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RINGS WITHOUT A GORENSTEIN ANALOGUE OF THE GOVOROV–LAZARD THEOREM

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Abstract

It was proved by Beligiannis and Krause that over certain Artin algebras, there are Gorenstein flat modules which are not direct limits of finitely generated Gorenstein projective modules. That is, these algebras have no Gorenstein analogue of the Govorov–Lazard theorem. We show that, in fact, there is a large class of rings without such an analogue. Namely, let R be a commutative local noetherian ring. Then the analogue fails for R if it has a dualizing complex, is henselian, not Gorenstein, and has a finitely generated Gorenstein projective module which is not free. The proof is based on a theory of Gorenstein projective preenvelopes. We show, among other things, that the finitely generated Gorenstein projective modules form an enveloping class in $\mathsf{mod}\, R$ if and only if R is Gorenstein or has the property that each finitely generated Gorenstein projective module is free. This is analogous to a recent result on covers by Christensen, Piepmeyer, Striuli and Takahashi, and their methods are an important input to our work.

Introduction

Gorenstein homological algebra was founded by Auslander and Bridger in [**1**]. Some of its main concepts are the so-called Gorenstein projective and Gorenstein flat modules, see [**10**, **12**]. These modules inhabit a theory parallel to classical homological algebra. For instance, just as projective modules can be used to define projective dimension, so Gorenstein projective modules can be used to define Gorenstein projective dimension. A commutative local noetherian ring is Gorenstein if and only if all its modules have finite Gorenstein projective dimension. A good introduction is given in [**5**]; in particular, the definitions of Gorenstein projective and Gorenstein flat modules can be found in [**5**, (4.2.1) and (5.1.1)].

The Govorov–Lazard theorem says that the closure under direct limits of the class of finitely generated projective modules is equal to the class of flat modules; see [**15**, **16**, Theorem 1.2]. It is natural to ask if this has a Gorenstein analogue. Namely, if $\mathscr{G}$ denotes the class of finitely generated Gorenstein projective modules, is $\varinjlim \mathscr{G}$ equal to the class of Gorenstein flat modules? In some cases the answer is yes, for instance over a ring which is Gorenstein in a suitable sense; this was established

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by Enochs and Jenda in [**11**, Theorem 10.3.8]. However, Beligiannis and Krause proved in [**2**, 4.2 and 4.3] that for certain Artin algebras, the answer is no.

We show for a considerably larger class of rings that there is no Gorenstein analogue of the Govorov–Lazard theorem. Namely, let R be a commutative local noetherian ring and let $\mathscr{F}$ be the class of finitely generated free modules. The following is our Theorem 2.7.

THEOREM A *If R has a dualizing complex, is henselian, not Gorenstein, and has $\mathscr{G} \neq \mathscr{F}$, then $\varinjlim \mathscr{G}$ is strictly contained in the class of Gorenstein flat modules.*

The proof is based on a theory of $\mathscr{G}$-preenvelopes, the development of which takes up most of the paper. The background is that the existence of $\mathscr{G}$-precovers has been considered at length. That is, if M is a finitely generated module, does there exist a homomorphism $G \xrightarrow{\gamma} M$ with G in $\mathscr{G}$ such that any other homomorphism $G' \to M$ with G' in $\mathscr{G}$ factors through γ? A breakthrough was achieved recently in [**7**] by Christensen *et al.* who proved, among other things, that if R is henselian, then $\mathscr{G}$-precovers exist for all finitely generated modules in precisely two cases: if R is Gorenstein, or if $\mathscr{G} = \mathscr{F}$.

We will consider the dual question: Existence of $\mathscr{G}$-preenvelopes. That is, if M is a finitely generated module, does there exist a homomorphism $M \xrightarrow{\mu} G$ with G in $\mathscr{G}$ such that any other homomorphism $M \to G'$ with G' in $\mathscr{G}$ factors through μ? We give criteria for the existence of various types of $\mathscr{G}$-preenvelopes in Theorem 2.5. One aspect is the following precise analogue of the precovering case.

THEOREM B *If R is henselian then all finitely generated R-modules have $\mathscr{G}$-preenvelopes if and only if R is Gorenstein or $\mathscr{G} = \mathscr{F}$.*

Note that the methods and results of [**7**] are an important input to our proof.

The paper is organized as follows: Section 1 prepares the ground by examining the connections between $\mathscr{G}$-precovers and $\mathscr{G}$-preenvelopes which are induced by the algebraic duality functor $(-)^* = \mathrm{Hom}_R(-, R)$. Section 2 proves Theorems A and B, among other things. Section 3 shows a method for constructing a Gorenstein flat module outside $\varinjlim \mathscr{G}$.

1. Algebraic duals of precovers and preenvelopes

This section proves Theorems 1.6 and 1.7 by which algebraic duals of various types of $\mathscr{G}$-precovers give the corresponding types of $\mathscr{G}$-preenvelopes, and vice versa.

SETUP 1.1 Throughout the paper, R is a commutative noetherian ring.

We denote the category of finitely generated R-modules by $\mathsf{mod}\, R$. Recall that $\mathscr{F}$ is the class of finitely generated free R-modules and $\mathscr{G}$ is the class of finitely generated Gorenstein projective R-modules.

REMARK 1.2 The following properties of $\mathscr{G}$ will be used below.

(i) $\mathrm{Ext}_R^{\geqslant 1}(\mathscr{G}, R) = 0$.
(ii) R is in $\mathscr{G}$.
(iii) The class $\mathscr{G}$ is closed under the algebraic duality functor $(-)^* = \mathrm{Hom}_R(-, R)$.

(iv) The biduality homomorphism $G \xrightarrow{\delta_G} G^{**}$, as defined in [5, (1.1.1)], is an isomorphism for each G in $\mathscr{G}$.

(v) Each module in $\mathscr{G}$ is isomorphic to a module G^* where G is in $\mathscr{G}$.

Here (i) and (iv) are part of the definition of $\mathscr{G}$, see [5, Definition (1.1.2)]. Property (ii) is by [5, Remark (1.1.3)] and (iii) is by [5, Observation (1.1.7)]. Property (v) is immediate from (iii) and (iv).

Lemma 1.3 *If C is an R-module satisfying* $\mathrm{Ext}^1_R(C, R) = 0$*, then* $\mathrm{Ext}^1_R(G, C^*) \cong \mathrm{Ext}^1_R(C, G^*)$ *for each G in $\mathscr{G}$.*

Proof. We have

$$\mathrm{H}^{<0}\mathrm{RHom}(C, R) = 0, \tag{1}$$

so $\mathrm{RHom}(C, R)$ can be represented in the derived category $\mathsf{D}(R)$ by a complex concentrated in non-negative cohomological degrees. Hence, there is a canonical morphism in $\mathsf{D}(R)$ from the zeroth cohomology $\mathrm{H}^0\mathrm{RHom}(C, R) \cong C^*$ to $\mathrm{RHom}(C, R)$. Complete it to a distinguished triangle,

$$C^* \xrightarrow{\chi} \mathrm{RHom}(C, R) \longrightarrow M \longrightarrow, \tag{2}$$

and consider the long exact cohomology sequence which consists of pieces

$$\mathrm{H}^i(C^*) \xrightarrow{\mathrm{H}^i\chi} \mathrm{H}^i\mathrm{RHom}(C, R) \longrightarrow \mathrm{H}^i M.$$

Since C^* is a module, $\mathrm{H}^i(C^*) = 0$ for $i \neq 0$. Combined with equation (1), the long exact sequence hence implies $\mathrm{H}^{\leqslant -2}M = 0$.

Moreover, $\mathrm{H}^0\chi$ is an isomorphism by the construction of χ, and by assumption, $\mathrm{H}^1\mathrm{RHom}(C, R) = \mathrm{Ext}^1(C, R) = 0$. So in fact, the long exact sequence also implies $\mathrm{H}^{-1}M = \mathrm{H}^0M = \mathrm{H}^1M = 0$.

Consequently, the complex M admits an injective resolution of the form $I = \cdots \to 0 \to I^2 \to I^3 \to \cdots$, and in particular,

$$\mathrm{H}^{\leqslant 1}\mathrm{RHom}(G, M) \cong \mathrm{H}^{\leqslant 1}\mathrm{Hom}(G, I) = 0 \tag{3}$$

for each R-module G.

Now let G be in $\mathscr{G}$. It follows from Remark 1.2(i) that there is an isomorphism $\mathrm{RHom}(G, R) \cong G^*$ in $\mathsf{D}(R)$, and hence by 'swap', [5, (A.4.22)], we get

$$\mathrm{RHom}(G, \mathrm{RHom}(C, R)) \cong \mathrm{RHom}(C, \mathrm{RHom}(G, R)) \cong \mathrm{RHom}(C, G^*).$$

Thus, by applying $\mathrm{RHom}(G, -)$ to the distinguished triangle (2) we obtain

$$\mathrm{RHom}(G, C^*) \longrightarrow \mathrm{RHom}(C, G^*) \longrightarrow \mathrm{RHom}(G, M) \longrightarrow .$$

Combining the long exact cohomology sequence of this with equation (3) proves the lemma.

Remark 1.4 We thank the referee for pointing out the following generalization: if $\mathrm{Ext}^i_R(C, R) = 0$ for $1 \leq i \leq n$, then $\mathrm{Ext}^i_R(G, C^*) \cong \mathrm{Ext}^i_R(C, G^*)$ for each $0 \leq i \leq n$ (sic) and G in $\mathscr{G}$. It is straightforward to adapt the proof of the lemma to show this.

LEMMA 1.5 *Let C be an R-module.*

(i) *If* $\mathrm{Ext}^1_R(C,\mathscr{G}) = 0$ *then* $\mathrm{Ext}^1_R(\mathscr{G}, C^*) = 0$.
(ii) *If* $\mathrm{Ext}^1_R(C, R) = 0$ *and* $\mathrm{Ext}^1_R(\mathscr{G}, C^*) = 0$*, then* $\mathrm{Ext}^1_R(C,\mathscr{G}) = 0$.

Proof. Combine Lemma 1.3 with Remark 1.2, parts (ii) and (iii), respectively, part (v).

Let $G \xrightarrow{\gamma} N$ be a $\mathscr{G}$-precover. For the following theorems, recall that γ is called a special $\mathscr{G}$-precover if $\mathrm{Ext}^1_R(\mathscr{G}, \mathrm{Ker}\,\gamma) = 0$, and that γ is called a cover if each endomorphism $G \xrightarrow{\varphi} G$ with $\gamma\varphi = \gamma$ is an automorphism. Special $\mathscr{G}$-preenvelopes and $\mathscr{G}$-envelopes are defined dually.

THEOREM 1.6 *Let M be in $\mathsf{mod}\,R$, let G be in $\mathscr{G}$, and let $G \xrightarrow{\gamma} M^*$ be a homomorphism. Consider the composition*

$$M \xrightarrow{\delta_M} M^{**} \xrightarrow{\gamma^*} G^*,$$

where δ denotes the biduality homomorphism again. Then

(i) *If γ is a $\mathscr{G}$-precover then $\gamma^*\delta_M$ is a $\mathscr{G}$-preenvelope.*
(ii) *If γ is a special $\mathscr{G}$-precover then $\gamma^*\delta_M$ is a special $\mathscr{G}$-preenvelope.*
(iii) *If γ is a $\mathscr{G}$-cover then $\gamma^*\delta_M$ is a $\mathscr{G}$-envelope.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
G \arrow[r, "\gamma"] \arrow[d, "\delta_G"', "\cong"] & M^* \arrow[d, hook, bend left, "\delta_{M^*}"] \\
G^{**} \arrow[r, "\gamma^{**}"'] & M^{***} \arrow[u, two heads, "(\delta_M)^*"]
\end{tikzcd}
\qquad
\begin{array}{l}
(1)\ \ \delta_{M^*}\gamma = \gamma^{**}\delta_G, \\
(2)\ \ \gamma = (\delta_M)^*\gamma^{**}\delta_G.
\end{array}$$

Here (1) just says that the biduality homomorphism is natural. By the proof of [5, Proposition (1.1.9)] we have $(\delta_M)^*\delta_{M^*} = 1_{M^*}$, so δ_{M^*} is (split) injective, $(\delta_M)^*$ (split) surjective. Now (2) follows from $\delta_{M^*}(\delta_M)^*\gamma^{**}\delta_G = \delta_{M^*}(\delta_M)^*\delta_{M^*}\gamma = \delta_{M^*}\gamma$ since δ_{M^*} is injective.

(i) Suppose that γ is a $\mathscr{G}$-precover and let $\widetilde{G}$ be in $\mathscr{G}$. Remark 1.2(iv) and 'swap' in the form [3, II. Exercise 4] give the following natural equivalences of functors,

$$\mathrm{Hom}(-, \widetilde{G}) \simeq \mathrm{Hom}(-, \widetilde{G}^{**}) \simeq \mathrm{Hom}(\widetilde{G}^*, (-)^*).$$

This gives the (top) two squares of the commutative diagram below, where we have abbreviated $\mathrm{Hom}(-,-)$ to $(-,-)$. The (bottom) commutative triangle comes from applying $\mathrm{Hom}(\widetilde{G}^*,-)$ to part (2) from the beginning of the proof.

$$
\begin{tikzcd}
(G^*, \widetilde{G}) \arrow[r, "{(\gamma^*, \widetilde{G})}"] \arrow[d, "\cong"] & (M^{**}, \widetilde{G}) \arrow[r, "{(\delta_M, \widetilde{G})}"] \arrow[d, "\cong"] & (M, \widetilde{G}) \arrow[d, "\cong"] \\
(\widetilde{G}^*, G^{**}) \arrow[r, "{(\widetilde{G}^*, \gamma^{**})}"] & (\widetilde{G}^*, M^{***}) \arrow[r, "{(\widetilde{G}^*, (\delta_M)^*)}"] & (\widetilde{G}^*, M^*) \\
 & (\widetilde{G}^*, G) \arrow[ul, "{(\widetilde{G}^*, \delta_G)}", "\cong"'] \arrow[ur, "{(\widetilde{G}^*, \gamma)}"'] &
\end{tikzcd}
$$

Since $\widetilde{G}^*$ is in $\mathscr{G}$ by Remark 1.2(iii), the map $\operatorname{Hom}(\widetilde{G}^*, \gamma)$ is surjective, and the diagram implies that so is $\operatorname{Hom}(\delta_M, \widetilde{G}) \circ \operatorname{Hom}(\gamma^*, \widetilde{G}) = \operatorname{Hom}(\gamma^*\delta_M, \widetilde{G})$. Hence $\gamma^*\delta_M$ is a $\mathscr{G}$-preenvelope.

(ii) Suppose that γ is a special $\mathscr{G}$-precover; in particular we have $\operatorname{Ext}^1(\mathscr{G}, \operatorname{Ker}\gamma) = 0$. Part (i) says that $\gamma^*\delta_M$ is a $\mathscr{G}$-preenvelope, and it remains to show $\operatorname{Ext}^1(C, \mathscr{G}) = 0$ where $C = \operatorname{Coker}(\gamma^*\delta_M)$. To prove this we use Lemma 1.5(ii). Thus, we need to show that $\operatorname{Ext}^1(\mathscr{G}, C^*) = 0$ and $\operatorname{Ext}^1(C, R) = 0$.

Applying $(-)^*$ to the exact sequence $M \xrightarrow{\gamma^*\delta_M} G^* \xrightarrow{\pi} C \longrightarrow 0$ gives the second exact row in

$$
\begin{tikzcd}
 & & G \arrow[r, "\gamma"] \arrow[d, "\delta_G"', "\cong"] & M^* \arrow[d, equal] \\
0 \arrow[r] & C^* \arrow[r, "\pi^*"'] & G^{**} \arrow[r, "(\delta_M)^*\gamma^{**}"'] & M^*
\end{tikzcd}
$$

where the square is commutative by part (2) at the beginning of the proof. It follows that $C^* \cong \operatorname{Ker}\gamma$, and hence $\operatorname{Ext}^1(\mathscr{G}, C^*) = 0$.

To prove $\operatorname{Ext}^1(C, R) = 0$, we will argue that each short exact sequence $0 \to R \to E \to C \to 0$ splits. Consider the diagram with exact rows,

$$
\begin{tikzcd}
 & M \arrow[r, "\gamma^*\delta_M"] \arrow[d, dashed, "\mu"'] & G^* \arrow[r, "\pi"] \arrow[d, dashed, "\nu"] \arrow[dl, dashed, "\varphi"'] & C \arrow[r] \arrow[d, equal] \arrow[dl, dashed, "\chi"'] & 0 \\
0 \arrow[r] & R \arrow[r, "\rho"'] & E \arrow[r, "\varepsilon"'] & C \arrow[r] & 0.
\end{tikzcd}
$$

By Remark 1.2, (i) and (iii), we have $\operatorname{Ext}^1(G^*, R) = 0$, so the functor $\operatorname{Hom}(G^*, -)$ preserves the exactness of the bottom row. In particular, there exists $G^* \xrightarrow{\nu} E$ with $\varepsilon\nu = \pi$. By the universal property of the kernel of ε, there exists a (unique) $M \xrightarrow{\mu} R$ with $\rho\mu = \nu\gamma^*\delta_M$.

Since $\gamma^*\delta_M$ is a $\mathscr{G}$-preenvelope and since R is in $\mathscr{G}$ by Remark 1.2(ii), there exists $G^* \xrightarrow{\varphi} R$ satisfying $\varphi\gamma^*\delta_M = \mu$. It follows that

$$(\nu - \rho\varphi)\gamma^*\delta_M = \nu\gamma^*\delta_M - \rho\varphi\gamma^*\delta_M = \nu\gamma^*\delta_M - \rho\mu = 0,$$

so by the universal property of the cokernel of $\gamma^*\delta_M$, there exists a (unique) $C \xrightarrow{\chi} E$ with $\chi\pi = \nu - \rho\varphi$. Consequently,

$$\varepsilon\chi\pi = \varepsilon(\nu - \rho\varphi) = \varepsilon\nu - \varepsilon\rho\varphi = \pi - 0 = \mathrm{id}_C\,\pi,$$

and since π is surjective we get $\varepsilon\chi = \mathrm{id}_C$. This proves that ε is a split epimorphism as desired.

(iii) Suppose that γ is a $\mathscr{G}$-cover. Part (i) says that $\gamma^*\delta_M$ is a $\mathscr{G}$-preenvelope, and it remains to show that each endomorphism $G^* \xrightarrow{\varphi} G^*$ with

$$\varphi\gamma^*\delta_M = \gamma^*\delta_M \tag{4}$$

is an automorphism. However, such an endomorphism has

$$\gamma\delta_G^{-1}\varphi^* = (\delta_M)^*\gamma^{**}\varphi^* = (\delta_M)^*\gamma^{**} = \gamma\delta_G^{-1}$$

where the first and third $=$ are by part (2) at the beginning of the proof while the second $=$ is $(-)^*$ of equation (4). Hence $\gamma(\delta_G^{-1}\varphi^*\delta_G) = \gamma$, and since γ is a $\mathscr{G}$-cover and $\delta_G^{-1}\varphi^*\delta_G$ is an endomorphism of G, it follows that $\delta_G^{-1}\varphi^*\delta_G$ is an automorphism.

Therefore φ^*, and hence also φ^{**}, is an automorphism. Applying Remark 1.2, (iii) and (iv), and naturality of the biduality homomorphism gives $\varphi = \delta_{G^*}^{-1}\varphi^{**}\delta_{G^*}$ whence φ is an automorphism as desired.

THEOREM 1.7 *Let M be in $\mathsf{mod}\,R$, let G be in $\mathscr{G}$, and let $M \xrightarrow{\mu} G$ be a homomorphism. Consider the algebraic dual $G^* \xrightarrow{\mu^*} M^*$. Then*

(i) *If μ is a $\mathscr{G}$-preenvelope then μ^* is a $\mathscr{G}$-precover.*
(ii) *If μ is a special $\mathscr{G}$-preenvelope then μ^* is a special $\mathscr{G}$-precover.*
(iii) *If μ is a $\mathscr{G}$-envelope then μ^* is a $\mathscr{G}$-cover.*

Proof. (i) We have $\mathrm{Hom}(G, \mu^*) \cong \mathrm{Hom}(\mu, G^*)$ by 'swap', [**3**, II. Exercise 4], and combined with Remark 1.2(iii) this implies the claim.

(ii) Suppose that μ is a special $\mathscr{G}$-preenvelope; in particular we have $\mathrm{Ext}^1(\mathrm{Coker}\,\mu, \mathscr{G}) = 0$. Part (i) says that μ^* is a $\mathscr{G}$-precover, and it remains to show $\mathrm{Ext}^1(\mathscr{G}, \mathrm{Ker}(\mu^*)) = 0$. But this follows from Lemma 1.5(i) because $\mathrm{Ker}(\mu^*) \cong (\mathrm{Coker}\,\mu)^*$.
(iii) Suppose that μ is a $\mathscr{G}$-envelope. Part (i) says that μ^* is a $\mathscr{G}$-precover, and it remains to show that each $G^* \xrightarrow{\varphi} G^*$ with $\mu^*\varphi = \mu^*$ is an automorphism.

The biduality homomorphism is natural so $\delta_G\mu = \mu^{**}\delta_M$, and since δ_G is an isomorphism by Remark 1.2(iv), it follows that $\mu = \delta_G^{-1}\mu^{**}\delta_M$. Applying $(-)^*$ to $\mu^*\varphi = \mu^*$ gives $\varphi^*\mu^{**} = \mu^{**}$. Combining these gives

$$(\delta_G^{-1}\varphi^*\delta_G)\mu = (\delta_G^{-1}\varphi^*\delta_G)(\delta_G^{-1}\mu^{**}\delta_M) = \delta_G^{-1}\varphi^*\mu^{**}\delta_M = \delta_G^{-1}\mu^{**}\delta_M = \mu.$$

Since μ is a $\mathscr{G}$-envelope and $\delta_G^{-1}\varphi^*\delta_G$ is an endomorphism of G, it follows that $\delta_G^{-1}\varphi^*\delta_G$ is an automorphism.

The argument used at the end of the proof of Theorem 1.6 now shows that φ is an automorphism as desired.

2. Existence of preenvelopes and the Govorov–Lazard theorem

This section proves Theorems A and B of the introduction; see Theorems 2.7 and 2.5.

SETUP 2.1 In this section, the commutative noetherian ring R is assumed to be local with residue class field k. We write $d = \operatorname{depth} R$.

In the following lemma, the case $d = 0$ is trivial, $d = 1$ is closely inspired by a proof of Takahashi, and $d \geqslant 2$ is classical. Recall that $\Omega^d(k)$ denotes the dth syzygy in a minimal free resolution of k over R.

LEMMA 2.2 *There exists an M in* $\mathsf{mod}\, R$ *such that $\Omega^d(k)$ is isomorphic to a direct summand of M^*.*

Proof. $d = 0$. We can use $M = k$ since $\Omega^d(k) = \Omega^0(k) = k$ and since $M^* = \operatorname{Hom}(k, R) \cong k^e$ with $e \neq 0$ because $d = 0$.

$d = 1$. We will show that $M = \Omega^d(k)^*$ works here; in fact, we will show that the biduality homomorphism for $\Omega^d(k)$ is an isomorphism so $\Omega^d(k) \cong \Omega^d(k)^{**} = M^*$.

There is a short exact sequence

$$0 \longrightarrow \mathfrak{m} \xrightarrow{\mu} R \longrightarrow k \longrightarrow 0, \tag{5}$$

where $\mathfrak{m}$ is the maximal ideal of R and μ is the inclusion, so $\Omega^d(k) = \Omega^1(k) = \mathfrak{m}$.

If R is regular then k has projective dimension 1 by the Auslander–Buchsbaum formula, so (5) shows that $\mathfrak{m}$ is projective whence the biduality homomorphism $\delta_{\mathfrak{m}}$ is an isomorphism as desired.

Assume that R is not regular. For reasons of clarity, we start by reproducing, in our notation, part of Takahashi's proof of [21, Theorem 2.8]. Applying $(-)^*$ and its derived functors to the short exact sequence (5) gives a long exact sequence containing

$$0 \longrightarrow R^* \xrightarrow{\mu^*} \mathfrak{m}^* \longrightarrow k^e \longrightarrow 0, \tag{6}$$

where we have written k^e instead of $\operatorname{Ext}^1(k, R)$, and where $e \neq 0$ since $d = 1$. Applying $(-)^*$ again gives a left exact sequence $0 \to (k^e)^* \to \mathfrak{m}^{**} \xrightarrow{\mu^{**}} R^{**}$; here $(k^e)^* = 0$ because $d = 1$, so μ^{**} is injective.

Consider the commutative square

$$\begin{tikzcd}
\mathfrak{m} \arrow[r, hook, "\mu"] \arrow[d, hook, "\delta_{\mathfrak{m}}"'] & R \arrow[d, "\delta_R", "\cong"'] \\
\mathfrak{m}^{**} \arrow[r, hook, "\mu^{**}"'] & R^{**}
\end{tikzcd}$$

where $\delta_{\mathfrak{m}}$ is injective because $\delta_R \mu$ is injective. There are inclusions

$$\operatorname{Im}(\mu^{**}\delta_{\mathfrak{m}}) \subseteq \operatorname{Im}(\mu^{**}) \subseteq R^{**}. \tag{7}$$

We have $R^{**}/\operatorname{Im}(\mu^{**}\delta_{\mathfrak{m}}) = R^{**}/\operatorname{Im}(\delta_R \mu) \cong R/\operatorname{Im}(\mu) \cong k$ where the first $\cong$ is because δ_R is an isomorphism. This quotient is simple so one of the inclusions (7) must be an equality; this means that either μ^{**} or $\delta_{\mathfrak{m}}$ is an isomorphism. Suppose that μ^{**} is an isomorphism; we will prove a contradiction whence $\delta_{\mathfrak{m}}$ is an isomorphism as desired.

To get the contradiction, we now depart from Takahashi's proof. Since μ^{**} is an isomorphism, so is $R^{***} \stackrel{\mu^{***}}{\to} \mathfrak{m}^{***}$, and so $\mathfrak{m}^{***} \cong R$. But $(\delta_{\mathfrak{m}})^* \delta_{\mathfrak{m}^*} = \mathrm{id}_{\mathfrak{m}^*}$ by the proof of [5, Proposition (1.1.9)], so $\mathfrak{m}^* \stackrel{\delta_{\mathfrak{m}^*}}{\to} \mathfrak{m}^{***}$ is a split monomorphism. It follows that $\mathfrak{m}^*$ is a direct summand of R, so $\mathfrak{m}^*$ is projective. Hence, the exact sequence (6) gives a projective resolution of k^e, and since $e \neq 0$ it follows that is $\mathrm{gldim}\, R \leqslant 1$ contradicting that R is not regular.

$d \geqslant 2$. Here we have $\Omega^d(k) = \Omega^2(\Omega^{d-2}(k))$, so it is enough to show that a second syzygy of a finitely generated module is a direct summand of some M^*. In fact, such a second syzygy Ω^2 is isomorphic to an M^*. Namely, Ω^2 sits in a short exact sequence $0 \to \Omega^2 \to P \stackrel{\pi}{\to} Q$ where P and Q are finitely generated projective modules. Consider the right-exact sequence $Q^* \stackrel{\pi^*}{\to} P^* \to M \to 0$ and apply $(-)^*$ to get a left-exact sequence $0 \to M^* \to P^{**} \stackrel{\pi^{**}}{\to} Q^{**}$. Since π^{**} is isomorphic to π, we get $\Omega^2 \cong M^*$.

The following lemma is implicitly in [7], but it is handy to make it explicit for reference. Recall from [7, Definition (2.1)] that if $\mathscr{B}$ is a full subcategory of $\mathsf{mod}\, R$, then a $\mathscr{B}$-approximation of an M in $\mathsf{mod}\, R$ is a short exact sequence $0 \to K \to B \to M \to 0$ where B is in $\mathscr{B}$ and $\mathrm{Ext}_R^{\geqslant 1}(\mathscr{B}, K) = 0$.

LEMMA 2.3 *Consider a special $\mathscr{G}$-precover and complete it with its kernel. The resulting short exact sequence* $0 \to K \to G \to M \to 0$ *is a $\mathscr{G}$-approximation of M.*

Proof. We know $\mathrm{Ext}^1(\mathscr{G}, K) = 0$. By [5, Corollary (4.3.5)(a)] each G in $\mathscr{G}$ sits in a short exact sequence $0 \to G' \to P \to G \to 0$ where P is a finitely generated projective module and G' is in $\mathscr{G}$, and it follows by an easy induction that $\mathrm{Ext}^{\geqslant 1}(\mathscr{G}, K) = 0$ as desired.

REMARK 2.4 Let us give a brief summary of a part of [7].

Recall from [7, (1.1)] that if $\mathscr{B}$ is a full subcategory of $\mathsf{mod}\, R$, then $\langle \mathscr{B} \rangle$ denotes the closure under direct summands and extensions. The class of finitely generated Gorenstein projective modules $\mathscr{G}$ is a so-called reflexive subcategory of $\mathsf{mod}\, R$ by [7, Definition (2.6)]. It follows from [7, Proposition (2.10)] that $\langle \widehat{R} \otimes_R \mathscr{G} \rangle$ is a reflexive subcategory of $\mathsf{mod}\, \widehat{R}$.

Now suppose that there is an $\langle \widehat{R} \otimes_R \mathscr{G} \rangle$-cover of $\Omega^d_{\widehat{R}}(k)$. The cover is an $\langle \widehat{R} \otimes_R \mathscr{G} \rangle$-approximation by [7, (2.2)(b)]. But when such an approximation exists, the proof of [7, Theorem (3.4)] gives that either, $\widehat{R}$ is Gorenstein, or $\langle \widehat{R} \otimes_R \mathscr{G} \rangle$ consists of free $\widehat{R}$-modules.

An important input to the proof of the next theorem are the methods and results developed by Christensen *et al.* in [7].

THEOREM 2.5 *The following three conditions are equivalent.*

(i) *Each module in* $\mathsf{mod}\, R$ *has a $\mathscr{G}$-envelope.*
(ii) *Each module in* $\mathsf{mod}\, R$ *has a special $\mathscr{G}$-preenvelope.*
(iii) *R is Gorenstein or $\mathscr{G} = \mathscr{F}$.*

They imply the following condition.

(iv) *Each module in* $\mathsf{mod}\, R$ *has a $\mathscr{G}$-preenvelope.*

Moreover, if R is henselian then (iv) *implies* (i), (ii) *and* (iii).

Proof. (i)⇒(ii) Holds by Wakamatsu's lemma, [24, Lemma 2.1.2].

(ii)⇒(iii) By Lemma 2.2, the module $\Omega^d_R(k)$ is a direct summand in a module of the form M^* where M is in $\mathsf{mod}\, R$. If (ii) holds then M has a special $\mathscr{G}$-preenvelope, and by Theorem 1.7(ii) it follows that M^* has a special $\mathscr{G}$-precover. Completing with the kernel gives a short exact sequence $0 \to K \to G \to M^* \to 0$ which is a $\mathscr{G}$-approximation of M^* by Lemma 2.3.

Tensoring the sequence with $\widehat{R}$ gives an $\langle \widehat{R} \otimes_R \mathscr{G} \rangle$-approximation of $\widehat{R} \otimes_R M^*$ by [7, Proposition 2.4]. In particular, there is an $\langle \widehat{R} \otimes_R \mathscr{G} \rangle$-precover of $\widehat{R} \otimes_R M^*$, and the same must hold for its direct summand $\widehat{R} \otimes_R \Omega^d_R(k) \cong \Omega^d_{\widehat{R}}(k)$. Hence, there is an $\langle \widehat{R} \otimes_R \mathscr{G} \rangle$-cover of $\Omega^d_{\widehat{R}}(k)$ by [22, Corollary 2.5].

But now the results of [7] imply that either, $\widehat{R}$ is Gorenstein, or $\langle \widehat{R} \otimes_R \mathscr{G} \rangle$ consists of free $\widehat{R}$-modules; see Remark 2.4. In the former case, R is Gorenstein by [18, Theorem 18.3]. In the latter case, in particular, $\widehat{R} \otimes_R G$ is a free $\widehat{R}$-module whenever G is in $\mathscr{G}$. But then G is a free R-module whence $\mathscr{G} = \mathscr{F}$; cf. [18, Corollary on p. 53, Exercise 7.1 and (3), p. 63].

(iii)⇒(i) First, suppose that R is Gorenstein. Then each finitely generated R-module has a $\mathscr{G}$-cover by unpublished work of Auslander; see [13, Theorem 5.5]. Existence of $\mathscr{G}$-envelopes now follows from Theorem 1.6(iii).

Secondly, suppose $\mathscr{G} = \mathscr{F}$. Then each finitely generated R-module has an $\mathscr{F}$-envelope by [23, Proposition 2.3(3)], which does not need that paper's assumption that the ring is henselian.

(i)⇒(iv) Trivial.

Now assume that R is henselian.

(iv)⇒(i) Suppose that (iv) holds. Then Theorem 1.7(i) implies that each R-module of the form M^* with M in $\mathsf{mod}\, R$ has a $\mathscr{G}$-precover. Since R is henselian, each M^* has a $\mathscr{G}$-cover by [22, Corollary 2.5], and so each M has a $\mathscr{G}$-envelope by Theorem 1.6(iii).

REMARK 2.6 As a consequence, the following conditions are equivalent.

(i) Each module in $\mathsf{mod}\, R$ has a $\mathscr{G}$-cover.
(ii) Each module in $\mathsf{mod}\, R$ has a special $\mathscr{G}$-precover.
(iii) Each module in $\mathsf{mod}\, R$ has a $\mathscr{G}$-envelope.
(iv) Each module in $\mathsf{mod}\, R$ has a special $\mathscr{G}$-preenvelope.
(v) R is Gorenstein or $\mathscr{G} = \mathscr{F}$.

Namely, (i)⇒(ii) is by Wakamatsu's lemma, [24, Lemma 2.1.1]. (ii)⇒(iv) follows from Theorem 1.6(ii). Conditions (iii), (iv) and (v) are equivalent by Theorem 2.5. And (v)⇒(i) follows from unpublished work by Auslander; see [13, Theorem 5.5].

Note that the equivalence of (i), (ii) and (v) was first established in [7], and that our proof depends on that paper.

Now assume that R is henselian. Combining with a result of Crawley–Boevey shows that the following conditions are also equivalent, where $\varinjlim \mathscr{G}$ denotes the closure of $\mathscr{G}$ under direct limits.

(i) Each module in $\mathsf{mod}\, R$ has a $\mathscr{G}$-precover.
(ii) Each module in $\mathsf{mod}\, R$ has a $\mathscr{G}$-preenvelope.
(iii) R is Gorenstein or $\mathscr{G} = \mathscr{F}$.
(iv) $\varinjlim \mathscr{G}$ is closed under set indexed direct products.

Namely, (i)⇒(iii) holds by [7, (2.8) and Theorem (3.4)]. (iii)⇒(i) follows from unpublished work by Auslander as above; see [13, Theorem 5.5]. (ii)⇔(iii) is by Theorem 2.5. And (ii)⇔(iv) holds by [8, (4.2)].

THEOREM 2.7 *If R has a dualizing complex, is henselian, not Gorenstein, and has $\mathscr{G} \neq \mathscr{F}$, then $\varinjlim \mathscr{G}$ is strictly contained in the class of Gorenstein flat modules.*

Proof. Each module in $\mathscr{G}$ is Gorenstein flat, cf. [5, Theorem (5.1.11)], and the class of Gorenstein flat modules is closed under direct limits by [14], so $\varinjlim \mathscr{G}$ is contained in the class of Gorenstein flat modules.

The class of Gorenstein flat modules is closed under set indexed products by [6, Theorem 5.7]. On the other hand, by the last four conditions of Remark 2.6, the assumptions on R imply that $\varinjlim \mathscr{G}$ is not closed under set indexed products.

EXAMPLE 2.8 It is easy to find rings of the type required by Theorem 2.7. For instance, let us show that the 1-dimensional ring

$$T = \mathbb{Q}[\![X, Y, Z, W]\!]/(X^2, Y^2, Z^2, XY)$$

satisfies the conditions of the theorem.

First note that since T is complete, it has a dualizing complex and is henselian.

Next consider $S = \mathbb{Q}[\![X, Y]\!]/(X^2, Y^2, XY)$ which is not Gorenstein. The ring T is $S[\![Z, W]\!]/(Z^2)$; that is, T is the ring of dual numbers over $S[\![W]\!]$. Since S is not Gorenstein, neither is $S[\![W]\!]$ nor T.

Finally, let z be the image of Z in T. Since T is the ring of dual numbers over $S[\![W]\!]$, the following complex is a complete projective resolution.

$$\cdots \longrightarrow T \xrightarrow{z\cdot} T \xrightarrow{z\cdot} T \longrightarrow \cdots$$

It shows that the non-projective module $T/(z)$ is Gorenstein projective, so $\mathscr{G} \neq \mathscr{F}$.

REMARK 2.9 Assume that R is artinian. Then it has a dualizing complex and is henselian (in fact, R is complete). Moreover, it is easy to prove that each Gorenstein flat module is Gorenstein projective.

If R is not Gorenstein and has $\mathscr{G} \neq \mathscr{F}$, then Theorem 2.7 shows that $\varinjlim \mathscr{G}$ is strictly contained in the class of Gorenstein projective modules. Hence [2, 4.2] shows that R is not a so-called virtually Gorenstein ring.

3. A special Gorenstein flat module

This short section shows a method for constructing a Gorenstein flat module outside $\varinjlim \mathscr{G}$.

CONSTRUCTION 3.1 Let $\{G_i\}_{i \in I}$ be a set of representatives of the isomorphism classes of indecomposable modules in $\mathscr{G}$. Let M be in $\mathsf{mod}\, R$. For each i in I, view $H(i) = \mathrm{Hom}_R(M, G_i)$ as a set and consider the direct product $G_i^{H(i)}$ indexed by that set. Define

$$\Lambda(M) = \prod_{i \in I} G_i^{H(i)}.$$

PROPOSITION 3.2 *Assume that R has a dualizing complex. Let M be in* $\mathsf{mod}\, R$ *and suppose that M does not have a $\mathscr{G}$-preenvelope. Then $\Lambda(M)$ is a Gorenstein flat module outside $\varinjlim \mathscr{G}$.*

Proof. As in the proof of Theorem 2.7, the modules in $\mathscr{G}$ are Gorenstein flat and the class of Gorenstein flat modules is closed under set indexed products, so $\Lambda(M)$ is Gorenstein flat.

For each i in I, consider the homomorphism

$$M \xrightarrow{\mu_i} G_i^{H(i)}, \quad m \longmapsto (h(m))_{h \in H(i)}.$$

Let $\Lambda(M) \stackrel{\pi_i}{\to} G_i^{H(i)}$ be the ith projection, and let $M \stackrel{\mu}{\to} \Lambda(M)$ be the unique homomorphism which satisfies $\pi_i \mu = \mu_i$ for each i in I. Then each homomorphism $M \stackrel{\eta}{\to} G$ with G in $\mathscr{G}$ factors through μ,

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "\eta"'] & \Lambda(M). \arrow[dl, dashed, "\lambda"] \\ G & \end{tikzcd}$$

Namely, we may assume $G = G_i$ for some i, since each G in $\mathscr{G}$ is isomorphic to a finite direct sum of modules from the set $\{G_i\}_{i \in I}$. But then η is an element of $H(i)$, and we can let λ equal the composition of the projections $\Lambda(M) \stackrel{\pi_i}{\to} G_i^{H(i)} \to G_i$ where the second one is onto the ηth copy of G_i.

Now, M is finitely presented, so if $\Lambda(M)$ were in $\varinjlim \mathscr{G}$ then [17, Proposition 2.1] would give that μ could be factored as $M \stackrel{\widetilde{\mu}}{\to} \widetilde{G} \to \Lambda(M)$ with $\widetilde{G}$ in $\mathscr{G}$. Since each homomorphism $M \stackrel{\eta}{\to} G$ factors through μ by the above, it would also factor through $\widetilde{\mu}$ which would hence be a $\mathscr{G}$-preenvelope of M. Since there is no such $\mathscr{G}$-preenvelope, $\Lambda(M)$ is outside $\varinjlim \mathscr{G}$.

REMARK 3.3 Let us end the paper with a remark about the approximation properties of some classes of modules in $\mathsf{Mod}\, R$, the category of all R-modules.

The class $\varinjlim \mathscr{G}$ is precovering by [9, Theorem 3.2]. On the other hand, suppose that $\mathscr{G}$ is not preenveloping in $\mathsf{mod}\, R$. Then $\varinjlim \mathscr{G}$ is not closed under set indexed products by [8, Theorem (4.2)]. Consequently $\varinjlim \mathscr{G}$ is not preenveloping in $\mathsf{Mod}\, R$ by [20, Theorem 3.3].

The class $\mathscr{P}$ of all Gorenstein projective modules is precovering in $\mathsf{Mod}\, R$ if R has finite Krull dimension by [19, Theorem 4.26]. On the other hand, $\mathscr{P}$ is not necessarily preenveloping. For a counterexample, let R be regular and local with depth $R \geq 1$. Since R is regular, $\mathscr{P}$ consists of the projective modules only, and since depth $R \geq 1$, it follows that R has no simple submodules, so $\mathscr{P}$ is not closed under set indexed products by [4, Theorems 3.2 and 3.3]. Consequently $\mathscr{P}$ is not preenveloping in $\mathsf{Mod}\, R$ by [20, Theorem 3.3] again.

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