

ALGEBRAS OF VARIABLE COEFFICIENT QUANTIZED DIFFERENTIAL OPERATORS

HANS PLESNER JAKOBSEN

ABSTRACT. In the framework of (vector valued) quantized holomorphic functions defined on non-commutative spaces, “quantized hermitian symmetric spaces”, we analyze what the algebras of quantized differential operators with variable coefficients should be. It is an immediate point that even 0th order operators, given as multiplications by polynomials, have to be specified as e.g. left or right multiplication operators since the polynomial algebras are replaced by quadratic, non-commutative algebras. In the settings we are interested in, there are bilinear pairings which allows us to define differential operators as duals of multiplication operators. Indeed, there are different choices of pairings which lead to quite different results. We consider three different pairings. The pairings are between quantized generalized Verma modules and quantized holomorphically induced modules. It is a natural demand that the corresponding representations can be expressed by (matrix valued) differential operators. We show that a quantum Weyl algebra $Weyl_q(n, n)$ introduced by T. Hyashi ([2]) plays a fundamental role. In fact, for one pairing, the algebra of differential operators, though inherently depending on a choice of basis, is precisely matrices over $Weyl_q(n, n)$.

We determine explicitly the form of the (quantum) holomorphically induced representations and determine, for the different pairings, if they can be expressed by differential operators.

1. INTRODUCTION

Suppose given 2 quadratic algebras \mathcal{A}_q^+ and \mathcal{A}_q^- and a non-degenerate bilinear form $(\cdot, \cdot)_X : \mathcal{A}_q^+ \times \mathcal{A}_q^- \rightarrow \mathbb{C}$. We will say that $(\cdot, \cdot)_X$ gives a pairing between the two algebras; the label X reflects that there will be several different pairings. A pairing displays the algebras as duals of each other, or more correctly, sets up identifications between one algebra and the dual of the other. While the dual space is unique, there may be considerable interest and usefulness in exhibiting the duals concretely in such manners. Even when the two algebras are given in advance, one may examine different pairings between them to optimize certain properties.

An algebra A is of course a left and a right module over itself. Thus, one has left multiplication operators ${}_aM$ and right multiplication operators M_b for all $a, b \in A$. Clearly, the algebra generated by the operators ${}_aM$ is isomorphic to A while the algebra determined by the operators M_b is isomorphic to A^o ; the opposite algebra.

In the case with two algebras in a pairing such as the previously mentioned algebras \mathcal{A}_q^\pm , we can define left and right “constant coefficient” differential operators ${}_c\partial_X, {}_X\partial_c$, on, say, \mathcal{A}_q^+ , as the operators obtained as the duals, via the pairing, of left and right multiplication operators on \mathcal{A}_q^- . These operators,

which depend upon the pairing X , may then be put together with the left and right multiplication operators to form an algebra $\mathcal{D}_{X,Full}^+$ of differential operators. Instances of such (left) operators are the Kashiwara derivations ([9]). In specific examples it is interesting to determine if $\mathcal{D}_{X,Full}^+$, for a specific index X , is generated by fewer operators, eg by left differential operators and left multiplication operators; $\mathcal{D}_{X,Full}^+ = Alg_{\mathbb{C}} \{dM, c\partial_X ; c \in \mathcal{A}_q^-, d \in \mathcal{A}_q^+\}$?

The algebras \mathcal{A}_q^{\pm} we consider are quadratic algebras that are specific subalgebras of $\mathcal{U}_q(\mathfrak{su}(n, n)^{\mathbb{C}})$. As quadratic algebras they are actually isomorphic. They are the algebras of the quantized generalized unit disk. Furthermore, they are modules for $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$, where \mathfrak{k} is a maximal compact subalgebra of $\mathfrak{su}(n, n)$, but as such, they are non-isomorphic; and not necessarily dual modules, either.

There is a further structure we need to include in our discussions: To each finite-dimensional $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ module V_{Λ} there is a quantized generalized Verma module over $\mathcal{U}_q(\mathfrak{su}(n, n)^{\mathbb{C}})$. As a $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ module it is given as $\mathcal{M}(V_{\Lambda}) = \mathcal{A}_q^- \otimes V_{\Lambda}$.

This extra structure leads to natural demands on the bilinear pairing. In this connection it is not profitable to consider a pairing between a Verma module and its ‘‘opposite’’ (interchanging positive and negative roots). The right notion of a dual of a generalized Verma module, in our context, is a holomorphically induced module.

It is natural to demand that an algebra of differential operators is rich enough that the operators in the holomorphically induced modules belong to it. To have any hope of that, one will of course need to include in the algebra the homomorphisms $hom_{\mathbb{C}}(V_{\Lambda}, V_{\Lambda})$, or, rather, the duals thereof.

About pairings: In the classical situation, the Killing form on a real semi-simple Lie algebra \mathfrak{g} , extended to $\mathcal{U}(\mathfrak{g}^{\mathbb{C}}) \times \mathcal{U}(\mathfrak{g}^{\mathbb{C}})$, gives the wanted pairing. M Rosso constructed the quantum analogue of this. After that, G. Lusztig ([14]) and M. Kashiwara ([9]) made valuable extensions and simplifications, and Kashiwara defined some derivations as duals of left or right multiplications. The history of this subject is very rich and we hope that we are not being too unfair in this sketchy summary. One should definitely also consult [8] and [19]. We follow here the book by J.C. Jantzen ([7]), not only for notation, but actually to the extent of copying directly several of his constructions and results.

We will study three bilinear forms, indexed by $X = J, K, L$. $(\cdot, \cdot)_J$ is the form considered by Jantzen, though he actually studies an additional form towards the end of his book. They reflect the three standard ways of quantizing integers:

$$\begin{aligned} [[a]]_q &= 1 + q^2 + \cdots + q^{2a-2} (J), \\ \{\{a\}\}_q &= 1 + q^{-2} + \cdots + q^{-(2a-2)} (K), \text{ and} \\ [a]_q &= q^{-a+1} + \cdots + q^{a-1} (L). \end{aligned}$$

Using fixed PBW bases, we define an auxiliary algebra $Weyl_q(n, n)$ - a quantization of the classical Weyl algebra in n^2 variables - as the algebra generated by n^2 commuting variables $D_{i,j}$ and n^2 commuting variables $M_{i,j}$, and where also $D_{i,j}$ commutes with $M_{s,t}$ if $(i, j) \neq (s, t)$ so that the only non-trivial relations are at fixed nodes. Here the relations are

$$\begin{aligned} D_{i,j}M_{i,j} - qM_{i,j}D_{i,j} &= H_{i,j}^{-1}, & D_{i,j}M_{i,j} - q^{-1}M_{i,j}D_{i,j} &= H_{i,j}, \\ H_{i,j}D_{i,j} &= q^{-1}D_{i,j}H_{i,j}, & \text{and} & & H_{i,j}M_{i,j} &= qM_{i,j}H_{i,j}. \end{aligned}$$

This is a very interesting algebra which was introduced by T. Hyashi ([2]). There has recently been renewed interest in it, see [11].

It turns out that there is a big difference between the three cases, where especially the case (J) leads to unpleasant results. The simplest case, on the other hand, is the case (L) where

$$\begin{aligned} Alg_{\mathbb{C}} \{ {}_dM, {}_c\partial_L ; c \in \mathcal{A}_q^-, d \in \mathcal{A}_q^+ \} &= Alg_{\mathbb{C}} \{ M_d, {}_L\partial_c ; c \in \mathcal{A}_q^-, d \in \mathcal{A}_q^+ \} \\ &= Weyl_q(n, n). \end{aligned}$$

For the case (K) there is a big subalgebra $\mathcal{KWeyl}_q(n, n)$ of $Weyl_q(n, n)$ with many pleasing properties such that

$$\begin{aligned} Alg_{\mathbb{C}} \{ {}_dM, M_f, {}_c\partial_K ; c \in \mathcal{A}_q^-, d, f \in \mathcal{A}_q^+ \} &= \\ Alg_{\mathbb{C}} \{ M_d, {}_fM, {}_K\partial_c ; c \in \mathcal{A}_q^-, d, f \in \mathcal{A}_q^+ \} &= \mathcal{KWeyl}_q(n, n). \end{aligned}$$

In both of the cases (K), (L), these algebras, augmented by (constant value) matrices, contain the generators of the holomorphically induced representations. To prove such a statement it suffices to determine the action of E_β , where β is the unique non-compact root and prove the statement in this special case. The mentioned action is given in Corollary 7.2.2 to Theorem 7.2.1; our first main result.

There is one related important study, namely that by L. Vaksman and his group [17]. In it, they extend substantially the quantized exterior derivative introduced in [1] and which already leads to derivatives. Vaksman et al. discovered a fundamental symmetry. Their method uses induction from the trivial $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ module. Here, the classically holomorphically induced module is annihilated by first order differential operators. In the quantized situation, there is a natural pairing and a natural algebra structure obtained from the tensor product. Proceeding like this, they have frozen the algebras at a specific weight which means that they do obtain interesting results, but not the general picture that we obtain. The extra symmetry is related to the fact that our algebras are bi-modules. Furthermore, their pairing is degenerate, though "mildly".

Quantized differential operators were also studied in [4], [5], [18].

In §2 we introduce the quantized Hermitian symmetric spaces (the case of $su(n, n)$) via the Lusztig operators. In §3 we study these spaces as $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ modules and for this purpose, $Weyl_q(n, n)$ is introduced. §4 contain many direct quotes from Jantzen's book. The bilinear pairing is introduced, the duals to left and right multiplication operators are determined, and the left action of

E_α (α a simple root) in \mathcal{U}_q^- is given. Duality considerations are continued in §5 where the various pairings we wish to study, are introduced. We also discuss various change-of-basis maps. One such is needed because Jantzen's form has a singularity at $q = 1$. In §6 we introduce the generalized Verma modules and the quantized holomorphically induced modules, and pairings between them.

Then, in §7 we obtain the dual of the action of E_β , β the unique non-compact simple root. This is given in Theorem 7.2.1. It should be observed how simple the result actually is and that it is given, essentially, by left and right multiplication operators. We also obtain the limit of the operator as $q \rightarrow 1$ and make sure that it agrees with the known "classical" operator.

Finally, in §8 we obtain the other main results about the algebras of polynomial coefficient differential operators; Theorem 8.6.3 and Theorem 8.6.5. As a bonus we obtain that $\mathcal{Weyl}_q(n, n)$, though manifestly defined via a PBW basis, actually is intrinsic.

The main technical part involves computing explicitly the multiplication operators ${}_cM, M_d$ which can be done using the defining quadratic relations. If we let $D_{i,j}$ and $M_{i,j}$ denote the generators of $\mathcal{Weyl}_q(n, n)$, $i, j \in \{1, 2, \dots, n\}$, and similarly let $H_{i,j}^{\pm 1}$ denote the operators $H^{\pm 1}$ at the node i, j then in all cases (J), (K), (L) the following holds: At each node we obtain from the left and right operators along with their duals, operators $D_{i,j}\psi_{i,j}$ and $M_{i,j}\phi_{i,j}$ for some elements $\psi_{i,j}$ and $\phi_{i,j}$ which are Laurent monomials in the elements $H_{1,1}, H_{2,1}, \dots, H_{n-1,n}, H_{n,n}$. The appearance of these factors is just one of the interesting consequences of working with a quantized Weyl algebra.

In the cases (L) and (K) we get sufficiently many such elements to find some simple generators. In the case (J), however, the generators remain complicated.

2. QUANTIZED HERMITEAN SYMMETRIC SPACES.

2.1. Basic definitions. We consider $\mathfrak{g} = su(n, n)$; $\mathfrak{g}^{\mathbb{C}}$ is a simple complex Lie algebra of type A_{2n-1} . We choose below a set of simple roots Π in the root space Ψ . The Weyl group is equal to S_{2n} . We denote the generators of the Weyl group by $s_\gamma, \gamma \in \Pi$ and denote by $E_\gamma, F_\gamma, K_\gamma^{\pm 1}$ for $\gamma \in \Pi$ the generators of $\mathcal{U}_q(\mathfrak{g}^{\mathbb{C}})$ is standard notation. The weight lattice is denoted by \mathcal{L} and we further extend the notation K_ξ to hold for any weight $\xi \in \mathcal{L}$ in the usual fashion.

The roots Ψ may be represented in \mathbb{R}^{2n} by the set

$$(1) \quad \Psi = \{\pm e_i \mp e_j \mid i, j = 1, 2, \dots, 2n \text{ and } i \neq j\},$$

where $\{e_1, e_2, \dots, e_{2n}\}$ is the standard basis of \mathbb{R}^{2n} . We then have

$$(2) \quad \Pi = \{e_i - e_{i+1} \mid i = 1, 2, \dots, 2n - 1\}.$$

Throughout, we let $\beta = e_n - e_{n+1}$ denote the unique **non-compact** simple root. The roots $\nu_i = e_{n-i} - e_{n-i+1}, i = 1, \dots, n - 1$ and the roots $\mu_j = e_{n+j} - e_{n+j+1}, j = n + 1, \dots, 2n$ are the **compact** simple roots of type A_{n-1} ; $\Pi = \{\mu_1, \dots, \mu_{n-1}\} \cup \{\beta\} \cup \{\nu_1, \dots, \nu_{n-1}\}$. We also set $\Pi_c = \{\mu_1, \dots, \mu_{n-1}\} \cup \{\nu_1, \dots, \nu_{n-1}\}$; the compact simple roots, and set $\Pi_L = \{\mu_1, \dots, \mu_{n-1}\}$, $\Pi_R = \{\nu_1, \dots, \nu_{n-1}\}$. Let $\mathfrak{k}_L^{\mathbb{C}}$ and $\mathfrak{k}_R^{\mathbb{C}}$ denote the subalgebras defined by the simple roots $\{\mu_1, \dots, \mu_{n-1}\}$ and $\{\nu_1, \dots, \nu_{n-1}\}$, respectively.

Finally, for $k = 1, \dots, n-1$, let δ_k^μ and δ_k^ν , respectively, denote the fundamental dominant weights for the roots $\{\mu_1, \dots, \mu_{n-1}\}$ and $\{\nu_1, \dots, \nu_{n-1}\}$, respectively.

A maximal compact subalgebra $\mathfrak{k}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ is

$$(3) \quad \mathfrak{k}^{\mathbb{C}} = su(n)^{\mathbb{C}} \oplus \mathbb{C} \oplus su(n)^{\mathbb{C}} = \mathfrak{k}_L^{\mathbb{C}} \oplus \zeta \oplus \mathfrak{k}_R^{\mathbb{C}},$$

where ζ is the center of $\mathfrak{k}^{\mathbb{C}}$ and is generated by an element h_β of the compact Cartan subalgebra. We have furthermore on the classical level

$$(4) \quad \mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+,$$

where \mathfrak{p}^\pm are abelian $\mathfrak{k}^{\mathbb{C}}$ modules, and

$$(5) \quad \mathcal{U}(\mathfrak{g}^{\mathbb{C}}) = \mathcal{P}(\mathfrak{p}^-) \cdot \mathcal{U}(\mathfrak{k}^{\mathbb{C}}) \cdot \mathcal{P}(\mathfrak{p}^+).$$

We have that $\mathcal{U}(\mathfrak{p}^\pm) = \mathcal{P}(\mathfrak{p}^\pm)$ are polynomial algebras.

On the quantized level we have

$$(6) \quad \mathcal{U}_q(\mathfrak{k}^{\mathbb{C}}) = \mathcal{U}_q(\mathfrak{k}_L^{\mathbb{C}}) \cdot \mathbb{C}[K_\beta^{\pm 1}] \cdot \mathcal{U}_q(\mathfrak{k}_R^{\mathbb{C}}).$$

There is an analogue of (5) for $\mathcal{U}_q(\mathfrak{g}^{\mathbb{C}})$,

$$(7) \quad \mathcal{U}_q(\mathfrak{g}^{\mathbb{C}}) = \mathcal{A}_q^- \cdot \mathcal{U}_q(\mathfrak{k}^{\mathbb{C}}) \cdot \mathcal{A}_q^+.$$

Here, \mathcal{A}_q^\pm are quadratic algebras which are furthermore $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$. We will describe these later.

We let $\mathcal{U}_q^0(\mathfrak{k}_L^{\mathbb{C}} \oplus \mathfrak{k}_R^{\mathbb{C}})$ denote the Laurent polynomials generated by the elements K_α for $\alpha \in \Pi_c$, and let $\mathcal{U}_q^0(\mathfrak{k}^{\mathbb{C}})$ denote the analogue where also $\alpha = \beta$ is allowed.

For use in the construction of the algebras \mathcal{A}_q^\pm we now consider some elements in the Weyl group.

Let $I = (1, 2, \dots, n, n+1, \dots, 2n)$. Consider the following elements in S_{2n} :

$$(8) \quad \omega_0(I) = (2n, 2n-1, \dots, n+1, n, \dots, 2, 1)$$

$$(9) \quad \omega^+(I) = (n+1, \dots, 2n-1, 2n, 1, 2, \dots, n-1, n)$$

$$(10) \quad \omega^L(I) = (n, n-1, \dots, 2, 1, n+1, n+2, \dots, 2n-1, 2n)$$

$$(11) \quad \omega^R(I) = (1, 2, \dots, n-1, n, 2n, 2n-1, \dots, n+2, n+1).$$

We have that ω_0 is the longest element, and

$$(12) \quad \omega_0 = \omega_0^L \omega_0^R \omega_0^+ = \omega_0^+ \omega_0^L \omega_0^R = \omega_0^L \omega_0^+ \omega_0^L,$$

and many more similar identities. We shall later use

$$(13) \quad \omega_0 = \omega_0^L \omega_0^R \omega_0^+$$

since this puts the elements from Π_c to the right according to the construction in [7, p. 163-168]. However, it is convenient first to consider $\omega_0 = \omega_0^+ \omega_0^L \omega_0^R$:

We shall in all cases use the following choice for a reduced expression for ω_0^+ :

$$(14) \quad \omega_0^+ = s_\beta s_{\mu_1} \cdots s_{\mu_{n-1}} s_{\nu_1} s_\beta s_{\mu_1} \cdots s_{\mu_{n-2}} s_{\nu_2} s_{\nu_1} s_\beta \cdots s_\beta s_{\mu_1} s_{\nu_{n-1}} \cdots s_{\nu_1} s_\beta.$$

Remark 2.1.1. *If we look at $n \times (n + r)$ we replace ω_0^+ by $\omega_0^+ \omega_E^+$ where*

$$(15) \quad \omega_E^+ = s_{\nu_n} \cdots s_{\nu_1} s_{\nu_{n+1}} \cdots s_{\nu_2} \cdots s_{\nu_{n+r-1}} \cdots s_{\nu_r}.$$

2.2. The Lusztig operators. G. Lusztig ([15]) has given a construction of braid operators T_α for each simple root α , and has extended these to operators T_ω for each element ω of the Weyl group. Indeed, there are two choices of such operators, the other usually denoted by T'_ω , and there is always a choice between q and q^{-1} . In the book ([7]), J. C. Jantzen describes, among many other things, these operators. We will throughout use the notation and choices from this book. ([7]).

Lemma 2.2.1. *[[15], [7][p. 156]] In the simply laced case with neighboring simple roots α, γ :*

$$(16) \quad T_\alpha(F_\gamma) = F_\gamma F_\alpha - q F_\alpha F_\gamma$$

$$(17) \quad T_\alpha(E_\gamma) = E_\alpha E_\gamma - q^{-1} E_\gamma E_\alpha = \text{ad}(E_\alpha)(E_\gamma)$$

$$(18) \quad T_\alpha^{-1}(F_\gamma) = F_\alpha F_\gamma - q F_\gamma F_\alpha$$

$$(19) \quad T_\alpha^{-1}(E_\gamma) = E_\gamma E_\alpha - q^{-1} E_\alpha E_\gamma.$$

Lemma 2.2.2. *[[15], [7][p. 156]] If α, γ are adjacent, then*

$$(20) \quad T_\gamma(E_\alpha) = T_\alpha^{-1}(E_\gamma)$$

$$(21) \quad T_\gamma(F_\alpha) = T_\alpha^{-1}(F_\gamma)$$

$$(22) \quad T_\alpha T_\gamma T_\alpha = T_\gamma T_\alpha T_\gamma$$

$$(23) \quad T_\alpha T_\gamma(E_\alpha) = E_\gamma$$

$$(24) \quad T_\alpha T_\gamma(F_\alpha) = F_\gamma.$$

If α, γ are not adjacent, then

$$(25) \quad T_\alpha T_\gamma = T_\gamma T_\alpha$$

Lusztig has further shown that if $s_\alpha s_\beta$ has order m then

$$(26) \quad \underbrace{T_\alpha T_\gamma \cdots}_m = \underbrace{T_\gamma T_\alpha \cdots}_m.$$

There is an important construction of a PBW type basis in \mathcal{U}^\pm for any reduced decomposition of $\omega_0 = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_i} \cdots s_{n(2n-1)}$. It is also due to Lusztig and is given as follows: Set, $\omega_i = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_i}$, $i = 1, \dots, 2n - 1$. Then,

$$(27) \quad \gamma_i = \omega_{i-1}(\alpha_i)$$

$$(28) \quad X_{\gamma_i} = T_{\omega_{i-1}}(E_{\alpha_i}) = T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_{i-1}}(E_{\alpha_i}) \in U_{\gamma_i}^+$$

$$(29) \quad Y_{\gamma_i} = T_{\omega_{i-1}}(F_{\alpha_i}) = T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_{i-1}}(F_{\alpha_i}) \in U_{\gamma_i}^-$$

We now introduce some intermediary bases. Set

$$(30) \quad Q_{ij} = T_{\mu_{i-1}}^{-1} T_{\mu_{i-2}}^{-1} \cdots T_{\mu_0}^{-1} T_{\nu_{j-1}}^{-1} T_{\nu_{j-2}}^{-1} \cdots T_{\nu_0}^{-1}(F_\beta)$$

$$(31) \quad P_{ij} = T_{\mu_{i-1}}^{-1} T_{\mu_{i-2}}^{-1} \cdots T_{\mu_0}^{-1} T_{\nu_{j-1}}^{-1} T_{\nu_{j-2}}^{-1} \cdots T_{\nu_0}^{-1}(E_\beta).$$

Set $\mathbf{a} = (a_{11}, \dots, a_{1,n}, \dots, a_{nn}) \in \mathbb{N}_0^{n^2}$, and set

$$(32) \quad P^{\mathbf{a}} = P_{11}^{a_{11}} \cdots P_{nn}^{a_{nn}}$$

$$(33) \quad Q^{\mathbf{b}} = Q_{11}^{b_{11}} \cdots Q_{nn}^{b_{nn}}.$$

Let $\{u_i^{\mathfrak{k}^+} \mid i \in I\}$ denote a PBW type basis of $\mathcal{U}^+(\mathfrak{k})$ and let $\{u_j^{\mathfrak{k}^-} \mid j \in J\}$ denote a PBW type basis of $\mathcal{U}^-(\mathfrak{k})$.

Proposition 2.2.3. *There is a basis*

$$(34) \quad \{u_i^{\mathfrak{k}^+} \cdot P^{\mathbf{a}} \mid i \in I; \mathbf{a} \in \mathbb{N}_0^{n^2}\}$$

of \mathcal{U}^+ , and a basis

$$(35) \quad \{u_j^{\mathfrak{k}^-} \cdot Q^{\mathbf{b}} \mid j \in J; \mathbf{b} \in \mathbb{N}_0^{n^2}\}$$

of \mathcal{U}^- .

Proof. This follows from Lusztig (see Jantzen §8.24) by using the following extra observations which are easily deduced from, in particular, Lemma 2.2.2.

$$1) T_\beta T_{\mu_1}(E_{\mu_2}) = T_\beta T_{\mu_2}^{-1}(E_{\mu_1}) = T_{\mu_2}^{-1} T_\beta(E_{\mu_1}) = T_{\mu_2}^{-1} T_{\mu_1}^{-1}(E_\beta).$$

$$2) T_\beta T_{\mu_1} T_{\mu_2} T_{\nu_1}(E_\beta) = T_\beta T_{\mu_1} T_{\nu_1}(E_\beta) = T_\beta T_{\mu_1} T_\beta^{-1}(E_{\nu_1}) = T_{\mu_1}^{-1} T_\beta T_{\mu_1}(E_{\nu_1}) = T_{\mu_1}^{-1} T_\beta(E_{\nu_1}) = T_{\mu_1}^{-1} T_{\nu_1}^{-1}(E_\beta). \quad \square$$

While these bases have many good properties, it is more natural to have \mathfrak{k}^+ to the right. To this end we employ $\omega_0 = \omega^L \omega^R \omega^+$:

Definition 2.2.4. *Set, for all $i, j \in \{1, 2, \dots, n\}$,*

$$(36) \quad Z_{n+1-i, n+1-j} = T_{\omega^\mu \omega^\nu}(P_{ij}) \text{ and } W_{n+1-i, n+1-j} = T_{\omega^\mu \omega^\nu}(Q_{ij}).$$

Let the roots γ_{ij} be defined by

$$(37) \quad \forall i, j \in \{1, 2, \dots, n\} : Z_{ij} = Z_{\gamma_{ij}} \text{ and } W_{ij} = W_{-\gamma_{ij}}.$$

Lemma 2.2.5.

$$(38) \quad Z_{i,j} = T_{\nu_{j-1}} T_{\nu_{j-2}} \cdots T_{\nu_0} \cdot T_{\mu_{i-1}} T_{\mu_{i-2}} \cdots T_{\mu_0}(E_\beta)$$

$$(39) \quad W_{i,j} = T_{\nu_{j-1}} T_{\nu_{j-2}} \cdots T_{\nu_0} \cdot T_{\mu_{i-1}} T_{\mu_{i-2}} \cdots T_{\mu_0}(F_\beta).$$

In particular,

$$(40) \quad Z_{i,j} = ad(E_{\nu_{j-1}}) ad(E_{\nu_{j-2}}) \cdots ad(E_{\nu_0}) \cdot ad(E_{\mu_{i-1}}) ad(E_{\mu_{i-2}}) \cdots ad(E_{\mu_0})(E_\beta).$$

Proof. Since all steps are similar, it suffices to prove that

$$(41) \quad T_{\omega^\mu}(T_{\mu_i}^{-1}T_{\mu_{i-1}}^{-1} \dots T_{\mu_0}^{-1}(E_\beta)) = T_{\mu_{n-i}}T_{\mu_{n-i-1}} \dots T_{\mu_0}(E_\beta).$$

Using

$$(42) \quad (\omega^L) = s_{\mu_1}s_{\mu_2}s_{\mu_1}s_{\mu_3}s_{\mu_2}s_{\mu_1} \dots s_{\mu_{n-1}}s_{\mu_{n-2}} \dots s_{\mu_1},$$

this follows from the above formulas, especially (22) in the form

$$(43) \quad T_{\mu_{k-1}}T_{\mu_k}^{-1}T_{\mu_{k-1}}^{-1} = T_{\mu_k}^{-1}T_{\mu_{k-1}}^{-1}T_{\mu_k}$$

and $T_{\mu_k}^{\pm 1}(X_\beta) = 0$ for $k \geq 2$ and $X = E$ or $X = F$. □

Observe that we now have an ordering

$$(44) \quad W_{1,1}, W_{2,1}, \dots, W_{n,1} W_{1,2}, \dots, W_{n,2}, \dots, W_{1,n}, \dots, W_{n,n}, \\ \dots W_{1,n+1}, \dots, W_{n,n+1}, \dots, W_{1,n+r}, \dots, W_{n,n+r},$$

and a similar ordering of the elements $Z_{i,j}$.

Now define

$$(45) \quad Z^{\mathbf{a}} := Z_{11}^{a_{11}} Z_{21}^{a_{21}} \dots Z_{n1}^{a_{n1}} Z_{12}^{a_{12}} \dots Z_{nn}^{a_{nn}},$$

$$(46) \quad W^{\mathbf{a}} := W_{11}^{a_{11}} W_{21}^{a_{21}} \dots W_{n1}^{a_{n1}} W_{12}^{a_{12}} \dots W_{nn}^{a_{nn}}.$$

Proposition 2.2.6. *There is a basis*

$$(47) \quad \{Z^{\mathbf{a}} \cdot u_i^{\mathfrak{k}^+} \mid i \in I; \mathbf{a} \in \mathbb{N}_0^{n^2}\}$$

of \mathcal{U}^+ , and a basis

$$(48) \quad \{W^{\mathbf{b}} \cdot u_j^{\mathfrak{k}^-} \mid j \in J; \mathbf{b} \in \mathbb{N}_0^{n^2}\}$$

of \mathcal{U}^- .

These are the bases we will use.

Example 2.2.7. $su(3, 3)$:

$$(49) \quad \omega_0 = (\beta\mu_1\mu_2\nu_1\beta\mu_1\nu_2\nu_1\beta)(\mu_1\mu_2\mu_1)(\nu_2\nu_1\nu_2).$$

Then,

$$(50) \quad \begin{aligned} \gamma_{11} &= \beta, & \gamma_{12} &= s_\beta s_{\mu_1} s_{\mu_2}(\nu_1), & \gamma_{13} &= s_\beta s_{\mu_1} s_{\mu_2} s_{\nu_1} s_\beta s_{\mu_1}(\nu_2), \\ \gamma_{21} &= s_\beta(\mu_1), & \gamma_{22} &= s_\beta s_{\mu_1} s_{\mu_2} s_{\nu_1}(\beta), & \gamma_{23} &= s_\beta s_{\mu_1} s_{\mu_2} s_{\nu_1} s_\beta s_{\mu_1} s_{\nu_2}(\nu_1), \\ \gamma_{31} &= s_\beta s_{\mu_1}(\mu_2), & \gamma_{32} &= s_\beta s_{\mu_1} s_{\mu_2} s_{\nu_1} s_\beta(\mu_1), & \gamma_{33} &= s_\beta s_{\mu_1} s_{\mu_2} s_{\nu_1} s_\beta s_{\mu_1} s_{\nu_2} s_{\nu_1}(\beta). \end{aligned}$$

Furthermore,

$$(51) \quad \begin{aligned} Z_{11} &= E_\beta, & Z_{12} &= E_{\nu_1} Z_{11} - q^{-1} Z_{11} E_{\nu_1}, & Z_{13} &= E_{\nu_2} Z_{12} - q^{-1} Z_{12} E_{\nu_2}, \\ Z_{21} &= E_{\mu_1} Z_{11} - q^{-1} Z_{11} E_{\mu_1}, & Z_{22} &= E_{\nu_1} Z_{21} - q^{-1} Z_{21} E_{\nu_1}, & Z_{23} &= E_{\nu_2} Z_{22} - q^{-1} Z_{22} E_{\nu_2}, \\ Z_{31} &= E_{\mu_2} Z_{21} - q^{-1} Z_{21} E_{\mu_2}, & Z_{32} &= E_{\nu_1} Z_{31} - q^{-1} Z_{31} E_{\nu_1}, & Z_{33} &= E_{\nu_2} Z_{32} - q^{-1} Z_{32} E_{\nu_2}. \end{aligned}$$

It follows easily from the quantized Serre relations that these elements generated the usual quantized 3×3 matrix algebra with “ q^{-1} relations” (see the Definition below).

2.3. The quadratic algebras $\mathcal{A}_q^+, \mathcal{A}_q^-$.

Definition 2.3.1. We let \mathcal{A}_q^+ and \mathcal{A}_q^- denote the algebras generated by the elements $Z_{ij}, i, j \in \{1, 2, \dots, n\}$ and $W_{ij}, i, j \in \{1, 2, \dots, n\}$, respectively.

Proposition 2.3.2. $\forall i, j, k, s, t \in \{1, 2, \dots, n\}$:

$$(52) \quad Z_{ij}Z_{ik} = q^{-1}Z_{ik}Z_{ij} \text{ if } j < k;$$

$$(53) \quad Z_{ij}Z_{kj} = q^{-1}Z_{kj}Z_{ij} \text{ if } i < k;$$

$$(54) \quad Z_{ij}Z_{st} = Z_{st}Z_{ij} \text{ if } i < s \text{ and } t < j;$$

$$(55) \quad Z_{ij}Z_{st} = Z_{st}Z_{ij} - (q - q^{-1})Z_{it}Z_{sj} = \text{ if } i < s \text{ and } j < t.$$

There are entirely identical relations for the elements W_{ij} .

Proof. These relations follow from Lemma 2.2.1, Lemma 2.2.2, and the quantum Serre relations. \square

Remark 2.3.3. We see that one gets the relations studied in e.g. ([6]) except for the replacement $q \rightarrow q^{-2}$. In ([6]) the methods were not related to the Lusztig operators.

Lemma 2.3.4. Set

$$(56) \quad Z_1 = Z_{11}, Z_2 = Z_{21}, Z_3 = Z_{12}, \text{ and } Z_4 = Z_{22}.$$

Then $\forall a \in \mathbb{N}$:

$$(57) \quad Z_4^a Z_1 = Z_1 Z_4^a + (q - q^{-1})q^{a-1}[a]_q Z_2 Z_3 Z_4^{a-1} \text{ and}$$

$$(58) \quad Z_4 Z_1^a = Z_1^a Z_4 + (q - q^{-1})q^{a-1}[a]_q Z_1^{a-1} Z_2 Z_3.$$

Proof. This follows easily by induction from Prop 2.3.2. \square

3. \mathcal{A}_q^\pm AS $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ MODULES

3.1. The quantized Weyl algebra. We wish to describe the natural left actions of $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ in these spaces in terms of some simple operators given by their matrix representation with respect to a given PBW basis. Specifically, introduce, for all $i, j \in \{1, 2, \dots, n\}$:

Definition 3.1.1.

$$(59) \quad M_{ij}^o(Z_{11}^{a_{11}} \cdots Z_{ij}^{a_{ij}} \cdots Z_{nn}^{a_{nn}}) = Z_{11}^{a_{11}} \cdots Z_{ij}^{a_{ij}+1} \cdots Z_{nn}^{a_{nn}}$$

$$(60) \quad D_{ij}^o(Z_{11}^{a_{11}} \cdots Z_{ij}^{a_{ij}} \cdots Z_{nn}^{a_{nn}}) = [a_{ij}] Z_{11}^{a_{11}} \cdots Z_{ij}^{a_{ij}-1} \cdots Z_{nn}^{a_{nn}}$$

$$(61) \quad H_{ij}^o(Z_{11}^{a_{11}} \cdots Z_{ij}^{a_{ij}} \cdots Z_{nn}^{a_{nn}}) = q^{a_{ij}} Z_{11}^{a_{11}} \cdots Z_{ij}^{a_{ij}+1} \cdots Z_{nn}^{a_{nn}}$$

$$(62) \quad M_{ij}^o(W_{11}^{a_{11}} \cdots W_{ij}^{a_{ij}} \cdots W_{nn}^{a_{nn}}) = W_{11}^{a_{11}} \cdots W_{ij}^{a_{ij}+1} \cdots W_{nn}^{a_{nn}}$$

$$(63) \quad D_{ij}^o(W_{11}^{a_{11}} \cdots W_{ij}^{a_{ij}} \cdots W_{nn}^{a_{nn}}) = [a_{ij}] W_{11}^{a_{11}} \cdots W_{ij}^{a_{ij}-1} \cdots W_{nn}^{a_{nn}}$$

$$(64) \quad H_{ij}^o(W_{11}^{a_{11}} \cdots W_{ij}^{a_{ij}} \cdots W_{nn}^{a_{nn}}) = q^{a_{ij}} W_{11}^{a_{11}} \cdots W_{ij}^{a_{ij}} \cdots W_{nn}^{a_{nn}}.$$

(Notice in particular (64).)

Lemma 3.1.2. *We have the following formulas for all $i, j \in \{1, 2, \dots, n\}$:*

$$(65) \quad D_{ij}^o M_{ij}^o - q M_{ij}^o D_{ij}^o = (H_{ij}^o)^{-1}$$

$$(66) \quad D_{ij}^o M_{ij}^o - q^{-1} M_{ij}^o D_{ij}^o = H_{ij}^o$$

$$(67) \quad H_{ij}^o D_{ij}^o = q^{-1} D_{ij}^o H_{ij}^o$$

$$(68) \quad H_{ij}^o M_{ij}^o = q M_{ij}^o H_{ij}^o.$$

Operators belonging to different nodes commute.

Definition 3.1.3. *We define $\text{Weyl}_q(n, n)$ to be the algebra generated by the operators M_{ij}^o, D_{ij}^o for all $(i, j) \in \{1, \dots, n\}^2$:*

$$(69) \quad \text{Weyl}_q(n, n) := \mathbb{C}[M_{ij}^o, D_{ij}^o; i, j = 1, \dots, n] = \text{Weyl}_q(1, 1)^{\times n^2}.$$

Observe that $(H_{ij}^o)^{\pm 1} \in \text{Weyl}_q(n, n)$ for all $i, j \in \{1, 2, \dots, n\}$.

Remark 3.1.4. *This algebra was first studied by T. Hiashi ([2]). Recently it was studied again in ([11]).*

3.2. The left actions of $\mathcal{U}_q(\mathfrak{k})$ on \mathcal{A}_q^\pm .

We denote the natural left action of a root vector X_μ on \mathcal{A}_q^\pm , with $X = E, F$, and $\mu \in \Pi$, by X_μ^\pm . We see this action as taking place in \mathcal{U}_q^\pm and for this reason introduce right multiplication operators $M_{X_\mu}: \mathcal{A}_q^\pm \ni u \rightarrow uX_\mu \in \mathcal{U}_q^\pm$, and analogous operators M_{K_μ} .

Proposition 3.2.1. *In \mathcal{A}_q^- :*

$$(70) \quad E_{\mu_k}^- = \sum_j (-q) M_{k,j}^o D_{k+1,j}^o H_{k,j+1}^o \cdots H_{k,n}^o (H_{k+1,j+1}^o)^{-1} \cdots (H_{k+1,n}^o)^{-1} M_{K_{\mu_k}} + M_{E_{\mu_k}}$$

$$(71) \quad F_{\mu_k}^- = \sum_j (-q^{-1}) M_{k+1,j}^o D_{k,j}^o H_{k+1,1}^o \cdots H_{k+1,j-1}^o (H_{k,1}^o)^{-1} \cdots (H_{k,j-1}^o)^{-1} + \prod_j (H_{k,j}^o)^{-1} H_{k+1,j}^o M_{F_{\mu_k}}.$$

Furthermore,

$$(72) \quad \text{In } \mathcal{A}_q^- : K_{\mu_k} = \prod_j H_{k,j}^o (H_{k+1,j}^o)^{-1}.$$

$$(73) \quad \text{In } \mathcal{A}_q^+ : K_{\mu_k} = \prod_j H_{k+1,j}^o (H_{k,j}^o)^{-1}.$$

Proof. We have the equations

$$(74) \quad E_{\mu_k} T_{\mu_j} T_{\mu_{j-1}} \cdots T_{\mu_1} (F_\beta) =$$

$$(75) \quad \begin{cases} T_{\mu_j} T_{\mu_{j-1}} \cdots T_{\mu_1} (F_\beta) E_{\mu_k} & \text{if } k \neq j \\ (-q) T_{\mu_{j-1}} T_{\mu_{j-2}} \cdots T_{\mu_1} (F_\beta) K_{\mu_k} + T_{\mu_j} T_{\mu_{j-1}} \cdots T_{\mu_1} (F_\beta) E_{\mu_k} & \text{if } k = j. \end{cases}$$

and

$$(76) \quad F_{\mu_k} T_{\mu_j} T_{\mu_{j-1}} \cdots T_{\mu_1} (F_\beta) = \begin{cases} T_{\mu_j} T_{\mu_{j-1}} \cdots T_{\mu_1} (F_\beta) F_{\mu_k} & \text{if } k \neq j, j+1 \\ -q^{-1} T_{\mu_{j+1}} T_{\mu_j} \cdots T_{\mu_1} (F_\beta) + q^{-1} T_{\mu_j} T_{\mu_{j-1}} \cdots T_{\mu_1} (F_\beta) F_{\mu_k} & \text{if } k = j+1 \\ q T_{\mu_j} T_{\mu_{j-1}} \cdots T_{\mu_1} (F_\beta) F_{\mu_j} & \text{if } k = j. \end{cases}$$

The first equation is clear if $k > j$ and if $k < j$ we can move E_{μ_k} past those T_ℓ for which $\ell > k$ and this reduces easily the case to $k = j$. Here it follows directly from the defining relations of the quantum group.

The other equation follows analogously. After that we get

$$(77) \quad E_{\mu_k} W_{i,j} = W_{i,j} E_{\mu_k} \text{ if } k \neq i-1$$

$$(78) \quad E_{\mu_k} W_{i,j}^a = (-q)[a] W_{i-1,j} W_{i,j}^{a-1} K_{\mu_k} + W_{i,j}^a E_{\mu_k} \text{ if } k = i-1$$

$$(79) \quad F_{\mu_k} W_{i,j} = W_{i,j} F_{\mu_k} \text{ if } k \neq i, i-1$$

$$(80) \quad F_{\mu_k} W_{i,j}^a = -q^{-1}[a] W_{i,j}^{a-1} W_{i+1,j} + q^{-a} W_{i,j}^a F_{\mu_k} \text{ if } k = i$$

$$(81) \quad F_{\mu_k} W_{i,j} = q W_{i,j} F_{\mu_k} \text{ if } k = i-1.$$

This leads to

$$(82) \quad E_{\mu_{i-1}} W_{i-1,1}^{a_{i-1,1}} W_{i-1,2}^{a_{i-1,2}} \cdots W_{i-1,j}^{a_{i-1,j}} \cdots W_{i-1,n}^{a_{i-1,n}} \cdot W_{i,1}^{a_{i,1}} W_{i,2}^{a_{i,2}} \cdots W_{i,j}^{a_{i,j}} \cdots \\ \cdots W_{i,n}^{a_{i,n}} \cdot W_{i+1,1}^{a_{i+1,1}} W_{i+1,2}^{a_{i+1,2}} \cdots W_{i+1,j}^{a_{i+1,j}} \cdots W_{i+1,n}^{a_{i+1,n}} = \\ \sum_{j=1}^n W_{i-1,1}^{a_{i-1,1}} W_{i-1,2}^{a_{i-1,2}} \cdots W_{i-1,j}^{a_{i-1,j}+1} \cdots W_{i-1,n}^{a_{i-1,n}} \cdot W_{i,1}^{a_{i,1}} W_{i,2}^{a_{i,2}} \cdots W_{i,j}^{a_{i,j}-1} \\ \cdots W_{i,n}^{a_{i,n}} \cdot W_{i+1,1}^{a_{i+1,1}} W_{i+1,2}^{a_{i+1,2}} \cdots W_{i+1,j}^{a_{i+1,j}} \cdots \\ \cdots W_{i+1,n}^{a_{i+1,n}} \cdot (-q) K_{\mu_{i-1}} [a_{i,j}] q^{a_{i-1,j}+1+\cdots+a_{i-1,n}+a_{i,j+1}-a_{i,j+1}-\cdots-a_{i,n}} \\ + W_{i-1,1}^{a_{i-1,1}} W_{i-1,2}^{a_{i-1,2}} \cdots W_{i-1,j}^{a_{i-1,j}} \cdots W_{i-1,n}^{a_{i-1,n}} \cdot W_{i,1}^{a_{i,1}} W_{i,2}^{a_{i,2}} \cdots W_{i,j}^{a_{i,j}} \\ \cdots W_{i,n}^{a_{i,n}} \cdot W_{i+1,1}^{a_{i+1,1}} W_{i+1,2}^{a_{i+1,2}} \cdots W_{i+1,j}^{a_{i+1,j}} \cdots W_{i+1,n}^{a_{i+1,n}} E_{\mu_{i-1}}.$$

Notice that the q exponents arise from the rearranging of terms (e.g. $W_{i-1,j}$) and K_{μ_k} .

There are analogous considerations for F_{μ_k} . □

The case of the Z_{ij} s is similar: First observe the very useful formula

$$(83) \quad ad(E_{\mu_1}) ad(E_{\mu_2}) ad(E_{\mu_1})(E_\beta) = 0.$$

This follows from a lengthy computation based on the Serre relations. Then observe that

$$(84) \quad E_{\mu_k} ad(E_{\mu_j}) ad(E_{\mu_{j-1}}) \cdots ad(E_{\mu_1})(E_\beta) \\ = \begin{cases} ad(E_{\mu_j}) ad(E_{\mu_{j-1}}) \cdots ad(E_{\mu_1})(E_\beta) E_{\mu_k} & \text{if } k \neq j, j+1 \\ ad(E_{\mu_{j+1}}) ad(E_{\mu_j}) \cdots ad(E_{\mu_1})(E_\beta) \\ + q^{-1} ad(E_{\mu_j}) \cdots ad(E_{\mu_1})(E_\beta) E_{\mu_k} & \text{if } k = j+1 \\ q \cdot ad(E_{\mu_j}) ad(E_{\mu_{j-1}}) \cdots ad(E_{\mu_1})(E_\beta) E_{\mu_j} & \text{if } k = j. \end{cases}$$

$$(85) \quad F_{\mu_k} \operatorname{ad}(E_{\mu_j}) \operatorname{ad}(E_{\mu_{j-1}}) \dots \operatorname{ad}(E_{\mu_1})(E_\beta) \\ = \begin{cases} \operatorname{ad}(E_{\mu_j}) \operatorname{ad}(E_{\mu_{j-1}}) \dots \operatorname{ad}(E_{\mu_1})(E_\beta) F_{\mu_k} & \text{if } k \neq j \\ \operatorname{ad}(E_{\mu_{j-1}}) \dots \operatorname{ad}(E_{\mu_1})(E_\beta) K_{\mu_k}^{-1} + \operatorname{ad}(E_{\mu_j}) \dots \operatorname{ad}(E_{\mu_1})(E_\beta) F_{\mu_k} & \text{if } k = j. \end{cases}$$

The form of the operators follow easily from this, in a way similar to the case of \mathcal{A}_q^- .

We have then proved

Proposition 3.2.2.

$$(86) \quad E_{\mu_k}^+ = \\ \sum_j D_{k,j}^o M_{k+1,j}^o (H_{k,1}^o)^{-1} \dots (H_{k,j-1}^o)^{-1} H_{k+1,1}^o \dots H_{k+1,j-1}^o \\ + (H_{k,1}^o)^{-1} \dots (H_{k,n}^o)^{-1} H_{k+1,1}^o \dots H_{k+1,n}^o M_{E_{\mu_k}},$$

$$(87) \quad F_{\mu_k}^+ =$$

$$(88) \quad \sum_j D_{k+1,j}^o M_{k,j}^o (H_{k+1,j+1}^o)^{-1} \dots (H_{k+1,n}^o)^{-1} H_{k,j+1}^o \dots H_{k,n}^o \cdot M_{K_{\mu_k}^{-1}}.$$

We use the antipode $S = S_J$ from ([7], p.34). Specifically

$$(89) \quad \forall \alpha \in \Pi : S_J(E_\alpha) = -K_\alpha^{-1} E_\alpha, \quad S_J(F_\alpha) = -F_\alpha K_\alpha, \quad S_J(K_\alpha) = K_\alpha^{-1}.$$

We shall later study dual modules. Here we recall the definition

$$(90) \quad (u^T v', v) := (v', S(u)v).$$

With this in mind, we observe that we have, modulo right actions by E_μ and F_μ ,

Corollary 3.2.3.

$$(91) \quad (-K_{\mu_k}^{-1}) E_{\mu_k}^- = \\ \sum_j q (H_{k,j}^o)^{-1} M_{k,j}^o H_{k+1,j}^o D_{k+1,j}^o H_{k+1,1}^o \dots H_{k+1,j-1}^o (H_{k,1}^o)^{-1} \dots (H_{k,j-1}^o)^{-1}$$

$$(92) \quad q \cdot F_{\mu_k}^- (-K_{\mu_k}) = \\ \sum_j M_{k+1,j}^o (H_{k+1,j}^o)^{-1} D_{k,j}^o H_{k,j}^o \cdot (H_{k+1,j+1}^o)^{-1} \dots (H_{k+1,n}^o)^{-1} H_{k,j+1}^o \dots H_{k,n}.$$

Of course, there are similar formulas for the actions of E_ν^\pm and F_ν^\pm . We omit those as they are entirely similar, easily deducible, and since, for our purposes, they do not add anything new.

4. DUALITY

4.1. The q Killing form d'après Jantzen. The q version of the Killing form was introduced by M. Rosso ([16]). Here we follow the comprehensive study offered in ([7]); a study that relies on the approaches offered in ([19]) and ([8]).

We cite:

Proposition 4.1.1 ([7] Proposition 6.12). *There exists a unique bilinear pairing $(\cdot, \cdot)_J: \mathcal{U}^{\geq 0} \times \mathcal{U}^{\leq 0} \rightarrow k$ such that for all $x, x' \in \mathcal{U}^{\geq 0}$, all $y, y' \in \mathcal{U}^{\leq 0}$, all $\mu, \nu \in \mathbf{Z}\Psi$, and all $\alpha, \beta \in \Pi$,*

$$(93) \quad (y, xx')_J = (\Delta(y), x' \otimes x)_J \quad , \quad (yy', x)_J = (y \otimes y', \Delta(x))_J,$$

$$(94) \quad (K_\mu, K_\nu)_J = q^{-(\mu, \nu)} \quad , \quad (F_\alpha, E_\beta)_J = -\delta_{\alpha, \beta} (q_\alpha - q_\alpha^{-1})^{-1}$$

$$(95) \quad (K_\mu, E_\alpha)_J = 0 \quad , \quad (F_\alpha, K_\mu)_J = 0.$$

(The form is extended to tensor products in the natural way).

Remark 4.1.2. *The form is unique for the given Δ .*

The following follows easily:

Corollary 4.1.3. *For all $x \in \mathcal{U}^{> 0}$, all $y \in \mathcal{U}^{< 0}$, all $\mu, \nu \in \mathbf{Z}\Psi$,*

$$(96) \quad (yK_\mu, xK_\nu)_J = (y, x)_J q^{-(\mu, \nu)}.$$

Proposition 4.1.4 ([7] Corollary in §8.30). *If q is not a root of unity, the form restricted to $\mathcal{U}_q^{-\mu} \times \mathcal{U}_q^\mu$ is non-degenerate for any $\mu \in \mathcal{L}^+$.*

The following also holds. It can be proved using the same argument as in ([7, Proposition 6.21]).

Proposition 4.1.5. *If q is not a root of unity,*

$$(97) \quad (K_\alpha, K_\gamma)_J = q^{-(\alpha, \gamma)}$$

extends to a non-degenerate bilinear form on $\mathcal{U}_q^0 \times \mathcal{U}_q^0$.

Proposition 4.1.6 ([7] 6.13 (3)).

$$(98) \quad (Ad(K_\alpha)x, y)_J = (x, Ad(K_\alpha^{-1})y)_J.$$

Proposition 4.1.7 ([7] Proposition 8.29). *Let $\omega = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_t}$ be a reduced representation. Then $T_{\alpha_1}(F_{\alpha_2}^b) = (T_{\alpha_1}(F_{\alpha_2}))^b$. Furthermore,*

$$(99) \quad \begin{aligned} & (T_{\alpha_1} T_{\alpha_2} \dots T_{\alpha_t}(F_{\alpha_t}^{b_t}) \dots T_{\alpha_1}(F_{\alpha_2}^{b_2}) F_{\alpha_1}^{b_1}, T_{\alpha_1} T_{\alpha_2} \dots T_{\alpha_t}(E_{\alpha_t}^{a_t}) \dots T_{\alpha_1}(E_{\alpha_2}^{a_2}) E_{\alpha_1}^{a_1})_J \\ &= \prod_i \delta_{a_i, b_i} \prod_{j=1}^t (F_{\alpha_j}^{a_j}, E_{\alpha_j}^{a_j})_J. \end{aligned}$$

Lemma 4.1.8 ([7] (4) p. 114).

$$(100) \quad (F_{\alpha_i}^{a_i}, E_{\alpha_i}^{a_i})_J = (-1)^{a_i} q_{\alpha_i}^{a_i(a_i-1)/2} [a_i]_{\alpha_i}! / (q_{\alpha_i} - q_{\alpha_i}^{-1})^{\alpha_i}.$$

According to our definitions, the following is immediate:

Corollary 4.1.9.

$$(101) \quad \begin{aligned} & \left(W_{11}^{b_{11}} W_{12}^{b_{12}} \dots W_{1n}^{b_{1n}} W_{21}^{b_{21}} \dots W_{nn}^{b_{nn}}, Z_{11}^{a_{11}} Z_{12}^{a_{12}} \dots Z_{1n}^{a_{1n}} Z_{21}^{a_{21}} \dots Z_{nn}^{a_{nn}} \right)_J \\ &= \prod_{(ij), (rs)} \delta_{(ij), (rs)} \prod_{(ij)} \left(\frac{-1}{q - q^{-1}} \right)^{a_{ij}} q^{a_i(a_i-1)/2} [a_i]! \end{aligned}$$

4.2. More formulas from Jantzen.

Definition 4.2.1 ([7] §6.14). *We here define the important operators r_α and r'_α in \mathcal{U}_q^+ :*

$$(102) \quad x \in U_\mu^+ : \Delta(x) = x \otimes 1 + \sum_{\alpha \in \Pi} r_\alpha(x) K_\alpha \otimes E_\alpha + (\text{rest}).$$

$$(103) \quad x \in U_\mu^+ : \Delta(x) = K_\mu \otimes x + \sum_{\alpha \in \Pi} E_\alpha K_{\mu-\alpha} \otimes r'_\alpha(x) + (\text{rest}).$$

Proposition 4.2.2 ([7] §6.14). *If $x \in \mathcal{U}_\mu$ then $r_\alpha(x), r'_\alpha(x) \in \mathcal{U}_{-\alpha+\mu}$. Let furthermore $x' \in \mathcal{U}_{\mu'}$. Then*

$$(104) \quad r_\alpha(1) = r'_\alpha(1) = 0 \quad ; \quad r_\alpha(E_\beta) = r'_\alpha(E_\beta) = \delta_{\alpha,\beta},$$

$$(105) \quad r'_\alpha(xx') = r'_\alpha(x)x' + q^{(\alpha,\mu)}xr'_\alpha(x'),$$

$$(106) \quad r_\alpha(xx') = q^{(\alpha,\mu')}r_\alpha(x)x' + xr_\alpha(x'),$$

$$(107) \quad (u^-F_\alpha, u^+)_J = (F_\alpha, E_\alpha)_J(u^-, r_\alpha(u^+))_J, \text{ and}$$

$$(108) \quad (F_\alpha u^-, u^+)_J = (F_\alpha, E_\alpha)_J(u^-, r'_\alpha(u^+))_J.$$

There are analogous operators r_α and r'_α in \mathcal{U}_q^- :

Definition 4.2.3 ([7] §6.15).

$$(109) \quad y \in U_{-\mu}^- : \Delta(y) = y \otimes K_\mu^{-1} + \sum_{\alpha \in \Pi} r_\alpha(y) \otimes F_\alpha K_{\mu-\alpha}^{-1} + (\text{rest}).$$

$$(110) \quad y \in U_{-\mu}^- : \Delta(y) = 1 \otimes y + \sum_{\alpha \in \Pi} F_\alpha \otimes r'_\alpha(y) K_\alpha^{-1} + (\text{rest}).$$

Proposition 4.2.4 ([7] §6.15). *If $y \in \mathcal{U}_{-\mu}^-$ then $r_\alpha(y), r'_\alpha(y) \in \mathcal{U}_{\alpha-\mu}^-$. Let furthermore $y' \in \mathcal{U}_{-\mu'}^-$. Then*

$$(111) \quad r_\alpha(1) = r'_\alpha(1) = 0 \quad ; \quad r_\alpha(F_\beta) = r'_\alpha(F_\beta) = \delta_{\alpha,\beta},$$

$$(112) \quad r_\alpha(yy') = r_\alpha(y)y' + q^{(\alpha,\mu)}yr_\alpha(y'),$$

$$(113) \quad r'_\alpha(yy') = q^{(\alpha,\mu')}r'_\alpha(y)y' + yr'_\alpha(y'),$$

$$(114) \quad \forall \alpha, \beta \in \Pi : r_\alpha \circ r'_\beta = r'_\beta \circ r_\alpha \text{ (Jantzen p. 218 l. 1)},$$

$$(115) \quad (y^-, E_\alpha y^+)_J = (F_\alpha, E_\alpha)_J(r_\alpha(y^-), y^+)_J \text{ and}$$

$$(116) \quad (y^-, y^+ E_\alpha)_J = (F_\alpha, E_\alpha)_J(r'_\alpha(y^-), y^+)_J.$$

Observe that

$$(117) \quad r_\nu(T_\nu(E_\beta)) = 0 = r'_\nu(T_\nu(F_\beta)).$$

From this we easily get the following special cases:

Lemma 4.2.5.

$$(118) \quad \begin{aligned} r_\beta W_{ij} &= \delta_{1,i} \delta_{1,j}, & r_\beta^n W_{11}^n &= [[n]]_q!, \\ r_{\mu_i} W_{k,j} &= \delta_{k,i+1} (-\gamma) W_{i,j}, & r_{\mu_i} &= 0 \text{ on } \mathcal{A}_q^+, \\ r'_{\mu_i} &= 0 \text{ on } \mathcal{A}_q^-, & r'_\beta Z_{ij} &= \delta_{1,i} \delta_{1,j}, \\ r'_\beta^n Z_{11}^n &= [[n]]_q!, & \text{and } r'_{\mu_i} Z_{k,j} &= \delta_{k,i+1} (q^{-1} \gamma) Z_{i,j}. \end{aligned}$$

Jantzen's formulas in Corollary 4.1.9 follows readily from these.

Remark 4.2.6. *The bilinear form $(\cdot, \cdot)_J$ has a singularity at $q = 1$. We will renormalize it later in a fixed PBW basis, but for now we keep it because of (115) and (116). Towards the end of his book, in §10.16, Jantzen introduces a renormalized bilinear form. This we will not use, since it, for our purposes, is more difficult to use.*

4.3. The left action of E_α in \mathcal{U}_q^- .

Proposition 4.3.1 ([7] Lemma 6.17). *With $\alpha \in \Pi$, $\mu \in \mathbb{Z}\Phi$, $y \in \mathcal{U}_{-\mu}^-$, and $x \in \mathcal{U}_\mu^+$:*

$$(119) \quad E_\alpha y - y E_\alpha = \frac{1}{q_\alpha - q_\alpha^{-1}} (K_\alpha r_\alpha(y) - r'_\alpha(y) K_\alpha^{-1})$$

$$(120) \quad x F_\alpha - F_\alpha x = \frac{1}{q_\alpha - q_\alpha^{-1}} (r_\alpha(x) K_\alpha - K_\alpha^{-1} r'_\alpha(x)).$$

(Jantzen proves this by a simple induction argument.)

From (100), (116), and (116) one easily deduces:

Corollary 4.3.2. *for elements $u_{-\mu}^- \in \mathcal{U}_{-\mu}^-$ and $u_{\mu-\alpha}^+ \in \mathcal{U}_{\mu-\alpha}^+$ we have*

$$(121) \quad (E_\alpha u_{-\mu}^- - u_{-\mu}^- E_\alpha, u_{\mu-\alpha}^+)_J = - (u_{-\mu}^-, (E_\alpha K_\alpha^{-1}) u_{\mu-\alpha}^+ - u_{\mu-\alpha}^+ (K_\alpha E_\alpha))_J.$$

5. DUALITY RECONSIDERED, ESPECIALLY THE $\mathcal{U}_q(\mathfrak{k})$ MODULES \mathcal{A}_q^\pm

We consider here non-singular pairings between a (highest weight) module and some other module. By this we mean in general a complex valued non-degenerate bilinear form taking inputs from two modules such that the second module is the dual module of the first according to the pairing. On some level, there is of course only one dual module, but they may be given in different realizations.

5.1. More on the Jantzen pairing. It is easy to see, since $r_\alpha = 0$ in \mathcal{A}_q^+ for any simple compact root α , that

Lemma 5.1.1. $\forall u_z^+ \in \mathcal{A}_q^+, u_w^- \in \mathcal{A}_q^-, u_{\mathfrak{k}}^+ \in \mathcal{U}_q^+, \text{ and } u_{\mathfrak{k}}^- \in \mathcal{U}_q^-:$

$$(122) \quad (u_w^- u_{\mathfrak{k}}^-, u_z^+ u_{\mathfrak{k}}^+)_J = (u_w^-, u_z^+)_J (u_{\mathfrak{k}}^-, u_{\mathfrak{k}}^+)_J.$$

We will use this version of the pairing because there are some very simple formulas for the duals of the operators r_α, r'_α to be studied later. However, there will also be modified pairings:

5.2. Pairings between \mathcal{A}_q^- and \mathcal{A}_q^+ .

Definition 5.2.1.

$$(123) \quad [[a]]_q = 1 + q^2 + \cdots + q^{2a-2},$$

$$(124) \quad [a]_q = q^{-a+1} + \cdots + q^{a-1}, \text{ and}$$

$$(125) \quad \{\{a\}\}_q = 1 + q^{-2} + \cdots + q^{-(2a-2)}.$$

If $\mathbf{a} = (a_{11}, a_{12}, \dots, a_{nn})$, $\mathbf{b} = (b_{11}, b_{12}, \dots, b_{nn})$, and $\mathbf{c} = (c_{11}, c_{12}, \dots, c_{nn})$ we let $[[\mathbf{c}]]! = \prod_{ij} ([[c_{ij}]]!)$, $\delta_{\mathbf{a}, \mathbf{b}} = \prod_{ij} \delta_{a_{ij}, b_{ij}}$, $|\mathbf{a}| = \sum_{ij} a_{ij}$, and $Z^{\mathbf{c}} = Z_{11}^{c_{11}} Z_{12}^{c_{12}} \cdots Z_{nn}^{c_{nn}}$.

Definition 5.2.2. *The following bilinear forms, indexed by J, K , and L , will be considered:*

$$(126) \quad (Z^{\mathbf{a}}, W^{\mathbf{b}})_J = \left(\frac{-1}{q - q^{-1}} \right)^{|\mathbf{a}|} \delta_{\mathbf{a}, \mathbf{b}} [[\mathbf{a}]]!,$$

$$(127) \quad (Z^{\mathbf{a}}, W^{\mathbf{b}})_L = \left(\frac{-1}{q - q^{-1}} \right)^{|\mathbf{a}|} \delta_{\mathbf{a}, \mathbf{b}} [\mathbf{a}]!, \text{ and}$$

$$(128) \quad (Z^{\mathbf{a}}, W^{\mathbf{b}})_K = \left(\frac{-1}{q - q^{-1}} \right)^{|\mathbf{a}|} \delta_{\mathbf{a}, \mathbf{b}} \{\{\mathbf{a}\}\}.$$

Corresponding to these forms we introduce two more families of differential operators (we include the old one for convenience):

Definition 5.2.3. $\forall i, j = 1, \dots, n$

$$(129) \quad \mathbb{D}_{ij}^o Z_{ij}^{a_{ij}} = [[a_{ij}]] Z_{ij}^{a_{ij}-1},$$

$$(130) \quad D_{ij}^o Z_{ij}^{a_{ij}} = [a_{ij}] Z_{ij}^{a_{ij}-1}, \text{ and}$$

$$(131) \quad \mathcal{D}_{ij}^o Z_{ij}^{a_{ij}} = \{\{a_{ij}\}\} Z_{ij}^{a_{ij}-1}.$$

Clearly, $\mathbb{D}_{ij}^o = H_{ij}^o D_{ij}^o$ and $\mathcal{D}_{ij}^o = (H_{ij}^o)^{-1} D_{ij}^o$.

Of course, we have formulas analogous to (65-68):

Lemma 5.2.4.

$$(132) \quad \mathbb{D}_{ij}^o M_{ij}^o - q^2 M_{ij}^o \mathbb{D}_{ij}^o = I,$$

$$(133) \quad \mathbb{D}_{ij}^o M_{ij}^o - M_{ij}^o \mathbb{D}_{ij}^o = (H_{ij}^o)^2,$$

$$(134) \quad H_{ij}^o \mathbb{D}_{ij}^o = q^{-1} \mathbb{D}_{ij}^o H_{ij}^o, \text{ and}$$

$$(135) \quad H_{ij}^o M_{ij}^o = q M_{ij}^o H_{ij}^o.$$

and

$$(136) \quad \mathcal{D}_{ij}^o M_{ij}^o - q^{-2} M_{ij}^o \mathcal{D}_{ij}^o = I,$$

$$(137) \quad \mathcal{D}_{ij}^o M_{ij}^o - M_{ij}^o \mathcal{D}_{ij}^o = (H_{ij}^o)^{-2},$$

$$(138) \quad H_{ij}^o \mathcal{D}_{ij}^o = q^{-1} \mathcal{D}_{ij}^o H_{ij}^o, \text{ and}$$

$$(139) \quad H_{ij}^o M_{ij}^o = q M_{ij}^o H_{ij}^o.$$

5.3. **Returning to the $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ modules \mathcal{A}_q^{\pm} .** We denote by \widehat{T}^X the dual operator of T according to some pairing $(\cdot, \cdot)_X$. Specifically,

$$(140) \quad \forall u, v \in \mathcal{A}_q^+ \times \mathcal{A}_q^- : (\widehat{T}^X v, u)_X = (v, Tu)_X, \text{ or}$$

$$(141) \quad (Tv, u)_X = (v, \widehat{T}^X u)_X.$$

We assume $\forall i, j \in \{1, 2, \dots, n\} : \widehat{H}_{ij}^o = H_{ij}^o$ and seek ‘‘self duality’’ in the sense of Jantzen p. 120 for the operators from Propositions 3.2.1 and 3.2.2.

Proposition 5.3.1. *There is the following equivalence:*

$$(142) \quad \forall \xi \in \Pi_c : \left. \begin{array}{l} \widehat{S}_J(E_{\xi}^-)^X = E_{\xi}^+ \\ \widehat{S}_J(F_{\xi}^-)^X = F_{\xi}^+ \\ \widehat{S}_J^{-1}(E_{\xi}^+)^X = E_{\xi}^- \\ \widehat{S}_J^{-1}(F_{\xi}^+)^X = F_{\xi}^- \end{array} \right\} \Leftrightarrow \forall i, j \in \{1, 2, \dots, n\} : \widehat{M}_{ij}^X = \kappa H_{ij}^o D_{ij}^o$$

holds, where κ is a complex constant.

Proof. We need to compare terms in (86) and (92). It follows easily, by looking at the operators involving $E_{\mu_k}^{\pm}$ that, for y representing any double index, $\widehat{D}_y^o = (H_y^o)^{-1} M_y^o q^{\alpha}$ and $\widehat{M}_y^X = H_y^o D_y^o q^{\beta}$ with $\alpha + \beta = 1$. It follows from these that $T \rightarrow \widehat{T}^X$ is an idempotent. Moreover, the equations are equivalent and are also equivalent to those arising from $F_{\mu_k}^{\pm}$.

Remark 5.3.2. *J.C. Jantzen introduces in §6.20 an altered bilinear form $\langle \cdot, \cdot \rangle$ on \mathcal{U}_q in which ad is self dual (his Proposition 6.20). This form does not restrict to $(\cdot, \cdot)_J$. It is not clear that self-duality of restrictions of ad to be desired. There is also a non-uniqueness in the sense that the co-product Δ may be altered. Furthermore, notice that*

$$(143) \quad E \rightarrow E_x = EK^x; \quad F \rightarrow F_x = K^{-x}F$$

defines an automorphism of the quantized enveloping algebra (preserves the q -Serre relations).

The following is straightforward:

Lemma 5.3.3. *If for some fixed i, j $\widehat{W}_{ij}^X = \kappa H_{ij}^o D_{ij}^o$ for some $\kappa \neq 0$ and some pairing X then*

$$(144) \quad \forall a, b \in \mathbb{N}_0 : (W_{ij}^a, Z_{ij}^b)_X = \delta_{a,b} \kappa^a [[a]]!$$

In other words, the form above is essentially the form $(\cdot, \cdot)_J$.

Notice that $\widehat{M}^L = D$. We shall see in later sections that there are difficulties with the form $(\cdot, \cdot)_J$, whereas the two other forms indexed by L and K , respectively, behave very nicely.

Remark 5.3.4. *We will also later define differential operators as duals of either left or right multiplication operators. Here we take the stance of using “flat” dualities without using antipodes. This is so because there is no natural antipode for, especially, right multiplication operators.*

5.4. **Change of basis.** Let $\mathbb{A}W^\alpha = q^{-\frac{\alpha(\alpha-1)}{2}}W^\alpha$.

$$(Z^\alpha, W^\beta)_L = (Z^\alpha, \mathbb{A}W^\beta)_J. \text{ Clearly, } \widehat{\mathbb{A}}^J = \mathbb{A}.$$

Then

$$(145) \quad \widehat{T}^L = (\widehat{\mathbb{A}T\mathbb{A}^{-1}})^J = \mathbb{A}^{-1}\widehat{T}^J\mathbb{A}.$$

Suppose we are given \widehat{T}^J in terms of sums of monomials in the operators L, M, H , then, to get \widehat{T}^L , we just need to make the replacements, in the given expression,

$$(146) \quad M \text{ by } (\mathbb{A}^{-1}M\mathbb{A}) = MH^o,$$

$$(147) \quad \text{and } D \text{ by } (\mathbb{A}^{-1}D\mathbb{A}) = (H^o)^{-1}D = \mathcal{D}.$$

Of course,

$$(148) \quad \widehat{T}^K = (\widehat{\mathbb{A}^2T\mathbb{A}^{-2}})^J = \mathbb{A}^{-2}\widehat{T}^J\mathbb{A}^2, \text{ and}$$

$$(149) \quad (\mathbb{A}^{-2}M\mathbb{A}^2) = M(H^o)^2,$$

$$(150) \quad (\mathbb{A}^{-2}D\mathbb{A}^2) = (H^o)^{-2}D.$$

Lemma 5.4.1. *The prescriptions (146 - 147) above extend to an automorphism of $\text{Weyl}_q(n, n)$. This is the change-of-basis automorphism.*

Proof: This follows immediately from (65 - 68). \square

Remark 5.4.2. *The change-of-basis map \mathbb{A} on \mathcal{A}_q^+ can also be viewed as a change-of-variable transformation, albeit in an enlarged algebra: Let*

$$(151) \quad \widetilde{\mathcal{A}}_q^+ = \mathcal{A} \times_s \mathbb{C}[H_{ij}^{\pm 1} \mid ij = 1, \dots, n].$$

Then

$$(152) \quad (q^{\frac{1}{2n}}Z_{ij}(H_{ij}^o)^{-1})^{a_{ij}} = q^{-\frac{a_{ij}(a_{ij}-1)}{2}}(Z_{ij})^{a_{ij}}q^{\frac{a_{ij}}{2n}}(H_{ij}^o)^{-a_{ij}}.$$

In a scalar module (where $\Lambda(\xi) = 0$ for all $\xi \in \Pi_c$ and $\Lambda(\beta) = \lambda \in \mathbb{R}$) one may set $H_{ij}v_\Lambda = q^{\frac{1}{2n}}v_\Lambda$ to enlarge the representation to these elements. Of course, further modifications will have to be added to our algebras for this to make sense - all operators of the form $D_{i,j}M_{i+1,j}^o$ should also be included.

This idea will be persued elsewhere.

6. THE FULL PAIRING

We are interested in studying duals of generalized Verma modules.

6.1. Highest Weights. A highest weight vector of a module of $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ has a highest weight Λ and a highest weight vector $v_\Lambda \neq \mathbf{0}$ for which

$$(153) \quad \forall i = 1, \dots, n-1 : K_{\mu_i}^{\pm 1} = q^{\pm \lambda_i^\mu} v_\Lambda, \quad K_{\nu_i}^{\pm 1} = q^{\pm \lambda_i^\nu} v_\Lambda, \quad \text{and} \quad K_\beta^{\pm 1} = q^{\pm \lambda} v_\Lambda.$$

We set $\Lambda = ((\lambda_1^\mu, \dots, \lambda_{n-1}^\mu), (\lambda_1^\nu, \dots, \lambda_{n-1}^\nu); \lambda) = (\Lambda_L, \Lambda_R, \lambda)$. We assume:

$$(154) \quad \forall i = 1, \dots, n-1 : \lambda_i^\nu, \lambda_j^\mu \in \mathbb{N}_0.$$

As a vector space, $V_\Lambda = V_{\Lambda_L} \otimes V_{\Lambda_R}$ where V_{Λ_L} and V_{Λ_R} are highest weight representations of $\mathcal{U}_q(\mathfrak{k}_L^{\mathbb{C}})$ and $\mathcal{U}_q(\mathfrak{k}_R^{\mathbb{C}})$, respectively, of highest weights $\Lambda_L = (\lambda_1^\mu, \dots, \lambda_{n-1}^\mu)$ and $\Lambda_R = (\lambda_1^\nu, \dots, \lambda_{n-1}^\nu)$, respectively. The highest weight vector can then be written as $v_\Lambda = v_{\Lambda_L} \otimes v_{\Lambda_R}$ with the stipulation that $K_\beta^{\pm 1} v_{\Lambda_L} \otimes v_{\Lambda_R} = q^{\pm \lambda} v_{\Lambda_L} \otimes v_{\Lambda_R}$.

The condition (154) is an integrality condition and gives rise to a finite-dimensional $\mathcal{U}_q(\mathfrak{k})$ module V_Λ in the following usual way:

$\mathcal{I}_L(\mathfrak{k})$ denote the left $\mathcal{U}_q(\mathfrak{k})$ invariant subspace in $\mathcal{U}_q(\mathfrak{k}^-)v_\Lambda$ generated by $\{(F_\gamma)^{\Lambda(\gamma)+1} v_\Lambda\}$ and let

$$(155) \quad V_\Lambda = \mathcal{U}_q(\mathfrak{k})v_\Lambda / \mathcal{I}_L(\mathfrak{k}).$$

The dual module V'_Λ is the highest weight module $V_{-\omega_0(\Lambda)}$. However, it is also convenient to view V'_Λ as a lowest weight module $V_{\Lambda'}^o$ characterized by a non-zero lowest weight vector $v_{\Lambda'}^o$ for which

$$(156) \quad \mathcal{U}(\mathfrak{k})_q^- v_{\Lambda'}^o = 0$$

$$(157) \quad \forall \alpha \in \Pi_c : K_\alpha^{\pm 1} v_{\Lambda'}^o = q^{\pm \Lambda'(\alpha)} v_{\Lambda'}^o$$

$$(158) \quad \forall \alpha \in \Pi_c : \Lambda'(\alpha) \in -\mathbb{N}_0.$$

The following is elementary

Lemma 6.1.1.

$$(159) \quad \Lambda' = -\Lambda.$$

Remark 6.1.2. One may consider a modified version $(\cdot, \cdot)_{mod}$ of $(\cdot, \cdot)_J$ on $\mathcal{U}_q^- \times \mathcal{U}_q^+$, where $\forall u_z^+ \in \mathcal{A}_q^+, \forall u_w^- \in \mathcal{A}_q^-, \forall u_\mathfrak{k}^\pm \in \mathcal{U}_q^\pm, \forall \lambda, \mu \in \mathfrak{L}$:

$$(160) \quad (u_w^- u_\mathfrak{k}^- K_\lambda, u_z^+ u_\mathfrak{k}^+ K_\mu)_{mod} = q^{(\Lambda(\lambda) - \Lambda(\mu))} (u_w^- u_\mathfrak{k}^-, u_z^+ u_\mathfrak{k}^+)_{mod}$$

such that the finite-dimensional modules V_Λ and $V_{\Lambda'}^o$ occur naturally as duals in this setting. Furthermore, it holds in the same generality as above that

$$(161) \quad (K_\lambda u_w^- u_\mathfrak{k}^-, u_z^+ u_\mathfrak{k}^+)_{mod} = (u_w^- u_\mathfrak{k}^-, K_\lambda^{-1} u_z^+ u_\mathfrak{k}^+)_{mod}.$$

6.2. Generalized Verma modules. Consider a finite dimensional module $V_\Lambda = V_{\Lambda_L, \Lambda_R, \lambda}$ over $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ with highest weight is defined by $\Lambda = (\Lambda_L, \Lambda_R, \lambda)$.

We extend such a module to a $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}}) \cdot \mathcal{A}_q^+$ module, by the same name, by letting \mathcal{A}_q^+ act trivially in V_Λ .

Definition 6.2.1. *The quantized generalized Verma module $\mathcal{M}(V_\Lambda)$ is given by*

$$(162) \quad \mathcal{M}(V_\Lambda) = \mathcal{U}_q(\mathfrak{g}^{\mathbb{C}}) \underset{\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})\mathcal{A}_q^+}{\otimes} V_\Lambda$$

with the natural action from the left. We denote the corresponding representation by $L_\Lambda(u)$ for $u \in \mathcal{U}_q$.

As a vector space, even as a $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$ module,

$$(163) \quad \mathcal{M}(V_\Lambda) = \mathcal{A}_q^- \otimes V_\Lambda.$$

We now consider pairings between $\mathcal{M}(V_\Lambda)$ and $\mathcal{A}_q^+ \otimes V_\Lambda^o$.

Definition 6.2.2. *Let $X = J, K, L$. $\forall u_z^+ \in \mathcal{A}_q^+, u_w^- \in \mathcal{A}_q^-, v \in V_\Lambda, v' \in V_\Lambda^o$:*

$$(164) \quad (u_w^- v, u_z^+ v')_X := (u_w^-, u_z^+)_X (v', v)_\Lambda.$$

Here, $(v', v)_\Lambda$ denotes the natural pairing between a module and its dual, and, as usual, the definition is extended by bilinearity to the full spaces.

Remark 6.2.3. *There seems to be some bias with this notation when $X = J$. However, we use the new form only when considering dual modules. Otherwise, it is the form $(\cdot, \cdot)_J$ in Proposition 4.1.1 that is considered. Furthermore, the two forms agree on $\mathcal{A}_q^- \times \mathcal{A}_q^+$. Hence we use the symbol J for both forms.*

By symmetry, the vector space $\mathcal{A}_q^+ V_\Lambda^o$ is also a left module for $\mathcal{U}_q(\mathfrak{g})$. **We will, however, consider another module structure on this space.**

We extend the notation from Subsection 5.3 as follows:

$$(165) \quad (\widehat{T}^X(u_z^+ v), u_w^- v')_X = (u_z^+ v, T(u_w^- v'))_X.$$

Recall that S denotes the antipode. We then define

$$(166) \quad \forall u \in \mathcal{U}_q : \left(\mathcal{O}_{-\omega(\Lambda)}^X(u)(u_z^+ v), u_w^- v' \right)_X = (u_z^+ v, L_\Lambda(S(u))(u_w^- v'))_X.$$

In other words,

$$(167) \quad \mathcal{O}_{\Lambda'}^X(u) = L_\Lambda(\widehat{S(u)})^X.$$

The pairings we consider result in duals of multiplication operators of the general form :

$$(168) \quad \widehat{M}^o{}^X = (H^o)^{\gamma_X} D^o.$$

Notice that $\gamma_J = 1$, $\gamma_L = 0$, and $\gamma_K = -1$.

6.3. Relation to holomorphically induced modules. Let us agree to write

$$(169) \quad \mathcal{U}_q = \mathcal{A}_q^+ \mathcal{U}_q(\mathfrak{k}) \mathcal{A}_q^-,$$

and consider functions $f : \mathcal{U}_q \rightarrow V'_\Lambda$, where V'_Λ is a finite dimensional $\mathcal{U}_q(\mathfrak{k})$ module. The subspace $\mathcal{H}(V'_\Lambda)$ of such functions which furthermore satisfy

$$(170) \quad \forall u \in \mathcal{U}_q, \forall u_{\mathfrak{k}} \in \mathcal{U}_q(\mathfrak{k}) : f(uu_{\mathfrak{k}}) = S(u_{\mathfrak{k}})f(u)$$

$$(171) \quad \forall u \in \mathcal{U}_q, \forall u_w \in \mathcal{A}_q^- : f(uu_w) = 0$$

is invariant under left action, and this module is equivalent to our dual module. The second condition can be interpreted as saying that the function should be annihilated by all anti-holomorphic (quantized) vector fields.

7. THE ACTIONS IN THE MODULE AND ITS DUAL

We consider $\mathcal{M}(V_\Lambda)$ and its dual. We only consider E_β , but a similar approach will work for any E_γ , $\gamma \in \Pi_c$.

7.1. Technical material. For use in the computation of the dual representation, we need to analyze in greater detail the commutator between $Z_{11} = E_\beta$ and an element of $\mathcal{U}_q^+(\mathfrak{k}_L^{\mathbb{C}})$:

We use (cf. (10))

$$(172) \quad \omega_0^L = s_{\mu_n} s_{\mu_{n-1}} s_{\mu_n} s_{\mu_{n-2}} s_{\mu_{n-1}} s_{\mu_n} \cdots \cdots s_{\mu_1} s_{\mu_2} \cdots s_{\mu_n}.$$

This leads to a PBW basis based on the ordered elements

$$(173) \quad E_{\mu_n}, (T_{\mu_n}(E_{\mu_{n-1}})), \dots, (T_{(\omega_0^L s_{\mu_n})}(E_{\mu_n})).$$

Let $u_L^+ \in \mathcal{U}_q^+(\mathfrak{k}_L^{\mathbb{C}})$. We are interested in terms of the form $u_L^+ E_\beta$, where we want to move E_β to the left.

In a straightforward manner, using Lemma 2.2.1 repeatedly, the right most element in our basis can be seen to be E_{μ_1} . Indeed, the right hand tail of the basis can is

$$(174) \quad T_{\mu_n} T_{\mu_{n-1}} \cdots T_{\mu_2}(E_{\mu_1}), T_{\mu_{n-1}} \cdots T_{\mu_2}(E_{\mu_1}), \dots, T_{\mu_2}(E_{\mu_1}), E_{\mu_1}.$$

We begin our computation by observing the equation

$$(175) \quad (E_{\mu_1})^k E_\beta = [k] T_{\mu_1}(E_\beta) E_{\mu_1}^{k-1} + q^{-k} E_\beta E_{\mu_1}^k.$$

Since $T_{\mu_i}(E_\beta) = E_\beta$ for $i \geq 2$, it follows that

$$(176) \quad T_{\mu_2}(E_{\mu_1}) E_\beta = T_{\mu_2} T_{\mu_1}(E_\beta) + q^{-1} E_\beta T_{\mu_2}(E_{\mu_1}).$$

Set $\check{X}_i = T_{\mu_i} T_{\mu_{i-1}} \cdots T_{\mu_2}(E_{\mu_1})$ for $i \geq 2$, $\check{X}_1 = E_{\mu_1}$, and $\check{X}_0 = 1$. Using the Serre relations and using that the operators T_α are automorphisms, one easily obtains, cf. a similar computation below,

$$(177) \quad \check{X}_i^{a_i} E_\beta = [a_i] Z_{i+1,1} \check{X}_i^{a_i-1} + q^{-k} E_\beta \check{X}_i^{a_i}.$$

From the Serre-relations we get (as in (84)):

$$(178) \quad E_{\mu_k} Z_{a,1} = \begin{cases} Z_{a+1,1} + qZ_{a,1}E_{\mu_k} & \text{If } a = k \\ q^{-1}Z_{a,1}E_{\mu_k} & \text{If } a = k + 1 \\ Z_{a,1}E_{\mu_k} & \text{If } a \neq k, k + 1 \end{cases}.$$

From this it follows that

$$(179) \quad \begin{aligned} \check{X}_k Z_{k+1,1} &= T_{\mu_k} \dots T_{\mu_2}(E_{\mu_1})T_{\mu_k} \dots T_{\mu_2}(T_{\mu_1}(E_\beta)) \\ &= T_{\mu_k} \dots T_{\mu_2}(E_{\mu_1}T_{\mu_1}(E_\beta)) = qT_{\mu_k} \dots T_{\mu_2}(T_{\mu_1}(E_\beta)E_{\mu_1}) \\ &= qZ_{k+1,1}\check{X}_k. \end{aligned}$$

More generally then,

$$(180) \quad \check{X}_\ell^k Z_{1,1} = [k]Z_{\ell+1,1}\check{X}_\ell^{k-1} + q^{-k}Z_{1,1}X_\ell^k.$$

Similarly, we get ($i \geq 1$)

$$(181) \quad \check{X}_{i+1,1}Z_{i+1,1} = (E_{\mu_{i+1}}\check{X}_{i,1} - q^{-1}\check{X}_{i,1}E_{\mu_{i+1}})Z_{i+1,1}$$

$$(182) \quad = (q - q^{-1})Z_{i+2,1}\check{X}_{i,1} + Z_{i+1,1}\check{X}_{i+1,1}.$$

There are many more formulas that can be derived in the same manner:

$$(183) \quad \check{X}_k Z_{k+1} = qZ_{k+1,1}\check{X}_k, \quad \check{X}_k Z_{k+2,1} = Z_{k+2,1}\check{X}_k, \quad \check{X}_1\check{X}_2 = q^{-1}\check{X}_2\check{X}_1,$$

$$(184) \quad \check{X}_2^k Z_{2,1} = (q - q^{-1})Z_{3,1}\check{X}_2^{k-1}\check{X}_1 + Z_{2,1}\check{X}_2^k, \quad \text{and}$$

$$(185) \quad \check{X}_2^k Z_{1,1} = Z_{3,1}\check{X}_2^{k-1} + q^{-k}Z_{1,1}\check{X}_2^k,$$

More generally, let $i \geq k \geq 1$ and $\ell \geq 1$. Then

$$(186) \quad \check{X}_{k+\ell,1}Z_{i+1,1} = (q - q^{-1})Z_{k+\ell+1,1}\check{X}_{k,1} + Z_{i+1,1}\check{X}_{k+\ell,1}.$$

It follows that, setting $X_1 := E_{\mu_1}$,

$$(187) \quad \begin{aligned} (\check{X}_k^{a_k} \dots \check{X}_i^{a_i} \dots \check{X}_1^{a_1})Z_{1,1} &= \sum_{i=1}^k q^{-a_1 - \dots - a_{i-1}} Z_{i+1,1}[a_i](\check{X}_k^{a_k} \dots \check{X}_i^{a_i-1} \dots \check{X}_1^{a_1}) \\ &+ q^{-\sum_i a_i} Z_{1,1}(\check{X}_k^{a_k} \dots \check{X}_i^{a_i} \dots \check{X}_1^{a_1}) + (q - q^{-1}) \cdot (\star\star\star) \end{aligned}$$

Above, we interpret $q^{-a_1 - \dots - a_{i-1}}$ as 1 when $i = 1$. The expression $(\star\star\star)$ in (187) is well behaved under $q \rightarrow 1$ and each monomial term in it is of the same total degree $1 + \sum_{i=1}^k a_i$ with exactly one factor of the form $Z_{r,1}$ for some $r = 2, \dots, n$. For use in Corollary 7.2.5 we observe that it thus will be unaffected by the change of basis.

We let \check{Y}_j denote the analogous terms in $\mathcal{U}_q^+(\mathfrak{k}_R^{\mathbb{C}})$.

We have in particular obtained a description of, to what extent $\text{ad}(E_\beta K_\beta^{-1})$, acting on $\mathcal{U}_q(\mathfrak{k}^{\mathbb{C}})$, can yield a component in \mathcal{A}_q^+ . This will be useful later.

Corollary 7.1.1. *If $E_\beta \text{Ad}(K_\beta^{-1})(\check{X}_\nu^{\mathfrak{c}} \check{Y}_\mu^{\mathfrak{d}}) - \check{X}_\nu^{\mathfrak{c}} \check{Y}_\mu^{\mathfrak{d}} E_\beta \in \mathcal{A}_q^+$ then $\check{X}_\nu^{\mathfrak{c}} \check{Y}_\mu^{\mathfrak{d}} = \check{X}_i \check{Y}_j$ for some $i \geq 0$ and some $j \geq 0$.*

Another easy result, which we will need later, is

Corollary 7.1.2.

$$(188) \quad \forall i = 1, \dots, n-1 : X_i Z_{11} = Z_{i+1,1} + q^{-1} Z_{11} X_i$$

$$(189) \quad \forall j = 1, \dots, n-1 : Y_j Z_{11} = Z_{1,j+1} + q^{-1} Z_{11} Y_j$$

$$(190) \quad \forall i, j = 1, \dots, n-1 : X_i Y_j Z_{11} = Z_{i+1,j+1} + q^{-1} Z_{i+1,1} Y_j$$

$$(191) \quad + q^{-1} Z_{1,j+1} X_i + q^{-2} Z_{11} X_i Y_j.$$

7.2. E_β acting in $\mathcal{U}_q^-, \mathcal{M}(V_\Lambda)$, and $\mathcal{M}'(V_\Lambda)$. First we introduce PBW bases $X_L^{\mathbf{a}} \in \mathcal{U}_q^-(\mathfrak{k}_L^{\mathbb{C}})$ and $Y_R^{\mathbf{b}} \in \mathcal{U}_q^-(\mathfrak{k}_R^{\mathbb{C}})$ by the same recipe as above. The symbol \mathbf{a} , and similarly for the others, as usual stands for an $n-1$ tuple of non-negative integers. We will often denote the $n-1$ tuple with all zeros as 0 and the corresponding term $X_\nu^{\mathbf{a}}$ will simply be denoted X_ν^0 - which is a complicated, but convenient, way of writing 1.

We maintain the identification $E_\beta = Z_{11}$. To compute the dual action of left multiplication by E_β in $\mathcal{U}_q^-(\mathfrak{g}^{\mathbb{C}})$, we need to compute \widehat{T}^J for $T = -K_\beta^{-1} L M_{Z_{11}}$. Here we use the form $(\cdot, \cdot)_J$ from Proposition 4.1.1 in combination with (121). First we determine $[E_\beta, g(w)u_{\mathfrak{k}}^-] = [E_\beta, g(w)]u_{\mathfrak{k}}^-$ for $g(w) \in \mathcal{A}_q^-$. It turns out to contain all the needed information.

Set

$$(192) \quad [E_\beta, g(w)] = \sum_{\mathbf{a}, \mathbf{b}, k} f_{\mathbf{a}, \mathbf{b}, k}^g X_L^{\mathbf{a}} Y_R^{\mathbf{b}} (K_\beta^k) p_{\mathbf{a}, \mathbf{b}, k}$$

where $p_{\mathbf{a}, \mathbf{b}, k} \in \mathcal{U}_q^0(\mathfrak{k}_L^{\mathbb{C}} \oplus \mathfrak{k}_R^{\mathbb{C}})$, and $f_{\mathbf{a}, \mathbf{b}, k}^g \in \mathcal{A}_q^-$. The summation is over all multiindices \mathbf{a}, \mathbf{b} and all integers k (though it is clear that only $k = -1, 0, 1$ will give rise to something non-zero). It will turn out below that if some $p_{\mathbf{a}, \mathbf{b}, k} \neq 0$ then, up to multiplication by a non-zero constant, $p_{\mathbf{a}, \mathbf{b}, k} = K_\beta^\ell$ for some integer $\ell = \ell(\mathbf{a}, \mathbf{b}, k)$. We make this choice and thereby remove the ambiguity in (192). Then, noting that r_β and r'_β map into \mathcal{U}_q^- , and noting Corollary (4.1.3), we have for all $\check{p}_{\mathbf{c}, \mathbf{d}} \in \mathcal{U}_q^0(\mathfrak{k}_L^{\mathbb{C}} \oplus \mathfrak{k}_R^{\mathbb{C}})$, and $\psi_{\mathbf{c}, \mathbf{d}}(z) \in \mathcal{A}_q^+$:

$$(193) \quad ([E_\beta, g(w)], \psi_{\mathbf{c}, \mathbf{d}}(z) \check{X}_L^{\mathbf{c}} \check{Y}_R^{\mathbf{d}} \check{p}_{\mathbf{c}, \mathbf{d}}) =$$

$$(194) \quad \frac{1}{\gamma} \left(K_\beta r_\beta(g(w)) - r'_\beta(g(w)) K_\beta^{-1}, \psi_{\mathbf{c}, \mathbf{d}}(z) \check{X}_L^{\mathbf{c}} \check{Y}_R^{\mathbf{d}} \check{p}_{\mathbf{c}, \mathbf{d}} \right) =$$

$$(195) \quad \frac{1}{\gamma(E_\beta, F_\beta)_J} \left(g(w), E_\beta \text{Ad}(K_\beta^{-1})(\psi_{\mathbf{c}, \mathbf{d}}(z) \check{X}_L^{\mathbf{c}} \check{Y}_R^{\mathbf{d}}) (K_\beta, \check{p}_{\mathbf{c}, \mathbf{d}}) \right)$$

$$(196) \quad - \frac{1}{\gamma(E_\beta, F_\beta)_J} \left(g(w), \psi_{\mathbf{c}, \mathbf{d}}(z) \check{X}_L^{\mathbf{c}} \check{Y}_R^{\mathbf{d}} E_\beta \right) (K_\beta^{-1}, \check{p}_{\mathbf{c}, \mathbf{d}}).$$

In (196) it follows from Corollary 7.1.1 that for this to be non-zero, $X_R^c \check{Y}_L^d = \check{X}_i \check{Y}_j$ for some $i, j \geq 0$. Using the same Corollary, we obtain

$$(197) \quad ([E_\beta, g(w)], \psi_{\mathbf{c}, \mathbf{d}}(z) \check{X}_\nu^c \check{Y}_\mu^d \check{p}_{\mathbf{c}, \mathbf{d}}) =$$

$$(198) \quad -\delta_{(\mathbf{c}, \mathbf{d}), (0, 0)} \left(g(w), E_\beta \text{Ad}(K_\beta^{-1})(\psi_{0,0})(z) \right) + \\ \delta_{(\mathbf{c}, \mathbf{d}), (0, 0)} (g(w), (\psi_{0,0})(z) E_\beta) + \sum_{i+j>0} \delta_{(\mathbf{c}, \mathbf{d}), (e_i, e_j)} (g(w), \psi_{ij} Z_{i+1, j+1}) =$$

$$(199) \quad -\delta_{(\mathbf{c}, \mathbf{d}), (0, 0)} \left(g(w), (Z_{11} M) \text{Ad}(K_\beta^{-1})(\psi_{0,0})(z) \right) (K_\beta, \check{p}_{0,0}) +$$

$$(200) \quad \delta_{(\mathbf{c}, \mathbf{d}), (0, 0)} (g(w), (M_{Z_{11}}(\psi_{0,0}))) (K_\beta^{-1}, \check{p}_{0,0}) +$$

$$(201) \quad \sum_{i+j>0} \delta_{(\mathbf{c}, \mathbf{d}), (e_i, e_j)} (g(w), M_{Z_{i+1, j+1}} \psi_{ij}) (K_\beta^{-1}, \check{p}_{i, j}).$$

On the other hand,

$$(202) \quad ([E_\beta, g(w)], \psi_{\mathbf{c}, \mathbf{d}}(z) \check{X}_\nu^c \check{Y}_\mu^d \check{p}_{\mathbf{c}, \mathbf{d}}) =$$

$$(203) \quad \left(\sum_{\mathbf{a}, \mathbf{b}, k} f_{\mathbf{a}, \mathbf{b}, k}^g X_\nu^{\mathbf{a}} Y_\mu^{\mathbf{b}} (K_\beta^k) p_{\mathbf{a}, \mathbf{b}, k}, \psi_{\mathbf{c}, \mathbf{d}}(z) \check{X}_\nu^c \check{Y}_\mu^d \check{p}_{\mathbf{c}, \mathbf{d}} \right),$$

where clearly $(\mathbf{a}, \mathbf{b}) = (\mathbf{c}, \mathbf{d})$.

In (199-201) we may introduce the duals of the left and right multiplication operators, whereby one can compare directly to (203). These duals are here denoted $O_1, O_2, O_{ij} : \mathcal{A}_q^- \rightarrow \mathcal{A}_q^-$.

Using the non-degeneracy of the form, we then reach:

Theorem 7.2.1.

$$(204) \quad [E_\beta, g(w)] = O_1(g) K_\beta + O_2(g) K_\beta^{-1} \\ + \sum_{i=1}^{n-1} O_{i1}(g) \frac{X_i}{(X_i, \check{X}_i)_J} K_\beta^{-1} + \sum_{j=1}^{n-1} O_{1j}(g) \frac{Y_j}{(Y_j, \check{Y}_j)_J} K_\beta^{-1} \\ + \sum_{i, j > 0} O_{ij}(g) \frac{X_i Y_j}{(X_i, \check{X}_i)_J (Y_j, \check{Y}_j)_J} K_\beta^{-1}, \text{ where}$$

$$(205) \quad \forall g, \forall \psi : (O_1(g), \psi)_J = -(g, Z_{11} \text{Ad}(K_\beta^{-1})(\psi))_J,$$

$$(206) \quad \forall g, \forall \psi : (O_2(g), \psi)_J = (g, \psi_{00} Z_{11})_J,$$

$$(207) \quad \forall g, \forall i, j; \forall \psi : (O_{ij}(g), \psi)_J = (g, \psi_{Z_{i+1, j+1}})_J.$$

Recall that that $\frac{1}{(X_i, \check{X}_i)_J} = \frac{1}{(Y_j, \check{Y}_j)_J} = -\gamma$.

Corollary 7.2.2.

$$(208) \quad \begin{aligned} \mathcal{O}_{\Lambda'}^J(E_\beta) &= (z_{11}M) \otimes 1 - M_{Z_{11}} \text{Ad}(K_\beta) \otimes K_\beta^2 \\ &+ \sum_{i>0} \gamma q^{-1} M_{Z_{i+1,1}} \text{Ad}(K_\beta) \otimes X_i^T K_\beta^2 + \sum_{j>0} \gamma q^{-1} M_{Z_{j+1,1}} \text{Ad}(K_\beta) \otimes Y_j^T K_\beta^2 \\ &- \sum_{i,j>0} \gamma^2 q^{-2} M_{Z_{i+1,j+1}} \text{Ad}(K_\beta) \otimes X_i^T Y_j^T K_\beta^2, \end{aligned}$$

where the operators X_i^T and Y_j^T are the dual operators to X_i, Y_j as acting in V_Λ .

We now address the issues of rescaling to avoid the singularity of the form at $q = 1$.

We let $\mathbb{B}(W^{\mathbf{b}}) = (-\gamma)^{|\mathbf{b}|} W^{\mathbf{b}}$, but do not rescale in V_Λ or its dual.

Definition 7.2.3.

$$(209) \quad \forall u \in \mathcal{U}_q : \mathcal{O}_{\Lambda'}^{\mathbb{B},J}(u) = \mathbb{B}^{-1} \mathcal{O}_{\Lambda'}^J(u) \mathbb{B}.$$

Corollary 7.2.4.

$$(210) \quad \mathbb{B}^{-1} \mathcal{O}_{\Lambda'}^J(F_\beta) \mathbb{B} = - \left(\text{Ad}(K_\beta^{-1}) \right) \circ r'_\beta.$$

This has a well-defined limit at $q = 1$.

Corollary 7.2.5.

$$(211) \quad \mathcal{O}_{\Lambda'}^{\mathbb{B},J}(E_\beta) = -\frac{1}{\gamma} \left((z_{11}M) \otimes 1 - M_{Z_{11}} \text{Ad}(K_\beta) \otimes K_\beta^2 \right)$$

$$(212) \quad - \sum_{i>0} q^{-1} M_{Z_{i1}} \text{Ad}(K_\beta) \otimes X_i^T K_\beta^2 - \sum_{j>0} q^{-1} M_{Z_{j1}} \text{Ad}(K_\beta) \otimes Y_j^T K_\beta^2$$

$$(213) \quad + \sum_{i,j>0} \gamma q^{-2} M_{Z_{ij}} \text{Ad}(K_\beta) \otimes X_i^T Y_j^T K_\beta^2.$$

7.3. Comparison to the classical result. The following is well known, see e.g. ([3]): If $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in su(n, n)^\mathbb{C}$, the infinitesimal action is given as

$$(214) \quad \begin{aligned} dU_\pi(X)f(z) &= d\pi \left(\begin{array}{cc} X_1 - zX_3 & 0 \\ 0 & X_3z + X_4 \end{array} \right) f(z) \\ &- (\delta(X_1z + X_2 - zX_3z - zX_4)f)(z). \end{aligned}$$

Here, $d\pi$ is the dual of the finite-dimensional representation of $\mathcal{U}_q(\mathfrak{k}^\mathbb{C})$ of highest weight $\Lambda = (\Lambda_L, \Lambda_R, \lambda)$. The version of $su(n, n)^\mathbb{C}$ we use is the one based on the Hermitian form $H_\beta = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. Finally, $\delta(Y)$ denotes the directional derivative $\sum_{ij} Y_{ij} \frac{\partial}{\partial Z_{ij}}$.

We now cite a special case of a result which will be proved in §8.1.1:

Lemma 7.3.1.

$$(215) \quad M_{Z_\beta} = (H_{n,1}^0 \cdots H_{2,1}^0 H_{1,2}^0 \cdots H_{1,n}^0) M_{11}^0 + \\ \gamma \sum_{c=2, d=2}^{n,n} P_{c,d}(H^0) \left(D_{c,d}^o M_{c,1}^o M_{1,d}^o \right) + \gamma^2 T_2$$

for monomials $P_{c,d}(H^0)$ and some term T_2 proportional to at least γ^2 .

It follows easily that

$$(216) \quad \mathcal{O}_{\Lambda'}^{\mathbb{B},J}(E_\beta)(Z^{\mathbf{a}}) = -\frac{1}{\gamma} \left(1 - q^{2(a_{11}+a_{21}+\cdots+a_{n1}+a_{12}+\cdots+a_{1n}-2\lambda)} \right) Z_{11} Z^{\mathbf{a}}$$

$$(217) \quad \sum_{i=2, j=2}^{n,n} q^{\alpha(\mathbf{a}, i, j, \lambda)} D_{i,j}^o M_{i,1}^o M_{1,j}^o(Z^{\mathbf{a}}) - \sum_{i>0} q^{-1} M_{Z_{i1}} \text{Ad}(K_\beta) \otimes X_i^T K_\beta^2 \\ - \sum_{j>0} q^{-1} M_{Z_{j1}} \text{Ad}(K_\beta) \otimes Y_j^T K_\beta^2 + \gamma T_2.$$

If we let $Z_1^{\mathbf{a}}$ denote the limit as $q \rightarrow 1$ then

$$(218) \quad \lim_{q \rightarrow 1} \frac{1}{\gamma} \left(1 - q^{2(a_{11}+a_{21}+\cdots+a_{n1}+a_{12}+\cdots+a_{1n}-2\lambda)} \right) Z_{11} Z^{\mathbf{a}} Z^{\mathbf{a}} = \\ (-Z_{11}^2 \frac{\partial}{\partial Z_{11}} - Z_{11} Z_{21} \frac{\partial}{\partial Z_{21}} - \cdots - Z_{11} Z_{n1} \frac{\partial}{\partial Z_{n1}})$$

$$(219) \quad -Z_{11} Z_{12} \frac{\partial}{\partial Z_{12}} - \cdots - Z_{11} Z_{1n} \frac{\partial}{\partial Z_{1n}} + Z_{11} \lambda \cdot I) Z_1^{\mathbf{a}}.$$

Corollary 7.3.2.

$$(220) \quad \lim_{q \rightarrow 1} \mathcal{O}_{\Lambda'}^{\mathbb{B},J}(E_\beta) = \sum_{i,j=1}^n Z_{i1} Z_{ij} \frac{\partial}{\partial Z_{ij}} - Z_{11} \lambda \cdot I$$

$$(221) \quad - \sum_{i>0} Z_{i1} \otimes X_i^T - \sum_{j>0} Z_{j1} \otimes Y_j^T.$$

Observation 7.3.3. *If we agree to write our matrix Z representing the elements Z_{ij} as $\sum_{ij} Z_{ij} E_{n+1-i,j}$, the matrix W representing the W_{ij} as $\sum_{ij} W_{ij} E_{i,n+1-j}$ and $X_3 = E_{1,n}$, then we get that the formula in Corollary 7.3.2 is the same as (214): Recall that we work with the dual representation on the \mathfrak{k} level. Specifically, $X_i = E_{n,n-i}$ and $Y_j = -E_{j+1,1}$ are exactly the correct expressions, bearing in mind the way the elements W_{ij} are defined, cf. (16) and (39). Also notice that*

$$(222) \quad -Z_{11} \lambda = (Z_{11}(-\Lambda(\beta)),$$

as must be the case in the dual module.

8. DIFFERENTIAL OPERATORS

8.1. Multiplication operators and their duals. We now introduce the fundamental multiplication operators. Together with their duals they form the foundation of any reasonable algebra of differential operators. Here we will be interested in examining how the different pairings may lead to different algebras. It should be noted that we use the “bare duality” to define differential operators. By this we mean that we do not use the antipode from any Hopf algebra that may be otherwise naturally affiliated with the situation.

Definition 8.1.1. *We define linear operators*

$$(223) \quad ({}_{W_{ij}}M), M_{W_{ij}} : \mathcal{U}_q^- = \mathcal{A}_q^- \mathcal{U}_q^-(\mathfrak{k}^{\mathbb{C}}) \rightarrow \mathcal{U}_q^- \text{ by}$$

$$(224) \quad \mathcal{A}_q^- \mathcal{U}_q^-(\mathfrak{k}^{\mathbb{C}}) \ni u_W u_{\mathfrak{k}^{\mathbb{C}}}^- \xrightarrow{M_{W_{ij}}} u_W W_{i,j} u_{\mathfrak{k}^{\mathbb{C}}}^-$$

$$(225) \quad \mathcal{A}_q^- \mathcal{U}_q^-(\mathfrak{k}^{\mathbb{C}}) \ni u_W u_{\mathfrak{k}^{\mathbb{C}}}^- \xrightarrow{({}_{W_{ij}}M)} W_{i,j} u_W u_{\mathfrak{k}^{\mathbb{C}}}^-.$$

Likewise, we define linear operators $({}_{Z_{ij}}M), M_{Z_{ij}} : \mathcal{U}_q^+ = \mathcal{A}_q^+ \mathcal{U}_q^+(\mathfrak{k}^{\mathbb{C}}) \rightarrow \mathcal{U}_q^+$ by

$$(226) \quad \mathcal{A}_q^+ \mathcal{U}_q^+(\mathfrak{k}^{\mathbb{C}}) \ni u_Z u_{\mathfrak{k}^{\mathbb{C}}}^+ \xrightarrow{M_{Z_{ij}}} u_Z Z_{i,j} u_{\mathfrak{k}^{\mathbb{C}}}^+,$$

$$(227) \quad \mathcal{A}_q^+ \mathcal{U}_q^+(\mathfrak{k}^{\mathbb{C}}) \ni u_Z u_{\mathfrak{k}^{\mathbb{C}}}^+ \xrightarrow{({}_{Z_{ij}}M)} Z_{i,j} u_Z u_{\mathfrak{k}^{\mathbb{C}}}^+.$$

Recall (126 - 128).

We define the linear operator $\widehat{M}_{W_{i,j}}^X$ acting on $\mathcal{A}_q^+ \mathcal{U}_q^+(\mathfrak{k}^{\mathbb{C}})$ by

$$(228) \quad (u_W u_{\mathfrak{k}^{\mathbb{C}}}^-, \widehat{M}_{W_{i,j}}^X (u_Z u_{\mathfrak{k}^{\mathbb{C}}}^+))_X = (M_{W_{i,j}}(u_W u_{\mathfrak{k}^{\mathbb{C}}}^-), u_Z u_{\mathfrak{k}^{\mathbb{C}}}^+)_X$$

$$(229) \quad = ((u_W W_{i,j} u_{\mathfrak{k}^{\mathbb{C}}}^-), u_Z u_{\mathfrak{k}^{\mathbb{C}}}^+)_X.$$

The linear operator $\widehat{({}_{W_{ij}}M)}^X$ is defined analogously.

We also use the notation

$$(230) \quad \frac{\partial}{\partial Z_{ij} X} = \widehat{M}_{W_{i,j}}^X \quad \text{and} \quad \frac{\partial}{X \partial Z_{ij}} = \widehat{({}_{W_{ij}}M)}^X.$$

Similarly, of course, for all the others.

From the behavior of the bilinear form, it follows that

$$(231) \quad \frac{\partial}{\partial Z_{ij} X} (Z^a u_{\mathfrak{k}^{\mathbb{C}}}^+) = \left(\frac{\partial}{\partial Z_{ij} X} Z^a \right) u_{\mathfrak{k}^{\mathbb{C}}}^+$$

$$(232) \quad \frac{\partial}{X \partial Z_{ij}} (Z^a u_{\mathfrak{k}^{\mathbb{C}}}^+) = \left(\frac{\partial}{X \partial Z_{ij}} Z^a \right) u_{\mathfrak{k}^{\mathbb{C}}}^+.$$

8.2. Four algebras of differential operators. On the way to defining differential operators on \mathcal{A}_q^+ we make some preliminary definitions:

Definition 8.2.1. *Set*

$$(233) \quad \mathcal{D}(\partial_{X,Z}M) = \mathbb{C} \left(\left(\frac{\partial}{\partial Z_{ij}} \right)_X, Z_{k,\ell}M \mid i, j, k, \ell \in \{1, 2, \dots, n\} \right);$$

the algebra of quantized right-left differential operators.

Similarly,

$$(234) \quad \mathcal{D}(\partial_X, M_Z) := \mathbb{C} \left(\left(\frac{\partial}{\partial Z_{ij}} \right)_X, M_{Z_{k,\ell}} \mid i, j, k, \ell \in \{1, 2, \dots, n\} \right)$$

$$(235) \quad \mathcal{D}({}_X\partial, {}_ZM) := \mathbb{C} \left(\left(\frac{\partial}{\partial Z_{ij}} \right)_X, Z_{k,\ell}M \mid i, j, k, \ell \in \{1, 2, \dots, n\} \right)$$

$$(236) \quad \mathcal{D}({}_X\partial, M_Z) := \mathbb{C} \left(\left(\frac{\partial}{\partial Z_{ij}} \right)_X, M_{Z_{k,\ell}} \mid i, j, k, \ell \in \{1, 2, \dots, n\} \right)$$

with analogous names.

8.3. Explicit formulas for the left multiplication operators. We will now compute $(W_{i,j}M)$ and $M_{W_{i,j}}$ explicitly. We do so by using (52 - 55) repeatedly. Notice that \mathcal{A}_q^+ has the same relations as \mathcal{A}_q^- , so that we by the same computations also compute $(Z_{i,j}M)$ and $M_{Z_{i,j}}$.

Specifically, we wish to expand $W_{ij}W_{11}^{a_{11}} \dots W_{xy}^{a_{xy}} \dots W_{ij}^{a_{ij}}$ in our basis (45) (cf. Proposition 2.2.6) by using these formulas. Unless $i = 1$ or $j = 1$ - in which case we can regroup using only the equations (52 - 54) - we have to use (55), and hence (57), already at the position (1, 1). Let us be more general and say that we have reached a position (x, y) where we use (57) to make the replacement $W_{ij}W_{xy}^{a_{xy}} \rightarrow W_{ij}W_{xy}^{a_{xy}}W_{ij} + (q - q^{-1})q^{a_{xy}-1}[a_{xy}]_q W_{xy}^{a_{xy}-1}W_{xj}W_{iy}$. Let us focus on the second term: W_{xj} is already in its right row and can be placed in its correct position using (52). W_{iy} however, may be in a wrong row and, if so, to bring it into its correct position we will have to use (55) again. This means that we have to keep track of the number of times we use either of the equations (52, 53), and (55). We omit the details of this cumbersome bookkeeping. The result we obtain is:

Lemma 8.3.1. *Let i, j be given. Let $r \in \mathbb{N}$, and let*

$$(237) \quad (\underline{a}, \underline{b}) = ((a_1, a_2, \dots, a_r), (b_1, b_2, \dots, b_r)) \in \mathbb{N}^r \times \mathbb{N}^r.$$

We say that $(\underline{a}, \underline{b})$ is a NW-partition of (i, j) if

$$(238) \quad a_0 = 1 \leq a_1 < a_2 < \dots < a_r < a_{r+1} := i \text{ and}$$

$$(239) \quad b_0 = 1 \leq b_1 < b_2 < \dots < b_r < b_{r+1} := j,$$

and we let \mathcal{P}_r^{NW} denote the set of all such. In the following formulas we use the convention that $(\prod_{j \in \emptyset} F_j) = 1$. The following formula holds:

$$(240) \quad \begin{aligned} (w_{i,j}M) &= M_{i,j}^o (H_{i,1}^o \cdots H_{i,j-1}^o) (H_{i-1,j}^o \cdots H_{1,j}^o) + \\ &\gamma^r \cdot \sum_{r \in \mathbb{N}} \sum_{(\underline{a}, \underline{b}) \in \mathcal{P}_r^{NW}} \prod_{k=1}^{r+1} (\prod_{y \in]b_{r-k+1}, b_{r-k+2}[} H_{a_k, y}^o) (\prod_{x \in]a_{k-1}, a_k[} H_{x, b_{r-k+2}}^o) \cdot \\ &\quad M_{a_{r+1}, b_1}^o \cdot \prod_{k=1}^r (H_{a_k, b_{r-k+1}}^o D_{a_k, b_{r-k+1}}^o M_{a_k, b_{r-k+2}}^o). \end{aligned}$$

We also set

$$(241) \quad B_r^{NW}(\underline{a}, \underline{b}) = M_{a_{r+1}, b_1}^o \cdot \prod_{k=1}^r (D_{a_k, b_{r-k+1}}^o M_{a_k, b_{r-k+2}}^o).$$

When we analyze further on (240), the terms in (241) are important, while the precise form of the monomials in the elements H_{st}^o are of no importance.

The following observation is very useful in later computations:

Observation 8.3.2. *If $k < r + 1$ and M_{a_k, y_1}^o is a factor in (241), then so is D_{a_k, y_2}^o for precisely one y_2 , and $y_2 < y_1$. M_{a_{r+1}, b_1}^o is a factor, but no factor $D_{a_{r+1}, y}^o$ occurs in (241). Similarly, If $k < r + 1$ and M_{x_1, b_k}^o is a factor in (241), then so is D_{x_2, b_k}^o for precisely one x_2 and $x_2 < x_1$. $M_{a_1, b_{r+1}}^o$ is a factor, but no factor $D_{x, b_{r+1}}^o$ occurs.*

Remark 8.3.3. $(z_{i,j}M)$ is given by exactly the same formula as $(w_{i,j}M)$. The operators M^o, H^o , and D^o must just be interpreted as operators in \mathcal{A}_q^+ .

8.4. The right multiplication operators. Here we get a similar result. In fact, using the anti-automorphism $\phi : Z_{ij} \rightarrow Z_{n+1-i, n+1-j}$ one gets the expression for $M_{W_{ij}}$ from that of $(w_{n+1-i, n+1-j}M)$ by replacing all factors $X_{a,b}, X = D^o, M^o, H^o$ by $X_{n+1-a, n+1-a}$ in the latter. We choose to state it with fewer details since it is only the explicit form of the term involving M_{ij}^o , together with Observation 8.4.2 below, that is important.

Lemma 8.4.1. *Let i, j be given. Let $r \in \mathbb{N}$, and let*

$$(242) \quad (\underline{a}, \underline{b}) = ((a_1, a_2, \dots, a_r), (b_1, b_2, \dots, b_r)) \in \mathbb{N}^r \times \mathbb{N}^r.$$

We say that $(\underline{a}, \underline{b})$ is an SE-partition of (i, j) if

$$(243) \quad i = a_0 < a_1 < \cdots < a_r \leq a_{r+1} = n \text{ and } j = b_0 < b_1 < \cdots < b_r \leq b_{r+1} = n\}.$$

We denote by \mathcal{P}_r^{SE} the set of all such.

For $(\underline{a}, \underline{b}) \in \mathcal{P}_r^{SE}$ set

$$(244) \quad B_R M^{SE}(\underline{a}, \underline{b}) = M_{a_r, b_0}^o \prod_{x=0}^{r-1} (D_{a_{r-x}, b_{x+1}}^o M_{a_{r-x-1}, b_{x+1}}^o) \text{ and}$$

$$(245) \quad C(\underline{a}, \underline{b}) = D_{a_r, b_0}^o \prod_{x=0}^{r-1} (M_{a_{r-x}, b_{x+1}}^o D_{a_{r-x-1}, b_{x+1}}^o).$$

Then

$$(246) \quad M_{W_{ij}} =$$

$$(247) \quad M_{i,j}^o(H_{i,j+1}^o \cdots H_{i,n}^o)(H_{i+1,j}^o \cdots H_{n,j}^o)$$

$$(248) \quad + \sum_{r \in \mathbb{N}} \sum_{(\underline{a}, \underline{b}) \in \mathcal{P}_r^{SE}} B_R M^{SE}(\underline{a}, \underline{b}) H_r^o(\underline{a}, \underline{b}),$$

where each $H_r^o(\underline{a}, \underline{b})$ is a Laurent monomial in some of the elements $H_{s,t}^o$.
Furthermore

$$(249) \quad \widehat{M_{W_{ij}}}^X = (H_{ij}^o)^{\gamma_X} D_{i,j}^o(H_{i,j+1}^o \cdots H_{i,n}^o)(H_{i+1,j}^o \cdots H_{n,j}^o) \sum_{(\underline{a}, \underline{b}) \in \mathcal{P}_r^{SE}} C(\underline{a}, \underline{b}) \widetilde{H}_r^o{}^X(\underline{a}, \underline{b}),$$

where $\widetilde{H}_r^o{}^X(\underline{a}, \underline{b}) = q^{\alpha_X} H_r^o(\underline{a}, \underline{b})$ and α_X depends on the case (as well as on $(\underline{a}, \underline{b})$). In the following analysis, the exact value is of no importance.

Observation 8.4.2. *Similarly to Observation 8.3.2, each row either has none or has exactly one D and one M except row $a_0 = i$. Each column has either none or exactly one D and one M except column $b_0 = j$.*

8.5. The various algebras of differential operators. We now combine left or right multiplication and left or right differential operators into bigger algebras of differential operators. To begin with, to understand their individual significance, we combine one kind of multiplication operator with one kind of differential operator. Before embarking on this, we need a special tool:

8.5.1. *Eigenspace decompositions in subalgebras.*

Definition 8.5.1.

$$(250) \quad \mathcal{L} := \mathbb{C}[(H_{ij}^o)^{\pm 1} \mid i, j = 1, \dots, n]$$

denotes the Laurent polynomial algebra generated by the commuting invertible elements H_{ij}^o ; $i, j = 1, \dots, n$. Furthermore, we set

$$(251) \quad \mathcal{LM} = \{\phi \in \mathcal{L} \mid \phi \text{ is a monomial}\}.$$

Proposition 8.5.2. *Let $O_0, O_1, \dots, O_N \in \text{Weyl}_q(n, n)$ and let $\Phi = \{\phi_1, \dots, \phi_N\} \subset \mathcal{LM}$. Assume that*

$$(252) \quad \forall i = 1, \dots, N, \forall j = 0, 1, \dots, N : \phi_i O_j = q^{k_{ij}} O_j \phi_i$$

$$(253) \quad \Leftrightarrow \text{Ad}\phi(O_j) = q^{k_{ij}} O_j \text{ for some } k_{ij} \in \mathbb{Z}.$$

Suppose that Φ distinguishes elements in the following sense: $k_{ij} \neq k_{ii}$ whenever $j \notin \{1, \dots, i\}$.

Let \mathcal{A}_0 be a subspace of $\text{Weyl}_q(n, n)$ which is invariant under right and left multiplication by $\phi_x; x = 0, 1, \dots, N$. Suppose that $O = \sum_{j=0}^N O_j$ belongs to \mathcal{A}_0 , then

$$(254) \quad O_0 \left(\prod_{i=1}^N \phi_i \right) \in \mathcal{A}_0.$$

Proof. We have by assumption that

$$(255) \quad \phi_1 O - q^{k_{11}} O \phi_1 = \sum_{j \in \{0, 2, \dots, N\}} (q^{k_{1j}} - q^{k_{11}}) O_j \phi_1 \in \mathcal{A}_0.$$

Up to irrelevant complex factors, we have thus removed the summand O_1 . By the further assumptions we can continue to remove summands until only O_0 remains. \square

Remark 8.5.3. We will use Proposition 8.5.2 repeatedly in the sequel. Here, \mathcal{A}_0 will even be a subalgebra, but we are not assuming that the inverses $\phi_x^{-1}, x = 1, \dots, N$ stabilize \mathcal{A}_0 , so we have to keep the factor $\prod_{i=1}^N \phi_i$ in (254).

Now we are ready for the algebras:

8.5.2. Right - Left.

First we consider $\mathcal{A}_0 = \mathcal{D}(\partial_X, {}_Z M)$. Here we have all operators $z_{ij} M$ and $\widehat{M}_{W_{ij}}^X$.

First observe the following simple fact which follows from formulas (65-66):

Lemma 8.5.4. *If $\mathcal{D}(\partial_X, {}_Z M)$ contains elements $\phi_{ij} M_{i,j}^o$ and $\psi_{ij} D_{i,j}^o$, with ϕ_{ij}, ψ_{ij} elements of \mathcal{LM} , then*

$$(256) \quad \phi_{ij} \psi_{ij} (H_{i,j}^o)^{\pm 1} \in \mathcal{D}(\partial_X, {}_Z M).$$

The signs in (256) is the source of an ambiguity which we try to control somewhat by the specific choices below (262).

We now begin to prove that for all $i, j \in \{1, 2, \dots, n\}$, $\mathcal{D}(\partial_X, {}_Z M)$ indeed does contain elements $\phi_{ij} M_{i,j}^o$ and $\psi_{ij} D_{i,j}^o$ and ϕ_{ij}, ψ_{ij} may be explicitly given in \mathcal{LM} .

First observe, for use here and later, that by definition,

$$(z_{1,j} M) = H_{1,1}^o \dots H_{1,j-1}^o M_{1,j}^o \in \mathcal{D}(\partial_X, {}_Z M)$$

and $\widehat{R M}_{W_{i,n}}^X \in \mathcal{D}(\partial_X, {}_Z M)$ where $M_{W_{i,n}} = H_{nn}^o \dots H_{i+1,n}^o M_{i,n}^o$. At the position $(1, n)$ we thus have

$$(257) \quad H_{1,1}^o \dots H_{1,n-1}^o M_{1,n}^o, H_{nn}^o \dots H_{2,n}^o (H_{1,n}^o)^{\gamma_X} D_{1,n}^o \in \mathcal{D}(\partial_X, {}_Z M),$$

and then, immediately from Lemma 8.5.4,

$$(258) \quad H_{1,1}^o \cdots H_{1,n-1}^o \cdot H_{nn}^o \cdots H_{2,n}^o (H_{1n}^o)^{\gamma_X \pm 1} \in \mathcal{D}(\partial_X, ZM).$$

There are 2 covariant first order elements in our quadratic algebra, namely $Z_{1,n}$ and $Z_{n,1}$. We have chosen to construct elements working out from position $(1, n)$, but of course one may as well use position $(n, 1)$, or a mix of the two.

We use an ordering

$$(259) \quad (1, n) < (2, n) < \cdots < (n, n) < (1, n-1) < (2, n-1) < \cdots \cdots < (n, 1),$$

but we still need some definitions before we get to a precise statement:

Definition 8.5.5. *Introduce the following elements in \mathcal{LM} :*

$$(260) \quad V_{ij}^{NW} = \left(\prod_{k < i} H_{kj}^o \right) \left(\prod_{s < j} H_{is}^o \right),$$

$$(261) \quad V_{ij}^{SE} = \left(\prod_{k > i} H_{kj}^o \right) \left(\prod_{s > j} H_{is}^o \right), \text{ and}$$

$$(262) \quad V_{ij} = V_{ij}^{NW} V_{ij}^{SE} (H_{ij}^o)^{\alpha_X}, \quad \alpha_X = \begin{cases} 0 & \text{in case } J \\ -2 & \text{in case } K \\ -1 & \text{in case } L \end{cases}.$$

Inductively set

$$(263) \quad \mathbb{V}_{in} := \left(\prod_{s < i} \mathbb{V}_{sn} \right) V_{in} = \left(\prod_{s < i} V_{sn}^{2^{i-1-s}} \right) V_{in}.$$

More generally, set

$$(264) \quad \mathbb{V}_{ij} = V_{ij} \left(\prod_{a=1}^{i-1} \mathbb{V}_{aj} \right) \left(\prod_{b=j+1}^n \mathbb{V}_{i,b} \right).$$

In particular,

$$(265) \quad \mathbb{V}_{1j} := \left(\prod_{s=1}^{(n-j)} V_{1,j+s}^{2^{s-1}} \right) V_{1j}.$$

Remark 8.5.6. *Below, we use heavily Observation 8.3.2 and Observation 8.4.2. Our main tool is Proposition 8.5.2 and here we need to construct a family Φ in \mathcal{LM} that distinguishes elements, e.g. $D_{a,j}^o M_{ij}^o M_{in}^o$, $a = 1, \dots, i-1$ from M_{in}^o . For this purpose we need to assume $\alpha_X \neq 2$. With γ_X as in (168), we have $\alpha_X = \gamma_X \pm 1$, so this only poses restrictions in case $X = J$ where we must pick $\alpha_X = 0$. In the other cases there is the previously mentioned ambiguity at each position i, n . This means that the elements constructed in the following lemmas are not unique, but they suffice for our purposes. Furthermore, it does not seem to give simplifications in the end results by allowing more general vales. The above choices are then made for the sake of specificity.*

Lemma 8.5.7. *It holds that*

$$(266) \quad \left(\prod_{s < i} \mathbb{V}_{sn} \right) V_{i,n}^{NW} M_{in}^o \in \mathcal{D}(\partial_X, ZM) \text{ and } \widehat{M_{W_{in}}^X} \in \mathcal{D}(\partial_X, ZM),$$

where $M_{W_{in}} = V_{in}^{SE} M_{in}^o$. Hence, in particular, the elements $\mathbb{V}_{i,n}$ belong to $\mathcal{D}(\partial_X, ZM)$.

Proof. This is proved by induction. The case $i = 1$ is just (257) and (258) with special choices.

Now look at a position (i, n) with $i \geq 2$. We have at our disposal \mathbb{V}_{kn} , $k = 1, \dots, i - 1$. Here we have that

$$(267) \quad (z_{in} M), H_{nn}^o \cdots H_{i-1,n}^o (H_{i,n}^o)^{\gamma_X} D_{i,n}^o \in \mathcal{D}(\partial_X, ZM),$$

where (use Lemma 8.4.1 in its equivalent (z_{in}) guise)

$$(268) \quad (z_{i,n} M) = O = H_{i1}^o \cdots H_{i,n-1}^o H_{1n}^o \cdots H_{i-1,n}^o M_{in}^o + \tilde{O} = V_{in}^{NW} M_{in}^o + \tilde{O}.$$

Observing that the operators \mathbb{V}_{kn} are monomials in the operators $V_{\ell,n}$, $\ell \leq k$, we notice that we could use the operators V_{kn} , $k = 1, \dots, i - 1$ as in Proposition 8.5.2 to remove the elements in \tilde{O} such that the element $V_{in}^{NW} M_{in}^o$ is obtained. Specifically, one could easily use $V_{1,n}, V_{2,n}, \dots$ to eliminate terms O_j for which $V_{k,n} O_j \neq q O_j V_{k,n}$ (here we must insist that $\alpha_j \neq 2$ as in (262)). This procedure works because each relevant row has a pair D, M as remarked after Lemma 8.4.1. First use $V_{1,n}$ then use $V_{2,n}$ on the remaining elements not distinguished by $V_{1,n}$, then $V_{3,n}$ and so forth. The difference between the elements $V_{a,n}$ and the (correct) elements \mathbb{V}_{an} is thus inconsequential, hence the latter elements also work. Hence $(\prod_{s < i} \mathbb{V}_{sn}) V_{i,n}^{NW} M_{in}^o \in \mathcal{D}_X(\partial_X, LM)$. We also have the element $\widehat{RM_{W_{1,n}}^-} = H_{nn}^o \cdots H_{2,n}^o (H_{in}^o)^{\gamma_X} D_{1,n}^o$, and using Lemma 8.5.4 the induction step is completed. \square

In complete analogy we get

Lemma 8.5.8. $\mathbb{V}_{1,j}$ belongs to $\mathcal{D}(\partial_X, ZM)$ for $j = n, n - 1, \dots, 1$. Indeed, $(\prod_{b=j+1}^n \mathbb{V}_{i,b}) V_{1,j}^{NW} M_{j,1}^o \in \mathcal{D}(\partial_X, ZM)$ and $\widehat{M_{W_{1,j}}^X} \in \mathcal{D}(\partial_X, ZM)$ where $M_{W_{1,j}} = V_{1,j}^{SE} M_{1,j}^o$.

Proposition 8.5.9. For all i, j , $V_{ij}^{NW} \left(\prod_{a=1}^{i-1} \mathbb{V}_{aj} \right) M_{ij}^o \in \mathcal{D}(\partial_X, ZM)$ and $V_{ij}^{SE} \left(\prod_{b=j+1}^n \mathbb{V}_{aj} \right) (H_{ij}^o)^{\gamma_X} D_{ij}^o \in \mathcal{D}(\partial_X, XM)$. As a consequence, $\mathbb{V}_{ij} \in \mathcal{D}(\partial_X, ZM)$.

Proof. This follows analogously while using the already established results. We can use $V_{1,j}, V_{2,j}, \dots, V_{i-1,j}$ for $(z_{ij} M)$ and $V_{i,j+1}, \dots, V_{i,n-1}, V_{i,n}$ for $M_{W_{ij}}$. \square

Remark: We could also use $V_{i1}, V_{i2}, \dots, V_{i,j-1}$ for $(z_{ij} M)$ and, independently, $V_{i+1,j}, \dots, V_{n-1,j}, V_{n,j}$ for $M_{W_{ij}}$.

8.5.3. Right - Right.

Here, in the terminology of Proposition 8.5.2, $\mathcal{A}_0 = \mathcal{D}(\partial_X, M_Z)$, and we have the operators $M_{Z_{ij}}$, and $\widehat{M_{W_{ij}}}^X$. We obtain a result analogous to the previous case, but for reasons that should become clear, we find it more reasonable to treat the 3 cases one by one.

The case J is quite complicated and is considered later. However, we get a clean result for each of the cases K, L:

Proposition 8.5.10. *We have the following:*

$$\begin{aligned} \mathcal{D}(\partial_D, M_Z) &= \text{Weyl}_q(n, n) \\ \mathcal{D}(\partial_K, M_Z) &= \mathbb{C}[V_{ij}^{SE} \mathcal{D}_{ij} \mid 1 \leq i, j \leq n] \cup \mathbb{C}[V_{ij}^{SE} M_{ij}^o \mid 1 \leq i, j \leq n]. \end{aligned}$$

In particular, $(H_{ij}^o)^{\pm 2} \in \mathcal{D}(\partial_K, M_Z)$ for $(i, j) \neq (1, 1)$, but only $(H_{11}^o)^{-2} \in \mathcal{D}(\partial_K, M_Z)$.

Proof: The induction again follows the ordering

$$(269) \quad (n, n) < (n, n-1) < \dots < (n, 1) < (n-1, n) < \dots < (1, 1).$$

and starts by observing that M_{nn}^o and $(H_{nn}^o)^{\gamma_X} D_{nn}^o \in \mathcal{D}(\partial_X, M_Z)$. This immediately gives $(H_{nn}^o)^{\pm 1}$ in case H, while it gives $(H_{nn}^o)^{-2}$ and $(H_{nn}^o)^0$ in case K. To finish case K, we observe that likewise, $H_{n,n}^o M_{n,n-1}^o$ and $H_{nn}^o (H_{n,n-1}^o)^{\gamma_X} D_{n,n-1}^o \in \mathcal{D}(\partial_X, M_Z)$. This gives that $(H_{nn}^o)^2 (H_{n,n-1}^o)^{-2} \in \mathcal{D}(\partial_K, M_Z)$ and $(H_{nn}^o)^2 \in \mathcal{D}(\partial_K, M_Z)$. After that the induction proceeds easily along the same lines with the exception of the restriction in case K when we reach the final point $(1, 1)$. Case L is the easiest since if we have all $(H_{ij}^o)^{\pm 1}$ below (i, j) then easily, $M_{ij}^o, D_{ij}^o \in \mathcal{D}(\partial_D, M_Z)$ hence the result. \square

Case $X = J$ is as complicated as in the previous Subsubsection 8.5.2, and the result is more complex:

Proposition 8.5.11. *At the general position i, j ,*

$$V_{ij}^{SE} \left(\prod_{a=i+1}^n \mathbb{Y}_{aj} \right) M_{ij}^o \text{ and } V_{ij}^{SE} \left(\prod_{a=i+1}^n \mathbb{Y}_{aj} \right) H_{ij} D_{ij}^o$$

belong to $\mathcal{DJ}(\partial_J, M_Z)$; $\mathbb{Y}_{nj} = \prod_{a=j+1}^n (H_{an}^o)^2$ and $\mathbb{Y}_{ij} = (V_{ij}^{SE})^2 \left(\prod_{a=i+1}^n \mathbb{Y}_{aj} \right)^2$. In particular, $\mathbb{Y}_{ij} \in \mathcal{D}_J(\partial_J, M_J)$.

Proof: This is again by induction using the ordering (269).

To begin with, at (n, n) , we have M_{nn}^o and $H_{nn}^o D^o$. This leads to I and $(H_{nn}^o)^2$. Then we get $H_{nn}^o M_{n,n-1}^o$ and $H_{nn}^o H_{n,n-1}^o D_{n,n-1}^o$. This leads to $(H_{nn}^o)^2 (H_{n,n-1}^o)^2$ (and, once again, $(H_{nn}^o)^2$.) After that it is clear that we have $(H_{nn}^o)^2 (H_{n,n-1}^o)^2 \dots (H_{nr}^o)^2$ for any $n > r \geq 1$. At a position $n-1, j$ we have

$$(270) \quad M_{Z_{n-1,r}} = V_{n-1,r}^{SE} M_{n-1,r}^o + \text{LOTS}$$

and we can use $H_{nn}^2(H_{n,n-1}^o)^2 \dots (H_{n,r+1}^o)^2$ to separate off the LOTs terms.

A similar result is obtained for $\widehat{M_{W_{n-1,r}}^J}$ and we then obtain elements of the form

$$(271) \quad ((H_{nn}^o)^4(H_{n,n-1}^o)^4 \dots (H_{n,r+1}^o)^4)(H_{n,r}^o)^2((H_{n-1,n}^o)^2(H_{n-1,n-1}^o)^2 \dots (H_{n-1,r+1}^o)^2).$$

This element together with the previous can now be used to attack a position $n-2, r$ using Observation 8.4.2 and Proposition 8.5.2. We use elements with 0 in the upper left corner since they do not affect the top term, and in general, we need one element per row. The result follows. \square

Remark 8.5.12. *This inductive formula above is easily solved:*

$$(272) \quad \mathbb{Y}_{ij} = V_{ij}^{SE} \left(\prod_{x=1}^{n-i} (Y_{i+x,j}^{SE})^{4 \cdot 3^{x-1}} \right) \Rightarrow$$

$$(273) \quad \mathbb{Y}_{ij} = \prod_{x=2}^{n-i} (H_{i+x,j}^o)^{2 \cdot 3^{x-2}} (H_{i+1,j}^o)^2 \prod_{y=j+1}^n (H_{iy}^o)^2 \prod_{y=j+1, x=1}^{n, n-i} (H_{i+x,y}^o)^{4 \cdot 3^{x-1}}.$$

We further remark that one can equally well use constructions based on columns instead of rows.

8.6. The algebras of differential operators. Conclusion. We have seen that the three cases $X = J, L, K$ lead to quite different results for $\mathcal{D}(\partial_X, M_X)$ and $\mathcal{D}(\partial_X, {}_Z M)$ though the following holds that in all 6 ($= 3 \times 2$) cases:

Theorem 8.6.1. *For any index i, j there are, case dependent, elements ψ_{ij} and ϕ_{ij} of \mathcal{LM} such that*

$$(274) \quad \psi_{ij} M^o \text{ and } \psi_{ij} D^o$$

belong to the algebra of the given case.

Definition 8.6.2. *Set*

$$(275) \quad \mathcal{D}_X^R = \mathcal{D}(\partial_X, M_Z, {}_Z M);$$

the algebra of quantized right differential operators, and set

$$(276) \quad \mathcal{D}_X^L = \mathcal{D}({}_X \partial, M_Z, {}_Z M);$$

the algebra of quantized left differential operators.

We now discuss the cases J, K, L one by one:

Theorem 8.6.3. *We have that*

$$(277) \quad \mathcal{D}_L^R = \mathcal{D}_L^L = \text{Weyl}_q(n, n).$$

The representation \mathcal{O}_Λ^L is given by differential operators together with $\text{End}(V'_\Lambda)$.

Proof: It is clear that

$$(278) \quad \mathcal{D}(\partial_H, {}_Z M) = \mathcal{D}(\partial_H, M_Z) = \text{Weyl}(n, n).$$

The rest is equally obvious. \square

In case J we do not get $(\widehat{{}_{W_{11}} M})^J$ from the right and we do not get $\widehat{M}_{W_{nn}}^J$ from the right. We do get parts of $(\widehat{{}_{Z_{11}} M})^J$ (on Z s) but miss out on eg. K_β^{-1} . We will refrain from stating further results for this case.

Definition 8.6.4. $\mathcal{KWeyl}_q(n, n)$ is set to be the smallest subalgebra of $\text{Weyl}_q(n, n)$ containing the following operators:

$$(279) \quad \forall i, j : V_{ij}^{NW} (H_{ij}^o)^{-1} D_{i,j}^o, V_{ij}^{NW} M_{i,j}^o, \text{ and } V_{ij}^{SE} M_{i,j}^o.$$

In particular, it contains all $(H_{ij}^o)^{\pm 2}$ and $V_{ij}^{NW} V_{ij}^{SE}$.

Theorem 8.6.5. We have that

$$(280) \quad \mathcal{D}_K^R = \mathcal{D}_K^L := \mathcal{KWeyl}_q(n, n).$$

The representation \mathcal{O}_Λ^K is given by operators from $\mathcal{KWeyl}_q(n, n)$ together with $\text{End}(V'_\Lambda)$.

Proof: The fact that the left and right algebras are identical is easy to see since by symmetry we easily get $\mathcal{D}(\partial_K, {}_Z M)$ while $\mathcal{D}(\partial_K, M_Z)$ is the dual of an analogous algebra $\mathcal{D}_K({}_W M, \partial_K)$ on the \mathcal{A}_q^- . The statements about the elements in the algebra follow from Proposition 8.5.10 and the following observation: The combined operators easily lead to e.g. $H_{11}^o V_{12}^{SE} \in \mathcal{D}_K^R$. By squaring this element we easily obtain $(H_{11}^o)^2$.

As for the the representation, by definition $\widehat{L}_{W_\beta}^K \in \mathcal{D}_K^R$. Next observe that $K_\beta = (H_{11}^o)^2 V^{11} NW V^{SE} \in \mathcal{D}_K^R$ and similarly, since we have all operators $(H_{ij}^o)^{-2}$ and thus can replace odd positive powers in elements H_{st}^o by odd negative powers, $K_\beta^{-1} \in \mathcal{D}_K^R$. Now consider $\mathcal{O}_\Lambda^K(E_\beta)$: It differs from $\mathcal{O}_\Lambda^J(E_\beta)$ by squares of elements H_{ij}^o , cf. (149-150). We may therefore equally well look at the latter which, by Corollary 7.2.2 is given by K_β^{-1} and left and right multiplication operators. We remark here that for the general expression in Corollary 7.2.2 we need to introduce $\text{End}(V'_\Lambda)$ (which is generated by the operators from $\mathcal{U}_1(\mathfrak{k}^{\mathbb{C}})$). We then only need to consider the representation restricted to $\mathcal{U}_1(\mathfrak{k}^{\mathbb{C}})$, and we can here, by similar arguments, restrict to consider the scalar case and we may as well just consider the expressions in Proposition 3.2.2 and easily get, up to factors of $(H_{st}^o)^{\pm 2}$,

$$(281) \quad E_{\mu_k}^Z = \sum_j (V_{k,j}^{NW} (H_{k,j}^o)^{-1} D_{k,j}^o V_{k+1,j}^{NW} M_{k+1,j}^o), \text{ and}$$

$$(282) \quad F_{\mu_k}^Z = \sum_j (V_{k+1,j}^{SE} (H_{k+1,j}^o)^{-1} D_{k+1,j}^o V_{k,j}^{SE} M_{k,j}^o).$$

Since these expressions are in the algebra, we are done. \square

We finish with the following result which of course is immediate from the way these algebras are invariantly defined from left and right actions:

Theorem 8.6.6. *Weyl $_q(n, n)$ as well as the various constructed subalgebras, though given manifestly in a fixed PBW basis, are intrinsically invariant.*

REFERENCES

- [1] V. Chari, A. Pressley. A Guide to Quantum Groups. Cambridge Univ. Press, 1995.
- [2] T. Hayashi, q -analogues of Clifford and Weyl Algebras–Spinor and Oscillator Representations of Quantum Enveloping Algebras, Comm. Math. Phys. Volume 127, Number 1 (1990), 129-144.
- [3] Hans P. Jakobsen and Michele Vergne, Restrictions and expansions of holomorphic representations, J. Functional Analysis 34, 29-53 (1979)
- [4] Hans P. Jakobsen, Q -differential operators. Preprint (19pp) 1999 (<http://xxx.lanl.gov/abs/math.QA/9907009>)
- [5] Hans P. Jakobsen, Quantized Dirac operators. Czech. J. Phys. 50, 1265–1270 (2000)
- [6] H. P. Jakobsen and H. Zhang, Double-partition Quantum Cluster Algebras J. Algebra 372 (2012), 172-203.
- [7] J.C. Jantzen, *Lectures on Quantum Groups*, A.M.S. Graduate Studies in Mathematics Vol. 6 (1996)
- [8] A. Joseph and G. Letzter, Rosso’s Form and Quantized Kac Moody algebras, Mathematische Zeitschrift, Vol. 222 (1996) 543-571.
- [9] M. Kashiwara, On crystal bases of the q -analogue of universal enveloping algebras, Duke Math. J. 63 (1991), 465–516.
- [10] M. Kashiwara, Global crystal bases of quantum groups, Duke Math. J. 69 (1993), 455–485.
- [11] A. P. Kitchin and S. Launois, On the Automorphisms of Quantum Weyl Algebras, arXiv:1511.01775v2 [math.QA] 12 Jul 2018, to appear in Journal of Pure and Applied Algebra.
- [12] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, Adv. in Math. 70 (1988), 237-249.
- [13] G. Lusztig, Canonical bases arising from quantized enveloping algebras, J Amer. Math. Soc. 3 (1990), 447-498.
- [14] G. Lusztig, Quantum groups at a root of 1, Geom. Dedicata, 35, 1990, p. 89-144.
- [15] G. Lusztig **Introduction to Quantum Groups**. Progress In Mathematics 110, Birkhäuser 1993.
- [16] M. Rosso, Certaines formes bilinéaires sur les groupes quantiques et une conjecture de Schechtman et Varchenko, C. R. Acad. Sci. Paris, t. 314, Série I, p. 5-8, 1992
- [17] S. Sinel’shchikov, L. Vaksman, Hidden symmetry of the differential calculus on the quantum matrix space, J. Phys. A., Math. Gen., 30 (1997), 23–26; in Lectures on q -analogs of Cartan domains and associated Harish-Chandra modules, L. Vaksman (ed.), math.QA/0109198, 136–140.
- [18] D. Shklyarov, G. Zhang, Covariant q -differential operators and unitary highest weight representations for $U_q(\mathfrak{su}(n, n))$, J. Math. Phys. **46** Issue 6 (2005). [dx.doi.org/10.1063/1.1927077](https://doi.org/10.1063/1.1927077)
- [19] T. Tanisaki, Killing forms, Harish-Chandra Isomorphisms, and universal R -matrices for quantum algebras, pp. 941-961 in: A. Tsuchiya, T. Eguchi, and M. Jimbo (eds.), **Infinite Analysis**, Part B, Proc. Kyoto 1991 (Advanced Series in Mathematical Physics 16), Rivers Edge, N. J., 1992 (World Scientific)

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, DENMARK

E-mail address: jakobsen@math.ku.dk