Gorenstein homological dimensions

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Abstract

In basic homological algebra, the projective, injective and flat dimensions of modules play an important and fundamental role. In this paper, the closely related Gorenstein projective, Gorenstein injective and Gorenstein flat dimensions are studied.

There is a variety of nice results about Gorenstein dimensions over special commutative noetherian rings; very often local Cohen–Macaulay rings with a dualizing module. These results are done by Avramov, Christensen, Enochs, Foxby, Jenda, Martsinkovsky and Xu among others. The aim of this paper is to generalize these results, and to give homological descriptions of the Gorenstein dimensions over *arbitrary* associative rings.
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0. Introduction

Throughout this paper, R denotes a non-trivial associative ring. All modules are—if not specified otherwise—*left R-modules*.

When R is two-sided and noetherian, Auslander and Bridger [2] introduced in 1969 the G-dimension, G-dim$_R M$, for every *finite*, that is, finitely generated, R-module M (see also [1] from 1966/67). They proved the inequality G-dim$_R M \leqslant \mathrm{pd}_R M$, with equality G-dim$_R M = \mathrm{pd}_R M$ when $\mathrm{pd}_R M$ is finite. Furthermore they showed the generalized Auslander–Buchsbaum formula (sometimes known as the Auslander–Bridger formula) for the G-dimension.

Over a general ring R, Enochs and Jenda defined in [9] a homological dimension, namely the *Gorenstein projective dimension*, Gpd$_R(-)$, for arbitrary (non-finite) modules. It is defined via resolutions with (the so-called) *Gorenstein projective modules*.

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Avramov, Buchweitz, Martsinkovsky and Reiten prove that a *finite* module over a noetherian ring is Gorenstein projective if and only if G-dim$_R M = 0$ (see the remark following [7, Theorem 4.2.6]).

Section 2 deals with this Gorenstein projective dimension, Gpd$_R(-)$. First we establish the following fundamental.

Theorem. *The class of all Gorenstein projective modules is resolving, in the sense that if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, where M'' is Gorenstein projective, then M' is Gorenstein projective if and only if M is Gorenstein projective.*

This result is a generalization of [10, Theorems 10.2.8 and 11.5.66], and of [7, Corollary 4.3.5], which all put restrictions on either the base ring, or on the modules. The result is also the main ingredient in the following important functorial description of the Gorenstein dimension.

Theorem. *Let M be a (left) R-module with finite Gorenstein projective dimension, and let $n \geqslant 0$ be an integer. Then the following conditions are equivalent.*

 (i) Gpd$_R M \leqslant n$.
 (ii) Ext$_R^i(M, L) = 0$ *for all $i > n$, and all R-modules L with finite* pd$_R L$.
 (iii) Ext$_R^i(M, Q) = 0$ *for all $i > n$, and all projective R-modules Q*.
 (iv) *For every exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, if $G_0, \ldots, G_{n-1}$ are Gorenstein projective, then also K_n is Gorenstein projective.*

Note that this theorem generalizes [7, Theorem 4.4.12], which is only proved for local noetherian Cohen–Macaulay rings admitting a dualizing module.

Next, we get the following generalization of [15, Theorem 5.5.6] (where the ring is assumed to be local, noetherian and Cohen–Macaulay with a dualizing module):

Theorem. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. If any two of the modules M', M or M'' have finite Gorenstein projective dimension, then so has the third.*

In Section 2 we also investigate *Gorenstein projective precovers*. Recall that a Gorenstein projective precover of a module M is a homomorphism of modules, $G \to M$, where G is Gorenstein projective, such that the sequence

$$\text{Hom}_R(Q, G) \to \text{Hom}_R(Q, M) \to 0$$

is exact for every Gorenstein projective module Q. We show that every module M with finite Gorenstein projective dimension admits a nice Gorenstein projective precover:

Theorem. *Let M be an R-module with finite Gorenstein projective dimension n. Then M admits a surjective Gorenstein projective precover $\varphi: G \twoheadrightarrow M$ where $K = \text{Ker}\, \varphi$ satisfies* pd$_R K = n - 1$ *(if $n = 0$, this should be interpreted as $K = 0$).*

Using these precovers, we show that there is an equality between the classical (left) finitistic projective dimension, FPD(R), and the related (*left*) *finitistic Gorenstein projective dimension*, FGPD(R), of the base ring R. The latter is defined as

$$\text{FGPD}(R) = \sup \left\{ \text{Gpd}_R M \; \middle| \; \begin{array}{l} M \text{ is a left } R\text{-module with finite} \\ \text{Gorenstein projective dimension.} \end{array} \right\}.$$

Important note. Above we have only mentioned the Gorenstein projective dimension for an R-module M. Dually one can also define the *Gorenstein injective dimension*, $\text{Gid}_R M$. All the results concerning Gorenstein projective dimension (with the exception of Proposition 2.16 and Corollary 2.21), have a Gorenstein injective counterpart.

With some exceptions, we do not state or prove these "dual" Gorenstein injective results. This is left to the reader.

Section 3 deals with *Gorenstein flat modules*, together with the *Gorenstein flat dimension*, $\text{Gfd}_R(-)$, in a way much similar to how we treated Gorenstein projective modules, and the Gorenstein projective dimension in Section 2.

For right coherent rings, a (left) R-module M is Gorenstein flat if, and only if, its Pontryagin dual $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a (right) Gorenstein injective R-module (please see Theorem 3.6). Using this we can prove the next generalization of [7, Theorem 5.2.14] and [10, Theorem 10.3.8].

Theorem. *If R is right coherent, $n \geq 0$ is an integer and M is a left R-module with finite Gorenstein flat dimension, then the following four conditions are equivalent*:

(i) $\text{Gfd}_R M \leq n$.
(ii) $\text{Tor}_i^R(L, M) = 0$ *for all right R-modules L with finite $\text{id}_R L$, and all $i > n$.*
(iii) $\text{Tor}_i^R(I, M) = 0$ *for all injective right R-modules I, and all $i > n$.*
(iv) *For every exact sequence $0 \to K_n \to T_{n-1} \to \cdots \to T_0 \to M \to 0$ if $T_0, \ldots, T_{n-1}$ are Gorenstein flat, then also K_n is Gorenstein flat.*

Besides the Gorenstein flat dimension of an R-module M, also the *large restricted flat dimension*, $\text{Rfd}_R M$, is of interest. It is defined as follows:

$$\text{Rfd}_R M = \sup \left\{ i \geq 0 \; \middle| \; \begin{array}{l} \text{Tor}_i^R(L, M) \neq 0 \text{ for some (right)} \\ R\text{-module with finite flat dimension.} \end{array} \right\}.$$

This numerical invariant is investigated in [8, Section 2] and in [7, Chapters 5.3–5.4]. It is conjectured by Foxby that if $\text{Gfd}_R M$ is finite, then $\text{Rfd}_R M = \text{Gfd}_R M$. Christensen [7, Theorem 5.4.8] proves this for local noetherian Cohen–Macaulay rings with a dualizing module. We have the following extension.

Theorem. *For any (left) R-module M there are inequalities*,

$$\text{Rfd}_R M \leq \text{Gfd}_R M \leq \text{fd}_R M.$$

Now assume that R is commutative and noetherian. If $\mathrm{Gfd}_R M$ *is finite, then we have equality* $\mathrm{Rfd}_R M = \mathrm{Gfd}_R M$. *If* $\mathrm{fd}_R M$ *is finite, then* $\mathrm{Rfd}_R M = \mathrm{Gfd}_R M = \mathrm{fd}_R M$.

Furthermore we prove that every module, M, with finite Gorenstein flat dimension admits a special *Gorenstein flat precover*, $G \twoheadrightarrow M$, and we show that the classical (left) finitistic flat dimension, $\mathrm{FFD}(R)$, is equal to the (*left*) *finitistic Gorenstein flat dimension*, $\mathrm{FGFD}(R)$ of R.

Notation. By $\mathcal{M}(R)$ we denote the class of all R-modules, and by $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$ we denote the classes of all projective, injective and flat R-modules respectively. Furthermore, we let $\tilde{\mathcal{P}}(R)$, $\tilde{\mathcal{I}}(R)$ and $\tilde{\mathcal{F}}(R)$ denote the classes of all R-modules with finite projective, injective and flat dimensions, respectively.

(Note that in the related paper [5] by Avramov and Martsinkovsky, studying finite modules, the symbol $\mathcal{F}(R)$ denotes the class of finite modules, $\mathcal{P}(R)$ the class of finite projective modules, and $\tilde{\mathcal{P}}(R)$ the class of finite modules with finite projective dimension).

1. Resolving classes

This section contains some general remarks about resolving classes, which will be important in our treatment of Gorenstein projective modules in the next section.

1.1. Resolving classes. Inspired by Auslanders–Bridger's result [2, (3.11)], we define the following terms for any class, $\mathcal{X}$, of R-modules.

(a) We call $\mathcal{X}$ *projectively resolving* if $\mathcal{P}(R) \subseteq \mathcal{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.
(b) We call $\mathcal{X}$ *injectively resolving* if $\mathcal{I}(R) \subseteq \mathcal{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{X}$ the conditions $X \in \mathcal{X}$ and $X'' \in \mathcal{X}$ are equivalent.

Note that we do not require that a projectively/injectively class is closed under direct summands, as in [2, (3.11)]. The reason for this will become clear in Proposition 1.4 below.

1.2. Orthogonal classes. For any class, $\mathcal{X}$, of R-modules, we define the associated *left orthogonal*, respectively, *right orthogonal*, class by

$$^{\perp}\mathcal{X} = \{M \in \mathcal{M}(R) \,|\, \mathrm{Ext}^i_R(M, X) = 0 \text{ for all } X \in \mathcal{X}, \text{ and all } i > 0\},$$

respectively,

$$\mathcal{X}^{\perp} = \{N \in \mathcal{M}(R) \,|\, \mathrm{Ext}^i_R(X, N) = 0 \text{ for all } X \in \mathcal{X}, \text{ and all } i > 0\}.$$

1.3. Example. It is well known that $\mathscr{P}(R) = {}^{\perp}\mathscr{M}(R)$, and that $\mathscr{P}(R)$ and $\mathscr{F}(R)$ both are projectively resolving classes, whereas $\mathscr{I}(R) = \mathscr{M}(R)^{\perp}$ is an injectively resolving class. Furthermore, it is easy to see the equalities,

$$^{\perp}\bar{\mathscr{P}}(R) = {}^{\perp}\mathscr{P}(R) \quad \text{and} \quad \bar{\mathscr{I}}(R)^{\perp} = \mathscr{I}(R)^{\perp}.$$

In general, the class ${}^{\perp}\mathscr{X}$ is projectively resolving, and closed under arbitrary direct sums. Similarly, the class $\mathscr{X}^{\perp}$ is injectively resolving, and closed under arbitrary direct products.

The next result is based on a technique of Eilenberg.

1.4. Proposition (Eilenberg's swindle). *Let X be a class of R-modules which is either projectively resolving, or injectively resolving. If $\mathscr{X}$ is closed under countable direct sums, or closed under countable direct products, then $\mathscr{X}$ is also closed under direct summands.*

Proof. Assume that Y is a direct summand of $X \in \mathscr{X}$. We wish to show that $Y \in \mathscr{X}$. Write $X = Y \oplus Z$ for some module Z. If $\mathscr{X}$ is closed under countable direct sums, then we define $W = Y \oplus Z \oplus Y \oplus Z \oplus \cdots$ (direct sum), and note that $W \cong X \oplus X \oplus \cdots \in \mathscr{X}$. If $\mathscr{X}$ is closed under countable direct products, then we put $W = Y \times Z \times Y \times Z \times \cdots$ (direct product), and note that $W \cong X \times X \times \cdots \in \mathscr{X}$. In either case we have $W \cong Y \oplus W$, so in particular the sum $Y \oplus W$ belongs to $\mathscr{X}$. If $\mathscr{X}$ is projectively resolving, then we consider the split exact sequence $0 \to Y \to Y \oplus W \to W \to 0$, and if $\mathscr{X}$ is injectively resolving, then we consider $0 \to W \to W \oplus Y \to Y \to 0$. In either case we conclude that $Y \in \mathscr{X}$. $\quad\square$

1.5. Resolutions. For any R-module M we define two types of resolutions.

(a) A *left $\mathscr{X}$-resolution* of M is an exact sequence $\boldsymbol{X} = \cdots \to X_1 \to X_0 \to M \to 0$ with $X_n \in \mathscr{X}$ for all $n \geqslant 0$.
(b) A *right $\mathscr{X}$-resolution* of M is an exact sequence $\boldsymbol{X} = 0 \to M \to X^0 \to X^1 \to \cdots$ with $X^n \in \mathscr{X}$ for all $n \geqslant 0$.

Now let $\boldsymbol{X}$ be any (left or right) $\mathscr{X}$-resolution of M. We say that $\boldsymbol{X}$ is *proper* (respectively, *co-proper*) if the sequence $\mathrm{Hom}_R(Y, \boldsymbol{X})$ (respectively, $\mathrm{Hom}_R(\boldsymbol{X}, Y)$) is exact for all $Y \in \mathscr{X}$.

In this paper we *only* consider proper left $\mathscr{X}$-resolutions, and co-proper right $\mathscr{X}$-resolutions (and *never* proper right $\mathscr{X}$-resolutions, or co-proper left $\mathscr{X}$-resolutions).

It is straightforward to show the next result.

1.6. Proposition. *Let $\mathscr{X}$ be a class of R-modules, and let $\{M_i\}_{i \in I}$ be a family of R-modules. Then the following hold*:

(i) *If $\mathscr{X}$ is closed under arbitrary direct products, and if each of the modules M_i admits a (proper) left $\mathscr{X}$-resolution, then so does the product $\prod M_i$.*

(ii) *If $\mathscr{X}$ is closed under arbitrary direct sums, and if each of the modules M_i admits a (co-proper) right $\mathscr{X}$-resolution, then so does the sum $\coprod M_i$.*
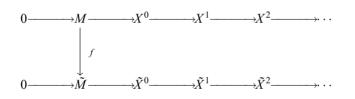
1.7. Horseshoe lemma. *Let $\mathscr{X}$ be a class of R-modules. Assume that $\mathscr{X}$ is closed under finite direct sums, and consider an exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules, such that*

$$0 \longrightarrow \operatorname{Hom}_R(M'', Y) \longrightarrow \operatorname{Hom}_R(M, Y) \longrightarrow \operatorname{Hom}_R(M', Y) \longrightarrow 0$$
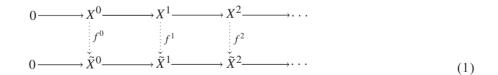
is exact for every $Y \in \mathscr{X}$. If both M' and M'' admits co-proper right $\mathscr{X}$-resolutions, then so does M.

Proof. Dualizing the proof of [10, Lemma 8.2.1], we can construct the co-proper resolution of M as the degreewise sum of the two given co-proper resolutions for M' and M''. $\quad\square$

1.8. Proposition. *Let $f : M \to \tilde{M}$ be a homomorphism of modules, and consider the diagram,*

$$\begin{array}{ccccccccc}
0 & \to & M & \to & X^0 & \to & X^1 & \to & X^2 & \to & \cdots \\
& & \downarrow f & & & & & & & & \\
0 & \to & \tilde{M} & \to & \tilde{X}^0 & \to & \tilde{X}^1 & \to & \tilde{X}^2 & \to & \cdots
\end{array}$$

where the upper row is a co-proper right $\mathscr{X}$-resolution of M, and the lower row is a right $\mathscr{X}$-resolution of $\tilde{M}$. Then $f : M \to \tilde{M}$ induces a chain map of complexes,

$$\begin{array}{ccccccc}
0 \to & X^0 & \to & X^1 & \to & X^2 & \to \cdots \\
& \downarrow f^0 & & \downarrow f^1 & & \downarrow f^2 & \\
0 \to & \tilde{X}^0 & \to & \tilde{X}^1 & \to & \tilde{X}^2 & \to \cdots
\end{array} \tag{1}$$

with the property that the square,

$$\begin{array}{ccc}
M & \to & X^0 \\
\downarrow f & & \downarrow f^0 \\
\tilde{M} & \to & \tilde{X}^0
\end{array}$$

commutes. Furthermore, the chain map (1) *is uniquely determined upto homotopy by this property.*

Proof. Please see [10, Exercise 2, p. 169], or simply "dualize" the argument following [10, Proposition 8.1.3]. □

2. Gorenstein projective and Gorenstein injective modules

In this section we give a detailed treatment of Gorenstein projective modules. The main purpose is to give functorial descriptions of the Gorenstein projective dimension.

2.1. Definition. A *complete projective resolution* is an exact sequence of projective modules, $\boldsymbol{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$, such that $\mathrm{Hom}_R(\boldsymbol{P}, Q)$ is exact for every projective R-module Q.

An R-module M is called *Gorenstein projective* (G-projective for short), if there exists a complete projective resolution $\boldsymbol{P}$ with $M \cong \mathrm{Im}(P_0 \to P^0)$. The class of all Gorenstein projective R-modules is denoted by $\mathscr{GP}(R)$.

Gorenstein injective (G-injective for short) modules are defined dually, and the class of all such modules is denoted by $\mathscr{GI}(R)$.

2.2. Observation. If $\boldsymbol{P}$ is a complete projective resolution, then by symmetry, all the images, and hence also all the kernels, and cokernels of $\boldsymbol{P}$ are Gorenstein projective modules. Furthermore, every projective module is Gorenstein projective.

Using the definitions, we immediately get the following characterization of Gorenstein projective modules.

2.3. Proposition. *An R-module M is Gorenstein projective if, and only if, M belongs to the left orthogonal class $^\perp \mathscr{P}(R)$, and admits a co-proper right $\mathscr{P}(R)$-resolution.*

Furthermore, if $\boldsymbol{P}$ is a complete projective resolution, then $\mathrm{Hom}_R(\boldsymbol{P}, L)$ is exact for all R-modules L with finite projective dimension. Consequently, when M is Gorenstein projective, then $\mathrm{Ext}_R^i(M, L) = 0$ for all $i > 0$ and all R-modules L with finite projective dimension.

As the next result shows, we can always assume that the modules in a complete projective resolution are free.

2.4. Proposition. *If M is a Gorenstein projective module, then there is a complete projective resolution, $\boldsymbol{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$, consisting of free modules F_n and F^n such that $M \cong \mathrm{Im}(F_0 \to F^0)$.*

Proof. Only the construction of the "right half", $0 \to M \to F^0 \to F^1 \to \cdots$ of $\boldsymbol{F}$ is of interest. By Proposition 2.3, M admits a co-proper right $\mathscr{P}(R)$-resolution, say

$$0 \to M \to Q^0 \to Q^1 \to \cdots .$$

We successively pick projective modules $P^0, P^1, P^2, \ldots$, such that all of the modules

$$F^0 = Q^0 \oplus P^0 \quad \text{and} \quad F^n = Q^n \oplus P^{n-1} \oplus P^n \text{ for } n > 0,$$

are free. By adding $0 \longrightarrow P^i \overset{=}{\longrightarrow} P^i \longrightarrow 0$ to the co-proper right $\mathscr{P}(R)$-resolution above in degrees i and $i + 1$, we obtain the desired sequence. $\square$

Next we set out to investigate how Gorenstein projective modules behave in short exact sequences. The following theorem is due to Foxby and Martsinkovsky, but the proof presented here differs somewhat from their original ideas. Also note that Enochs and Jenda in [10, Theorems 10.2.8 and 11.5.66], have proved special cases of the result.

2.5. Theorem. *The class $\mathscr{GP}(R)$ of all Gorenstein projective R-modules is projectively resolving. Furthermore, $\mathscr{GP}(R)$ is closed under arbitrary direct sums and under direct summands.*

Proof. The left orthogonal class $^{\perp}\mathscr{P}(R)$ is closed under arbitrary direct sums, by Example 1.3, and so is the class of modules which admit a co-proper right $\mathscr{P}(R)$-resolution, by Proposition 1.6 (ii). Consequently, the class $\mathscr{GP}(R)$ is also closed under arbitrary direct sums, by Proposition 2.3.

To prove that $\mathscr{GP}(R)$ is projectively resolving, we consider any short exact sequence of R-modules, $0 \to M' \to M \to M'' \to 0$, where M'' is Gorenstein projective.

First assume that M' is Gorenstein projective. Again, using the characterization in Proposition 2.3, we conclude that M is Gorenstein projective, by the Horseshoe Lemma 1.7, and by Example 1.3, which shows that the left orthogonal class $^{\perp}\mathscr{P}(R)$ is projectively resolving.

Next assume that M is Gorenstein projective. Since $^{\perp}\mathscr{P}(R)$ is projectively resolving, we get that M' belongs to $^{\perp}\mathscr{P}(R)$. Thus, to show that M' is Gorenstein projective, we only have to prove that M' admits a co-proper right $\mathscr{P}(R)$-resolution. By assumption, there exists co-proper right $\mathscr{P}(R)$-resolutions,

$$\boldsymbol{M} = 0 \to M \to P^0 \to P^1 \to \cdots \quad \text{and} \quad \boldsymbol{M}'' = 0 \to M'' \to P''^0 \to P''^1 \to \cdots .$$

Proposition 1.8 gives a chain map $\boldsymbol{M} \to \boldsymbol{M}''$, lifting the homomorphism $M \to M''$. We let $\boldsymbol{C}$ denote the mapping cone of $\boldsymbol{M} \to \boldsymbol{M}''$, and we note the following properties:

Since $\boldsymbol{M} \to \boldsymbol{M}''$ is a quasi-isomorphism (both $\boldsymbol{M}$ and $\boldsymbol{M}''$ are exact), the long exact sequence of homology for the mapping cone shows that $\boldsymbol{C}$ is exact. Furthermore, if Q is any projective module, then $\operatorname{Hom}_R(\boldsymbol{C}, Q)$ is isomorphic to (a shift of) the mapping cone of the quasi-isomorphism,

$$\operatorname{Hom}_R(\boldsymbol{M}'', Q) \to \operatorname{Hom}_R(\boldsymbol{M}, Q),$$

and thus, also $\mathrm{Hom}_R(\boldsymbol{C}, Q)$ is exact. Next note that we have a short exact sequence of complexes,

$$
\begin{array}{ccccccccc}
& & \vdots & & \vdots & & \vdots & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P''^0 \oplus P^1 & =\!=\!= & P''^0 \oplus P^1 & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^0 & \longrightarrow & M'' \oplus P^0 & \longrightarrow & M'' & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & & \\
& & \| & & \| & & \| & & \\
0 & \longrightarrow & \boldsymbol{M'} & \longrightarrow & \boldsymbol{C} & \longrightarrow & \boldsymbol{D} & \longrightarrow & 0
\end{array}
\qquad (2)
$$

We claim that the first column, $\boldsymbol{M'}$, is a co-proper right $\mathscr{P}(R)$-resolution of M'. Since both $\boldsymbol{C}$ and $\boldsymbol{D}$ are exact, the long exact sequence in homology shows that $\boldsymbol{M'}$ is exact as well. Thus $\boldsymbol{M'}$ is a right $\mathscr{P}(R)$-resolution of M'.

To see that it is co-proper, we let Q be any projective module. Applying $\mathrm{Hom}_R(-, Q)$ to (2) we obtain another exact sequence of complexes,

$$0 \to \mathrm{Hom}_R(\boldsymbol{D}, Q) \to \mathrm{Hom}_R(\boldsymbol{C}, Q) \to \mathrm{Hom}_R(\boldsymbol{M'}, Q) \to 0.$$

For the first row,

$$0 \to \mathrm{Hom}_R(M'', Q) \to \mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(M', Q) \to 0,$$

exactness follows from Proposition 2.3, since M'' is Gorenstein projective, and for the remaining rows exactness is obvious. As already noticed, $\mathrm{Hom}_R(\boldsymbol{C}, Q)$ is exact, and obviously, so is $\mathrm{Hom}_R(\boldsymbol{D}, Q)$. Thus, another application of the long exact sequence for homology shows that $\mathrm{Hom}_R(\boldsymbol{M'}, Q)$ is exact as well. Hence $\boldsymbol{M'}$ is co-proper.

Finally we have to show that the class $\mathscr{GP}(R)$ is closed under direct summands. Since $\mathscr{GP}(R)$ is projectively resolving, and closed under arbitrary direct sums, the desired conclusion follows from Proposition 1.4. $\square$

Here is the first exception to the "Important note" on page 2. We state, but do not prove, the Gorenstein injective version of Theorem 2.20 above (as we will need it in Section 3, when we deal with Gorenstein flat modules).

2.6. Theorem. *The class $\mathscr{GI}(R)$ of all Gorenstein injective R-modules is injectively resolving. Furthermore $\mathscr{GI}(R)$ is closed under arbitrary direct products and under direct summands.*

2.7. Proposition. *Let M be any R-module and consider two exact sequences,*

$$0 \longrightarrow K_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

$$0 \longrightarrow \tilde{K}_n \longrightarrow \tilde{G}_{n-1} \longrightarrow \cdots \longrightarrow \tilde{G}_0 \longrightarrow M \longrightarrow 0,$$

where $G_0,\ldots,G_{n-1}$ and $\tilde{G}_0,\ldots,\tilde{G}_{n-1}$ are Gorenstein projective modules. Then K_n is Gorenstein projective if and only if $\tilde{K}_n$ is Gorenstein projective.

Proof. Since the class of Gorenstein projective modules is projectively resolving and closed under arbitrary sums, and under direct summands, by Theorem 2.5, the stated result is a direct consequence of [2, Lemma 3.12].   □

At this point we introduce the Gorenstein projective dimension:

2.8. Definition. The *Gorenstein projective dimension*, $\mathrm{Gpd}_R M$, of an R-module M is defined by declaring that $\mathrm{Gpd}_R M \leqslant n$ ($n \in \mathbb{N}_0$) if, and only if, M has a Gorenstein projective resolution of length n. We use $\overline{\mathscr{GP}}(R)$ to denote the class of all R-modules with *finite* Gorenstein projective dimension.

Similarly, one defines the *Gorenstein injective dimension*, $\mathrm{Gid}_R M$, of M, and we use $\overline{\mathscr{GI}}(R)$ to denote the class of all R-modules with finite Gorenstein injective dimension.

Hereafter, we immediately deal with Gorenstein projective precovers, and proper left $\mathscr{GP}(R)$-resolutions. We begin with a definition of precovers.

2.9. Precovers. Let $\mathscr{X}$ be any class of R-modules, and let M be an R-module. An $\mathscr{X}$-*precover* of M is an R-homomorphism $\varphi\colon X \to M$, where $X \in \mathscr{X}$, and such that the sequence,

$$\mathrm{Hom}_R(X',X) \xrightarrow{\mathrm{Hom}_R(X',\varphi)} \mathrm{Hom}_R(X',M) \longrightarrow 0$$

is exact for every $X' \in \mathscr{X}$. ($\mathscr{X}$-preenvelopes of M are defined "dually".)

For more details about precovers (and preenvelopes), the reader may consult [10, Chapters 5 and 6] or [15, Chapter 1]. Instead of saying $\mathscr{GP}(R)$-precover, we shall use the term *Gorenstein projective precover*.
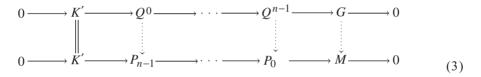
In the case where $(R, \mathfrak{m}, k)$ is a local noetherian Cohen–Macaulay ring admitting a dualizing module, special cases of the results below can be found in [12, Theorems 2.9 and 2.10].

2.10. Theorem. *Let M be an R-module with finite Gorenstein projective dimension n. Then M admits a surjective Gorenstein projective precover, $\varphi : G \twoheadrightarrow M$, where $K = \operatorname{Ker} \varphi$ satisfies $\operatorname{pd}_R K = n - 1$ (if $n = 0$, this should be interpreted as $K = 0$).*
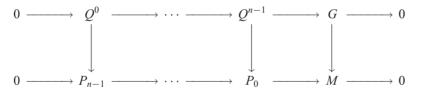
In particular, M admits a proper left Gorenstein projective resolution (that is, a proper left $\mathscr{GP}(R)$-resolution) of length n.

Proof. Pick an exact sequence, $0 \to K' \to P_{n-1} \to \cdots \to P_0 \to M \to 0$, where $P_0, \ldots, P_{n-1}$ are projectives. Then K' is Gorenstein projective by Proposition 2.7. Hence there is an exact sequence $0 \to K' \to Q^0 \to \cdots \to Q^{n-1} \to G \to 0$, where $Q^0, \ldots, Q^{n-1}$ are projectives, G is Gorenstein projective, and such that the functor $\operatorname{Hom}_R(-, Q)$ leaves this sequence exact, whenever Q is projective.

Thus there exist homomorphisms, $Q^i \to P_{n-1-i}$ for $i = 0, \ldots, n-1$, and $G \to M$, such that the following diagram is commutative.

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & K' & \longrightarrow & Q^0 & \longrightarrow & \cdots & \longrightarrow & Q^{n-1} & \longrightarrow & G & \longrightarrow & 0 \\
& & \big\| & & \big\downarrow & & & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}
\tag{3}
$$

This diagram gives a chain map between complexes,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Q^0 & \longrightarrow & \cdots & \longrightarrow & Q^{n-1} & \longrightarrow & G & \longrightarrow & 0 \\
& & \big\downarrow & & & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

which induces an isomorphism in homology. Its mapping cone is exact, and all the modules in it, except for $P_0 \oplus G$ (which is Gorenstein projective), are projective. Hence the kernel K of $\varphi : P_0 \oplus G \twoheadrightarrow M$ satisfies $\operatorname{pd}_R K \leqslant n - 1$ (and then necessarily $\operatorname{pd}_R K = n - 1$).

Since K has finite projective dimension, we have $\operatorname{Ext}^1_R(G', K) = 0$ for any Gorenstein projective module G', by Proposition 2.3, and thus the homomorphism

$$\operatorname{Hom}_R(G', \varphi) \colon \operatorname{Hom}_R(G', P_0 \oplus G) \to \operatorname{Hom}_R(G', M)$$

is surjective. Hence $\varphi : P_0 \oplus G \twoheadrightarrow M$ is the desired precover of M. $\quad\square$

2.11. Corollary. *Let $0 \to G' \to G \to M \to 0$ be a short exact sequence where G and G' are Gorenstein projective modules, and where $\operatorname{Ext}^1_R(M, Q) = 0$ for all projective modules Q. Then M is Gorenstein projective.*

Proof. Since $\operatorname{Gpd}_R M \leqslant 1$, Theorem 2.10 above gives the existence of an exact sequence $0 \to Q \to \tilde{G} \to M \to 0$, where Q is projective, and $\tilde{G}$ is Gorenstein projective. By our assumption $\operatorname{Ext}^1_R(M, Q) = 0$, this sequence splits, and hence M is Gorenstein projective by Theorem 2.5. $\quad\square$

2.12. Remark. If R is left noetherian and M is finite, then all the modules appearing in the proof of Theorem 2.10 can be chosen to be finite. Consequently, the module G in the Gorenstein projective precover $\varphi{:}G \twoheadrightarrow M$ of Theorem 2.10 (and hence also K) can be chosen to be finite. Let us write it out:

2.13. Corollary. *Every finite R-module N with finite Gorenstein projective dimension has a finite surjective Gorenstein projective precover, $0 \to K \to G \to N \to 0$, such that the kernel K has finite projective dimension.*

2.14. Observation. Over a local noetherian ring $(R, \mathfrak{m}, k)$ admitting a dualizing module, Auslander and Buchweitz introduce in their paper [3]

 (i) a *maximal Cohen–Macaulay approximation*, $0 \to I_N \to M_N \to N \to 0$, and
(ii) a *hull of finite injective dimension*, $0 \to N \to I^N \to M^N \to 0$

for every finite R-module N. Here M_N and M^N are finite maximal Cohen–Macaulay modules, and I_N, I^N have finite injective dimension.

Note how the sequence $0 \to K \to G \to N \to 0$ from Corollary 2.13 resembles their maximal Cohen–Macaulay approximation.

2.15. Theorem. *Let N be an R-module with finite Gorenstein injective dimension n. Then N admits an injective Gorenstein injective preenvelope, $\varphi{:}N \hookrightarrow H$, where $C = $ Coker φ satisfies $\mathrm{id}_R C = n - 1$ (if $n = 0$, this should be interpreted as $C = 0$).*
In particular, N admits a co-proper right Gorenstein injective resolution (that is, a co-proper right $\mathscr{GI}(R)$-resolution) of length n.

Using completely different methods, Enochs and Jenda proved in [9, Theorem 2.13] the Gorenstein injective dual version of Proposition 2.11 above. However, Proposition 2.11 itself is only proved for (left) coherent rings and *finitely presented* (right) modules in [10, Theorem 10.2.8].

We now wish to prove how the Gorenstein projective dimension, which is defined in terms of resolutions, can be measured by the Ext-functors in a way much similar to how these functors measure the ordinary projective dimension.

2.16. Proposition. *Assume that R is left noetherian, and that M is a finite (left) R-module with Gorenstein projective dimension m (possibly $m = \infty$). Then M has a Gorenstein projective resolution of length m, consisting of finite Gorenstein projective modules.*

Proof. Simply apply Proposition 2.7 to a resolution of M by finite projective modules. □

Using Propositions 2.3 and 2.7 together with standard arguments, we immediately obtain the next two results.

2.17. Lemma. *Consider an exact sequence* $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ *where* $G_0, \ldots, G_{n-1}$ *are Gorenstein projective modules. Then*

$$\operatorname{Ext}_R^i(K_n, L) \cong \operatorname{Ext}_R^{i+n}(M, L)$$

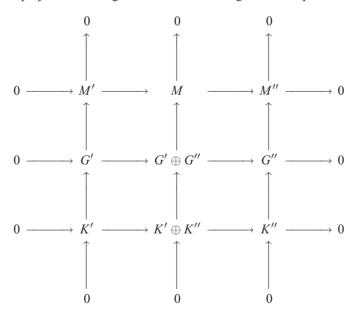*for all R-modules L with finite projective dimension, and all integers $i > 0$.*

2.18. Proposition. *Let* $0 \to K \to G \to M \to 0$ *be an exact sequence of R-modules where G is Gorenstein projective. If M is Gorenstein projective, then so is K. Otherwise we get* $\operatorname{Gpd}_R K = \operatorname{Gpd}_R M - 1 \geqslant 0$.

2.19. Proposition. *If* $(M_\lambda)_{\lambda \in \Lambda}$ *is any family of R-modules, then we have an equality*

$$\operatorname{Gpd}_R \left(\coprod M_\lambda \right) = \sup\{\operatorname{Gpd}_R M_\lambda \mid \lambda \in \Lambda\}.$$

Proof. The inequality '$\leqslant$' is clear since $\mathscr{GP}(R)$ is closed under direct sums by Theorem 2.5. For the converse inequality '$\geqslant$', it suffices to show that if M' is any direct summand of an R-module M, then $\operatorname{Gpd}_R M' \leqslant \operatorname{Gpd}_R M$. Naturally we may assume that $\operatorname{Gpd}_R M = n$ is finite, and then proceed by induction on n.

The induction start is clear, because if M is Gorenstein projective, then so is M', by Theorem 2.5. If $n > 0$, we write $M = M' \oplus M''$ for some module M''. Pick exact sequences $0 \to K' \to G' \to M' \to 0$ and $0 \to K'' \to G'' \to M'' \to 0$, where G' and G'' are projectives. We get a commutative diagram with split-exact rows,

Applying Proposition 2.18 to the middle column in this diagram, we get that $\operatorname{Gpd}_R(K' \oplus K'') = n - 1$. Hence the induction hypothesis yields that $\operatorname{Gpd}_R K' \leqslant n - 1$, and thus

the short exact sequence $0 \to K' \to G' \to M' \to 0$ shows that $\mathrm{Gpd}_R M' \leqslant n$, as desired. $\square$

2.20. Theorem. *Let M be an R-module with finite Gorenstein projective dimension, and let n be an integer. Then the following conditions are equivalent*:

(i) $\mathrm{Gpd}_R M \leqslant n$.
(ii) $\mathrm{Ext}_R^i(M, L) = 0$ *for all $i > n$, and all R-modules L with finite $\mathrm{pd}_R L$.*
(iii) $\mathrm{Ext}_R^i(M, Q) = 0$ *for all $i > n$, and all projective R-modules Q.*
(iv) *For every exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ where $G_0, \ldots, G_{n-1}$ are Gorenstein projectives, then also K_n is Gorenstein projective.*

Consequently, the Gorenstein projective dimension of M is determined by the formulas:

$$\mathrm{Gpd}_R M = \sup\{i \in \mathbb{N}_0 \mid \exists L \in \bar{\mathscr{P}}(R): \ \mathrm{Ext}_R^i(M, L) \neq 0\}$$

$$= \sup\{i \in \mathbb{N}_0 \mid \exists Q \in \mathscr{P}(R): \ \mathrm{Ext}_R^i(M, Q) \neq 0\}.$$

Proof. The proof is 'cyclic'. Obviously (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i), so we only have to prove the last two implications.

To prove (i) $\Rightarrow$ (ii), we assume that $\mathrm{Gpd}_R M \leqslant n$. By definition there is an exact sequence, $0 \to G_n \to \cdots \to G_0 \to M \to 0$, where $G_0, \ldots, G_n$ are Gorenstein projectives. By Lemma 2.17 and Proposition 2.3, we conclude the equalities $\mathrm{Ext}_R^i(M, L) \cong \mathrm{Ext}_R^{i-n}(G_n, L) = 0$ whenever $i > n$, and L has finite projective dimension, as desired.

To prove (iii) $\Rightarrow$ (iv), we consider an exact sequence,

$$0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0, \tag{4}$$

where $G_0, \ldots, G_{n-1}$ are Gorenstein projectives. Applying Lemma 2.17 to this sequence, and using the assumption, we get that $\mathrm{Ext}_R^i(K_n, Q) \cong \mathrm{Ext}_R^{i+n}(M, Q) = 0$ for every integer $i > 0$, and every projective module Q. Decomposing (4) into short exact sequences, and applying Proposition 2.18 successively n times, we see that $\mathrm{Gpd}_R K_n < \infty$, since $\mathrm{Gpd}_R M < \infty$. Hence there is an exact sequence,

$$0 \to G_m' \to \cdots \to G_0' \to K_n \to 0,$$

where $G_0', \ldots, G_m'$ are Gorenstein projectives. We decompose it into short exact sequences, $0 \to C_j' \to G_{j-1}' \to C_{j-1}' \to 0$, for $j = 1, \ldots, m$, where $C_m' = G_m'$ and $C_0' = K_n$. Now another use of Lemma 2.17 gives that

$$\mathrm{Ext}_R^1(C_{j-1}', Q) \cong \mathrm{Ext}_R^j(K_n, Q) = 0$$

for all $j = 1, \ldots, m$, and all projective modules Q. Thus Proposition 2.11 can be applied successively to conclude that $C_m', \ldots, C_0'$ (in that order) are Gorenstein projectives. In particular $K_n = C_0'$ is Gorenstein projective.

The last formulas in the theorem for determination of $\mathrm{Gpd}_R M$ are a direct consequence of the equivalence between (i)–(iii). $\square$

2.21. Corollary. *If R is left noetherian, and M is a finite* (*left*) *module with finite Gorenstein projective dimension, then*

$$\mathrm{Gpd}_R M = \sup\{i \in \mathbb{N}_0 \mid \mathrm{Ext}^i_R(M,R) \neq 0\}.$$

Proof. By Theorem 2.20, it suffices to show that if $\mathrm{Ext}^i_R(M,Q) \neq 0$ for some projective module Q, then also $\mathrm{Ext}^i_R(M,R) \neq 0$. We simply pick another module P, such that $Q \oplus P \cong R^{(\Lambda)}$ for some index set Λ, and then note that $\mathrm{Ext}^n_R(M,R)^{(\Lambda)} \cong \mathrm{Ext}^n_R(M,Q) \oplus \mathrm{Ext}^n_R(M,P) \neq 0$. $\square$

2.22. Theorem. *Let N be an R-module with finite Gorenstein injective dimension, and let n be an integer. Then the following conditions are equivalent*:

(i) $\mathrm{Gid}_R N \leqslant n$.
(ii) $\mathrm{Ext}^i_R(L,N) = 0$ *for all* $i > n$, *and all R-modules L with finite* $\mathrm{id}_R L$.
(iii) $\mathrm{Ext}^i_R(I,N) = 0$ *for all* $i > n$, *and all injective R-modules I*.
(iv) *For every exact sequence* $0 \to N \to H^0 \to \cdots \to H^{n-1} \to C^n \to 0$ *where* $H^0, \ldots, H^{n-1}$ *are Gorenstein injectives, then also C^n is Gorenstein injective.*

Consequently, the Gorenstein injective dimension of N is determined by the formulas:

$$\mathrm{Gid}_R N = \sup\{i \in \mathbb{N}_0 \mid \exists L \in \bar{\mathscr{I}}(R): \ \mathrm{Ext}^i_R(L,N) \neq 0\}$$

$$= \sup\{i \in \mathbb{N}_0 \mid \exists I \in \mathscr{I}(R): \ \mathrm{Ext}^i_R(I,N) \neq 0\}.$$

Comparing Theorem 2.22 above with Matlis' Structure Theorem on injective modules we get the next result.

2.23. Corollary. *If R is commutative and noetherian, and N is a module with finite Gorenstein injective dimension, then*

$$\mathrm{Gid}_R N = \sup\{i \in \mathbb{N}_0 \mid \exists \mathfrak{p} \in \mathrm{Spec}\, R : \mathrm{Ext}^i_R(E_R(R/\mathfrak{p}),N) \neq 0\}.$$

Here $E_R(R/\mathfrak{p})$ denotes the injective hull of $R/\mathfrak{p}$.

2.24. Theorem. *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of R-modules. If any two of the modules M, M', or M'' have finite Gorenstein projective dimension, then so has the third.*

Proof. The proof of [5, Proposition 3.4] shows that this theorem is a formal consequence of Proposition 2.7. $\square$

2.25. Theorem. *Let* $0 \to N' \to N \to N'' \to 0$ *be a short exact sequence of R-modules. If any two of the modules N, N', or N'' have finite Gorenstein injective dimension, then so has the third.*

2.26. Remark. The theory of Gorenstein projective modules is particularly nice when the ring $(R, \mathfrak{m}, k)$ is local, noetherian, Cohen–Macaulay and has a dualizing module.

In that case we can consider the Auslander class $\mathscr{A}(R)$, defined in [4, (3.1)]. See also [15, Definition 5.5.1].

From [12, Corollary 2.4], the following implications are known for any R-module M:

$$M \in \mathscr{A}(R) \iff \mathrm{Gpd}_R M < \infty \iff \mathrm{Gpd}_R M \leqslant \dim R.$$

In this case the previous Theorem 2.24 is trivial, as it is easy to see that $\mathscr{A}(R)$ is closed under short exact sequences (this can be found in e.g. [15, Theorem 5.5.6]).

Similar remarks are to be said about the Bass class $\mathscr{B}(R)$ and the Gorenstein injective dimension.

It is only natural to investigate how much the usual projective dimension differs from the Gorenstein projective one. The answer follows easily from Theorem 2.20.

2.27. Proposition. *If M is an R-module with finite projective dimension, then $\mathrm{Gpd}_R M = \mathrm{pd}_R M$. In particular there is an equality of classes $\mathscr{GP}(R) \cap \bar{\mathscr{P}}(R) = \mathscr{P}(R)$.*

Proof. Assume that $n = \mathrm{pd}_R M$ is finite. By definition, there is always an inequality $\mathrm{Gpd}_R M \leqslant \mathrm{pd}_R M$, and consequently, we also have $\mathrm{Gpd}_R M \leqslant n < \infty$. In order to show that $\mathrm{Gpd}_R M = n$, we need, by Theorem 2.20, the existence of a projective module P, such that $\mathrm{Ext}_R^n(M, P) \neq 0$.

Since $\mathrm{pd}_R M = n$, there is some module, N, with $\mathrm{Ext}_R^n(M, N) \neq 0$. Let P be any projective module which surjects onto N. From the long exact homology sequence, it now follows that also $\mathrm{Ext}_R^n(M, P) \neq 0$, as desired. $\quad\square$

Using relative homological algebra, Enochs and Jenda have shown similar results to Proposition 2.27 above in [10, Propositions 10.1.2 and 10.2.3].

We end this section with an application of Gorenstein projective precovers. We compare the (*left*) *finitistic Gorenstein projective dimension* of the base ring R,

$$\mathrm{FGPD}(R) = \sup \left\{ \mathrm{Gpd}_R M \,\middle|\, \begin{array}{l} M \text{ is a (left) } R\text{-module with finite} \\ \text{Gorenstein projective dimension.} \end{array} \right\},$$

with the usual, and well-investigated, (left) finitistic projective dimension, $\mathrm{FPD}(R)$.

2.28. Theorem. *For any ring R there is an equality $\mathrm{FGPD}(R) = \mathrm{FPD}(R)$.*

Proof. Clearly $\mathrm{FPD}(R) \leqslant \mathrm{FGPD}(R)$ by Proposition 2.27. Note that if M is a module with $0 < \mathrm{Gpd}_R M < \infty$, then Theorem 2.10 in particular gives the existence of a module K with $\mathrm{pd}_R K = \mathrm{Gpd}_R M - 1$, and hence we get $\mathrm{FGPD}(R) \leqslant \mathrm{FPD}(R) + 1$. Proving the inequality $\mathrm{FGPD}(R) \leqslant \mathrm{FPD}(R)$, we may therefore assume that

$$0 < \mathrm{FGPD}(R) = m < \infty.$$

Pick a module M with $\mathrm{Gpd}_R M = m$. We wish to find a module L with $\mathrm{pd}_R L = m$. By Theorem 2.10 there is an exact sequence $0 \to K \to G \to M \to 0$ where G is Gorenstein projective, and $\mathrm{pd}_R K = m - 1$. Since G is Gorenstein projective, there exists

a projective module Q with $G \subseteq Q$, and since also $K \subseteq G$, we can consider the quotient $L = Q/K$. Note that $M \cong G/K$ is a submodule of L, and thus we get a short exact sequence $0 \to M \to L \to L/M \to 0$.

If L is Gorenstein projective, then Proposition 2.18 will imply that $\mathrm{Gpd}_R(L/M) = m + 1$, since $\mathrm{Gpd}_R M = m > 0$. But this contradict the fact that $m = \mathrm{FGPD}(R) < \infty$. Hence L is *not* Gorenstein projective, in particular, L is not projective. Therefore the short exact sequence $0 \to K \to Q \to L \to 0$ shows that $\mathrm{pd}_R L = \mathrm{pd}_R K + 1 = m$. $\quad\square$

For the (left) finitistic Gorenstein injective dimension, $\mathrm{FGID}(R)$, and the usual (left) finitistic injective dimension, $\mathrm{FID}(R)$, we of course also have:

2.29. Theorem. *For any ring R there is an equality* $\mathrm{FGID}(R) = \mathrm{FID}(R)$.

3. Gorenstein flat modules

The treatment of Gorenstein flat R-modules is different from the way we handled Gorenstein projective modules. This is because Gorenstein flat modules are defined by the tensor product functor $- \otimes_R -$ and not by $\mathrm{Hom}_R(-, -)$. However, over a right coherent ring there is a connection between Gorenstein flat left modules and Gorenstein injective right modules, and this allow us to get good results.

3.1. Definition. A *complete flat resolution* is an exact sequence of flat (left) R-modules, $\boldsymbol{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$, such that $I \otimes_R \boldsymbol{F}$ is exact for every injective right R-module I.

An R-module M is called *Gorenstein flat* (*G-flat* for short), if there exists a complete flat resolution $\boldsymbol{F}$ with $M \cong \mathrm{Im}(F_0 \to F^0)$. The class of all Gorenstein flat R-modules is denoted $\mathscr{GF}(R)$.

There is a nice connection between Gorenstein flat and Gorenstein injective modules, and this enable us to prove that the class of Gorenstein flat modules is projectively resolving. We begin with:

3.2. Proposition. *The class $\mathscr{GF}(R)$ is closed under arbitrary direct sums.*

Proof. Simply note that a (degreewise) sum of complete flat resolutions again is a complete flat resolution (as tensorproducts commutes with sums). $\quad\square$

3.3. Remark. From Bass [6, Corollary 5.5], and Gruson–Raynaud [14, Seconde partie, Theorem 3.2.6], we have that $\mathrm{FPD}(R) = \dim R$, when R is commutative and noetherian.

3.4. Proposition. *If R is right coherent with finite left finitistic projective dimension, then every Gorenstein projective (left) R-module is also Gorenstein flat.*

Proof. It suffices to prove that if $\boldsymbol{P}$ is a complete projective resolution, then $I \otimes_R \boldsymbol{P}$ is exact for all injective right modules I. Since R is right coherent, $F = \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is

a flat (left) R-module by [15, Lemma 3.1.4]. Since FPD(R) is finite, Jensen [13, Proposition 6] implies that F has finite projective dimension, and consequently $\mathrm{Hom}_R(\boldsymbol{P}, F)$ is exact by Proposition 2.3. By adjointness,

$$\mathrm{Hom}_{\mathbb{Z}}(I \otimes_R \boldsymbol{P}, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(\boldsymbol{P}, F),$$

and the desired result follows. □

3.5. Example. Let R be any integral domain which is not a field, and let K denote the field of fractions of R. Then K is a flat (and hence Gorenstein flat) R-module which is not contained in any free R-module, in particular, K cannot be Gorenstein projective.

3.6. Theorem. *For any* (*left*) R-*module* M, *we consider the following conditions.*

 (i) *M is a Gorenstein flat* (*left*) *R-module.*
 (ii) *The Pontryagin dual* $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ *is a Gorenstein injective right R-module.*
(iii) *M admits a co-proper right flat resolution* (*that is, a co-proper right $\mathscr{F}(R)$-resolution*), *and* $\mathrm{Tor}_i^R(I, M)=0$ *for all injective right R-modules I, and all integers $i > 0$.*

Then (i) $\Rightarrow$ (ii). *If R is right coherent, then also* (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), *and hence all three conditions are equivalent.*
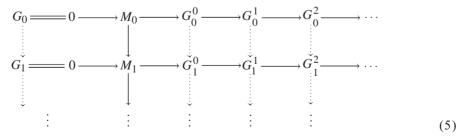
Proof. As the theorem is stated, it is an extended non-commutative version of [7, Theorem 6.4.2], which deals with commutative, noetherian rings. However, a careful reading of the proof, compared with basic facts about the Pontryagin dual, gives this stronger version. □

3.7. Theorem. *If R is right coherent, then the class $\mathscr{GF}(R)$ of Gorenstein flat R-modules is projectively resolving and closed under direct summands.*

Furthermore, if $M_0 \to M_1 \to M_2 \to \cdots$ is a sequence of Gorenstein flat modules, then the direct limit $\varinjlim M_n$ is again Gorenstein flat.

Proof. Using Theorem 2.6 together with the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 3.6 above, we see that $\mathscr{GF}(R)$ is projectively resolving. Now, comparing Proposition 3.2 with Proposition 1.4, we get that $\mathscr{GF}(R)$ is closed under direct summands.

Concerning the last statement, we pick for each n a co-proper right flat resolution $\boldsymbol{G}_n$ of M_n (which is possible by Theorem 3.6 (iii)), as illustrated in the next diagram.

$$\begin{array}{ccccccccccc}
G_0 & = & 0 & \longrightarrow & M_0 & \longrightarrow & G_0^0 & \longrightarrow & G_0^1 & \longrightarrow & G_0^2 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_1 & = & 0 & \longrightarrow & M_1 & \longrightarrow & G_1^0 & \longrightarrow & G_1^1 & \longrightarrow & G_1^2 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array} \qquad (5)$$

By Proposition 1.8, each map $M_n \to M_{n+1}$ can be lifted to a chain map $\boldsymbol{G}_n \to \boldsymbol{G}_{n+1}$ of complexes. Since we are dealing with *sequences* (and not arbitrary direct systems), each column in (5) is again a direct system. Thus it makes sense to apply the exact functor $\varinjlim$ to (5), and doing so, we obtain an exact complex,

$$\boldsymbol{G} = \varinjlim \boldsymbol{G}_n = 0 \to \varinjlim M_n \to \varinjlim G_n^0 \to \varinjlim G_n^1 \to \cdots,$$

where each module $G^k = \varinjlim G_n^k$, $k = 0, 1, 2, \ldots$ is flat. When I is injective right R-module, then $I \otimes_R \boldsymbol{G}_n$ is exact because: since $F = \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is a flat (left) R-module (recall that R is right coherent), we get exactness of

$$\mathrm{Hom}_R(\boldsymbol{G}_n, F) = \mathrm{Hom}_R(\boldsymbol{G}_n, \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(I \otimes_R \boldsymbol{G}_n, \mathbb{Q}/\mathbb{Z}),$$

and hence of $I \otimes_R \boldsymbol{G}_n$, since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module. Since $\varinjlim$ commutes with the homology functor, we also get exactness of

$$I \otimes_R \boldsymbol{G} \cong \varinjlim (I \otimes_R \boldsymbol{G}_n).$$

Thus we have constructed the "right half", $\boldsymbol{G}$, of a complete flat resolution for $\varinjlim M_n$. Since M_n is Gorenstein flat, we also have

$$\mathrm{Tor}_i^R(I, \varinjlim M_n) \cong \varinjlim \mathrm{Tor}_i^R(I, M_n) = 0$$

for $i > 0$, and all injective right modules I. Thus $\varinjlim M_n$ is Gorenstein flat. $\quad\square$

3.8. Proposition. *Assume that R is right coherent, and consider a short exact sequence of (left) R-modules $0 \to G' \to G \to M \to 0$, where G and G' are Gorenstein flats. If $\mathrm{Tor}_1^R(I, M) = 0$ for all injective right modules I, then M is G-flat.*

Proof. Define $H = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$ and $H' = \mathrm{Hom}_{\mathbb{Z}}(G', \mathbb{Q}/\mathbb{Z})$, which are Gorenstein injectives by the general implication (i) $\Rightarrow$ (ii) in Theorem 3.6. Applying the dual of Corollary 2.11 (about Gorenstein injective modules) to the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \to H \to H' \to 0,$$

and noting that we have an isomorphism,

$$\mathrm{Ext}_R^1(I, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}_1^R(I, M), \mathbb{Q}/\mathbb{Z}) = 0$$

for all injective right modules I, we see that $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective. Since R is right coherent, we conclude that M is Gorenstein flat. $\quad\square$

Next we introduce the Gorenstein flat dimension via resolutions, and show how the Tor-functors can be used to measure this dimension when R is right coherent.

3.9. Gorenstein flat dimension. As done in [11] (and similar to the Gorenstein projective case), we define the *Gorenstein flat dimension*, $\mathrm{Gfd}_R M$, of a module M by declaring that $\mathrm{Gfd}_R M \leqslant n$ if, and only if, M has a resolution by Gorenstein flat modules of length n. We let $\overline{\mathscr{G}\mathscr{F}}(R)$ denote the class of all R-modules with finite Gorenstein flat dimension.

3.10. Proposition (Flat base change). *Consider a flat homomorphism of commutative rings $R \to S$ (that is, S is flat as an R-module). Then for any (left) R-module M we have an inequality,*

$$\mathrm{Gfd}_S(S \otimes_R M) \leqslant \mathrm{Gfd}_R M.$$

Proof. If $\boldsymbol{F}$ is a complete flat resolution of R-modules, then $S \otimes_R \boldsymbol{F}$ is an exact (since S is R-flat) sequence of flat S-modules. If I is an injective S-module, then, since S is R-flat, I is also an injective R-module. Thus we have exactness of

$$I \otimes_S (S \otimes_R \boldsymbol{F}) \cong (I \otimes_S S) \otimes_R \boldsymbol{F} \cong I \otimes_R \boldsymbol{F},$$

and hence $S \otimes_R \boldsymbol{F}$ is a complete flat resolution of S-modules. $\square$

3.11. Proposition. *For any (left) R-module M there is an inequality,*

$$\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leqslant \mathrm{Gfd}_R M.$$

If R is right coherent, then we have the equality,

$$\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \mathrm{Gfd}_R M.$$

Proof. The inequality follows directly from the implication (i) $\Rightarrow$ (ii) in Theorem 3.6. Now assume that R is right coherent. For the converse inequality, we may assume that $\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = m$ is finite. Pick an exact sequence,

$$0 \to K_m \to G_{m-1} \to \cdots \to G_0 \to M \to 0,$$

where $G_0, \ldots, G_{m-1}$ are Gorenstein flats. Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to this sequence, we get exactness of

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \to H^0 \to \cdots \to H^{m-1} \to C^m \to 0,$$

where we have defined $H^i = \mathrm{Hom}_{\mathbb{Z}}(G_i, \mathbb{Q}/\mathbb{Z})$ for $i = 0, \ldots, m-1$, together with $C^m = \mathrm{Hom}_{\mathbb{Z}}(K_m, \mathbb{Q}/\mathbb{Z})$. Since $H^0, \ldots, H^{m-1}$ are Gorenstein injectives, Theorem 2.22 implies that $C^m = \mathrm{Hom}_{\mathbb{Z}}(K_m, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective. Now another application of Theorem 3.6 gives that K_m is Gorenstein flat (since R is right coherent), and consequently $\mathrm{Gfd}_R M \leqslant m = \mathrm{Gid}_R \mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$. $\square$

Using the connection between Gorenstein flat and Gorenstein injective dimension, which Proposition 3.11 establishes, together the Gorenstein injective versions of Propositions 2.18 and 2.19, we get the next two results.

3.12. Proposition. *Assume that R is right coherent. Let $0 \to K \to G \to M \to 0$ be a short exact sequence of R-modules where G is Gorenstein flat, and define $n = \mathrm{Gfd}_R M$. If M is Gorenstein flat, then so is K. If otherwise $n > 0$, then $\mathrm{Gfd}_R K = n - 1$.*

3.13. Proposition. *Assume that R is right coherent. If $(M_\lambda)_{\lambda \in \Lambda}$ is any family of (left) R-modules, then we have an equality,*

$$\mathrm{Gfd}_R \left(\coprod M_\lambda \right) = \sup\{ \mathrm{Gfd}_R M_\lambda \mid \lambda \in \Lambda \}.$$

The next theorem is a generalization of [7, Theorem 5.2.14], which is proved only for (commutative) local, noetherian Cohen–Macaulay rings with a dualizing module.

3.14. Theorem. *Assume that R is right coherent. Let M be a (left) R-module with finite Gorenstein flat dimension, and let $n \geqslant 0$ be an integer. Then the following four conditions are equivalent*:

  (i) $\operatorname{Gfd}_R M \leqslant n$.
 (ii) $\operatorname{Tor}_i^R(L, M) = 0$ *for all right R-modules L with finite $\operatorname{id}_R L$, and all $i > n$.*
(iii) $\operatorname{Tor}_i^R(I, M) = 0$ *for all injective right R-modules I, and all $i > n$.*
 (iv) *For every exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, where $G_0, \ldots, G_{n-1}$ are Gorenstein flats, then also K_n is Gorenstein flat.*
     Consequently, the Gorenstein flat dimension of M is determined by the formulas:

$$\operatorname{Gfd}_R M = \sup\{i \in \mathbb{N}_0 \mid \exists L \in \bar{\mathscr{I}}(R): \ \operatorname{Tor}_i^R(L, M) \neq 0\}$$

$$= \sup\{i \in \mathbb{N}_0 \mid \exists I \in \mathscr{I}(R): \ \operatorname{Tor}_i^R(I, M) \neq 0\}.$$

Proof. Combine the adjointness isomorphism,

$$\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Tor}_i^R(L, M), \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}_R^i(L, \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}))$$

for right R-modules L, together with the identity from Proposition 3.11,

$$\operatorname{Gid}_R \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \operatorname{Gfd}_R M,$$

and use Theorem 2.22. $\square$

3.15. Theorem. *Assume that R is right coherent. If any two of the modules M, M' or M'' in a short exact sequence $0 \to M' \to M \to M'' \to 0$ have finite Gorenstein flat dimension, then so has the third.*

Proof. Consider $0 \to \operatorname{Hom}_{\mathbb{Z}}(M'', \mathbb{Q}/\mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(M', \mathbb{Q}/\mathbb{Z}) \to 0$. Using Proposition 3.11 together with Theorem 2.25, the desired conclusion easily follows. $\square$

Next, we examine the large restricted flat dimension, and relate it to the usual flat dimension, and to the Gorenstein flat dimension.

3.16. Large restricted flat dimension. For a R-module M, we consider the *large restricted flat dimension*, which is defined by

$$\operatorname{Rfd}_R M = \sup \left\{ i \geqslant 0 \,\middle|\, \begin{array}{l} \operatorname{Tor}_i^R(L, M) \neq 0 \text{ for some (right)} \\ R\text{-module with finite flat dimension.} \end{array} \right\}.$$
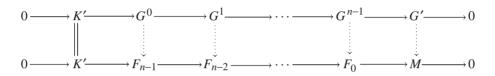
3.17. Lemma. *Assume that R is right coherent. Let M be any R-module with finite Gorenstein flat dimension n. Then there exists a short exact sequence $0 \to K \to G \to$*

$M \to 0$ *where G is Gorenstein flat, and where* $\mathrm{fd}_R K = n - 1$ *(if $n = 0$, this should be interpreted as $K = 0$).*

Proof. We may assume that $n > 0$. We start by taking an exact sequence,

$$0 \to K' \to F_{n-1} \to \cdots \to F_0 \to M \to 0,$$

where $F_0, \ldots, F_{n-1}$ are flats. Then K' is Gorenstein flat by Theorem 3.14, and hence Theorem 3.6 (iii) gives an exact sequence $0 \to K' \to G^0 \to \cdots \to G^{n-1} \to G' \to 0$, where $G^0, \ldots, G^{n-1}$ are flats, G' is Gorenstein flat, and such that the functor $\mathrm{Hom}_R(-, F)$ leaves this sequence exact whenever F is a flat R-module. Consequently, we get homomorphisms, $G^i \to F_{n-1-i}$, $i = 0, \ldots, n - 1$, and $G' \to M$, giving a commutative diagram:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & K' & \longrightarrow & G^0 & \longrightarrow & G^1 & \longrightarrow \cdots \longrightarrow & G^{n-1} & \longrightarrow & G' & \longrightarrow 0 \\
& & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & K' & \longrightarrow & F_{n-1} & \longrightarrow & F_{n-2} & \longrightarrow \cdots \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow 0
\end{array}
$$

The argument following diagram (3) in the proof of Theorem 2.10 finishes the proof. □

3.18. Remark. As noticed in the proof of Theorem 2.10, the homomorphism $G \twoheadrightarrow M$ in a short exact sequence $0 \to K \to G \to M \to 0$ where $\mathrm{pd}_R K$ is finite, is necessarily a Gorenstein projective precover of M.

But the homomorphism $G \twoheadrightarrow M$ in the exact sequence $0 \to K \to G \to M \to 0$ established above in Lemma 3.17, where $\mathrm{fd}_R K$ is finite, is not necessarily a Gorenstein flat cover of M, since it is not true that $\mathrm{Ext}_R^1(T, K) = 0$ whenever T is Gorenstein flat and $\mathrm{fd}_R K$ is finite.

We make up for this loss in Theorem 3.23. Meanwhile, we have the application below of the simpler Lemma 3.17.

The large restricted flat dimension was investigated in [8, Section 2] and in [7, Chapters 5.3–5.4]. It is conjectured by Foxby that if $\mathrm{Gfd}_R M$ is finite, then $\mathrm{Rfd}_R M = \mathrm{Gfd}_R M$. Christensen [7, Theorem 5.4.8] proves this for local noetherian Cohen–Macaulay rings with a dualizing module. We have the following extension:

3.19. Theorem. *For any R-module M, we have two inequalities,*

$$\mathrm{Rfd}_R M \leqslant \mathrm{Gfd}_R M \leqslant \mathrm{fd}_R M.$$

Now assume that R is commutative and noetherian. If $\mathrm{Gfd}_R M$ is finite, then

$$\mathrm{Rfd}_R M = \mathrm{Gfd}_R M.$$

If $\mathrm{fd}_R M$ is finite, then we have two equalities

$$\mathrm{Rfd}_R M = \mathrm{Gfd}_R M = \mathrm{fd}_R M.$$

Proof. The last inequality $\mathrm{Gfd}_R M \leqslant \mathrm{fd}_R M$ is clear. Concerning $\mathrm{Rfd}_R M \leqslant \mathrm{Gfd}_R M$, we may assume that $n = \mathrm{Gfd}_R M$ is finite, and then proceed by induction on $n \geqslant 0$.

If $n = 0$, then M is Gorenstein flat. We wish to prove that $\mathrm{Tor}_i^R(L, M) = 0$ for all $i > 0$, and all right modules L with finite flat dimension. Therefore assume that $\ell = \mathrm{fd}_R L$ is finite. Since M is Gorenstein flat, there exists an exact sequence,

$$0 \to M \to G^0 \to \cdots \to G^{\ell-1} \to T \to 0,$$

where $G^0, \ldots, G^{\ell-1}$ are flats (and T is Gorenstein flat). By this sequence we conclude that $\mathrm{Tor}_i^R(-, M) \cong \mathrm{Tor}_{i+\ell}^R(-, T)$ for all $i > 0$, in particular we get that $\mathrm{Tor}_i^R(L, M) \cong \mathrm{Tor}_{i+\ell}^R(L, T) = 0$ for all $i > 0$, since $i + \ell > \mathrm{fd}_R L$.

Next we assume that $n > 0$. Pick a short exact sequence $0 \to K \to T \to M \to 0$ where T is Gorenstein flat, and $\mathrm{Gfd}_R K = n - 1$. By induction hypothesis we have

$$\mathrm{Rfd}_R K \leqslant \mathrm{Gfd}_R K = n - 1,$$

and hence $\mathrm{Tor}_j^R(L, K) = 0$ for all $j > n - 1$, and all (right) R-modules L with finite flat dimension. For such an L, and an integer $i > n$, we use the long exact sequence,

$$0 = \mathrm{Tor}_i^R(L, T) \to \mathrm{Tor}_i^R(L, M) \to \mathrm{Tor}_{i-1}^R(L, K) = 0,$$

to conclude that $\mathrm{Tor}_i^R(L, M) = 0$. Therefore $\mathrm{Rfd}_R M \leqslant n = \mathrm{Gfd}_R M$.

Now assume that R is commutative and noetherian. If $\mathrm{fd}_R M$ is finite, then [7, Proposition 5.4.2] implies that $\mathrm{Rfd}_R M = \mathrm{fd}_R M$, and hence also $\mathrm{Rfd}_R M = \mathrm{Gfd}_R M = \mathrm{fd}_R M$.

Next assume that $\mathrm{Gfd}_R M = n$ is finite. We have to prove that $\mathrm{Rfd}_R M \geqslant n$. Naturally we may assume that $n > 0$. By Lemma 3.17 there exists a short exact sequence, say $0 \to K \to T \to M \to 0$, where T is Gorenstein flat and $\mathrm{fd}_R K = n - 1$. Since T is Gorenstein flat, we have a short exact sequence $0 \to T \to G \to T' \to 0$ where G is flat and T' is Gorenstein flat. Since $K \subseteq T \subseteq G$, we can consider the residue class module $Q = G/K$.

Because G is flat and $\mathrm{fd}_R K = n - 1$, exactness of $0 \to K \to G \to Q \to 0$ shows that $\mathrm{fd}_R Q \leqslant n$. Note that $M \cong T/K$ is a submodule of $Q = G/K$ with $Q/M \cong (G/K)/(T/K) \cong G/T \cong T'$, and thus we get a short exact sequence $0 \to M \to Q \to T' \to 0$. Since $\mathrm{Gfd}_R M = n$, Theorem 3.14 gives an injective module I with $\mathrm{Tor}_n^R(I, M) \neq 0$. Applying $I \otimes_R -$ to $0 \to M \to Q \to T' \to 0$, we get

$$0 = \mathrm{Tor}_{n+1}^R(I, T') \to \mathrm{Tor}_n^R(I, M) \to \mathrm{Tor}_n^R(I, Q),$$

showing that $\mathrm{Tor}_n^R(I, Q) \neq 0$. Since $\mathrm{Gfd}_R Q \leqslant \mathrm{fd}_R Q \leqslant n < \infty$, Theorem 3.14 gives that $\mathrm{Gfd}_R Q \geqslant n$. Therefore $\mathrm{fd}_R Q = n$, and consequently $\mathrm{Rfd}_R Q = \mathrm{fd}_R Q = n$.

Thus we get the existence of an R-module L with finite flat dimension, such that $\operatorname{Tor}_n^R(L, Q) \neq 0$. Since T' is Gorenstein flat, then $\operatorname{Rfd}_R T' \leqslant 0$, and so the exactness of $\operatorname{Tor}_n^R(L, M) \to \operatorname{Tor}_n^R(L, Q) \to \operatorname{Tor}_n^R(L, T') = 0$ proves that also $\operatorname{Tor}_n^R(L, M) \neq 0$. Hence $\operatorname{Rfd}_R M \geqslant n$, as desired. $\square$

Our next goal is to prove that over a right coherent ring, every (left) module M with finite $\operatorname{Gfd}_R M$, admits a Gorenstein flat precover.

This result can be found in [12, Theorem 3.5] for local noetherian Cohen–Macaulay rings $(R, \mathfrak{m}, k)$, admitting a dualizing module. Actually the proof presented there almost works in the general case, when we use as input the strong results about Gorenstein flat modules from this section.

3.20. Cotorsion modules. Xu [15, Definition 3.1.1], calls an R-module K for *cotorsion*, if $\operatorname{Ext}_R^1(F, K) = 0$ for all flat R-modules F. In [15, Lemma 2.1.1] it is proved that if $\varphi \colon F \to M$ is a flat cover of any module M, then the kernel $K = \operatorname{Ker} \varphi$ is cotorsion. Furthermore, if R is right coherent, and M is a left R-module with finite flat dimension, then M *has* a flat cover by [15, Theorem 3.1.11].

3.21. Pure injective modules. Recall that a short exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

of (left) modules is called *pure exact* if $0 \to X \otimes A \to X \otimes B \to X \otimes C \to 0$ is exact for every (right) module X. In this case we also say that A is a *pure submodule* of B. A module H is called *pure injective* if the sequence

$$0 \longrightarrow \operatorname{Hom}_R(C, H) \longrightarrow \operatorname{Hom}_R(B, H) \longrightarrow \operatorname{Hom}_R(A, H) \longrightarrow 0$$

is exact for every pure exact sequence $0 \to A \to B \to C \to 0$. By [15, Theorem 2.3.8], every R-module M has a pure injective envelope, denoted $\operatorname{PE}(M)$, such that $M \subseteq \operatorname{PE}(M)$. If R is right coherent, and F is flat, then both $\operatorname{PE}(F)$ and $\operatorname{PE}(F)/F$ are flat too, by [15, Lemma 3.1.6]. Also note that every pure injective module is cotorsion.

3.22. Proposition. *Assume that R is right coherent. If T is a Gorenstein flat R-module, then $\operatorname{Ext}_R^i(T, K) = 0$ for all integers $i > 0$, and all cotorsion R-modules K with finite flat dimension.*

Proof. We use induction on the finite number $\operatorname{fd}_R K = n$. If $n = 0$, then K is flat. Consider the Pontryagin duals $K^* = \operatorname{Hom}_{\mathbb{Z}}(K, \mathbb{Q}/\mathbb{Z})$, and $K^{**} = \operatorname{Hom}_{\mathbb{Z}}(K^*, \mathbb{Q}/\mathbb{Z})$. Since R is right coherent, and K^* is injective, then K^{**} is flat, by [15, Lemma 3.1.4]. By [15, Proposition 2.3.5], K is a pure submodule of K^{**}, and hence K^{**}/K is flat. Since K is cotorsion, $\operatorname{Ext}_R^1(K^{**}/K, K) = 0$, and consequently,

$$0 \to K \to K^{**} \to K^{**}/K \to 0$$

is split exact. Therefore, K is a direct summand of K^{**}, which implies that $\operatorname{Ext}_R^i(T, K)$ is a direct summand of

$$\operatorname{Ext}_R^i(T, K^{**}) \cong \operatorname{Ext}_R^i(T, \operatorname{Hom}_{\mathbb{Z}}(K^*, \mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Tor}_i^R(K^*, T), \mathbb{Q}/\mathbb{Z}) = 0,$$

where $\operatorname{Tor}_i^R(K^*, T) = 0$, since T is Gorenstein flat, and K^* is injective.

Now assume that $n = \operatorname{fd}_R K > 0$. By the remarks in 3.20, we can pick a short exact sequence $0 \to K' \to F \to K \to 0$, where $F \to K$ is a flat cover of K, and K' is cotorsion with $\operatorname{fd}_R K' = n - 1$. Since both K' and K are cotorsion, then so is F, by [15, Proposition 3.1.2]. Applying the induction hypothesis, the long exact sequence,
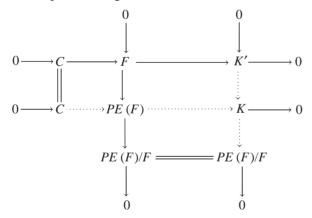
$$0 = \operatorname{Ext}_R^i(T, F) \to \operatorname{Ext}_R^i(T, K) \to \operatorname{Ext}_R^{i+1}(T, K') = 0,$$

gives the desired conclusion. □

3.23. Theorem. *Assume that R is right coherent ring R, and that M is an R-module with finite Gorenstein flat dimension n. Then M admits a surjective Gorenstein flat precover $\varphi\colon T \twoheadrightarrow M$, where $K = \operatorname{Ker}\varphi$ satisfies $\operatorname{fd}_R K = n - 1$ (if $n = 0$, this should be interpreted as $K = 0$).*

In particular, M admits a proper left Gorenstein flat resolution (that is, a proper left $\mathscr{GF}(R)$-resolution) of length n.
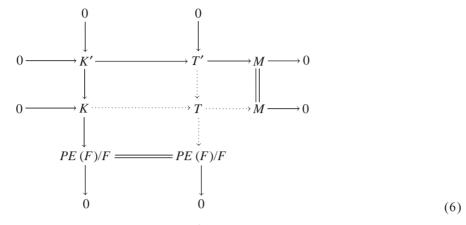
Proof. We may assume that $n > 0$. By Proposition 3.22, it suffices to construct an exact sequence $0 \to K \to T \to M \to 0$ where K is cotorsion with $\operatorname{fd}_R K = n - 1$. By Lemma 3.17 there exists a short exact sequence $0 \to K' \to T' \to M \to 0$ where T' is Gorenstein flat and $\operatorname{fd}_R K' = n - 1$. Since $\operatorname{fd}_R K'$ is finite, then K' has a flat cover by the remarks in 3.20, say $\psi\colon F \twoheadrightarrow K'$, and the kernel $C = \operatorname{Ker}\psi$ is cotorsion. Now consider the pushout diagram,

In the sequence $0 \to C \to \operatorname{PE}(F) \to K \to 0$, both C and $\operatorname{PE}(F)$ are cotorsion, and hence also K is cotorsion by [15, Proposition 3.1.2]. Furthermore, since $\operatorname{PE}(F)/F$ is flat, the short exact sequence $0 \to K' \to K \to \operatorname{PE}(F)/F \to 0$ shows that

$$\operatorname{fd}_R K = \operatorname{fd}_R K' = n - 1.$$

Finally we consider the pushout diagram,

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K' & \longrightarrow & T' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \parallel & & \\
0 & \longrightarrow & K & \cdots\cdots\cdots\cdots & T & \cdots\cdots & M & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & & & \\
 & & PE\,(F)/F & =\!=\!=\!=\!= & PE\,(F)/F & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{6}
$$

In the second column in (6), both T' and $PE(F)/F$ are Gorenstein flat, and hence also T is Gorenstein flat, since the class $\mathscr{GF}(R)$ is projectively resolving by Theorem 3.7. Therefore the lower row in diagram (6), $0 \to K \to T \to M \to 0$, is the desired sequence. $\square$

Finally we may compare the (*left*) *finitistic Gorenstein flat dimension* of the base ring R, defined by

$$
\mathrm{FGFD}(R) = \sup \left\{ \mathrm{Gpd}_R M \; \middle| \; \begin{array}{l} M \text{ is a (left) } R\text{-module with} \\ \text{finite Gorenstein flat dimension.} \end{array} \right\},
$$

with the usual (left) finitistic flat dimension, $\mathrm{FFD}(R)$.

3.24. Theorem. *If R is right coherent, then* $\mathrm{FGFD}(R) = \mathrm{FFD}(R)$.

Proof. Analogous to the proof of Theorem 2.28, using Proposition 3.12 instead of 2.18, and Theorem 3.23 above instead of 2.10. $\square$

Acknowledgements

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