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RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

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ABSTRACT. In this paper we prove that for any associative ring R, and for any left R-module M with finite projective dimension, the Gorenstein injective dimension $\mathrm{Gid}_R M$ equals the usual injective dimension $\mathrm{id}_R M$. In particular, if $\mathrm{Gid}_R R$ is finite, then also $\mathrm{id}_R R$ is finite, and thus R is Gorenstein (provided that R is commutative and Noetherian).

1. Introduction

It is well known that among the commutative local Noetherian rings $(R, \mathfrak{m}, k)$, the *Gorenstein rings* are characterized by the condition $\mathrm{id}_R R < \infty$. From the dual of [10, Proposition (2.27)] ([6, Proposition 10.2.3] is a special case) it follows that the *Gorenstein injective dimension* $\mathrm{Gid}_R(-)$ is a *refinement* of the usual injective dimension $\mathrm{id}_R(-)$ in the following sense:

For any R-module M there is an inequality $\mathrm{Gid}_R M \leqslant \mathrm{id}_R M$, and if $\mathrm{id}_R M < \infty$, then there is an equality $\mathrm{Gid}_R M = \mathrm{id}_R M$.

Now, since the injective dimension $\mathrm{id}_R R$ of R measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension $\mathrm{Gid}_R R$ of R measure? As a consequence of Theorem (2.1) below, it turns out that:

An associative ring R with $\mathrm{Gid}_R R < \infty$ also has $\mathrm{id}_R R < \infty$ (and hence R is Gorenstein, provided that R is commutative and Noetherian).

This result is proved by Christensen [2, Theorem (6.3.2)] in the case where $(R, \mathfrak{m}, k)$ is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem (2.1), together with a series of related results. Among these results is Theorem (3.2), which has the nice, and easily stated, Corollary (3.3):

Assume that $(R, \mathfrak{m}, k)$ is a commutative local Noetherian ring, and let M be an R-module of finite depth, that is, $\mathrm{Ext}_R^m(k, M) \neq 0$ for some $m \in \mathbb{N}_0$ (this happens for example if $M \neq 0$ is finitely generated). If either

$$(i)\ \ \mathrm{Gfd}_R M < \infty \ \text{ and } \ \mathrm{id}_R M < \infty \qquad \text{or} \qquad (ii)\ \ \mathrm{fd}_R M < \infty \ \text{ and } \ \mathrm{Gid}_R M < \infty,$$

then R is Gorenstein.

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This corollary is also proved by Christensen [2, Theorem (6.3.2)] in the case where $(R, \mathfrak{m}, k)$ is Cohen-Macaulay with a dualizing module. However, Theorem (3.2) itself (dealing not only with local rings) is a generalization of [8, Proposition 2.10] (in the module case) by Foxby from 1979.

We should briefly mention the history of Gorenstein injective, projective and flat modules: *Gorenstein injective* modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in [3]. The dual concept, *Gorenstein projective* modules, was already introduced by Auslander and Bridger [1] in 1969, but only for finitely generated modules over a two-sided Noetherian ring. *Gorenstein flat* modules were also introduced by Enochs and Jenda; please see [5].

1.1. Setup and notation. Let R be any associative ring with a nonzero multiplicative identity. All modules are—if not specified otherwise—*left* R-modules. If M is any R-module, we use $\mathrm{pd}_R M$, $\mathrm{fd}_R M$, and $\mathrm{id}_R M$ to denote the usual projective, flat, and injective dimension of M, respectively. Furthermore, we write $\mathrm{Gpd}_R M$, $\mathrm{Gfd}_R M$, and $\mathrm{Gid}_R M$ for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimension of M, respectively.

2. Rings with finite Gorenstein injective dimension

Theorem 2.1. *If M is an R-module with $\mathrm{pd}_R M < \infty$, then $\mathrm{Gid}_R M = \mathrm{id}_R M$. In particular, if $\mathrm{Gid}_R R < \infty$, then also $\mathrm{id}_R R < \infty$ (and hence R is Gorenstein, provided that R is commutative and Noetherian).*

Proof. Since $\mathrm{Gid}_R M \leqslant \mathrm{id}_R M$ always, it suffices to prove that $\mathrm{id}_R M \leqslant \mathrm{Gid}_R M$. Naturally, we may assume that $\mathrm{Gid}_R M < \infty$.

First consider the case where M is Gorenstein injective, that is, $\mathrm{Gid}_R M = 0$. By definition, M is a kernel in a complete injective resolution. This means that there exists an exact sequence $\boldsymbol{E} = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots$ of injective R-modules, such that $\mathrm{Hom}_R(I, \boldsymbol{E})$ is exact for every injective R-module I, and such that $M \cong \mathrm{Ker}(E_1 \to E_0)$. In particular, there exists a short exact sequence $0 \to M' \to E \to M \to 0$, where E is injective, and M' is Gorenstein injective. Since M' is Gorenstein injective and $\mathrm{pd}_R M < \infty$, it follows by [4, Lemma 1.3] that $\mathrm{Ext}^1_R(M, M') = 0$. Thus $0 \to M' \to E \to M \to 0$ is split-exact; so M is a direct summand of the injective module E. Therefore, M itself is injective.

Next consider the case where $\mathrm{Gid}_R M > 0$. By [10, Theorem (2.15)] there exists an exact sequence $0 \to M \to H \to C \to 0$ where H is Gorenstein injective and $\mathrm{id}_R C = \mathrm{Gid}_R M - 1$. As in the previous case, since H is Gorenstein injective, there exists a short exact sequence $0 \to H' \to I \to H \to 0$ where I is injective and H' is Gorenstein injective. Now consider the pull-back diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & H & \longrightarrow & C & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & P & \longrightarrow & I & \longrightarrow & C & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & H' & = & H' & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since I is injective and $\mathrm{id}_R C = \mathrm{Gid}_R M - 1$ we get $\mathrm{id}_R P \leqslant \mathrm{Gid}_R M$ by the second row. Since H' is Gorenstein injective and $\mathrm{pd}_R M < \infty$, it follows (as before) by [4, Lemma 1.3] that $\mathrm{Ext}^1_R(M, H') = 0$. Consequently, the first column $0 \to H' \to P \to M \to 0$ splits. Therefore $P \cong M \oplus H'$, and hence $\mathrm{id}_R M \leqslant \mathrm{id}_R P \leqslant \mathrm{Gid}_R M$. $\square$

The theorem above has, of course, a dual counterpart:

Theorem 2.2. *If M is an R-module with $\mathrm{id}_R M < \infty$, then $\mathrm{Gpd}_R M = \mathrm{pd}_R M$.* $\square$

Theorem (2.6) below is a "flat version" of the two previous theorems. First recall the following.

Definition 2.3. The *left finitistic projective dimension* $\mathsf{Left}\mathrm{FPD}(R)$ of R is defined as

$$\mathsf{Left}\mathrm{FPD}(R) = \sup\{\ \mathrm{pd}_R M \mid M \text{ is a } \textit{left } R\text{-module with } \mathrm{pd}_R M < \infty\ \}.$$

The right finitistic projective dimension $\mathsf{Right}\mathrm{FPD}(R)$ of R is defined similarly.

Remark 2.4. When R is commutative and Noetherian, we have that $\mathsf{Left}\mathrm{FPD}(R)$ and $\mathsf{Right}\mathrm{FPD}(R)$ equals the Krull dimension of R, by [9, Théorème (3.2.6) (Seconde partie)].

Furthermore, we will need the following result from [10, Proposition (3.11)]:

Proposition 2.5. *For any (left) R-module M the inequality*

$$\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leqslant \mathrm{Gfd}_R M$$

holds. If R is right coherent, then we have $\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \mathrm{Gfd}_R M$. $\square$

We are now ready to state:

Theorem 2.6. *For any R-module M, the following conclusions hold:*

(i) *Assume that $\mathsf{Left}\mathrm{FPD}(R)$ is finite. If $\mathrm{fd}_R M < \infty$, then $\mathrm{Gid}_R M = \mathrm{id}_R M$.*
(ii) *Assume that R is left and right coherent with finite $\mathsf{Right}\mathrm{FPD}(R)$. If $\mathrm{id}_R M < \infty$, then $\mathrm{Gfd}_R M = \mathrm{fd}_R M$.*

Proof. (i) If $\mathrm{fd}_R M < \infty$, then also $\mathrm{pd}_R M < \infty$, by [11, Proposition 6] (since $\mathsf{Left}\mathrm{FPD}(R) < \infty$). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since R is left coherent, we have that $\mathrm{fd}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leqslant \mathrm{id}_R M < \infty$, by [12, Lemma 3.1.4]. By assumption, $\mathsf{Right}\mathrm{FPD}(R) < \infty$, and therefore also

$\mathrm{pd}_R\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}) < \infty$, by [11, Proposition 6]. Now Theorem (2.1) gives that $\mathrm{Gid}_R\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}) = \mathrm{id}_R\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$. It is well known that

$$\mathrm{fd}_R M = \mathrm{id}_R\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$$

(without assumptions on R), and by Proposition (2.5) above, we also get $\mathrm{Gfd}_R M = \mathrm{Gid}_R\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$, since R is right coherent. The proof is done. □

3. A theorem on Gorenstein rings by Foxby

We end this paper by generalizing a theorem [8, Proposition 2.10] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

3.1. The small support. Assume that R is commutative and Noetherian. For an R-module M, an integer n, and a prime ideal $\mathfrak{p}$ in R, we write $\beta_n^R(\mathfrak{p},M)$, respectively, $\mu_R^n(\mathfrak{p},M)$, for the nth Betti number, respectively, nth Bass number, of M at $\mathfrak{p}$.

Foxby [8, Definition p. 157] or [7, (14.8)] defines the *small (or homological) support* of an R-module M to be the set

$$\mathrm{supp}_R M = \{\ \mathfrak{p} \in \mathrm{Spec}\, R \mid \exists n \in \mathbb{N}_0 \colon \beta_n^R(\mathfrak{p},M) \neq 0\ \}.$$

Let us mention the most basic results about the small support, all of which can be found in [8, pp. 157 − 159] and [7, Chapter 14]:

- (a) The small support, $\mathrm{supp}_R M$, is contained in the usual (large) support, $\mathrm{Supp}_R M$, and $\mathrm{supp}_R M = \mathrm{Supp}_R M$ if M is finitely generated. Also, if $M \neq 0$, then $\mathrm{supp}_R M \neq \emptyset$.
- (b) $\mathrm{supp}_R M = \{\ \mathfrak{p} \in \mathrm{Spec}\, R \mid \exists n \in \mathbb{N}_0 \colon \mu_R^n(\mathfrak{p},M) \neq 0\ \}$.
- (c) Assume that $(R,\mathfrak{m},k)$ is local. If M is an R-module with finite depth, that is,
$$\mathrm{depth}_R M := \inf\{\ m \in \mathbb{N}_0 \mid \mathrm{Ext}_R^m(k,M) \neq 0\ \} < \infty$$
(this happens for example if $M \neq 0$ is finitely generated), then $\mathfrak{m} \in \mathrm{supp}_R M$, by ($b$) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of [8, Proposition 2.10] is immediate:

Theorem 3.2. *Assume that R is commutative and Noetherian. Let M be any R-module, and assume that any of the following four conditions is satisfied:*

- (i) $\mathrm{Gpd}_R M < \infty$ *and* $\mathrm{id}_R M < \infty$,
- (ii) $\mathrm{pd}_R M < \infty$ *and* $\mathrm{Gid}_R M < \infty$,
- (iii) *R has finite Krull dimension, and* $\mathrm{Gfd}_R M < \infty$ *and* $\mathrm{id}_R M < \infty$,
- (iv) *R has finite Krull dimension, and* $\mathrm{fd}_R M < \infty$ *and* $\mathrm{Gid}_R M < \infty$.

Then $R_{\mathfrak{p}}$ is a Gorenstein local ring for all $\mathfrak{p} \in \mathrm{supp}_R M$. □

Corollary 3.3. *Assume that $(R,\mathfrak{m},k)$ is a commutative local Noetherian ring. If there exists an R-module M of finite depth, that is,*

$$\mathrm{depth}_R M := \inf\{\ m \in \mathbb{N}_0 \mid \mathrm{Ext}_R^m(k,M) \neq 0\ \} < \infty,$$

and which satisfies either

- (i) $\mathrm{Gfd}_R M < \infty$ *and* $\mathrm{id}_R M < \infty$, *or*
- (ii) $\mathrm{fd}_R M < \infty$ *and* $\mathrm{Gid}_R M < \infty$,

then R is Gorenstein. □

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