CONFORMAL HARMONIC ANALYSIS AND INTERTWINING DIFFERENTIAL OPERATORS

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INTRODUCTION

Earlier in this conference, we have heard Professor I.E. Segal's lecture "Causality and Symmetry in Cosmology and the Conformal Group". I refer to that for motivation, general background, and references. Here shall be described a few results concerning the development of the theory towards elementary particle physics; in particular some results of a study, jointly with Michele Vergne (1), of the interplay between representations of the conformal group (in the form of SU(2,2)) and classical mass-zero-equations, will be given.

GENERALITIES

For technical reasons it is convenient to work with the group

$$G = \{g \in SL(4,\mathbb{C}) \mid g^*\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} g = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\}.$$

This group is isomorphic to the group

$$SU(2,2) = \{g \in SL(4,\mathbb{C}) \mid g^*\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} g = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\}$$

which is used in Segal's theory, and it is straightforward to translate results from G to SU(2,2).

G acts on the generalized upper half-plane $\mathcal{D} = \{2 \times 2$ complex matrices $z \mid \frac{z-z^*}{2i}$ is positive definite$\}$ by, for $G \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; a, b, c, d, 2×2 complex matrices,

$$g \cdot z = \frac{az+b}{cz+d}.$$

One can prove that $(g \cdot z^*)^* = g \cdot z$, and thus G leaves the boundary of $\mathcal{D}$, $\partial\mathcal{D} = H(2) = \{2 \times 2$ complex matrices $h \mid h = h^*\}$, invariant, and this gives rise to the well-known local action of SU(2,2) on Minkowski space M, identified with H(2) by

$$M \ni (t,x,y,z) \Leftrightarrow h = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \in H(2).$$

Finally, let $c: \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \to \begin{pmatrix} \lambda_4 & -\lambda_2 \\ -\lambda_3 & \lambda_1 \end{pmatrix}$ denote the co-factor mapping on the space of 2×2 complex matrices and observe that c's restriction to H(2) is the operation of space-reversal.

THE REPRESENTATIONS

Let ψ be a function from H(2) to $\mathbb{C}$, $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, and $n \in \mathbb{Z}$ and define

$$(U_n(g)\psi)(h) = \det(ch+d)^{-(n+2)}\psi(g^{-1}h).$$

Moreover, let φ be a function from H(2) to $\mathbb{C}^2$ and define

$$(D_n^+(g)\varphi)(h) = (ch+d)^{-1}\det(ch+d)^{-(n+2)}\varphi(g^{-1}h),$$

$$(D_n^-(g)\varphi)(h) = (hc^*+d^*)\det(hc^*+d^*)^{-(n+3)}\varphi(g^{-1}h),$$

and finally define, on functions from H(2) to $\mathbb{C}^4$,

$$D_n(g) = D_n^+(g) \oplus D_n^-(g).$$

U_n and D_n are then well-defined actions on measurable functions.

It is proved in (1), but has also been observed elsewhere (see References in (1)), that one has

Theorem 1. For $n \geq 0$ there exist Hilbert spaces H_n and K_n, obtained as completions of spaces of positive energy and positive mass functions, such that U_n is unitary and irreducible in H_n, and D_n is the direct dum of two unitary irreducible representations in K_n. Moreover, there exists a Hilbert space H_{-1} (K_{-1}) consisting of solutions to $\Box\psi = 0$ ($\not\partial\varphi = 0$) in which U_{-1} (D_{-1}) is unitary.

The significance of the representations U_n and D_n for integers $n \leq -1$ and the relation of these to positive mass functions is illustrated by

THE INTERTWINING RELATIONS

For the sake of being explicit, let

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2},$$

$$\sigma = \begin{pmatrix} \frac{\partial}{\partial t} + \frac{\partial}{\partial z} & \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & \frac{\partial}{\partial t} - \frac{\partial}{\partial z} \end{pmatrix}, \qquad \tilde{\sigma} = \begin{pmatrix} \frac{\partial}{\partial t} - \frac{\partial}{\partial z} & -\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & \frac{\partial}{\partial t} + \frac{\partial}{\partial z} \end{pmatrix},$$

and $\not\partial = \begin{pmatrix} 0 & \sigma \\ \tilde{\sigma} & 0 \end{pmatrix}$. (Then $\not\partial^2 = \Box$.)

Theorem 2. (1) $\forall\ g \in G$:

$$\Box^n U_{-n}(g) = U_n(g)\Box^n$$

$$\not\partial^{2n+1} D_{-1-n}(g) = D_n(g)\not\partial^{2n+1}$$

whenever both sides make sense.

Let, for $n \in \mathbb{N}$, two functions $\psi_i : H(2) \to \mathbb{C}$ $(i=1,2)$ be equivalent if $\square^n(\psi_1-\psi_2) = 0$ and denote the equivalence classes by $[\cdot]_n^0$. Likewise define, for non-negative integers n, two functions $\varphi_i:H(2) \to \mathbb{C}^4$ (i=1,2) to be equivalent if $\not{\partial}^{2n+1}(\varphi_1-\varphi_2) = 0$ and denote the equivalence classes by $[\cdot]_n^{\frac{1}{2}}$.

Theorem 3. (1) It is possible to put a Hilbert space structure H_n (K_n) on a space of equivalence classes $[\cdot]_n^0([\cdot]_n^{\frac{1}{2}})$ of positive mass function, such that U_{-n} (D_{-1-n}) is a unitary representation in H_n (K_n).

Remark. If one restricts the representations U_n and D_n to the Poincaré group together with the scale-transformations, the proofs of the above mentioned facts are quite simple. Also, theorem 3 can then be sharpened, in the sense that for each $U_{-n}(D_{-1-n})$ (n positive (or zero)) there is a natural Hilbert space H_{-n} (K_{-n}) of positive mass functions, on which this restriction is unitary. However, this is not the case for the full group G. In fact, for the one-parameter group

$$\tau(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \subseteq G$$

(which is the G-version of the uni-time translation subgroup of SU(2,2)), $U_{-n}(\tau(\theta))$ ($D_{-1-n}(\tau(\theta))$) does not leave H_{-n} (K_{-n}) invariant, as can be seen easily by computations on special functions.

REFERENCE

1. H.P. JAKOBSEN and M. VERGNE, Wave and Dirac operators, and representations of the conformal group. (To appear in J. Functional Analysis.)