and $\text{End}_A(H_a)$ are isomorphic rings, cf. Proposition (4.2).

(3.8) Proposition. Let $x, y \in A$ be an exact pair of zero divisors, and let $a \in A$.

(a) If $a$ is weakly regular on $A/(y)$ then $G_a$ is isomorphic to the ideal $(y, a)$.

(b) If $a$ is weakly regular on $A/(x)$ then $H_a$ is isomorphic to the ideal $(x, a)$.

Proof. We only prove part (a) as the proof of (b) is similar. Since $G_a = \text{Coker} \gamma_a$, the assertion will follow once we have established exactness of the sequence,

$$
A^2 \xrightarrow{\gamma_a} A^2 \xrightarrow{(y-a)} (y, a) \longrightarrow 0.
$$

Clearly, the image of $(y-a)$ is all of $(y, a)$, and its composite with $\gamma_a$ is zero:

$$(y-a)\gamma_a = (y-a) \begin{pmatrix} x & a \\ 0 & y \end{pmatrix} = (0 \ 0).$$

It remains to see that $\text{Ker}(y-a) \subseteq \text{Im} \gamma_a$. Thus, assume that $(b_1, b_2) \in A^2$ satisfies

$$b_1y - b_2a = (y-a) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0.$$

As $b_2a = b_1y \in (y)$ and since $a$ is weakly regular on $A/(y)$, it follows that $b_2 \in (y)$, i.e. $b_2 = c_2y$ for some $c_2$. Now $(b_1 - c_2a)y = b_1y - b_2a = 0$, and as $\text{Ann}_A(y) = (x)$ we conclude that $b_1 - c_2a = c_1x$ for some $c_1$. Consequently,

$$\gamma_a \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x & a \\ 0 & y \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1x + c_2a \\ c_2y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

that is, $(b_1, b_2) \in \text{Im} \gamma_a$, as desired. \hfill \Box

(3.9) Corollary. Let $x, y \in A$ be an exact pair of zero divisors, and let $a \in A$.

(a) If $a$ is a unit then $G_a \cong H_a \cong A$.

(b) If $a$ is not a unit then neither $G_a$ nor $H_a$ is free.

Proof. (a): Immediate from Proposition (3.8).

(b): If $a$ is not a unit then all the entries in $\gamma_a$ and $\eta_a$ belong to $A$. Consequently, the homomorphisms $k \otimes_A \gamma_a$ and $k \otimes_A \eta_a$ are zero, and it follows that

$$
\cdots \longrightarrow A^2 \xrightarrow{\gamma_a} A^2 \xrightarrow{\eta_a} A^2 \xrightarrow{\gamma_a} A^2 \longrightarrow G_a \longrightarrow 0
$$

is an augmented minimal free resolution of $G_a$. Hence $G_a$ has infinite projective dimension. Analogously one sees that $H_a$ has infinite projective dimension. \hfill \Box
4. Indecomposability

For a general ring element $a$, the $A$-modules $G_a$ and $H_a$ are not indecomposable, cf. Example (4.6). However, under suitable assumptions we will prove that $\text{End}_A(G_a)$ and $\text{End}_A(H_a)$ are both isomorphic to $A$, in particular, the endomorphism rings are (commutative, noetherian and) local. Hence $G_a$ and $H_a$ are indecomposable in a quite strong sense.

We begin by establishing a relation between the endomorphism rings of $G_a$ and $H_a$ which is valid for any ring element $a$.

(4.1) **Observation.** Let $F$ be an $A$-linear contravariant endofunctor on the category of $A$-modules, and let $M$ be an $A$-module. Then there is an induced homomorphism of $A$-algebras, $\text{End}_A(M)^{\text{op}} \to \text{End}_A(FM)$ given by $\alpha \mapsto F\alpha$.

(4.2) **Proposition.** Let $M$ be a reflexive $A$-module. Then the induced homomorphism $\varepsilon_M : \text{End}_A(M)^{\text{op}} \to \text{End}_A(\text{Hom}_A(M,A))$ of $A$-algebras is an isomorphism. In particular, $M$ is indecomposable if and only if $\text{Hom}_A(M,A)$ is indecomposable.

**Proof.** For every $A$-module $M$ there is a commutative diagram of $A$-modules,

\[
\begin{array}{ccc}
\text{Hom}(M,M) & \xrightarrow{\varepsilon_M} & \text{Hom}(\text{Hom}(M,A),\text{Hom}(M,A)) \\
\delta_M \circ \gamma & \downarrow & \Sigma \\
\text{Hom}(M,\text{Hom}(\text{Hom}(M,A),A)) & \end{array}
\]

where $\delta_M : M \to \text{Hom}(\text{Hom}(M,A),A)$ is the biduality homomorphism, and $\Sigma$ is the so-called swap isomorphism given by $\Sigma(\theta)(\omega)(m) = \theta(m)(\omega)$ for $\omega \in \text{Hom}(M,A)$ and $\theta \in \text{Hom}(M,\text{Hom}(M,A),A))$ and $m \in M$. By definition, $M$ is reflexive if $\delta_M$ is an isomorphism, in which case the map $\delta_M \circ \gamma$ is an isomorphism, and the commutative diagram above establishes the desired $A$-algebra isomorphism.

The last assertion in the proposition follows from the already established ring isomorphism, as an $A$-module is indecomposable if and only if its endomorphism ring has no non-trivial idempotents. \hfill $\Box$

(4.3) **Corollary.** Let $x, y \in A$ be an exact pair of zero divisors, and let $a \in A$. Then there is an isomorphism $\text{End}_A(G_a)^{\text{op}} \cong \text{End}_A(H_a)$ of $A$-algebras. In particular, $G_a$ is indecomposable if and only if $H_a$ is indecomposable.

**Proof.** Apply Proposition (4.2) to the (totally) reflexive module $M = G_a$, and use the fact that $\text{Hom}_A(G_a,A) \cong H_a$ by Proposition (3.5). \hfill $\Box$

The following result is probably folklore. However, since the author was not able to find a reference, a proof has been included.

(4.4) **Lemma.** Let $M$ be a finitely generated $A$-module. If $A$ and $\text{Hom}_A(M,M)$ are isomorphic as $A$-modules, then the canonical homomorphism of $A$-algebras, given by $\chi : A \to \text{End}_A(M)$, $\chi(a) = a1_M$, is also an isomorphism.

**Proof.** Let $\zeta : A \to \text{Hom}_A(M,M)$ be an isomorphism of $A$-modules. As $\zeta$ is surjective, there exists $b \in A$ such that $b\zeta(1) = \zeta(b) = 1_M$. It follows that $bM = M$. Furthermore, $M \neq 0$ since $\text{Hom}_A(M,M)$ is non-zero. Hence Nakayama’s Lemma [10, Theorem 2.2] implies that $b$ is a unit in $A$, and thus $b\zeta$ is also an isomorphism. It remains to note that $b\zeta = \chi$ since $b\zeta(a) = ba\zeta(1) = a\zeta(b) = a1_M = \chi(a)$. \hfill $\Box$
(4.5) **Theorem.** Let \(x, y \in A\) be a regular exact pair of zero divisors, and let \(a \in A\) be weakly regular on \(A/(x, y)\). Then there are isomorphisms of \(A\)-algebras,

\[ \text{End}_A(G_a) \cong A \cong \text{End}_A(H_a). \]

In particular, the endomorphism rings \(\text{End}_A(G_a)\) and \(\text{End}_A(H_a)\) are (commutative, noetherian and) local, and thus the \(A\)-modules \(G_a\) and \(H_a\) are indecomposable.

**Proof.** Taking \(b = 1\) in Theorem (7.7)(a), we get the first two isomorphism of \(A\)-modules in the chain \(\text{Hom}_A(H_a, H_a) \cong \text{Hom}_A(G_a, G_a) \cong H_1 \cong A\). The last isomorphism is by Corollary (3.9)(a). Now Lemma (4.4) completes the proof. \(\square\)

(4.6) **Example.** For a general ring element \(a\), the conclusion in Theorem (4.5) fails. For example, if \(a\) belongs to \((x)\), say \(a = qx\), then the commutative diagram,

\[
\begin{array}{ccc}
A^2 & \xrightarrow{(x \ a \ y)} & A^2 \\
(1 \ q \ 0 \ 1) & \cong & (1 \ 0 \ q \ 0) \\
A^2 & \xrightarrow{(x \ 0 \ y)} & A^2
\end{array}
\]

shows that

\[ G_a = \text{Coker} \gamma_a = \text{Coker} \begin{pmatrix} x & a \\ 0 & y \end{pmatrix} \cong \text{Coker} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \cong A/(x) \oplus A/(y), \]

which is not indecomposable.

5. **Isomorphism Classes**

In this section, we investigate whether two given modules from the joint family \((G_a)_{a \in A} \cup (H_a)_{a \in A}\) are isomorphic. Of course, they might very well be exactly that since for example \(G_1 \cong A \cong H_1\) by Corollary (3.9)(a), and furthermore \(G_a \cong G_u\) whenever \(u\) is a unit, see Observation (3.3).

(5.1) **Theorem.** Let \(x, y \in A\) be a regular exact pair of zero divisors. Furthermore, let \(a, b\) be elements in \(A\) such that:

1. \(a\) or \(b\) is weakly regular on \(A/(x, y)\), and
2. \(a\) and \(b\) are not both units.

Then the \(A\)-modules \(G_a\) and \(H_b\) are not isomorphic.

**Proof.** From the assumption (1) and Theorem (4.5) we have \(\text{Hom}_A(G_a, G_a) \cong A\) or \(\text{Hom}_A(H_b, H_b) \cong A\). Thus, to prove the result it suffices to argue that \(\text{Hom}_A(G_a, H_b)\) is not isomorphic to \(A\). From the assumption (1) and Theorem (7.5)(b) we have an isomorphism \(\text{Hom}_A(G_a, H_b) \cong H_{ab}\). It follows from the assumption (2) that \(ab\) is not a unit, and hence \(H_{ab}\) is not free by Corollary (3.9)(b). \(\square\)

(5.2) **Theorem.** Let \(x, y \in A\) be a regular exact pair of zero divisors. Let \(a \in A\) be weakly regular on the \(A\)-module \(A/(x, y)\), and let \(b \in A\) be a non-unit. Then

1. The \(A\)-modules \(G_a\) and \(G_{ab}\) are not isomorphic.
2. The \(A\)-modules \(H_a\) and \(H_{ab}\) are not isomorphic.
Proof. We will only prove part (a), as the proof of (b) is similar.

It follows from Theorem (4.5) that \( \text{Hom}_A(G_a, G_a) \cong A \). Thus, to prove the result it suffices to argue that \( \text{Hom}_A(G_{ab}, G_a) \) is not isomorphic to \( A \). However, from Theorem (7.7)(a) we get that \( \text{Hom}_A(G_{ab}, G_a) \cong H_b \) which is not free as \( b \) is not a unit, see Corollary (3.9)(b).

\[ \square \]

6. The main theorem

This section contains the main theorem of this paper. Although the proof of our main result is quite short, it uses the machinery from the previous sections and from Section 7 below. For the last claim in the theorem, it is useful to keep in mind that \( H_1 \cong A \cong G_1 \) by Corollary (3.9)(a).

(6.1) Theorem. Let \( x, y \in A \) be a regular exact pair of zero divisors. Let \( (b_n)_{n \geq 1} \) be a sequence of (not necessarily distinct) elements in \( A \) which are regular on the \( A \)-module \( A/(x, y) \). If we define \( a_n = b_1 \cdots b_n \) then

\[
G_1, G_2, G_3, G_4, \ldots, H_1, H_2, H_3, H_4, \ldots
\]

is a (double) infinite family of indecomposable, non-free, totally reflexive \( A \)-modules, which are pairwise non-isomorphic. Furthermore, one has the following identities:

\[
\begin{align*}
\text{Hom}_A(H_{am}, G_{an}) & \cong \text{Hom}_A(H_{an}, G_{am}) \cong G_{am \cdot an}, \\
\text{Hom}_A(G_{am}, H_{an}) & \cong \text{Hom}_A(G_{an}, H_{am}) \cong H_{am \cdot an}, \\
\text{Hom}_A(G_{an}, G_{am}) & \cong \text{Hom}_A(H_{an}, H_{am}) \cong \begin{cases} H_{am \cdot an} & \text{for } m \geq n, \\
G_{am \cdot an} & \text{for } m \leq n. \end{cases}
\end{align*}
\]

Proof. It follows from Proposition (3.5) that \( G_{an} \) and \( H_{an} \) are totally reflexive. As \( a_n \) is not a unit, \( G_{an} \) and \( H_{an} \) are non-free by Corollary (3.9)(b). Since \( a_n \) is regular on \( A/(x, y) \), it follows from Theorem (4.5) that \( G_{an} \) and \( H_{an} \) are indecomposable.

By Theorem (5.1), the modules \( G_{an} \) and \( H_{an} \) are not isomorphic. If \( m < n \) then \( a_n = a_m \cdot b \) where \( b = b_{n+1} \cdots b_n \) (which is not a unit). Thus Theorem (5.2) gives that \( G_{an} \) and \( G_{m \cdot a_n} \) are not isomorphic, and furthermore that \( H_{am} \) and \( H_{an} \) are not isomorphic. Hence the modules in the given list are pairwise non-isomorphic.

The \( \text{Hom} \)-identities are immediate from Theorems (7.5) and (7.7).

\[ \square \]

7. Computation of Hom-modules

In this section, we will explicitly compute certain Hom-modules. For example, we will prove that \( \text{Hom}_A(H_b, G_a) \) is isomorphic to \( G_{ab} \). In order to carry out such computations, we need a concrete way to represent the module of homomorphisms between two given finitely generated \( A \)-modules. The representation we will use can be found in e.g. Greuel and Pfister [7 Example 2.1.26]; we give a recap in (7.2).

(7.1) Observation. Consider the following commutative diagram of \( A \)-modules with exact rows and columns:

\[
\begin{array}{ccc}
0 & \rightarrow & K \\
\downarrow & & \downarrow \\
L & \rightarrow & L' \\
\downarrow & \alpha' & \downarrow \\
M & \rightarrow & M' \\
\downarrow & \beta' & \downarrow \\
N & \rightarrow & N' \\
\end{array}
\]

\[
\begin{array}{cc}
0 & 0 \\
\lambda & \lambda' \\
\alpha & \alpha' \\
\beta & \beta' \\
\nu & \nu' \\
\end{array}
\]
Define submodules \( I \subseteq Q \subseteq M \) by \( I = \text{Im} \beta \) and \( Q = \mu^{-1}(\text{Im} \beta') \). Then \( \alpha \) maps \( Q \) onto \( \text{Ker} \lambda = \text{Im} \iota \), and the kernel of \( \alpha \)'s restriction \( \alpha: Q \to \text{Im} \iota \) is exactly \( I \). Consequently, there is an isomorphism of \( A \)-modules,

\[
Q/I \cong K,
\]

which maps \([q]_I \in Q/I\) to the unique \( x \in K \) with \( \iota(x) = \alpha(q) \).

(7.2) Proposition. Let \( \rho_i \) be an \( m_i \times n_i \) matrix with entries in \( A \), where \( i = 1, 2 \). Consider \( \rho_i: A^{m_i} \to A^{m_i} \) as an \( A \)-linear map which acts on column vectors by multiplication from the left. Then

\[
Q_{\rho_1, \rho_2} = \{ \psi \in M_{m_2 \times m_1}(A) \mid \psi \rho_1 = \rho_2 \xi \text{ for some } \xi \in M_{n_2 \times n_1}(A) \}
\]

is an \( A \)-submodule of \( M_{m_2 \times m_1}(A) \) which contains \( \rho_2 M_{n_2 \times n_1}(A) \) as a submodule. Furthermore, there is an isomorphism of \( A \)-modules,

\[
\text{Hom}_A(\text{Coker} \rho_1, \text{Coker} \rho_2) \cong Q_{\rho_1, \rho_2}/(\rho_2 M_{n_2 \times n_1}(A)).
\]

Proof. Set \( C_i = \text{Coker} \rho_i \). From the canonical exact sequences,

\[
A^{n_i} \xrightarrow{\rho_i} A^{m_i} \xrightarrow{\pi_i} C_i \to 0,
\]

we get a commutative diagram of \( A \)-modules with exact rows and columns,

\[
\begin{array}{cccc}
0 & \to & \text{Hom}_A(C_1, C_2) & \to & \text{Hom}_A(A^{m_1}, C_2) & \to & \text{Hom}_A(A^{n_1}, C_2) & \to & 0 \\
\downarrow & & \downarrow \pi_2 & & \downarrow \pi_2 & & \downarrow \pi_2 & & \\
& & \text{Hom}_A(A^{m_1}, A^{m_2}) & \to & \text{Hom}_A(A^{n_1}, A^{m_2}) & \to & \text{Hom}_A(A^{n_1}, A^{n_2}) & \to & \\
& & \downarrow \rho_2 & & \downarrow \rho_2 & & \downarrow \rho_2 & & \\
& & \text{Hom}_A(A^{m_1}, A^{n_2}) & \to & \text{Hom}_A(A^{n_1}, A^{n_2}) & & & & \\
\end{array}
\]

We identify \( \text{Hom}_A(A^n, A^n) \) with the \( A \)-module \( M_{v \times u}(A) \) of \( v \times u \) matrices with entries in \( A \) (a matrix is viewed as an \( A \)-linear map which acts on column vectors by multiplication from the left). With this identification, the maps \( \circ \rho_1 \), respectively, \( \circ \rho_2 \), in the bottom square in the diagram above become multiplication by the matrix \( \rho_1 \) from the right, respectively, multiplication by \( \rho_2 \) from the left.

The proposition is now immediate from Observation (7.1). The isomorphism

\[
Q_{\rho_1, \rho_2}/(\rho_2 M_{n_2 \times n_1}(A)) \cong \text{Hom}_A(C_1, C_2)
\]

given by sending \([\psi]\) to the unique homomorphism \( \eta \in \text{Hom}_A(C_1, C_2) \) such that \( \eta \circ \pi_1 = \pi_2 \circ \psi \). Here \([\cdot]\) denotes coset with respect to \( \rho_2 M_{n_2 \times n_1}(A) \).

\[\square\]

(7.3) Lemma. Let \( x, y \in A \) be a regular exact pair of zero divisors. Furthermore, let \( a, b \in A \) be elements such that \( a \) or \( b \) is weakly regular on the \( A \)-module \( A/(x, y) \). Then the \( A \)-submodule,

\[
Q_{\eta_b, \gamma_a} = \{ \psi \in M_2(A) \mid \psi \eta_b = \gamma_a \xi \text{ for some } \xi \in M_2(A) \}
\]

of \( M_2(A) \) is generated by the following five matrices,

\[
\psi_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0 & 0 \\ x & b \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \psi_4 = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad \psi_5 = \begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix}.
\]
Proof. First note that if we define,
\[ \xi_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_4 = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}, \quad \xi_5 = \begin{pmatrix} 0 & 0 \\ y & -b \end{pmatrix}, \]
then \( \psi_1, \psi_b = \gamma_a \xi_1 \), and hence \( \psi_1 \) belongs to \( Q_{\psi_b, \gamma_a} \).

Next we show that \( \psi_1, \ldots, \psi_5 \) generate all of \( Q_{\psi_b, \gamma_a} \). To this end, let \( \psi = (b_{ij}) \) be any matrix in \( Q_{\psi_b, \gamma_a} \), that is, there exists \( \xi = (c_{ij}) \) such that \( \psi \xi = \psi_b = \gamma_a \xi \); i.e.
\[ (\star) \quad \begin{bmatrix} b_{11} y & -b_{11} b + b_{12} x \\ b_{21} y & -b_{21} b + b_{22} x \end{bmatrix} = \begin{bmatrix} c_{11} x + c_{21} a & c_{12} x + c_{22} a \\ c_{21} y & c_{22} y \end{bmatrix}. \]
We must prove the existence of \( f_1, \ldots, f_5 \) in \( A \) such that \( \psi = \sum_{i=1}^5 f_i \psi_i \), that is,
\[ (\star\star) \quad \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} f_1 x + f_5 a & f_1 \\ f_2 x + f_3 y & f_2 b + f_3 y \end{bmatrix}. \]

Of course, if we define \( f_1 = b_{12} \) then entry \((1, 2)\) in \((\star\star)\) holds. Also note that from entry \((2, 1)\) in \((\star)\) we get \((b_{21} - c_{21}) y = 0\), and hence \( b_{21} - c_{21} = qx \) for some \( q \).

Below, we construct \( f_2 \) and \( f_3 \) such that entry \((2, 1)\) in \((\star\star)\) holds, that is,
\[ f_2, f_3 \text{ such that } b_{21} = f_2 x + f_3 y. \]

However, first we demonstrate how \( f_3 \) and \( f_4 \) can be constructed from \((\star)\) and \((\dagger)\) such that entries \((1, 1)\) and \((2, 2)\) in \((\star\star)\) hold:

**Existence of \( f_3 \):** From entry \((2, 2)\) in \((\star)\) we get \(-b_{21} b + b_{22} x = c_{22} y\). Combining this with \((\dagger)\) we obtain \((b_{22} - f_2 b)x = (c_{22} + f_3 b)y\). As the exact pair of zero divisors \( x, y \) is regular, see Definition \((2.3)\), it follows that \( b_{22} - f_2 b = f_3 y \) for some \( f_3 \). Thus entry \((2, 2)\) in \((\star\star)\) holds.

**Existence of \( f_4 \):** From entry \((1, 1)\) in \((\star)\), from the equation \( b_{21} - c_{21} = qx \) found above, and from \((\dagger)\), we get the following equalities,
\[ b_{11} y = c_{11} x + c_{21} a = c_{11} x + (b_{21} - qx)a = (c_{11} + f_2 a - qa)x + f_3 ay. \]

Consequently, \((b_{11} - f_5 a)y \in (x)\). As the pair \( x, y \) is regular, see Definition \((2.3)\), it follows that \( b_{11} - f_5 a = f_4 x \) for some \( f_4 \). This shows that entry \((1, 1)\) in \((\star\star)\) holds.

It remains to prove that \((\dagger)\) holds, i.e. that \( b_{21} \) belongs to \((x, y)\). The proof is divided into two cases:

**Existence of \((\dagger)\) in the case where \( a \) is weakly regular on \( A/(x, y) \):**

From entry \((1, 1)\) in \((\star)\), we have \( c_{21} a = -c_{11} x + b_{11} y \in (x, y) \). As \( a \) is weakly regular on \( A/(x, y) \), it follows that \( c_{21} \in (x, y) \). Combining this with the equation \( b_{21} - c_{21} = qx \) found above, it follows that \( b_{21} \in (x, y) \).

**Existence of \((\dagger)\) in the case where \( b \) is weakly regular on \( A/(x, y) \):**

From entry \((2, 2)\) in \((\star)\), we have \( b_{22} b = b_{22} x - c_{22} y \in (x, y) \). As \( b \) is weakly regular on \( A/(x, y) \), it follows that \( b_{22} \in (x, y) \) as desired. \( \square \)

(7.4) **Observation.** Of course, the modules \( G_a \) and \( H_a \) do not only depend on the ring element \( a \), but also on the exact pair of zero divisors \( x, y \). A more precise notation would therefore be \( G_{a, x, y}^x \) and \( H_{a, x, y}^y \).

Note that if \( x, y \) is a (regular) exact pair of zero divisors, then so is \( y, x \). By Definition \((3.2)\) and Observation \((3.3)\), it follows that \( G_{a, y, x}^x \cong H_{a, y, x}^x \) and \( H_{a, y, x}^y \cong G_{a, y, x}^x \).

(7.5) **Theorem.** Let \( x, y \in A \) be a regular exact pair of zero divisors. Furthermore, let \( a, b \in A \) be elements such that \( a \) or \( b \) is weakly regular on the \( A \)-module \( A/(x, y) \). Then there are isomorphisms of \( A \)-modules,
(a) $\text{Hom}_A(H_b, G_b) \cong \text{Hom}_A(H_b, G_a) \cong G_{ab}$.
(b) $\text{Hom}_A(G_a, H_b) \cong \text{Hom}_A(G_b, H_a) \cong H_{ab}$.

Proof. First note that by applying part (a) to the regular exact pair of zero divisors $y, x$ (instead of $x, y$), then part (b) follows from Observation 7.4.

In part (a), the first isomorphism is by Corollary 3.6(a). To prove the second isomorphism, we first apply Proposition 7.2 to get the representation,

$$\text{Hom}_A(H_b, G_a) \cong Q_{\eta_b, \gamma_a}/(\gamma_a M_2(A)).$$

Using this identification, we can consider the $A$-linear map,

$$A^2 \xrightarrow{\pi} \text{Hom}_A(H_b, G_a) \ , \ \begin{pmatrix} s \\ t \end{pmatrix} \longmapsto s[\psi_1] + t[\psi_2].$$

Here $[\cdot]$ denotes coset with respect to the submodule $\gamma_a M_2(A)$, and $\psi_i$ are the matrices introduced in Lemma 7.3. Recall that $G_{ab} = \text{Coker} \gamma_{ab}$; thus to prove the theorem, it suffices to show exactness of the sequence,

$$A^2 \xrightarrow{\gamma_{ab}} A^2 \xrightarrow{\pi} \text{Hom}_A(H_b, G_a) \longrightarrow 0.$$

To show that $\pi$ is surjective, it suffices by Lemma 7.3 to see that each $[\psi_i]$ is in the image of $\pi$. However, this is clear since $[\psi_3] = -a[\psi_1]$ and $[\psi_4] = [\psi_5] = [0]$.

To see that $\pi \gamma_{ab} = 0$, we must show that if $(s, t) = \gamma_{ab}(u, v)$, where $(u, v) \in A^2$, then $s\psi_1 + t\psi_2$ is in $\gamma_a M_2(A)$. But as

$$\begin{pmatrix} s \\ t \end{pmatrix} = \gamma_{ab} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x & ab \\ 0 & y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} ux + vab \\ vy \end{pmatrix},$$

the desired conclusion follows since,

$$s\psi_1 + t\psi_2 = (ux + vab) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + vy \begin{pmatrix} 0 & 0 \\ x & b \end{pmatrix} = \begin{pmatrix} 0 & ux + vab \\ 0 & vyb \end{pmatrix} = \gamma_a \begin{pmatrix} 0 & u \\ 0 & vb \end{pmatrix}.$$

It remains to prove that $\text{Ker} \pi \subseteq \text{Im} \gamma_{ab}$. To this end, assume that $(s, t) \in A^2$ satisfies $\pi(s, t) = 0$. This means that $s\psi_1 + t\psi_2$ belongs to $\gamma_a M_2(A)$, that is, there exists $\xi = (c_{ij})$ such that $s\psi_1 + t\psi_2 = \gamma_a \xi$; equivalently,

$$\begin{pmatrix} 0 & s \\ tx & tb \end{pmatrix} = \begin{pmatrix} c_{11}x + c_{21}a & c_{12}x + c_{22}a \\ c_{21}y & c_{22}y \end{pmatrix}.$$

From entry (2,1) in (†) we get that $tx = c_{21}y$. Since the exact pair of zero divisors $x, y$ is regular, see Definition 2.3, it follows that $t = vy$ for some $v$.

From entry (2,2) in (†) we get that $tb = c_{22}y$. Combining this with $t = vy$ one gets $(vb - c_{22})y = 0$. Consequently, $vb - c_{22} = px$ for some $p$. Now, inserting $c_{22} = vb - px$ in the equation $s = c_{12}x + c_{22}a$ coming from entry (1,2) in (†), we get that $s = (c_{12} - pa)x + vab = ux + vab$, where $u = c_{12} - pa$.

Since $s = ux + vab$ and $t = vy$, we see from (†) that $(s, t) = \gamma_{ab}(u, v) \in \text{Im} \gamma_{ab}$. □

It is also desirable to know what the module $\text{Hom}_A(G_a, G_a) \cong \text{Hom}_A(H_c, H_a)$ (see Corollary 3.6(c) for this isomorphism) looks like for every combination of $u$ and $v$. Since the author was not able to figure this out, we restrict ourselves to the case where $u$ is in the ideal generated by $v$, or vice versa, see Theorem 7.7.
(7.6) **Lemma.** Let $x, y \in A$ be a regular exact pair of zero divisors. Furthermore, let $a \in A$ be weakly regular on the $A$-module $A/(x, y)$, and let $b \in A$ be any element. Then the $A$-submodule,

$$Q_{\gamma_+ \gamma_\ast} = \{ \psi \in M_2(A) \mid \psi \gamma_{ab} = \gamma_a \xi \text{ for some } \xi \in M_2(A) \}$$

of $M_2(A)$ is generated by the following five matrices,

$$\psi_1 = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \quad \psi_4 = \begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix}, \quad \psi_5 = \begin{pmatrix} 0 & a \\ y & 0 \end{pmatrix}.$$

**Proof.** First note that if we define,

$$\xi_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_4 = \begin{pmatrix} a & 0 \\ 0 & ab \end{pmatrix}, \quad \xi_5 = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix},$$

then $\psi_i \gamma_{ab} = \gamma_a \xi_i$, and hence $\psi_i$ belongs to $Q_{\gamma_+ \gamma_\ast}$.

Next we show that $\psi_1, \ldots, \psi_5$ generate all of $Q_{\gamma_+ \gamma_\ast}$. To this end, let $\psi = (b_{ij})$ be any matrix in $Q_{\gamma_+ \gamma_\ast}$, that is, there exists $\xi = (c_{ij})$ such that $\psi \gamma_{ab} = \gamma_a \xi$; i.e.

$$(*) \quad \begin{pmatrix} b_{11}x & b_{11}ab + b_{12}y \\ b_{21}x & b_{21}ab + b_{22}y \end{pmatrix} = \begin{pmatrix} c_{11}x + c_{12}a & c_{12}x + c_{22}a \\ c_{21}y & c_{22}y \end{pmatrix}.$$  

We must prove the existence of $f_1, \ldots, f_5$ in $A$ such that $\psi = \sum_{i=1}^5 f_i \psi_i$, that is,

$$(**) \quad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} f_2 + f_4a & f_3x + f_5a \\ f_4y & f_1x + f_2b + f_5y \end{pmatrix}.$$

From entry (2, 1) in (*) we get that $b_{21}x = c_{21}y$. By assumption, the exact pair of zero divisors $x, y$ is regular, see Definition (2.3), so it follows that $b_{21} = f_4y$ for some $f_4$, which gives entry (2, 1) in (**).

From entry (1, 2) in (*) we get that $(b_{11}b - c_{22})a = c_{12}x - b_{12}y \in (x, y)$. As $a$ is weakly regular on $A/(x, y)$, there exist $p$ and $f_5$ such that

$$(\dagger) \quad b_{11}b - c_{22} = px - f_5y.$$  

From entry (2, 2) in (*) we get $b_{21}ab + b_{22}y = c_{22}y$. Since $b_{21} = f_4y$, it follows that $(f_4ab + b_{22} - c_{22})y = 0$. Consequently, there exists $q$ such that

$$(\ddagger) \quad f_4ab + b_{22} - c_{22} = qx.$$  

Now define $f_1 = q - p$ and $f_2 = b_{11} - f_4a$. Clearly, entry (1, 1) in (**) holds. Furthermore, subtracting (\dagger) from (\ddagger) and rearranging terms gives,

$$b_{22} = (q - p)x + (b_{11} - f_4a)b + f_5y = f_1x + f_2b + f_5y,$$

which shows that also entry (2, 2) in (**) holds.

Finally, $c_{12}x - b_{12}y = (b_{11}b - c_{22})a = (px - f_5y)a$. The first equality is from entry (1, 2) in (*), and the second is by (\dagger). Consequently, $(b_{12} - f_5a)y = (c_{12} - pa)x$. Since the exact pair of zero divisors $x, y$ is regular, see Definition (2.3), it follows that $b_{12} - f_5a = f_3x$ for some $f_3$. Hence entry (1, 2) in (**) holds. \qed

(7.7) **Theorem.** Let $x, y \in A$ be a regular exact pair of zero divisors. Furthermore, let $a \in A$ be weakly regular on the $A$-module $A/(x, y)$, and let $b \in A$ be any element. Then there are isomorphisms of $A$-modules,

(a) $\text{Hom}_A(H_a, H_{ab}) \cong \text{Hom}_A(G_{ab}, G_a) \cong H_b$.

(b) $\text{Hom}_A(G_a, G_{ab}) \cong \text{Hom}_A(H_{ab}, H_a) \cong G_b$. 

Proof. First note that by applying part (a) to the regular exact pair of zero divisors $y, x$ (instead of $x, y$), then part (b) follows from Observation \([7.4]\).

In part (a), the first isomorphism is by Corollary \([3.6]\)(c). To prove the second isomorphism, we apply Proposition \([7.2]\) to get the representation,

$$\text{Hom}_A(G_{ab}, G_a) \cong Q_{\gamma b, \gamma a}/(\gamma_a M_2(A)).$$

Using this identification, we can consider the matrices introduced in Lemma \([7.6]\). Recall that $\psi$ satisfies $\pi M_2 = 0$. However, this is clear since $\psi \psi = 0$. This means that $\psi \psi = 0$. This means that $\psi \psi$ is surjective, it suffices by Lemma \([7.6]\) to see that each $\psi \psi$ belongs to $\gamma_a M_2(A)$.

To see that $\pi \psi$ is surjective, it suffices by Lemma \([7.6]\) to see that each $\psi \psi$ belongs to $\gamma_a M_2(A)$. To this end, assume that $(s, t) \in \gamma_a M_2(A)$.

As the exact pair of zero divisors $x, y$ is regular, see Definition \([2.3]\), it follows that $s + t = vx$ for some $u, v$, that is, $s = uy - vb$.

From entry (2, 1) in \((\xi)\) we get $c_{21} y = 0$, and consequently $c_{21} = px$ for some $p$. Combining this with entry (1, 1) in \((\xi)\), we get $t = (c_{11} + pa)x = px$ for $v = c_{11} + pa$.

Since $t = vx$ we get from entry (2, 2) in \((\xi)\) that $(s + vb)x = c_{22} y$. As the exact pair of zero divisors $x, y$ is regular, see Definition \([2.3]\), it follows that $s + vb = uy$ for some $u$, that is, $s = uy - vb$.

As $s = uy - vb$ and $t = vx$, we see from \((\xi)\) that $(s, t) = \eta u, v) \in \text{Im} \eta_b$. \(\square\)

References


Department of Basic Sciences and Environment, Faculty of Life Sciences, University of Copenhagen, Thorvaldensvej 40, DK-1871 Frederiksberg C, Denmark

E-mail address: hholm@life.ku.dk
URL: http://www.matdat.life.ku.dk/~hholm/