1. Topology

Here are some basic definitions concerning topological spaces. You may think of the topology as coming from a metric, but other topologies may occur.

Definition 1.1. *A set X is called a topological space with topology $\mathcal{T}$ provided there is a family $\mathcal{T}$ of subsets of X for which the following holds:*

- $X \in \mathcal{T}$ *and* $\emptyset \in \mathcal{T}$.
- $O_1, \dots, O_r \in \mathcal{T} \Rightarrow O_1 \cap \cdots \cap O_r \in \mathcal{T}$.
- *If* $\{O_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, *then* $\cup_{\alpha \in A} O_\alpha \in \mathcal{T}$ *(A is any index set).*

The sets in $\mathcal{T}$ are called the **open** *subsets of X.*

If $x \in X$, a **neighborhood of x** *is by definition any open subset O with $x \in O$.*

The topological space X is **Hausdorff** *if*

$$\forall x_1 \neq x_2 \in X : \exists O_1, O_2 \in \mathcal{T} : x_1 \in O_1, x_2 \in O_2 \text{ and } O_1 \cap O_2 = \emptyset. \tag{1}$$

A set X can always be made into a topological space by choosing $\mathcal{T} = \{X, \emptyset\}$. This is the **trivial** topology. At the other extreme, $\mathcal{T} = \mathbb{P}(X)$ (all subsets of X) is also a topology. This is also of no practical use. We will write $(X, \mathcal{T})$ when we are in the situation of Definition 1.1.

Definition 1.2. *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces. A map $f : X_1 \to X_2$ is* **continuous** *(with respect to the given topologies) if*

$$\forall U \in \mathcal{T}_2 : f^{-1}(U) \in \mathcal{T}_1. \tag{2}$$

Definition 1.3. *A subfamily $\mathcal{B} \subseteq \mathcal{T}$ is called a* **basis for the topology** *or a* **basis for the open sets**, *if any $O \in \mathcal{T}$ can be written as a union of some sets from $\mathcal{B}$. The space X is said to be* **second countable** *if there is a countable basis for the open sets.*

Definition 1.4. *A topological space $(X, \mathcal{T})$ is* **disconnected** *if there are two open non-empty sets O_1, O_2 of X such that*

$$X = O_1 \cup O_2 \text{ and } O_1 \cap O_2 = \emptyset. \tag{3}$$

If X is not disconnected we say that it is **connected**.

Definition 1.5. *If $(X, \mathcal{T})$ is a topological space and $Y \subset X$,*

$$\mathcal{T}_Y = \{O \cap Y \mid O \in \mathcal{T}\} \tag{4}$$

defines a topology on Y called the **relative** *topology or the subset topology.*

2. Differentiable manifolds

Definition 2.1. *Let M be a second countable topological space. A* **differentiable structure on M (of dimension n**) *is a family $\mathcal{A} = \{(\chi_i, O_i)\}_{i \in I}$, where I is an index set, such that*

M1 $\forall i \in I : O_i$ *is an open subset of M and χ is a homeomorphism of O_i onto the open set $\chi(O_i) \subseteq \mathbb{R}^n$*
M2 $M = \cup_{i \in I} O_i$.
M3 $\forall i, j \in I \ \chi_i \circ \chi_j^{-1} : \chi_j(O_i \cap O_j) \to \chi_i(O_i \cap O_j)$ *is a C^∞ map.*

Remark 2.2. *Actually, it is a "differentiable structure of class C^∞" we have defined. In a similar fashion one may define structures of class C^k for all $k = 0, 1, \ldots,$ as well as class C^ω (analytic), but here we are only concerned with C^∞.*

Definition 2.3. *A set M is a* **differentiable manifold** *(an n-dimensional differentiable manifold) if M is a second countable Hausdorff topological space with differentiable structure $\mathcal{A} = \{(\chi, O_i)\}_{i \in I}$ (of dimension n). Furthermore, $\mathcal{A}$ is an* **atlas**, *and the individual elements (χ_i, O_i) are called* **charts**. *One also refers to (χ_i, O_i) as "local coordinates" on M, or as a local parametrization. The n in the definition is called the dimension of M. Occasionally we will write M^n.*

3. Lie Groups

Definition 3.1. *A* **Lie Group** *G is is an abstract group which is also a differentiable manifold and where the two structures are compatible in the sense that the maps*

$$\begin{aligned} G \times G \ni (g, h) &\mapsto g \cdot h \in G \\ G \ni g &\mapsto g^{-1} \in G \end{aligned}$$

are both smooth.

3.1. Matrix Lie Groups.

Definition 3.2. *A Matrix Lie Group is a closed subgroup of $Gl(n, \mathbb{C})$ for some n.*

In the case of a matrix Lie Group, the fundamental observation is that there exists an open ball $U_\varepsilon = B_0(\varepsilon) = \{X \in M_n(\mathbb{C}) \mid |X| < \varepsilon\}$ such that $V_\varepsilon = \exp(U_\varepsilon)$ is an open set in $GL(n, \mathbb{C})$ with the property that $G \cap V_\varepsilon = \exp(U_\varepsilon \cap \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.

For any $g \in G$ one may define a chart $(\chi_g, g \cdot V_\varepsilon)$ where the V_ε is as above, and where

$$(g \cdot V_\varepsilon) \ni h \mapsto \chi_g(h) := \log(g^{-1}h).$$

Since the operations of taking inverse and of multiplication from the left are smooth in $GL(n, \mathbb{C})$, and since log is smooth, it follows easily that the "change of coordinates" $(\chi_a) \circ (\chi_b)^{-1}$ are smooth for all $a, b \in G$ where ever they are defined. Thus, a Matrix Lie Group *is* indeed a Lie Group.

4. The Matrix Lie Groups, their Lie algebras, and the exponential map

We give here some fundamental definitions and results - the latter without proofs - relating to certain families of Lie groups and Lie algebras. Some of the definitions are particular for this course.

Definition 4.1. *By a matrix Lie group we understand a closed subgroup of either $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.*

Definition 4.2. *A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ satisfying*

- *(Skew symmetry)*
  $\forall X, Y \in \mathfrak{g} : [X, Y] = -[Y, X].$
- *(The Jacobi identity)*
  $\forall X, Y, Z \in \mathfrak{g} : [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$

The map $[\cdot, \cdot]$ is called **the Lie bracket**. *The linear structures may in fact be over any field $\mathbb{F}$. If $\mathbb{F} = \mathbb{R}$ we say that the Lie algebra is real, if $\mathbb{F} = \mathbb{C}$ we say that it is complex.*

Definition 4.3. *The Lie algebras $gl(n, \mathbb{R})$ and $gl(n, \mathbb{C})$ are defined to be the sets of all $n \times n$ real and complex,respectively, matrices equipped with the Lie bracket*

$$[X, Y] = XY - YX.$$

Here XY is given by matrix multiplication, etc.

Remark 4.4. *Any real subspace of $gl(n,\mathbb{R})$ – or of $gl(n,\mathbb{C})$(!) – which is* invariant *under the Lie bracket is a real Lie algebra. Any complex subspace of $gl(n,\mathbb{R})$ which is* invariant *under the Lie bracket is a complex Lie algebra.*

Example 4.5. $u(n) = \{H \in gl(n,\mathbb{C}) \mid H^* = -H\}$ *is a real Lie algebra.* $su(n) = \{H \in gl(n,\mathbb{C}) \mid H^* = -H \text{ and } Tr(H) = 0\}$ *is a real Lie algebra.*

Remark 4.6. *By the famous Ado's Theorem, any finite dimensional real Lie algebra is equivalent to one obtained from a subspace of $gl(n,\mathbb{R})$ for some n. In contrast, there are very many finite-dimensional abstract Lie groups that do not have isomorphic images inside some $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$.*

Definition 4.7. *The exponential map $gl(n,\mathbb{C}) \mapsto Gl(n,\mathbb{C})$ is given as*

$$\forall A \in gl(n,\mathbb{C}) : \exp(A) = e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!}.$$

Observe that $\| \sum_{i=0}^{\infty} \frac{A^i}{i!} \| \leq \sum_{i=0}^{\infty} \frac{\|A\|^i}{i!} = e^{\|A\|}$, where $\|A\|$ denotes the operator norm of A. A similar estimate holds for the Hilbert-Schmidt norm $\| \cdot \|_{HS}$; c.f. Exercise 2.3.2.

Proposition 4.8. *The following useful facts hold:*

i) $\forall s,t \in \mathbb{R} : \exp((s+t)A) = \exp(sA)\exp(tA)$.
ii) *More generally, if* $[A,B] = 0$ *then* $\exp(A+B) = \exp(A)\exp(B)$.
iii) $\det(e^A) = e^{Tr(A)}$.
iv) $\exp(A)$ *is invertible;* $(\exp(A))^{-1} = \exp(-A)$.
v) $(\exp(A))^* = (\exp(A^*))$.

Moreover, we mention without proof

Lemma 4.9. *If $\mathbb{R} \ni t \mapsto a(t) \in GL(n,\mathbb{C})$ and $\mathbb{R} \ni t \mapsto b(t) \in GL(n,\mathbb{C})$ are differentiable functions, then*

$$\frac{d}{dt}(a(t)\cdot b(t)) = \frac{da(t)}{dt} \cdot b(t) + a(t) \cdot \frac{db(t)}{dt} \qquad (= a'(t)b(t) + a(t)b'(t)).$$

Moreover,

$$\frac{d\exp(t\cdot A)}{dt} = A\exp(t\cdot A) = \exp(t\cdot A)A.$$

The first major theorem relating Lie groups and Lie algebras is the following:

Theorem 4.10. *Let G be a matrix Lie group. Then*

$$\mathfrak{g} := \{H \in gl(n, \mathbb{R}) \mid \forall t \in \mathbb{R} : \exp(tH) \in G\}$$

is a Lie subalgebra. A similar result is true for $gl(n, \mathbb{C})$.[1] *We say that $\mathfrak{g}$ is the algebra of G.*

Example 4.11. *To find the Lie algebra $su(2)$ of $SU(2)$ we observe that for X to be in $su(2)$ we must have $\forall t : (\exp(tX))(\exp(tX))^* = 1$ and $\forall t : \det(\exp(tX)) = 1$. If we use v) in Proposition 4.8 and differentiate the first equation at $t = 0$ we obtain: $X + X^* = 0$, i.e. X is skew adjoint (iX is self adjoint). The second equation becomes $e^{tTr(X)} = 1$ for all t which implies that $Tr(X) = 0$. Thus,*

$$su(2) = \left\{ \begin{pmatrix} ix & \lambda \\ -\overline{\lambda} & -ix \end{pmatrix} \mid x \in \mathbb{R}, \lambda \in \mathbb{C} \right\}.$$

A basis of $su(2)$ is given by e.g.

$$X = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; Y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \; Z = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5}$$

Notice that

$$[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = Y. \tag{6}$$

Notice that "the same" computations give that $u(n) = \{X \in gl(n, \mathbb{C}) \mid X + X^* = 0\}$ and $su(n) = \{X \in gl(n, \mathbb{C}) \mid X + X^* = 0,$ and $Tr(X) = 0\}$.

Example 4.12. *The Lie algebra $so(3)$ of $SO(3)$ is given as*

$$so(3) = \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

A basis of $so(3)$ is given by e.g.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

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[1] The condition is still that $\forall t \in \mathbb{R} : \exp(tH) \in G$

Definition 4.13. *Let $\mathfrak{g}, \mathfrak{h}$ be real Lie algebras. A real linear map $d\pi : \mathfrak{g} \mapsto \mathfrak{h}$ that satisfies*

$$\forall X, Y \in \mathfrak{g} : \; d\pi([X, Y]) = [d\pi(X); d\pi(Y)],$$

where the Lie brackets are computed in the relevant Lie algebras, is called a **homomorphism**. *The definition for the complex case is analogous.*

In case $\mathfrak{h} = gl(n, \mathbb{C})$ (viewed as a real Lie algebra) we say that $d\pi$ is a **Lie algebra representation**. *In the latter case, if $d\pi(\mathfrak{g}) \subseteq u(n)$ we say that $d\pi$ is* **infinitesimally unitary**.

The second major theorem relating Lie groups and Lie algebras is:

Theorem 4.14. *Let $\phi : G \mapsto H$ be a homomorphism between two matrix Lie groups G and H. Let the Lie algebras be denoted $\mathfrak{g}, \mathfrak{h}$, respectively. Then there exists a Lie algebra homomorphism $d\phi$: $\mathfrak{g} \mapsto \mathfrak{h}$ such that*

$$\forall t \in \mathbb{R}, \forall X \in \mathfrak{g} : \phi(\exp_G(tX)) = \exp_H(td\phi(X)).$$

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_H} \\ \mathfrak{g} & \xrightarrow{d\phi} & \mathfrak{h} \end{array} \tag{7}$$

In the special case where $\phi = \pi$ is an n-dimensional complex (unitary) representation of a matrix Lie group G we obtain a Lie algebra homomorphism $d\pi : \mathfrak{g} \mapsto gl(n, \mathbb{C})$ $(u(n))$. In this case, $d\pi(X)$ is called the **infinitesimal generator** corresponding to X.

Remark 4.15. *It is of course not at all clear* a priori *why a Lie group homomorphism ϕ, which to begin with only is assumed to be continuous, is differentiable and hence can give rise to the map $d\phi$ - not even in the special case of a continuous representation.*

For later use we mention that any homomorphism $d\phi : \mathfrak{g} \mapsto \mathfrak{h}$ between real Lie algebras can be extended to the complexifications, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ simply by setting

$$d\pi^{\mathbb{C}}(X_1 + iX_2) = d\pi(X_1) + id\pi(X_2).$$

This is extremely useful. Since any matrix $A \in gl(n, \mathbb{C})$ can be written uniquely as $A = K_1 + iK_2$ with K_1, K_2 skew adjoint, it follows that any representation $d\pi$ of $u(n)$ can be extended (complexified) to $gl(n, \mathbb{C})$.

But even more so, any representation of $gl(n,\mathbb{R})$ can be complexifed to arepresentation of $gl(n,\mathbb{C})$. Hence, to any finite dimensional irreducible (contemplate this!) representation of $gl(n,\mathbb{R})$ there corresponds an irreducible representation of $u(n)$ (in the same space) and vice versa.

5. The classical groups

5.1. **Bilinear forms and pseudo-orthogonal groups.** We let $B \in \mathcal{B}(V_1, V_2, \dots V_n)$ denote the set of *multi linear* maps $V_1 \times V_2 \times \cdots \times V_n \mapsto \mathbb{F}$ in analogy to the case with the bilinear maps. Further, we set $\mathcal{B}(V,V) = \mathcal{B}_2(V)$, and, in general, $\mathcal{B}(\underbrace{V,V,\dots,V}_{n}) = \mathcal{B}_n(V)$.

Definition 5.1. *Let $B \in \mathcal{B}(V,W)$. We say that B is* non degenerate *if*

$$\forall v \in V : B(v,w) = 0 \Rightarrow w = 0$$
$$\forall w \in W : B(v,w) = 0 \Rightarrow v = 0.$$

Let, as usual, S_n denote the symmetric group in n letters and let $\mathrm{Sgn}(\sigma) = (-1)^\sigma$ denote the sign of the permutation $\sigma \in S_n$.

Definition 5.2. *Let $B \in \mathcal{B}_n(V)$. We say that B is* symmetric *if*

$$\forall v_1, v_2, \dots, v_n \in V \;\; og \; \forall \sigma = (\sigma_1, \dots, \sigma_n) \in S_n : \quad B(v_{\sigma_1}, \dots, v_{\sigma_n}) = B(v_1, \dots, v_n).$$

Definition 5.3. *Let $B \in \mathcal{B}_n(V)$. We say that B is* alternating *if*

$$\forall v_1, v_2, \dots, v_n \in V \;\; og \; \forall \sigma = (\sigma_1, \dots, \sigma_n) \in S_n : \quad B(v_{\sigma_1}, \dots, v_{\sigma_n}) = Sgn(\sigma) B(v_1, \dots, v_n).$$

Assume that $B \in \mathcal{B}_2(V)$ is non degenerate and symmetric. Then we set

$$O(B,\mathbb{F}) = \{g \in \mathrm{Aut}_{\mathbb{F}}(V) \mid \forall v_1, v_2 \in V : B(gv_1, gv_2) = B(v_1, v_2)\}. \tag{8}$$

One can say that $O(B,\mathbb{F})$ is the invariance group of B. We also refer to $O(B,\mathbb{F})$ as a *pseudo-orthogonal group* though this terminology is used mostly for the case with $\mathbb{F} = \mathbb{R}$.

A real vector space V with a positive definite inner product (dot product) $(\cdot,\cdot)$ (here after just called an inner product) is, as is well known, called a (pre) Euclidean space. If $V \simeq \mathbb{R}^n$ we denote by $(\cdot,\cdot)_E$ the usual inner product

$$(v,w)_E = x_1 y_1 + x_2 y_2 \cdots x_n y_n, \tag{9}$$

where $v = (x_1, x_2, \cdots, x_n)$ and $w = (y_1, y_2, \cdots, y_n)$ and we will also use this terminology if $V \simeq \mathbb{C}^n$.

As is well known, if $(V, (\cdot, \cdot))$ is a real Euclidean vector space, a linear operator $o : V \mapsto V$ is said to be **orthogonal** if

$$\forall x, y \in V : (o(x), o(y)) = (x, y). \tag{10}$$

We set $O(V) = \{o \mid o \,\text{orthogonal}\}$, but if $V = \mathbb{R}^n$ we denote it as $O(V) = O(n)$ and we refer to it as the "orthogonal group in n dimensions". In all cases, $O(V)$ is a group.

Now let B be a symmetric non degenerate form on $\mathbb{R}^n$.

It is straightforward to show (see the problem sessions) that there exists a symmetric linear operator J_B so that

$$\forall x, y \in \mathbb{R}^n : B(x, y) = (x, J_B(y))_E. \tag{11}$$

One must have $\det(J_B) \neq 0$. Conversely, any such operator J (matrix) defines a symmetric non degenerate form B_J by

$$\forall x, y \in \mathbb{R}^n : B_J(x, y) = (x, J(y))_E. \tag{12}$$

Definition 5.4. *If J is symmetric and has p strictly positive eigenvalues and q strictly negative eigenvalues with $p + q = n$ we define*

$$\begin{aligned} O_J(p,q) &= \{o \in Gl(n,\mathbb{R}) \mid \forall x, y \in \mathbb{R}^n : B_J(o(x), o(y)) = B_J(x, y)\} \\ &= \{o \in Gl(n,\mathbb{R}) \mid \forall x, y \in \mathbb{R}^n : (Jo(x), o(y))_E = (Jx, y)_E\} \\ &= \{o \in Gl(n,\mathbb{R}) \mid o^* J o = J\}^2. \end{aligned}$$

The groups thus defined depend explicitly on J but are all isomorphic. For this reason, the J is usually dropped and one just writes $O(p, q)$.

Likewise, if $B_a \in \mathcal{B}_2(V)$ is non degenerate and alternating (skew symmetric)

$$Sp(B_a, \mathbb{F}) = \{g \in \mathrm{Aut}_{\mathbb{F}}(V) \mid \forall v_1, v_2 \in V : B_a(gv_1, gv_2) = B_a(v_1, v_2)\}. \tag{13}$$

Again one could say that $Sp(B_a, \mathbb{F})$ is the invariance group of B_a. These are the (real or complex, depending on whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.) *symplectic groups*.

There is a skew symmetric linear operator J_{B_a} such that

$$\forall x, y \in V : B_a(x, y) = (x, J_{B_a}(y))_E. \tag{14}$$

[2] Notice that one has: $[\forall x, y \in \mathbb{R}^n : (A(x), y) = 0] \Leftrightarrow A = 0$ - one can merely look at $x = e_i, y = e_j$ for all the vectors e_i in the canonical basis.

To be completely specific, if $V = \mathbb{R}^{2n}$ (or $V = \mathbb{C}^{2n}$)[3] we set

$$J_{B_a} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{15}$$

- where I_n is the identity on $\mathbb{R}^n$ (or $\mathbb{C}^n$) - and we set

$$Sp(n, \mathbb{R}) = \{g \in GL(2n, \mathbb{R}) \mid \forall v_1, v_2 \in V : (J_{B_a}\, gv_1, gv_2) = (J_{B_a}\, v_1, v_2)\}, \tag{16}$$

with an analogous definition in the complex case.

5.2. **Hermitean forms.** We remind you that a **sesquilinear form** H_s in a complex vector space V is a map

$$V \times V \ni v, w \mapsto H_s(v, w) \in \mathbb{C}, \tag{17}$$

satisfying:

$$\begin{gathered} \forall \lambda \in \mathbb{C} : \forall v_1, v_2, w, \in V : \\ H_s(v_1 + \lambda v_2, w) = H_s(v_1, w) + \lambda H_s(v_2, w) \text{ (linearity in the 1. var.)} \\ \text{and} \\ \forall \lambda \in \mathbb{C} : \forall v, w_1, w_2, \in V : \\ H_s(v, w_1 + \lambda w_2) = H_s(v, w_1) + \overline{\lambda} H_s(v, w_2) \text{ (anti-linearity in the 2. var.)} \end{gathered}$$

We say that the sesquilinear form H_s (henceforth denoted H) is *hermitian* provided

$$\forall v, w \in V : H(w, v) = \overline{H(v, w)}. \tag{18}$$

(and call it skew hermitian if $\forall v, w \in V : H(w, v) = -\overline{H(v, w)}$ - but we will not get involved with those here...)

Finally, a hermitian form H is *positive definite* if

$$\forall v \in V : v \neq 0 \Rightarrow H(v, v) > 0. \tag{19}$$

As is well known, a complex vector space V with a positive definite hermitian form $\langle \cdot, \cdot \rangle$ (hereafter just an inner product) is called a (pre) Hilbert space. If $V \simeq \mathbb{C}^n$, we denote by $\langle \cdot, \cdot \rangle_0$ the usual inner product;

$$\langle v, w \rangle_0 = x_1\overline{y_1} + x_2\overline{y_2} \cdots x_n\overline{y_n}, \tag{20}$$

if $v = (x_1, x_2, \cdots, x_n)$ and $w = (y_1, y_2, \cdots, y_n)$.

We set $\mathcal{U}(\mathcal{H}) = \{U \mid U \text{ unitary}\}$, but if $\mathcal{H} = \mathbb{C}^n$ we usually write $\mathcal{U}(\mathcal{H}) = U(n)$ and call it the "unitary group in n dimensions". Quite generally, $\mathcal{U}(\mathcal{H})$ is a group.

[3] Notice that it says $2n$!

Now, let H be a Hermitean form on $\mathbb{C}^n$. If we do not maintain the assumption about it being positive definite, it is natural to assume in stead that H is **non degenerate**.[4] So, this is what we do.

It is straightforward to prove /see the problem sessions) that there is a Hermitean (or self adjoint) operator J_H[5]

$$\forall x, y \in \mathbb{C}^n : H(x,y) = \langle x, J_H(y)\rangle_0. \tag{21}$$

One must have that $\det(J_H) \neq 0$. Conversely, any such operator operator (matrix) J defines a Hermitean form H_J by

$$\forall x, y \in \mathbb{C}^n : H_J(x,y) = \langle x, J(y)\rangle_0. \tag{22}$$

Definition 5.5. *If J is Hermitean and has p strictly positive eigenvalues and q strictly negative eigenvalues with $p+q=n$ we define*

$$\begin{aligned} U_J(p,q) &= \{u \in Gl(n,\mathbb{C}) \mid \forall x,y \in \mathbb{C}^n : H_J(u(x),u(y)) = H_J(x,y)\} \\ &= \{u \in Gl(n,\mathbb{C}) \mid \forall x,y \in \mathbb{C}^n : \langle Ju(x),u(y)\rangle_0 = \langle Jx,y\rangle_0\} \\ &= \{u \in Gl(n,\mathbb{C}) \mid u^* J u = J\}. \end{aligned}$$

As in Definition 5.4, one usually drops the J and just writes $U(p,q)$.

6. The Canonical Commutation Relations

Let B_a be a non-degenerate skew-symmetric form on a finite-dimensional real vector space V. We know that V then is even dimensional; $\dim V = 2n$

Definition 6.1. *The Heisenberg Algebra $\mathfrak{h}_V$, or just $\mathfrak{h}(n)$, based on V is the vector space $V \times \mathbb{R}$ equipped with the Lie bracket*

$$\forall (v,c),(v_1,c_1) \in V \times \mathbb{R} : [(v,c),(v_1,c_1)] = (0, B_a(v,v_1)). \tag{23}$$

The Heisenberg Group $H(V)$, or just $H(n)$, based on V is the vector space $V \times \mathbb{R}$ equipped with the product

$$\forall (v,c),(v_1,c_1) \in V \times \mathbb{R} : (v,c) \star (v_1,c_1) = (v+v_1, c+c_1+\frac{1}{2}B_a(v,v_1)). \tag{24}$$

One can easily bring the skew-symmetric form into 'canonical form': Specifically, there is a basis where one may write the elements v of V as $v=(q,p)$ with $q,p \in \mathbb{R}^n$ and such that

[4] This defined completely analogous to the case with bilinear forms.

[5] So. J_H satisfies: $\forall x,y \in \mathbb{C}^n : \langle J_H(x),y\rangle = \langle x, J_H(y)\rangle$. More generally, the adjoint T^* to the linear operator T on the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is defined by the equation $\forall x,y \in V$: $\langle T^*(x),y\rangle = \langle x, T(y)\rangle$. T is then Hermitian if $T^* = T$.

$$B_a(\begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}) = q \cdot p_1 - q_1 \cdot p \tag{25}$$

where we use the usual $\cdot$-product in $\mathbb{R}^n$. We shall use this form in the sequel and both on the algebra and on the group level.

Consider

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}}q_r & \frac{1}{\sqrt{2}}p_r & c \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}}p_s \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}}q_s \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{26}$$

where the vectors in the first row are written in row form and the vectors in the last column are written in column form. This formula defines a faithful representation of $\mathfrak{h}(n)$ as can be easily seen.

Likewise,

$$\begin{pmatrix} 1 & \frac{1}{\sqrt{2}}q_r & \frac{1}{\sqrt{2}}p_r & c \\ 0 & 1 & 0 & \frac{1}{\sqrt{2}}p_s \\ 0 & 0 & 1 & -\frac{1}{\sqrt{2}}q_s \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

is a faithful $2+2n$-dimensional representation of $H(n)$.

Other matrix versions of $\mathfrak{h}(n)$ and $H(n)$ are also in use, for instance

$$\widetilde{\mathfrak{h}(n)} = \left\{ \begin{pmatrix} 0 & q_1 & q_2 & \cdots & q_n & z \\ 0 & 0 & 0 & \dots & 0 & p_1 \\ 0 & & \ddots & \dots & 0 & \vdots \\ \vdots & & & & 0 & p_{n-1} \\ 0 & 0 & 0 & \dots & 0 & p_n \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \mid q_i, p_i, z \in \mathbb{R} \right\}.$$

The analogous formula with 1's in the diagonal is then an isomorphic version of $H(n)$. In the sequel we will use directly the definitions (23) and (24) together with (25).

The following formula[6] gives the Stone-von Neumann representation of the canonical commutation relations.

Theorem 6.2. *The following formula defines a strongly continuous unitary representation of $H(n)$ in $L^2(\mathbb{R}^n)$:*

$$(U(q,p,c)f)(x) = e^{i\hbar(x\cdot q + c - \frac{1}{2}q\cdot p)} f(x-p). \tag{28}$$

[6]Notice two small changes from the lectures which is introduced to make the formula more natural.

Proof: The group property follows easily. To prove continuity, it suffices to prove that the map

$$(q,p,c) \mapsto U(q,p,c)f$$

is continuous for f in a dense subspace. Here one may choose the Schwartz Space $\mathcal{S}(\mathbb{R}^n)$, or the space $C_c^\infty(\mathbb{R}^n)$ of smooth functions with compact support. For such functions the result follows easily by Lebesgue's Theorem on Dominated Convergence. □

We will later indicate a proof of the Stone-von Neumann Uniqueness Theorem which states that this representation is irreducible and essentially unique[7]. However, we wish to use this result immediately and for this reason we now turn to the Symplectic Group:

Theorem 6.3. *Let, for $g \in Sp(n,\mathbb{R})$, $g \bullet (q_1,\cdots,q_n,p_1,\cdots,p_n)$ denote the natural linear action of g on $\mathbb{R}^{2n}$. The function*

$$\begin{aligned} Sp(n,\mathbb{R}) \times H(n) &\mapsto H(n): \\ \big(g, \big((q_1,\cdots,q_n,p_1,\cdots,y_p),c\big)\big) &\mapsto g \star \big((q_1,\cdots,q_n,p_1,\cdots,p_n),c\big) \\ &= \big(g \bullet (q_1,\cdots,q_n,p_1,\cdots,p_n),c\big) \end{aligned}$$

defines an action of $Sp(n,\mathbb{R})$ on $H(n)$ by automorphisms.

[7]Two different choice of the positive constant $\hbar$ will give inequivalent representations, but besides this, the representation is unique.