GROUP THEORETICAL ASPECTS OF THE CHRONOMETRIC THEORY.*

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Below is given a brief outline of one particular point, namely that of chronometric quantum-numbers for mass-zero equations. A complete description of these and related matters, which results from the joint investigations of I.E. Segal, B. Speh, B. Ørsted and myself, will appear elsewhere. In this connection I would like to thank Segal, Speh, Ørsted and M. Harris for valuable discussions. The other aspects touched upon in the talk are described in [1], and [2]. The chronometric theory is described in [5]. For additional background and further results, c.f. [3] and literature cited there.

1. Notation

$\tau = \tau_1$ denotes the defining representation of $Gl(2,\mathbb{C})$, i.e. $\tau\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$. τ_n denotes the n-th fold symmetrized tensor product of τ, acting on $V_n = \overset{n}{\underset{s}{\otimes}} \mathbb{C}^2$, and τ_0 denotes the trivial representation. The letters a,b,c,d,h,k,u,v,z,w denote 2 x 2 matrices.

2. Basics

The unitary representations of SU(2,2) corresponding to spin $\frac{n}{2}$ and mass o are

$$\begin{aligned}(U_n^{+}(g)f)\,(z) &= \tau_n(cz+d)^{-1}\det(cz+d)^{-1}\,f(g^{-1}z), \text{ and}\\ (U_n^{-}(g)f)\,(z) &= \tau_n(a-(g^{-1}z)c)^{-1}\det(a-(g^{-1}z)c)^{n+1}\,f(g^{-1}z),\end{aligned} \tag{2.1}$$

where $g^{-1} = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SU(2,2)$, $g^{-1}z = \frac{az+b}{cz+d}$, and $f(z) \in V_n$.

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Let

$$SU(2,2)_D = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; g^*\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

$$SU(2,2)_B = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; g^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

$$D = \left\{ z \;\middle|\; \frac{z-z^*}{2i} \text{ is strictly positive definite} \right\}, \text{ and}$$

$$B = \{ z \mid z^* z < 1 \}.$$

Of course, the groups $SU(2,2)_D$ and $SU(2,2)_B$ are isomorphic. D is the generalized upper half-plane and B is the generalized unit disk. The Shilov boundary of D is $H(2) = \{$ 2 x 2 hermitian matrices $\}$ = Minkowski space, whereas the Shilov boundary of B is $U(2) = \{$ 2 x 2 unitary matrices $\}$. $\widetilde{U(2)}$ = the universal covering group of $U(2) = \mathbb{R} \times SU(2) = \mathbb{R} \times S^3$ is the Segal cosmos.

The formulas (2.1) define representations of $SU(2,2)_D$ as well as $SU(2,2)_B$ provided that for $SU(2,2)_D$, f is a holomorphic function on D, and that for $SU(2,2)_B$, f is a holomorphic function on B. These representations are equivalent. In both versions one can pass to the Shilov boundary and thus realize the representations on the respective space-times. One reason that the following claims are true is that the transition from $SU(2,2)_D$ to $SU(2,2)_B$ (and from D to B) can be obtained through the action of an element in the complexification of SU(2,2), and that all the expressions are analytic in the variables g and z. Observe that once a choice of domain has been made the term $(a - (g^{-1}z)c)^{-1}$ may be simplified: For $g^{-1} \in SU(2,2)_D$ and $z \in D$ it is equal to $zc^* + d^*$ and in the B- situation it is equal to $(a^* + zb^*)$.

3. Chronometric quantum numbers

The problem may be handled by means of the "ladder representations" [4] (see also [1]). In contrast, the approach presented here is intrinsic in the sense that it deals directly with the relevant function spaces and differential equations. In the following we shall only consider the representations U_n^+ since the treatment of the U_n^-'s follows from this by obvious modifications.

The domain D is particularly nice for describing the Hilbert space H_n^+ that carries the unitary irreducible representation U_n^+: H_n^+ is a reproducing kernel Hilbert space and in the D-version the kernel is given as

$$K_n^+(z,w) = \tau_n\left(\frac{z-w^*}{2i}\right)^{-1} \det\left(\frac{z-w^*}{2i}\right)^{-1} = \int_{b(C^+)} \tau_n(k) e^{i\,\mathrm{tr}(z-w^*)k}\, dm(k), \qquad (3.1)$$

where $b(C^+) = \{ k = k^* \mid \mathrm{tr}\, k \geq 0 \text{ and } \det k = 0 \}$ is the boundary of the solid forward light-cone C^+, and $dm(k)$ is the usual Lorentz-invariant measure.

We wish to determine a complete set of quantum numbers for the representations U_n^+. From the point of view of Segal's cosmological theory as well as from that of group theory, the natural way to do this is to use the maximal compact subgroup K of SU(2,2). K coincides with the maximal subgroup of isometries of SU(2,2) under its action on U(2). In terms of group theory we shall thus determine the decomposition of the restriction of U_n^+ to K as a direct sum of finite-dimensional representations of K; the so-called K-types.

In $SU(2,2)_B$ we thus choose K as

$$K = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \;\middle|\; u,v \in U(2),\ \det(u\ v) = 1 \right\} \qquad (3.2)$$

In particular, K = U(1) x SU(2) x SU(2).

$T = \left\{ \begin{pmatrix} e^{i\tau/2} & 0 \\ 0 & e^{-i\tau/2} \end{pmatrix} \;\middle|\; \tau \in \mathbb{R} \right\}$ is Segal's time-translation subgroup of K.

The natural domain to use for the determination of the K-types is B. On B,

$$(U_n^+(u,v)f)\ (z) = \left(U_n^+\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} f\right)(z) = \tau_n(v) \det v\ f(u^{-1} z\ v), \qquad (3.3)$$

hence the finite-dimensional K-irreducible subspaces consist of V_n-valued polynomials. (3.3) clearly defines an action of K on the set of all V_n-valued polynomials on B. We denote this action by $\widetilde{U}_n^+$.

Consider the representation P of K on the space of all $\mathbb{C}$-valued polynomials defined by

$$(P(u,v)\,p)\ (z) = p\,(u^{-1} z\ v) \qquad (3.4)$$

<u>Proposition 3.1</u> $P(u,v) = \bigoplus_{l=0}^{\infty} \bigoplus_{r=0}^{\infty} \dfrac{\tau_l(v) \otimes \tau_l(\bar{u})}{\det u^{2r}}$

For later use we observe that P leaves the ideal I = I(det z) of polynomials proportional to det z, invariant. Clearly then

<u>Proposition 3.2</u> $P(u,v)\Big|_I = \bigoplus_{l=0}^{\infty} \bigoplus_{r=1}^{\infty} \frac{\tau_l(v) \otimes \tau_l(\bar{u})}{\det u^{2r}}$

The K-types of U_n^+ are, of course, to be found among the summands of $\widetilde{U}_n^+(u,v) = \tau_n(v) \det v \otimes P(u,v)$.

<u>Proposition 3.3</u>

$$\tau_n(v) \det v \otimes P(u,v) = \bigoplus_{l=0}^{\infty} \bigoplus_{r=0}^{\infty} \bigoplus_{\gamma=0}^{\min\{n,l\}} \frac{\tau_{n+l-2\gamma}(v) \otimes \tau_l(\bar{u})}{\det u^{\gamma+l+2r}}$$

However, since the Hilbert space of U_n^+ consists of solutions to certain differential equations, some of the summands must be excluded. To see exactly which do not occur we return to the domain D. By expanding the functions on D around the point $i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ and by choosing K in this version as the subgroup of $SU(2,2)_D$ that leaves i fixed, it follows (e.g. by a Gram-Schmidt argument) that the finite-dimensional subspaces of functions on D that carry the irreducible K-representations are built up from functions of the form

$$F_{\underline{q}}(z) = \int_{b(C^+)} \tau_n(k)\, e^{i\, \mathrm{tr}(z+i)k}\, \underline{q}(k)\, dm\,(k), \tag{3.5}$$

where $\underline{q}$ is a V_n-valued polynomial. However, $\underline{q}$ determines a non-zero function $F_{\underline{q}}$ if and only if there is an x in V_n such that $\langle \tau_n(k)\underline{q}(k), x \rangle$ is not identically zero on $b(C^+)$. That is: Exactly those $\underline{q}$'s for which for all x in V_n, $\langle \tau_n(k)\underline{q}(k), x \rangle = 0$ on $b(C^+)$ do not contribute to the K-types in U_n^+. Now observe that $b(C^+)$ is a sufficiently big subset of the variety of the prime ideal I(det z) that we may conclude:

$$\forall x \in V_n \ :\ \langle \tau_n(k)\underline{q}(k), x \rangle = 0 \quad \text{on } b(C^+) \iff$$

$$\forall x \in V_n \ :\ \langle \tau_n(z)\underline{q}(z), x \rangle \in I\ (\det z).$$

Let $A = \{ \underline{q} \mid \forall x \in V_n \ :\ \langle \tau_n(z)\underline{q}(z), x \rangle \in I\ (\det z) \}$

A is clearly invariant under the action of $\widetilde{U}_n^+$, and even though it originally was determined from the domain D, it follows by analyticity that

<u>Proposition 3.4</u> The K-types in A are exactly those in $\widetilde{U}_n^+$ that do not occur in U_n^+.

Let $\mathcal{S}$ denote the linear map of $A \otimes V_n$ into the space of $\mathbb{C}$-valued polynomials,

$$(\mathcal{S}(\underline{q} \otimes x))(z) = \langle \tau_n(z)\underline{q}(z), \bar{x} \rangle \tag{3.7}$$

Then

$$P(u,v)\mathcal{S} = \mathcal{S}(\widetilde{U}_n^{+}(u,v) \otimes \det \bar{v}\ \tau_n(\bar{u})) \tag{3.8}$$

Now choose a summand $\dfrac{\tau_{n+1-2\gamma}(v) \otimes \tau_1(\bar{u})}{\det u^{\gamma+1+2r}}$

from $\widetilde{U}_n^{+}$ and assume that $\underline{q} \in A$ transforms according to this representation. It follows from (3.8) and Proposition 3.1 that $\langle \tau_n(z)\underline{q}(z), \bar{x} \rangle$ transforms according to

$\dfrac{\tau_{n+1-2\gamma}(v) \otimes \tau_{n+1-2\gamma}(\bar{u})}{\det u^{2\gamma+2r}}$.

Since $\underline{q} \in A$ it follows from Proposition 3.2 that either $\gamma > 0$ or $r > 0$. Thus

Proposition 3.5

$$U_n^{+}(u,v) = \bigoplus_{l=o}^{\infty} \frac{\tau_{n+l}(v) \otimes \tau_l(\bar{u})}{\det u}$$

By analogous reasoning it follows that

Proposition 3.6

$$U_n^{-}(u,v) = \bigoplus_{l=0}^{\infty} \frac{\tau_l(v) \otimes \tau_{l+n}(\bar{u})}{\det u}$$

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