A NEW CLASS OF UNITARIZABLE HIGHEST WEIGHT REPRESENTATIONS OF INFINITE DIMENSIONAL LIE ALGEBRAS.

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0. Introduction.

The representation theory of infinite-dimensional Lie algebras has emerged in the past few years as a field that has remarkable applications to many areas of mathematics and mathematical physics. All these applications show that the following two assumptions about the representation in question are fundamental :

1) unitarizability ;
2) existence of a highest weight vector.

In more detail, let $\mathfrak{g}$ be a complex (possibly infinite-dimensional) Lie algebra, let $\mathcal{U}(\mathfrak{g})$ denote its universal enveloping algebra, let $\mathfrak{p}$ be a subalgebra of $\mathfrak{g}$ and let ω be an antilinear anti-involution of $\mathfrak{g}$ (i.e. $\omega.[x,y] = [\omega.y, \omega.x]$ and $\omega.(\lambda x) = \bar{\lambda}(\omega.x)$) such that

$$(0.1) \qquad \mathfrak{p} + \omega.\mathfrak{p} = \mathfrak{g} \quad .$$

Let $\lambda : \mathfrak{p} \to \mathbb{C}$ be a 1-dimensional representation of $\mathfrak{p}$. A representation $\pi : \mathfrak{g} \to gl(V)$ is called a *highest weight representation* with highest weight λ if there exists a vector $v_\lambda \in V$ with the following properties :

$$(0.2) \qquad \pi(\mathcal{U}(\mathfrak{g}))\, v_\lambda = V \;;$$

(0.3) $\pi(b)v_\lambda = \lambda(b)v_\lambda$ for any $b \in \mathfrak{p}$

(Of course, (0.2) is satisfied automatically if the representation π is irreducible) A Hermitian form H on V such that

(0.4) $H(v_\lambda, v_\lambda) = 1$,

(0.5) $H(\pi(g)u, v) = H(u, \pi(\omega.g)v)$ for all $g \in \mathfrak{g}$ and $u, v \in V$

is called *contravariant* (it is determined uniquely by (0.4) and (0.5)). It is easy to show that, under some natural assumptions, for any highest weight $\lambda : \mathfrak{p} \longrightarrow \mathbb{C}$ there exists a unique highest weight representation with a non-degenerate contravariant Hermitian form. The non-trivial problem is whether this form is positive definite; if this is the case, the representation π is called *unitarizable*.
For example, let $\mathfrak{g}$ be the infinite-dimensional Heisenberg algebra, i.e. a Lie algebra with a basis p_i, q_i $(i \in \mathbb{Z})$ and c, with commutation relations : $[p_i, q_i] = c$ and all the other brackets zero. Put $\mathfrak{p} = \mathbb{C}c + \sum \mathbb{C}p_i$ and let $\lambda : \mathfrak{p} \to \mathbb{C}$ be defined by $\lambda(c) = a \in \mathbb{C}^\times$ and $\lambda(p_i) = 0$.
Then any representation of $\mathfrak{g}$ with highest weight λ is irreducible and equivalent to the canonical commutation relations representation L(a) (i.e. $q_i \longrightarrow x_i$, $p_i \longrightarrow a\frac{\partial}{\partial x_i}$, $c \longrightarrow a$). Let ω be an antilinear anti-involution of $\mathfrak{g}$ defined by $\omega(p_i) = q_i$, $\omega(q_i) = p_i$, $\omega(c) = c$. Then L(a) is unitarizable if and only if a is a positive real number.
The unitarizable highest weight representations of finite-dimensional semisimple Lie algebras have been classified quite recently, and the answer is highly non-trivial [1], [3].
The present paper grew out from an attempt to solve the analogous problem for affine Kac-Moody algebras. Recall that, given a simple finite-dimensional Lie algebra $\dot{\mathfrak{g}}$, the associated affine Kac-Moody algebra is

$$\mathfrak{g} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \dot{\mathfrak{g}} + \mathbb{C}c ,$$

with the following commutation relations :

(0.6) $[z^m \otimes a, z^k \otimes b] = z^{m+k} \otimes [a, b] + m\delta_{m,-k}(a, b)c \; ; \; [\mathfrak{g}, c] = 0$

Here a,b $\in \dot{\mathfrak{g}}$, (a,b) is the Killing form on $\dot{\mathfrak{g}}$, and $m, k \in \mathbb{Z}$. Let $\dot{\mathfrak{b}}$ be a Borel subalgebra of $\dot{\mathfrak{g}}$ and $\dot{\omega}$ a compact antilinear anti-involution (i.e. the real subalgebra $\{x \in \dot{\mathfrak{g}} \mid \dot{\omega}.x = -x\}$ is the compact form of $\dot{\mathfrak{g}}$) such that (0.1) holds. The conventional choice of the "Borel subalgebra" $\mathfrak{b}$ of $\mathfrak{g}$ is

(0.7) $\mathfrak{b} = \mathbb{C}c \oplus (1 \otimes \dot{\mathfrak{b}}) \oplus (z \otimes \dot{\mathfrak{g}}) \oplus (z^2 \otimes \dot{\mathfrak{g}}) \oplus \dots$

Let ω be the compact antilinear anti-involution of $\mathfrak{g}$, i.e. $\omega\cdot(z^m \otimes a + \lambda c) = z^{-m} \otimes \dot{\omega}(a) + \bar{\lambda} c$ and let $\mathfrak{p} = \mathfrak{b}$. Then the affine algebra $\mathfrak{g}$ admits a remarkable family of unitarizable highest weight representations, called integrable highest weight representations. An exposition of the theory of these representations along with some of its beautiful applications may be found in the book [4].

On the other hand, a simple computation shows that for $\mathfrak{p}$ =(conventional $\mathfrak{b}$)and any other choice of ω there is no unitarizable highest weight modules except the trivial one, in sharp contrast to the finite-dimensional theory.

However, again in contrast to the finite-dimensional theory, an affine Lie algebra has several conjugacy classes of Borel subalgebras, and the next natural step is to try "non-conventional" Borel subalgebras.

As a result, we found the following unitarizable highest weight representations of the Lie algebra $\mathfrak{g} = sl_2(\mathbb{C}[z, z^{-1}]) = \mathbb{C}[z, z^{-1}] \otimes sl_2(\mathbb{C})$ (the central charge, i.e. the eigenvalue of c, is trivial) :

Let $V = \mathbb{C}[x_k ; k \in \mathbb{Z}]$ be the space of polonomials in indeterminates x_k . Put $f_k = \begin{pmatrix} 0 & 0 \\ z^k & 0 \end{pmatrix}$, $h_k = \begin{pmatrix} z^k & 0 \\ 0 & -z^k \end{pmatrix}$, $e_k = \begin{pmatrix} 0 & z^k \\ 0 & 0 \end{pmatrix}$.

Let $\mathfrak{p} = \left\{ \begin{pmatrix} a(z) & b(z) \\ 0 & -a(z) \end{pmatrix} \middle| \ a(z), b(z) \in \mathbb{C}[z, z^{-1}] \right\}$,

and let ω be an antilinear anti-involution of the Lie algebra $sl_2(\mathbb{C}[z, z^{-1}])$ defined by

$$\omega\cdot f_k = -e_{-k} \ , \quad \omega\cdot e_k = -f_{-k} \ , \quad \omega\cdot h_k = h_{-k} \ . \tag{0.8}$$

Let m be a finite measure on the circle S^1, not concentrated in a finite number of points ; put $\lambda_k = \int_{S^1} z^k dm$ (e.g. $\lambda_k = \delta_{k,0}$ if m is the Lebesgue measure). Then the map

$$\begin{aligned} f_k &\to x_k \ ; \ h_k \to -\Big(\lambda_k + 2\sum_{j\in\mathbb{Z}} x_{j+k} \frac{\partial}{\partial x_j}\Big), \\ e_k &\to -\Big(\sum_{i} \lambda_{i+k} \frac{\partial}{\partial x_i} + \sum_{i,j\in\mathbb{Z}} x_{i+j+k} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}\Big) \end{aligned} \tag{0.9}$$

defines a unitarizable irreducible representation π_m of the Lie algebra $sl_2(\mathbb{C}[z,z^{-1}])$ on the space V, the polynomial 1 being an eigenvector for $\mathfrak{b}$, if m is positive. The real form $\{x \in sl_2(\mathbb{C}[z, z^{-1}]) \mid \omega\cdot x = -x\}$ of $\mathfrak{g}$ is the real Lie algebra $su(1,1)^{S^1}$ of polynomial maps of the circle into su(1,1). Thus, we have a unitary representation of $su(1,1)^{S^1}$ on the space V, and this construction can easily be generalized to the case of $su(n,1)^{S^1}$. Moreover, we show that these representations together with integrable highest weight representations and representations "concentrated" in a finite number of points, are the only unitarizable highest weight representations of all affine Kac-Moody algebras.

Finally, in a similar fashion, we can construct unitarizable highest weight representations of the Lie algebra $su(n,1)^X$, where X is a set with a finite measure. The corresponding formula for the Hermitian form is identical to that for the truncated

correlation function in quantum field theory.
We will discuss elsewhere the question of integrability of these representations to the corresponding group $SU(n,1)^X$.

1. Generalities.

Let $\mathfrak{g}$ be a Lie algebra with an antilinear anti-involution ω ; then ω extends uniquely to an antilinear anti-involution of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. Let $\mathfrak{p}$ be a subalgebra of $\mathfrak{g}$ satisfying (0.1). Choose a subspace $\mathfrak{n} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}$. Then we have the decompostition into a direct sum of vector spaces : $\mathcal{U}(\mathfrak{g}) = \mathfrak{n}\,\mathcal{U}(\mathfrak{g}) \oplus \mathcal{U}(\mathfrak{p})$. Denote by β the projection on the second summand. Let $\lambda : \mathfrak{p} \to \mathbb{C}$ be a 1-dimensional representation of $\mathfrak{b}$ and extend it to the whole $\mathfrak{U}(\mathfrak{p})$. Put $\mathfrak{U}^\lambda(\mathfrak{p}) = \{ b \in \mathfrak{p} \mid \lambda(b) = 0 \}$. Define a sesquilinear form H on $\mathfrak{U}(\mathfrak{g})$ by

$$(1.1) \qquad H(u,v) = \lambda\big(\beta((\omega.v)u)\big)$$

It is straightforward that

$$(1.2) \quad H(u'u, v) = H(u, (\omega.u')v) \text{ and,}$$

$$(1.3) \quad H(\mathcal{U}^\lambda(\mathfrak{p}), \mathcal{U}(\mathfrak{g})) = 0 \quad .$$

In order to have the Hermitian property of H $\left(\text{i.e. } H(u,u') = \overline{H(u',u)}\right)$ we need to assume

$$(1.4) \quad \lambda(\beta(u)) = \overline{\lambda(\beta(\omega.u))} \quad \text{for } u \in \mathcal{U}(\mathfrak{g})$$

We put

$$(1.5) \quad M(\lambda) = \mathcal{U}(\mathfrak{g}) \,/\, \mathcal{U}(\mathfrak{g})\,\mathcal{U}^\lambda(\mathfrak{p})$$

and define a representation $\tilde{\pi}_\lambda$ of $\mathfrak{g}$ on $M(\lambda)$ via left multiplication. The representation $(M(\lambda), \tilde{\pi}_\lambda)$ is called a (generalized) *Verma module*. We denote by $v_\lambda \in M(\lambda)$ the image of 1. Then (0.2) and (0.3) are satisfied, so that $M(\lambda)$ is a highest weight representation with highest weight λ . Furthermore, since, by (1.3), the kernel of the Hermitian form H contains $\mathcal{U}^\lambda(\mathfrak{p})$, we obtain a contravariant Hermitian form on $M(\lambda)$, also denoted by H. By uniqueness of the contravariant form, H is independent of the choice of $\mathfrak{n}$ (satisfying (1.4)).
Let $I(\lambda)$ be the Kernel of H on $M(\lambda)$ and put $L(\lambda) = M(\lambda)/I(\lambda)$. Then $\tilde{\pi}_\lambda$ induces a highest weight representation π_λ of $\mathfrak{g}$ on the space $L(\lambda)$ and H induces a non-

degenerate contravariant Hermitian form on $L(\lambda)$, also denoted by H. It is clear that conversely, if $L(\lambda)$ is a highest weight representation with a non-degenerate Hermitian form, it is obtained from the Verma module $M(\lambda)$ as above.

Thus, we arrive at the following

Lemma 1.1. Let $\mathfrak{g}$ *be a Lie algebra with an antilinear anti-involution* ω, *and let* $\mathfrak{p}$ *be a subalgebra of* $\mathfrak{g}$ *satisfying (0.1). Let* λ *be a 1-dimensional representation of* $\mathfrak{p}$. *Suppose that we can choose a subspace* $\mathfrak{n} \subset \mathfrak{g}$, *such that* $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}$, *and such that the corresponding projection* $\beta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{p})$ *satisfies (1.4). Then there exists a unique highest weight representation* π_λ *with highest weight* λ *of* $\mathfrak{g}$ *on a vector space* $L(\lambda)$ *with a non-degenerate contravariant Hermitian form.* $\square$

We shall sometimes write $M_{\mathfrak{p},\omega}(\lambda)$, $L_{\mathfrak{p},\omega}(\lambda)$, $\pi_{\lambda;\mathfrak{p}}$ and H_ω instead of $M(\lambda)$, $L(\lambda)$, π_λ, and H, in order to emphasize the dependence on $\mathfrak{p}$ and ω.

Remark : Let $\mathfrak{g}$, $\mathfrak{p}$, ω, λ and $\mathfrak{n}$ be as in Lemma 1.1, and let $\mathfrak{b} \subset \mathfrak{p}$ be a subalgebra such that $\mathfrak{b} + \omega\mathfrak{b} = \mathfrak{g}$ and such that there exists a subspace $\mathfrak{u} \subset \mathfrak{p}$ with $\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{u}$ and

$$(1.6) \qquad \lambda|_{\mathfrak{u}} = 0 \quad .$$

Let $\bar{\lambda} = \lambda|_{\mathfrak{b}}$. Then the highest weight representations $\pi_{\lambda;\mathfrak{p}}$ and $\pi_{\bar{\lambda},\mathfrak{b}}$ are equivalent. Indeed, since $\mathcal{U}^{\lambda}(\mathfrak{p}) \supset \mathcal{U}^{\bar{\lambda}}(\mathfrak{b})$, there exists a surjective $\mathfrak{g}$-map $\tilde{\varphi} : M_{\mathfrak{b},\omega}(\bar{\lambda}) \to M_{\mathfrak{p},\omega}(\lambda)$. Using (1.6), one easily checks that Ker $\tilde{\varphi}$ is contained in the Kernel of H_ω. It follows that $\tilde{\varphi}$ induces an equivalence of representations

$$\varphi : L_{\mathfrak{b},\omega}(\bar{\lambda}) \xrightarrow{\sim} L_{\mathfrak{p},\omega}(\lambda) \quad . \square$$

Example : The canonical commutation relations representation L(a) of the infinite-dimensional Heisenberg algebra (see the introduction) is a Verma module. Choose $\mathfrak{u} = \sum_i \mathbb{C} q_i$; then condition (1.4) holds whenever $a \in \mathbb{R}$. Hence L(a) carries a Hermitian form whenever $a \in \mathbb{R}$ (it is clear that this condition is also necessary). The representation L(a) is irreducible iff $a \neq 0$. Thus L(a) is an (irreducible) highest weight representation with a contravariant Hermitian form if $a \in \mathbb{R} \setminus \{0\}$. $\square$

2. Involutions and Borel subalgebras.

For basic definitions and facts of the theory of Kac-Moody algebras we refer to the book [4].

Let $\mathfrak{g} = \mathfrak{g}'(A)$ be a Kac-Moody algebra associated to the generalized Cartan matrix A. This is a Lie algebra with *Chevalley generators* $e_i, f_i, \alpha_i^\vee$ $(i = 1, \dots, n)$ satisfying certain well-known relations. Let Δ be the set of roots of $\mathfrak{g}$ and let

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}_\alpha$$

be the *root space decomposition* of $\mathfrak{g}$.

We call a subset Δ_+ of Δ a *set of positive roots* if the following three properties hold :

(2.1) if $\alpha, \beta \in \Delta_+$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta_+$;

(2.2) if $\alpha \in \Delta$, then either α or $-\alpha$ lie in Δ_+ ;

(2.3) if $\alpha \in \Delta_+$, then $-\alpha \notin \Delta_+$.

Example : Let Π^{st} be the set of roots corresponding to the generators e_i $(i=1,\dots n)$ of $\mathfrak{g}$; we call Π^{st} the *standard set of simple roots*. Let $\Delta_+^{st} = \{ \sum_i k_i \alpha_i \mid k_i \in \mathbb{Z}_+ = \{0,1,2,\dots\}$ and $\alpha_i \in \Pi^{st} \}$. Then Δ_+^{st} is the conventional set of positive roots (see [4]), which we call the *standard set of positive roots*. □
Given a set of positive roots Δ_+, one associates to it the *Borel subalgebra* $\mathfrak{b} = \bigoplus_{\alpha \in \Delta_+ \cup \{0\}} \mathfrak{g}_\alpha$. The Borel subalgebra associated to Δ_+^{st} is denoted by $\mathfrak{b}^{st}$.
A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{b}$ is called a *parabolic subalgebra.*
An antilinear anti-involution ω of $\mathfrak{g}$ is called *consistent* if $\omega . \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$
It is clear that, replacing e_i by λe_i and f_i by $\lambda^{-1} f_i$, one can bring ω to the following form :

$$\omega \cdot e_i = \pm f_i \qquad (i = 1, \dots, n)$$

An important example is the *compact antilinear anti-involution* ω_c defined by :

$$\omega_c \cdot e_i = f_i \qquad (i = 1, \dots, n)$$

Note that if ω is a consistent antilinear anti-involution and $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, then condition (0.1) holds. One can show that conversely, if (0.1) holds for some $\mathfrak{p}$ and ω , then ω is conjugate to a consistent antilinear anti-involution (cf. [5]).
Let W be the Weyl group of the Kac-Moody algebra $\mathfrak{g}$; for a real root α, let $r_\alpha \in W$ denotes the reflection with respect to α .
We start with the classification of sets of positive roots in the finite-dimensional case.

Lemma 2.1. If Δ is a finite root system, then a set of positive roots Δ_+ is W-conjugate to Δ_+^{st} .
Proof. If $\Pi^{st} \subset \Delta_+$, then $\Delta_+ = \Delta_+^{st}$ and there is nothing to prove. Otherwise, there exists $\alpha \in \Pi^{st} \setminus \Delta_+$, and $| (r_\alpha . \Delta_+) \cap - \Delta_+^{st} | < | \Delta_+ \cap - \Delta_+^{st} |$. After a finite number of such steps we get $\Delta_+ = \Delta_+^{st}$. □
By Lemma 2.1, a Borel subalgebra of a finite-dimensional simple Lie algebra $\dot{\mathfrak{g}}$ is conjugate to $\mathfrak{b}^{st}$.

The situation is different for infinite-dimensional Kac-Moody algebras. For example, let $\mathfrak{g}$ be an affine Kac-Moody algebra associated to $\dot{\mathfrak{g}}$ ("non-twisted" case) ; then the subalgebra $\mathfrak{b}$ defined by (0.7) is $\mathfrak{b}^{st}$. On the other hand, putting $\dot{\mathfrak{b}} = \dot{\mathfrak{h}} \oplus \dot{\mathfrak{n}}$, where $\dot{\mathfrak{h}}$ is a Cartan subalgebra and $\dot{\mathfrak{n}}$ a maximal nilpotent subalgebra of $\dot{\mathfrak{g}}$, we have another Borel subalgebra of $\mathfrak{g}$, which is not conjugate to $\mathfrak{b}^{st}$, namely the *natural* Borel subalgebra :

$$\mathfrak{b}^{nat} = \mathbb{C}c \oplus (\mathbb{C}[z] \otimes_{\mathbb{C}} \dot{\mathfrak{h}}) \oplus (\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathfrak{n}).$$

A "twisted" affine algebra is a fixed point set in $\mathfrak{g}$ of a non-trivial symmetry of Chevalley generators, and we take $\mathfrak{b}^{nat}$ to be the intersection with this set of the natural Borel subalgebra of $\mathfrak{g}$.

To show that $\mathfrak{b}^{st}$ and $\mathfrak{b}^{nat}$ are not conjugate, note that (similar fact holds for any Kac-Moody algebra) :

$$\mathfrak{b}^{st} = (\dot{\mathfrak{h}} + \mathbb{C}c) \oplus [\mathfrak{b}^{st}, \mathfrak{b}^{st}] \;;$$

on the other hand one has

$$\mathfrak{b}^{nat} = (\mathbb{C}c + \mathbb{C}[z] \otimes_{\mathbb{C}} \dot{\mathfrak{h}}) \oplus [\mathfrak{b}^{nat}, \mathfrak{b}^{nat}] .$$

Now we turn to the classification, up to W-equivalence, of the subsets of positive roots of an affine root system ; this is equivalent to the classification of Borel subalgebras of an affine Kac-Moody algebra $\mathfrak{g}$ up to conjugation.

Let Δ be the root system of an affine Kac-Moody algebra $\mathfrak{g}$ and let $\Pi^{st} = \{\alpha_0, \alpha_1, \dots, \alpha_\ell\}$ be the standard set of simple roots (the ordering of simple roots is that of [4]).

Put $\dot{\Pi} = \{\alpha_1, \dots, \alpha_\ell\}$, $\dot{\Delta} = \{\alpha \in \Delta \mid \alpha = \sum_{i=1}^{\ell} k_i \alpha_i\}$,

$\dot{\Delta}_+ = \{\alpha \in \dot{\Delta} \mid \alpha = \sum_{i=1}^{\ell} k_i \alpha_i$ with $k_i \geqslant 0\}$.

Then $\dot{\Delta}$ is the root system of the "underlying" finite-dimensional simple Lie algebra $\dot{\mathfrak{g}}$. Let δ be the unique indivisible imaginary root from Δ_+^{st} . Recall that the sets Δ and Δ_+^{st} can be easily reconstructed in terms of the finite root system $\dot{\Delta}$ and the root δ [4, Chapter 6] :

Example. Let $\mathfrak{g}$ be an affine Lie algebra associated to $\dot{\mathfrak{g}}$. Then :

$$(2.4)\quad \Delta = \{\alpha + n\delta \mid \alpha \in \dot{\Delta} \cup \{0\} , n \in \mathbb{Z}\} \setminus \{0\} ,$$

$$(2.5)\quad \Delta_+^{st} = \{\alpha + n\delta \mid \alpha \in \dot{\Delta} \cup \{0\} , n > 0\} \cup \dot{\Delta}_+ . \quad \square$$

Given a root space decomposition $\dot{\mathfrak{g}} = (\bigoplus_{\alpha \in \dot{\Delta}} \dot{\mathfrak{g}}_\alpha) \oplus \dot{\mathfrak{h}}$ we have :

$\mathfrak{g}_{\alpha + k\delta} = z^k \otimes \mathfrak{g}_\alpha$ if $\alpha \in \dot{\Delta}$; $\mathfrak{g}_{k\delta} = z^k \otimes \dot{\mathfrak{h}}$.

For a "twisted" affine Lie algebra, the set of roots is a subset of $(\Delta \cup \frac{1}{2}\Delta) \cap Q$, invariant under the shift by $k\delta$ (k = 2 or 3), where Δ is defined by (2.4) (see [4, Chapter 6] for details).

Let Δ_+ be a set of positive roots of an affine root system Δ . Replacing Δ_+ by $-\Delta_+$ if necessary, we can assume that $\delta \in \Delta_+$. A root $\alpha \in \Delta_+$ is called *bad* if all the roots of the form $\alpha+n\delta$ $(n\in\mathbb{Z})$ lie in Δ_+ ; otherwise, a root $\alpha\in\Delta_+$ is called *good.* It is clear that δ is a good root for any Δ_+ and that all the roots from Δ_+^{st} are good.

Lemma 2.2. Let Δ_+ be a set of positive roots of an affine root system Δ, such that $\delta\in\Delta_+$.

(a) If $\alpha\in\Delta_+$ is good, then there exists $s\in\mathbb{Z}_+$ such that a root $\alpha+n\delta$ lies in Δ_+ iff $n\geq -s$.

(b) If $\alpha,\beta,\alpha+\beta\in\Delta_+$ and α is bad, then $\alpha+\beta$ is bad.

(c) If $\alpha,\beta,\alpha-\beta\in\Delta_+$ and α and β are good, then $\alpha-\beta$ is good.

(d) If $\alpha,\beta,\alpha+\beta\in\Delta_+$ and α and β are good, then $\alpha+\beta$ is good.

Proof. (*a*) and (*b*) are obvious and (*c*) follows from (*b*). If α and β are good roots for Δ_+, but all the roots of the form $\alpha+\beta+n\delta$ $(n\in\mathbb{Z})$ lie in Δ_+, then all the roots of the form $-\alpha-\beta+n\delta$ lie in $-\Delta_+$. But $\alpha+s\delta\in-\Delta_+$ for some s since α is good, hence all the roots of the form $-\beta+n\delta$ lie in $-\Delta_+$ and all the roots of the form $\beta+n\delta$ lie in Δ_+. This contradiction proves (*d*). $\square$

Let X be a subset of $\dot{\Pi}$. We associate to X a subset of positive roots Δ_+^X of Δ as follows. In the non-twisted case we put :

$$\Delta_+^X=\{\alpha+n\delta \mid \alpha\in\dot{\Delta}_+\setminus\mathbb{Z}X,\ n\in\mathbb{Z}\}\cup \{\alpha+n\delta \mid \alpha\in(\dot{\Delta}\cap\mathbb{Z}X)\cup\{0\},\ n>0\}\cup\dot{\Delta}_+ \qquad (2.6)$$

In the twisted case, we put $\Delta_+^X=(\Delta_+\cup\frac{1}{2}\Delta_+)\cap\Delta$, where Δ_+ is the set defined as in (2.6).

Proposition 2.1. If Δ is an affine root system, then every set of positive roots is $W\times\{\pm1\}$-conjugate to one of the sets Δ_+^X.

Proof. Let Δ_+ be a subset of positive roots of Δ ; we can assume that $\delta\in\Delta_+$. Let X $=\{\alpha_i\in\dot{\Pi} \mid$ either α_i or $-\alpha_i$ is a good root of $\Delta_+\}$. Put $\dot{\Delta}_X=\mathbb{Z}X\cap\dot{\Delta}$,

$$\Delta_X=\{\alpha+n\delta \mid \alpha\in\dot{\Delta}_X\cup\tfrac{1}{2}\dot{\Delta}_X\cup\{0\},\ n\in\tfrac{1}{2}\mathbb{Z}\}\cap\Delta,$$

$$\Delta_{X+}=\Delta_X\cap\Delta_+.$$

By lemma 2.2, for each $\alpha\in\Delta_X$, either α or $-\alpha$ is a good root of Δ_+. It follows that $|\Delta_{X+}\cap-\Delta_+^{st}|<\infty$. Applying the same argument as that in the proof of Lemma 2.1, we may assume that $\Delta_{X+}\subset\Delta_+^{st}$. Applying again this argument we may assume that $\dot{\Pi}\subset\Delta_+^{st}$. But then, by Lemma 2.2, Δ_{X+} is the set of good roots of Δ_+. It follows that $\Delta_+=\Delta_+^X$. $\square$

3. A list of unitarizable highest weight representations.

Let $\mathfrak{g}$ be an arbitrary Kac-Moody algebra, let $\mathfrak{b}$ be the standard Borel subalgebra, of $\mathfrak{g}$, let $\omega=\omega_c$ and let $\lambda:\mathfrak{b}\to\mathbb{C}$ be a 1-dimensional representation of $\mathfrak{b}$

defined by

$$\lambda(e_i) = 0 \quad , \quad \lambda(\alpha_i^\vee) = m_i \in \mathbb{Z}_+ \quad (i = 1,\dots,n) \ .$$

Then the representation $\pi_{\lambda;\mathfrak{b},\omega_c}$ is unitarizable ([4, Chapter 11]). These representations are called *integrable* highest weight representations. In particular, if $\mathfrak{g}$ is finite-dimensional, thesse representations are precisely all finite-dimensional irreducible representations of $\mathfrak{g}$.

Furthermore, it is well-known that if $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, then an infinte-dimensional highest weight representation $\pi_{\lambda;\mathfrak{b},\omega}$ is unitarizable only if ω is a consistent antilinear anti-involution which corresponds to a Hermitian symmetric space, and all possibilities for λ are listed in [1],[3].

There is the following "elementary" way to construct a unitarizable highest weight representation π_λ of an affine Lie algebra $\mathfrak{g}$. First, we take $\pi_\lambda(c) = 0$ so that π can be viewed as a representation of the Lie algebra $\mathbb{C}[Z,Z^{-1}]\otimes_{\mathbb{C}}\dot{\mathfrak{g}}$ in the "non twisted case" or its subalgebra in the "twisted" case. Furthermore, fix N non-zero complex numbers $c_1,\dots,c_N$ and denote by $\varphi_i : \mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}}\dot{\mathfrak{g}} \to \dot{\mathfrak{g}}$ the evaluation map at c_i i.e. $\varphi_i(z^k \otimes g) = c_i^k g$. Fix a Borel subalgebra $\dot{\mathfrak{b}}$ of $\dot{\mathfrak{g}}$ and a consistent antilinear anti-involution $\dot{\omega}$ of $\dot{\mathfrak{g}}$. Let π_{λ_i} $(i=1,\dots,N)$ be a unitarizable highest weight representation of $\dot{\mathfrak{g}}$ on L (λ_i) = $L_{\dot{\mathfrak{b}},\dot{\omega}}(\lambda_i)$. Then $\pi_{\lambda_i}\cdot\varphi_i$ is a unitarizable highest weight representation of $\mathfrak{g}$.

Let $\mathfrak{p} = \mathbb{C}[z,z^{-1}]\otimes\dot{\mathfrak{b}}$, define a representation $\mathfrak{p}\to\mathbb{C}$ by $\lambda(z^k\otimes b) = \sum_i c_i^k \lambda_i(b)$ and an antilinear anti-involution ω of $\mathfrak{g}$ by $\omega.(z^k\otimes g) = z^{-k}\otimes(\dot{\omega}.g)$

Then we have :

$$L_{\mathfrak{p},\omega}(\lambda) = L(\lambda_1)\otimes\dots\dots\otimes L(\lambda_N) \ ,$$

$$\pi_{\lambda;\mathfrak{p}} \quad = (\pi_{\lambda_1}\cdot\varphi_1)\otimes\dots\otimes(\pi_{\lambda_N}\cdot\varphi_N)$$

Thus, the representation $\pi_{\lambda;\mathfrak{p}}$ of $\mathfrak{g}$ on the space $L_{\mathfrak{p},\omega}(\lambda)$ is unitarizable. We call these representations *elementary*.

Finally, let $\mathfrak{g} = s\ell_{\ell+1}(\mathbb{C}[z,z^{-1}])$ (i.e. we assume that the center $\mathbb{C}c$ acts trivially), let $\mathfrak{p} = \{(a_{ij}(z))\in\mathfrak{g} \mid a_{ij}=0 \text{ for } i>j\} \supset \mathfrak{b}^{nat}$, and let $\omega.(a_{ij}(z)) = (\varepsilon_{ij}\,\bar{a}_{ji}(z^{-1}))$, where $\varepsilon_{ij}=1$ if $i\neq 1$ or $j\neq 1$ or $i=j=1$, and $\varepsilon_{ij}=-1$ otherwise (for a(z) = $\sum c_i\, z^i \in \mathbb{C}[z,z^{-1}]$ we write $\bar{a}$ (z) = $\sum \bar{c}_i\, z^i$).

Let m be a finite positive measure on the unit circle $S^1\subset\mathbb{C}$. Define a linear functional $e_m : \mathbb{C}[z,z^{-1}]\to\mathbb{C}$ by $e_m(a(z)) = \int_{S^1} a(z)\,dm$. Define a representation $\lambda_m : \mathfrak{p}\to\mathbb{C}$ by $\lambda_m((a_{ij}(z)) = -e_m(a_{11}(z)))$.

Then the representation π_{λ_m} of $s\ell_{\ell+1}(\mathbb{C}[z,z^{-1}])$ on the vector space $L_{\mathfrak{p},\omega}(\lambda_m)$ is called *exceptional*. We will show, as a part of a more general result, that they are unitarizable.

Now we can state our first main result.

Theorem 3.1. Let $\mathfrak{g}$ *be an affine Lie algebra, let* ω *be a consistent antilinear anti-involution of* $\mathfrak{g}$ *and let* $\mathfrak{b}$ *be a Borel subalgebra of* $\mathfrak{g}$*. Let* $\lambda : \mathfrak{b} \to \mathbb{C}$ *be a 1-dimensional representation of* $\mathfrak{b}$*. Then the representation* $\pi_{\lambda;\mathfrak{b},\omega}$ *of* $\mathfrak{g}$ *on the space* $L_{\mathfrak{b},\omega}(\lambda)$ *is unitarizable if and only if it is equivalent to either an integrable representation, or an elementary representation, or an exceptional representation.*

4. Elimination.

We now begin to prove Theorem 3.1. This section is devoted to the negative, i.e. non-unitarizable, aspects.

Let us first look at the affine Lie algebra $\hat{L}(sl_2)$ associated to $sl_2(\mathbb{C})$. We have that $sl_2(\mathbb{C}) = \mathrm{span}\,\{e, f, h\}$ with commutation relations

$$[e,f] = h \qquad ; \qquad [h,e] = 2e \qquad ; \qquad [h,f] = -2f \tag{4.1}$$

We write elements $z^n \otimes X$ of $\mathbb{C}[z, z^{-1}] \otimes sl_2$ as $z^n X$. We have that

$$\hat{L}(sl_2) = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} sl_2 \oplus \mathbb{C}c \quad , \tag{4.2}$$

where, in particular,

$$[z^n e, z^m f] = n\,\delta_{n,-m}\, c \tag{4.3}$$

According to Proposition 2.1 there are, up to conjugacy, two distinct sets of positive roots. In either case, a consistent involution ω is of the form

$$\omega(e) = \varepsilon_1 f \qquad ; \qquad \omega(zh) = \varepsilon_2\, zh \tag{4.4}$$

where $\varepsilon_1^2 = \varepsilon_2^2 = 1$. There are essensially four distinct parabolic subalgebras compatible with (some of) the involutions above :

$$\begin{aligned} \mathfrak{b}^{st} &= h \oplus \mathrm{span}\,\{z^k e \mid k \geq 0\} \oplus \mathrm{span}\,\{z^k f,\ z^k h \mid k > 0\}, \\ \mathfrak{p}^{nat} &= \mathrm{span}\,\{z^k e,\ z^k h \mid k \in \mathbb{Z}\}, \\ \mathfrak{b}^{nat} &= \mathrm{span}\,\{z^k h \mid k \geq 0\} \oplus \mathrm{span}\,\{z^k e \mid k \in \mathbb{Z}\}, \text{ and} \end{aligned} \tag{4.5}$$

$$\mathbf{p}^{kt} = \operatorname{span}\{ z^k h, z^k e, z^k f \mid k \geqslant 0 \}$$

Assuming that $L(\lambda) = L_{b,\omega}(\lambda)$ is unitarizable, we now analyze the restrictions this requirement imposes on the data. First we observe, as remarked after Lemma 1.1, that $\mathbf{p}^{kt}$ does not give other unitarizable modules than does $\mathbf{b}^{st}$. A second simplifying observation is, that due to the symmetry of the Hermitian form, in the case of $\mathbf{b}^{nat}$, $\lambda(z^n h) = -\delta_{n,0}\, a$ for some $a \in \mathbb{R}$. Hence, the treatment of this case is covered by that of $\mathbf{p}^{nat}$.
Returning to (4.4), it follows that

$$(4.6) \quad \omega(z^n h) = \varepsilon_2^n z^{-n} h \qquad\qquad \omega(z^n f) = \varepsilon_1 \varepsilon_2^n z^{-n} e.$$

We let $(z^n h)\cdot v_\lambda = \lambda(z^n h)\, v_\lambda = \lambda_n\, v_\lambda$, and $c \cdot v_\lambda = c\, v_\lambda$.
Let us turn to an examination of the standard set of positive roots:
Firstly, the symmetry of H forces $\lambda_n = \lambda\, \delta_{n,0}$. Secondly, we have that for all $k, n \in \mathbb{N}$

$$(4.7) \quad \begin{aligned} &H\left((z^{-n} f)(z^{-2} e)^k v_\lambda,\ (z^{-n} f)(z^{-2} e)^k v_\lambda\right) = \\ &\varepsilon_1 (\varepsilon_2)^n (\lambda + 2k + nc)\, H\left((z^{-2} e)^k v_\lambda,\ (z^{-2} e)^k v_\lambda\right). \end{aligned}$$

It is no loss of generality to assume that $(z^{-2} e)^k v_\lambda \neq 0$ since we may otherwise just replace it with some $(z^{-j} e)^k v_\lambda$, $j \in \mathbb{N}$. Thus if the module is unitarizable and non-trivial, then $\varepsilon_1 = \varepsilon_2 = 1$. This is the compact antilinear anti-involution, and hence by the sl_2-theory, $L(\lambda)$ is an integrable representation (observe that $\lambda, c \in \mathbb{Z}_+$, and $c \geqslant \lambda$).
In the natural case we have

$$(4.8) \quad H(z^n f\, v_\lambda,\ z^n f\, v_\lambda) = \varepsilon_1 \varepsilon_2^n (-nc + \lambda) \quad \text{for all } n \in \mathbb{Z}.$$

Thus, positivity of H implies that $c = 0$ and $\varepsilon_2 = 1$. Further we observe that for a general $a = \sum a_n z^n \in \mathbb{C}[z, z^{-1}]$

$$(4.9) \quad H(a f v_\lambda,\ a f v_\lambda) = \varepsilon_1 \lambda (a a^*)$$

where $a^* = \sum \bar{a}_n z^{-n}$. Thus, $\varepsilon_1 \lambda$ is a positive linear functional on $\mathbb{C}[z, z^{-1}]$
It is then natural to represent $\mathbb{C}[z, z^{-1}]$ as the set of functions on S^1,
$\mathbb{C}[z, z^{-1}] = \left\{ \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \mid \text{all but a finite number of the } a_n\text{'s are non-zero} \right\}$
The positivity of $\varepsilon_1 \lambda$ implies continuity and thus it extends to a positive Radon measure μ (i.e. locally finite) on S^1. It follows from the general result ; Proposition 6.2 below, that the module $L_{\mathbf{b}^{nat}, \omega}(\lambda)$ is irreducible except in the case where

supp (μ) is contained in a finite number of points.
Consider then the case $\varepsilon_1 = 1$. This corresponds to a compact (su(2)) situation. Thus there must exist an integer $i_0 \in \mathbb{N}$ and a non-trivial invariant subspace S_{i_0} such that $\forall i \geqslant i_0 : f^i \in S_{i_0}$. Thus, the measure μ must be finitely supported. It is straightforward that in this case we do have unitarity.
Finally, as we shall see below, the case $\varepsilon_1 = -1$ leads to unitarity.
Let us summarize :
Lemma 4.1. For $\hat{L}(sl_2)$ *only the following situations may lead to unitarity.*

$$(4.10)\quad \mathfrak{b}^{st} : \omega(e) = f \ , \quad \omega(f) = e \ , \quad \omega(z h) = z^{-1} h \ \text{and} \quad c \geqslant |\lambda|$$

$$\text{and} \quad \mathfrak{p}^{nat} : \omega(e) = \varepsilon f \ , \quad \omega(f) = \varepsilon e \ , \quad \omega(z h) = z^{-1} h$$

$$\text{and} \quad \lambda(z^n n) = \lambda_n = \varepsilon \int_{S^1} e^{in\theta} \, d\mu(\theta)$$

where if $\varepsilon = 1$, μ *is finitely supported, and always,* $\varepsilon\mu$ *is a positive Radon measure.* ☐

An immediate consequence of this lemma for an arbitrary affine Lie algebra is that for the standard system of positive roots only the compact involution may lead to unitarity (and hence $L(\lambda)$ must be an integrable representation).Turning now to the natural or partly natural ("non-standard") situation we will assume that the measures involved are not finitely supported since this case is easily dealt with (cf. §3). It follows from Lemma 4.1 that now c=0, and $\omega(e_\alpha) = -e_{-\alpha}$ ("non-compact") on the non-standard part.

Let $\dot{\mathfrak{g}}$ denote a simple complex Lie algebra of finite diemnsion. It is well-known [2] and quite straight forward to see that besides conjugation in a compact real form, the only situations that lead to unitarizable highest weight modules are those where $\dot{\mathfrak{g}}$ has a real form corresponding to a Hermitian symmetric space, and where ω is conjugation with respect to this.
Specifically, let $\dot{\mathfrak{g}} = (\dot{\mathfrak{g}}_0)^{\mathbb{C}}$ and let $\dot{\mathfrak{g}}_0 = \dot{\mathfrak{k}} \oplus \dot{\mathfrak{p}}$ be a Cartan decomposition of the real Lie algebra $\dot{\mathfrak{g}}_0$. Then $\dot{\mathfrak{k}}$ has a one-dimensional center $\mathfrak{z} = \mathbb{R} h_0$, where h_0 is chosen such that its eigenvalues under the adjoint action on $\dot{\mathfrak{p}}^{\mathbb{C}}$ are $\pm i$. Let

$$(4.11)\quad \dot{\mathfrak{p}}^{+} = \left\{ X \in \dot{\mathfrak{p}}^{\mathbb{C}} \ \middle|\ [h_0, X] = i X \right\}$$

$$\text{and} \quad \dot{\mathfrak{p}}^{-} = \left\{ X \in \dot{\mathfrak{p}}^{\mathbb{C}} \ \middle|\ [h_0, X] = -i X \right\}$$

Let $\dot{\mathfrak{k}}_1 = [\dot{\mathfrak{k}}, \dot{\mathfrak{k}}]$ and let i $\dot{\mathfrak{h}}$ be a maximal abelian subalgebra of $\dot{\mathfrak{k}}$. Then $\dot{\mathfrak{k}} = \dot{\mathfrak{k}}_1 \oplus R h_0$
$i\dot{\mathfrak{h}} = (i\dot{\mathfrak{h}} \cap \dot{\mathfrak{k}}_1) \oplus \mathbb{R} h_0$, $(i\dot{\mathfrak{h}} \cap \dot{\mathfrak{k}}_1)^{\mathbb{C}}$ is a Cartan subalgebra of $\dot{\mathfrak{k}}_1^{\mathbb{C}}$, and $\dot{\mathfrak{h}} = (i\dot{\mathfrak{h}})^{\mathbb{C}}$ is a Cartan subalgebra of $\dot{\mathfrak{g}}$. The sets of compact and non-compact roots of $\dot{\mathfrak{g}}$ relative to $\dot{\mathfrak{h}}$ are denoted Δ_c and Δ_n , respectively ; $\Delta = \Delta_c \cup \Delta_n$. We choose an ordering

of Δ such that $\dot{\mathfrak{p}}^{+} = \sum_{\alpha \in \Delta_n^+} \dot{\mathfrak{g}}_\alpha$

For $\gamma \in \Delta$ let H_γ be the unique element of $\dot{\mathfrak{h}} \cap [\dot{\mathfrak{g}}_\gamma, \dot{\mathfrak{g}}_{-\gamma}]$ for which $\gamma(H_\gamma) = 2$.
For $\alpha \in \Delta_n^+$ choose $e_\alpha \in \dot{\mathfrak{g}}_\alpha$ such that

$$[e_\alpha, \omega(e_\alpha)] = H_\alpha, \tag{4.12}$$

and let $e_{-\alpha} = \omega(e_\alpha)$.
Since in any non-standard situation c = 0, and since by the assumptions on the measures we are looking at irreducible modules (cf. §6), it follows from the preceding analysis that $\mathcal{U}(\mathbb{C}[z, z^{-1}] \otimes \dot{\mathfrak{k}}_1^{\mathbb{C}}) \cdot v_\lambda$ must be in the Kernel of the Hermitian form. Thus our module is "scalar", that is, of the form

$$M(\lambda) = \mathcal{U}(\mathbb{C}[z, z^{-1}] \otimes \dot{\mathfrak{p}}^{-}) \cdot v_\lambda \tag{4.13}$$

where $(\mathbb{C}[z, z^{-1}] \otimes \dot{\mathfrak{p}}^{+}) \cdot v_\lambda = 0$, $(\mathbb{C}[z, z^{-1}] \otimes \dot{\mathfrak{k}}_1) \cdot v_\lambda = 0$

and $(z^n h_0) \cdot v_\lambda = -\left(\int_{S^1} e^{in\theta} d\mu(\theta)\right) v_\lambda$

for some positive measure μ which is not supported by a finite number of points.
Observe that

$$\lambda_0 = -\int_{S^1} d\mu(\theta) < 0. \tag{4.14}$$

We now recall a result about the scalar modules $\mathcal{U}(\dot{\mathfrak{p}}^{-}) \cdot v_\lambda$ with v_λ as above ([6],[7]). *If there are at least two perpendicular non-compact roots (i.e. the real rank is greater than one) then for (a finite number of) critical values $c_1, c_2, \dots$ $\dots$, $0 > c_1 > c_2 > \dots$, the module $\mathcal{U}(\dot{\mathfrak{p}}^{-}) \cdot v_\lambda$ with $\lambda_0 = c_i$, $i = 1, 2, \dots$ is reducible. In fact, the Hermitian form restricted to this space is degenerate.* Thus, when λ_0 equals one of these critical values, by the irreducibility of $M(\lambda)$ (Proposition 6.2) there can be no unitarity. Finally, for any $\lambda_0 < 0$, it is easy to deform the measure μ, without destroying the irreducibility, into a measure $\tilde{\mu}$ which yields a λ_0 among these critical walues. We may thus state:

Proposition 4.2. In case the real rank of $\dot{\mathfrak{g}}$ is greater than one there can be no unitarizable module based on a natural parabolic subalgebra and on infinitely supported measures. $\square$

Remark : Hence only $\dot{\mathfrak{g}} = su(n,1)$ remains. $\square$

To bring our analysis to a conclusion we now turn to the "twisted" affine Lie algebras

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad \text{or} \quad \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \tag{4.15}$$

We maintain the assumption that the measures are not finitely supported. In a non-standard situation, c = 0, and thus, restricted to $\mathfrak{g}_0$ one must be in a non-standard or trivial situation. However, if the module is trivial on $\mathfrak{g}_0$, it is trivial on $\mathfrak{g}$. Thus, by the preceding results, the only case that needs to be considered is $A_2^{(2)}$ with a natural parabolic subalgebra in $\mathfrak{g}_0$:

(4.16) $A_2^{(2)} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and

$\dot{\mathfrak{g}}_0 = sl_2 = \text{span}\{e, f, h\}$ as in (3.1).

Observe that the non-standardness of $\mathfrak{g}_0$ forces a non-standardness on the full algebra. Inside $\mathfrak{g}_1$ one can easily find elements u_1^+, u_1^-, u_2^+, and u_2^- such that

$$(4.17)\qquad \text{span}\{u_1^+, u_1^-, h\} \cong \text{span}\{u_2^+, u_2^-, h\} \cong sl_2$$

as Lie algebras, and such that

$$[e, u_1^+] = -u_2^+, \quad \text{and} \quad [f, u_1^-] = u_2^-.$$

Since we must have $\omega(e) = -f$ and $\omega(u_1^+) = -u_1^-$ it follows that $\omega(u_2^+) = u_2^-$, and this is impossible (it is compact). Thus at this level, there are no unitarizable modules.

Thus, the only highest weight representations which may be unitarizable are those listed in Theorem 3.1.

5. Unitarity.

Let R denote a (non-commutative) associative algebra over $\mathbb{C}$ and let φ be a trace on R, i.e. a linear map of R into $\mathbb{C}$ which satisfies

$$(5.1)\qquad \varphi(ab) = \varphi(ba) \qquad \text{for all } a, b \in R.$$

We define the Lie algebra

$$(5.2)\qquad sl_2(R, \varphi) = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \,\middle|\, a_i \in R,\ i = 1,2,3,4 \text{ and } \varphi(a_1 + a_4) = 0 \right\},$$

and we consider the Verma module $M(\varphi)$ defined by the property that there exists a non-zero vector v_φ such that

$$(5.3)\qquad \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix} v_\varphi = \varphi(a_1) \cdot v_\varphi$$

Let $i \in \mathbb{N}$. We say that $\gamma = (\gamma_1, \dots, \gamma_s) \in \mathbb{N}^s$ is an s-*partition of* i if

$$(5.4)\qquad i = \gamma_1 + \gamma_2 + \cdots + \gamma_s, \quad \text{and}$$

$$\gamma_1 \geq \gamma_2 \geq \cdots\cdots \geq \gamma_s > 0 .$$

We let $\mathrm{Par}_s(i)$ denote the set of all such s-tuples. Further, if $\gamma \in \mathrm{Par}_s(i)$ we let $D_\gamma(i)$ denote the set of distributions of i objects in the Young diagram of γ. If $(d_1, \dots, d_i) \in R^i$ and if $t_\gamma \in D_\gamma(i)$ we let, for $j = 1, \dots, s$, $d(t_\gamma)_j$ denote the product of the elements in the jth row.

Let $\gamma \in \mathrm{Par}_s(N)$. Utilizing the fact that φ is a trace, we will say that $\pi_1^1 \times \pi_2^1 \in S_N \times S_N$ is equivalent to $\pi_1 \times \pi_2 \in S_N \times S_N$, where S_N denotes the group of permutations of N letters, if for all $z_1, \dots, z_i$, $\omega_1, \dots \omega_i \in R$,

$$(5.5)\ \varphi(z_{\pi_1'(1)}\, \omega_{\pi_2'(1)} \cdots z_{\pi_1'(\gamma_1)}\, \omega_{\pi_2'(\gamma_1)}) \cdots\cdots \varphi(z_{\pi_1'(\gamma_1+\cdots+\gamma_{s-1}+1)}\, \omega_{\pi_2'(\gamma_1+\cdots+\gamma_{s-1}+1)} \cdots)$$

can be obtained from the analogous expression for $\pi_1 \times \pi_2$ by a permutation of the s factors $\varphi(\cdots)$ and/or by cyclic permutation of the variables (e.q. $\varphi(z_3 \omega_3\, z_1 \omega_1\, z_2 \omega_2) = \varphi(z_2 \omega_2\, z_3 \omega_3\, z_1 \omega_1)$.

The set of equivalence classes is denoted by $(S_N \times S_N)(\gamma)$.

Lemma 5.1. *Let* $z_1, \dots, z_N, \omega_1, \dots, \omega_N \in R$. *Then in* $M(\varphi)$,

$$(-1)^N \begin{pmatrix} 0 & z_1 \\ 0 & 0 \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 & z_N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \omega_1 & 0 \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 & 0 \\ \omega_N & 0 \end{pmatrix} \cdot v_\varphi =$$

$$(5.6)\ \sum_{s=1}^{N} \sum_{\gamma \in \mathrm{Par}_s(N)} \sum_{[\pi_1 \times \pi_2] \in (S_N \times S_N)(\gamma)} (-1)^{\gamma_1} \varphi(z_{\pi_1(1)}\, \omega_{\pi_2(1)} \cdots z_{\pi_1(\gamma_1)}\, \omega_{\pi_2(\gamma_1)}) \cdots (-1)^{\gamma_s} \varphi(\cdots z_{\pi_1(N)} \omega_{\pi_2(N)}) \cdot v_\varphi$$

Proof : We proceed by induction. N = 1 is trivial, so assume (5.6) is true up to N. We have, by (5.3),

$$(5.7)\ (-1)^{N+1} \begin{pmatrix} 0 & z_1 \\ 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & z_{N+1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \omega_1 & 0 \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 & 0 \\ \omega_{N+1} & 0 \end{pmatrix} \cdot v_\varphi =$$

$$= \sum_{i=1}^{N+1} (-1)^N \begin{pmatrix} 0 & z_1 \\ 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & z_N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \omega_1 & 0 \end{pmatrix} \cdots \widehat{\begin{pmatrix} 0 & 0 \\ \omega_i & 0 \end{pmatrix}} \cdots \begin{pmatrix} 0 & 0 \\ \omega_{N+1} & 0 \end{pmatrix} (-1)\, \varphi(z_{N+1} \omega_i) \cdot v_\varphi +$$

$$+ (-1)^{N+1} \sum_{j>i} \begin{pmatrix} 0 & z_1 \\ 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & z_N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \omega_1 & 0 \end{pmatrix} \cdots \widehat{\begin{pmatrix} 0 & 0 \\ \omega_i & 0 \end{pmatrix}} \cdots \begin{pmatrix} 0 & 0 \\ -\omega_i z_{N+1} \omega_j - \omega_j z_{N+1} \omega_i & 0 \end{pmatrix} \cdots$$

$$\cdots \begin{pmatrix} 0 & 0 \\ \omega_{N+1} & 0 \end{pmatrix} \cdot v_\varphi .$$

This, then, can be evaluated using (5.6), and as a result one will get an expression analogous to this. To treat the constants rigorously it is, by symmetry, enough to examine terms of the form

$$(5.8)\ (-1)^{\gamma_1} \varphi(z_1 \omega_1 \cdots z_{\gamma_1} \omega_{\gamma_1}) \cdots (-1)^{\gamma_s} \varphi(z_{(\gamma_1+\cdots\gamma_{s-1}+1)} \cdots z_{N+1}\, \omega_{N+1})$$

Assuming $\gamma_s > 1$ ($\gamma_s = 1$ is trivial), the only way in which such a term can emerge is clearly by replacing ω_N by $-\omega_N z_{N+1} \omega_{N+1}$ in the analogous expression

$$(-1)^{\gamma_1}\varphi(z_1\omega_1 \dots z_{\gamma_1}\omega_{\gamma_1}) \cdots (-1)^{\gamma_s-1}\varphi(z_{\gamma_1+\dots+\gamma_{s-1}+1} \cdots z_N\omega_N)$$

in (5.6). □

As a first application of this result we get the unitarizability for su($r,1$) : Let R denote the algebra of polynomial functions from the circle into $gl(r,\mathbb{C})$. Let tr denote the usual trace on $gl(r,\mathbb{C})$ and define

$$(5.9) \qquad \varphi_\mu(p(\theta)) = -\int_0^{2\pi} tr\,(p(\theta))\,d\mu$$

for some positive Radon measure μ . Observe that $-\varphi_\mu(p_1(\theta)\,p_2(\theta)^*)$ defines a positive defintie inner product on R. Take the z's of (5.6) to be of the form

$$(5.10) \qquad z = \begin{bmatrix} c_1 e^{in_1\theta} & c_2 e^{in_2\theta} & \cdots & c_r e^{in_r\theta} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and the w's to be of the form z*, and observe that any product $z_1 \cdot z_2{}^*$ gives a matrix with at most one non-zero entry, namely the upper left hand corner. It follows that any expression $z_1 z_2{}^* z_3 z_4{}^* \cdots z_{2N-1} z_{2N}{}^*$ can be written as a product $z_a \cdot z_b{}^*$ of just two elements of R. Thus (5.6) expresses the Hermitian form on the module $M(\lambda)$ (with $\lambda = \varphi_\mu$, see (1.5)) as a sum of tensor products of positive definite Hermitian forms, and thus as something positive. This observation completes the proof of Theorem 3.1.

Consider now the case where R is a commutative algebra over $\mathbb{C}$ and assume that $a \to a^*$ is an antilinear involution of R such that

$$(5.11) \qquad \varphi(a^*) = \overline{\varphi(a)} \qquad \text{for all } a \in R$$

Define an antilinear anti-involution ω of $sl_2(R,\varphi)$ by

$$(5.12) \qquad \omega\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix}$$

and identify elements $\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ with the entry $c \in R$. An easy consequence of Lemma 5.1 is then,

Corollary 5.2 : For a,b $\in$ *R put* $(a,b) = -\varphi(b^*a)$. *Then*

$$(5.13) \qquad H_\varphi(c_1 \dots c_N . v_\varphi\,,\, c'_1 \dots c'_{N'} . v_\varphi) = \\ = \delta_{NN'} \sum_{s=1}^{N} \sum_{\gamma \in Par_s(N)} \sum_{t_\gamma, t_{\gamma'} \in D_\gamma(N)} K_\gamma \cdot \prod_{i=1}^{s} (\gamma_i)^{-1} \cdot (d(t_\gamma)_i\,,\, d(t'_\gamma)_i)$$

defines the contravariant Hermitian form on $M(\varphi)$*, where, if*

$$\gamma_1 = \cdots = \gamma_{n_1} > \gamma_{n_1+1} = \cdots \in \gamma_{n_1+n_2} > \gamma_{n_1+n_2+1} = \cdots\,, \quad K_\gamma = ((n_1!)(n_2!)\cdots)^{-1} . \;\square$$

Corollary 5.3 : H_φ is positive semi-definite if and only if the form

$$(a, b) = -\varphi(b^* a) \tag{5.14}$$

is positive semi-definite on R. $\square$

Remark 1. In the case $R = \mathbb{C}[z, z^{-1}]$ with $\varphi \equiv$ evaluation at 1, the Hermitian form has a (large) kernel. Thus, in particular $M(\varphi)$ need not be irreducible (cf §3 and below). $\square$

Remark 2. Let R be an arbitrary commutative algebra with a basis $\{a_\beta\}_{\beta\in B}$. Define the structure constants $C^{\gamma}_{\alpha\beta}$ by

$$a_\alpha a_\beta = C^{\gamma}_{\alpha\beta} a_\gamma$$

where the usual summation convention is used. The elements of the form

$$(a_{\alpha_1} f) \ \dots \ (a_{\alpha_i} f) \cdot v_\varphi ,$$

where $af = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$, form a basis of $M(\varphi)$, and $(a_\alpha . h) . v_\varphi = \varphi(a_\alpha) v = \lambda_\alpha . v$, and $(ae) v_\varphi = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} v_\varphi = 0$.

Due to the commutativity of the af's it is natural to represent $M(\varphi)$ as a space of polynomials $\mathbb{C}[X_\beta ; \beta\in B]$, and it follows easily that the action by left multiplication on $M(\varphi)$ is transformed into the following action on $\mathbb{C}[X_\beta ; \beta\in B]$:

$a_{\alpha_0} f$: Multiplication by X_{α_0}

$a_{\alpha_0} h$: $\lambda_{\alpha_0} - 2\, C^{\gamma}_{\alpha_0\alpha} X_\gamma \frac{\partial}{\partial X_\alpha}$

$a_{\alpha_0} e$: $\lambda_\gamma C^{\gamma}_{\alpha_0\alpha} \frac{\partial}{\partial X^\alpha} - X_\delta C^{\gamma}_{\alpha_0\alpha} C^{\delta}_{\gamma\beta} \frac{\partial}{\partial X_\alpha} \frac{\partial}{\partial X_\beta}$.

In particular, for $\mathbb{C}[z, z^{-1}]$ we have that $C^{k}_{n,m} = \delta_{n+m,k}$ and the formulas (0.9) in the introduction thus follow immediately . $\square$

6. Irreducibility.

Let $\dot{\mathfrak{g}}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, $\dot{\mathfrak{h}}$ a Cartan subalgebra, Δ the set of roots, $\Pi = \{\alpha_1, \dots, \alpha_r\}$ a basis of Δ, and Δ^+ the corresponding set of positive roots. Consider a subset Y of Π and let

$$\Delta_Y = \Delta \cap \sum_{\alpha_i \in Y} \mathbb{Z} \cdot \alpha_i , \tag{6.1}$$

with Δ_Y^+, Δ_Y^- defined analogously. Further, let

$$\dot{\mathfrak{g}} = \dot{\mathfrak{g}}_{-1} \oplus \dot{\mathfrak{g}}_0 \oplus \dot{\mathfrak{g}}_1 \tag{6.2}$$

be a decomposition of $\dot{\mathfrak{g}}$ such that $\dot{\mathfrak{g}}_0 \oplus \dot{\mathfrak{g}}_1$, is a compatible parabolic. Let $\{e_\alpha\}_{\alpha\in\Delta}$,

$\{H_\alpha\}_{\alpha\in\Pi}$ be a Chevalley basis of $\dot{\mathfrak{g}}$. We assume that $Y \neq \Pi$ and consider an element Λ of $\dot{\mathfrak{h}}^*$ satisfying $\Lambda(H_{\alpha_i}) = \lambda_i = 0$ for $\alpha_i \in Y$ and $\Lambda(H_{\alpha_j}) = \lambda_j \neq 0$ for $\alpha_j \notin Y$. For later use we now prove a lemma about (generalized) Verma modules. Let $M(\Lambda)$ be the left $\mathcal{U}(\dot{\mathfrak{g}})$-module $\mathcal{U}(\dot{\mathfrak{g}}_{-1}).v_\Lambda$ where v_Λ is a non-zero vector in the 1-dimensional space of the $\mathcal{U}(\dot{\mathfrak{g}}_0 \oplus \dot{\mathfrak{g}}_1)$ -module defined by Λ.

Lemma 6.1 Let $v \in M(\Lambda)$ *be a weight vector such that* $e_\alpha . v = 0$ *for all* $\alpha \in \Delta^+$. *Let* $v = u . v_\lambda$ *for some* $u \in \mathcal{U}(\dot{\mathfrak{g}}_{-1})$. *Then, if* u *is expanded on a standard Poincaré-Birkhoff-Witt basis, at least one of the summands yielding non-zero coordinates contains a factor of some* $e_{-\alpha_i}$, $\alpha_i \in \Pi \setminus Y$.

Proof. Let t_0 denote the minimum height of the α's for which $e_{-\alpha}$ occur in u. Suppose $t_0 > 1$ and let $\alpha_i \in \Pi$ be chosen such that $[e_{\alpha_i}, u]$ contains a term with a factor of height $t_0 - 1$. (Choose e.g. the ordering compatible with height). Since $[e_{\alpha_i}, u] = 0$, another term is needed to remove this again, but this can evidently only be done through the negative of the expression that yielded the first term. Thus, $t_0 > 1$ contradicts the non-triviality of u. $\square$

Let R be a commutative algebra and let $\{\varphi_i\}$ be a finite family of linear functionals on R. To avoid instant degeneracy we assume that

(6.3a) $a \in R$, $\varphi_i(Ra) = 0$ *implies* $a = 0$ *for each* i.

(6.3b) *R contains no finite-dimensional non-trivial ideals.*

We let v_0 define a 1-dimensional $\mathcal{U}(\dot{\mathfrak{g}}_0 \oplus \dot{\mathfrak{g}}_1)$-module through a regular element Λ. We now define a linear map $\Lambda_\varphi : R \otimes \dot{\mathfrak{h}} \to \mathbb{C}$ by

$$(6.4)\quad \Lambda_\varphi(a \otimes H_{\alpha_i}) = \varphi_i(a)\lambda_i$$

with the λ_i's as before, and corresponding to this we consider the left $\mathcal{U}(R\otimes\dot{\mathfrak{g}})$ - module

$$(6.5)\quad M(\Lambda_\varphi) = \mathcal{U}(\dot{\mathfrak{g}}_{-1} \otimes R) . v_\varphi ,$$

where we now have $(a \otimes h) v_\varphi = \Lambda_\varphi(a \otimes h) \cdot v_\varphi$, $a \in R$, $h \in \dot{\mathfrak{h}}$

Proposition 6.2. $M(\Lambda_\varphi)$ *is irreducible.*

Proof. Suppose that S is a non-trivial invariant subspace. As in the finite-dimensional case there is at least one non-zero element v of S which satisfies

$$(6.6)\quad \forall a \in R,\ \forall \alpha \in \Delta^+ :\quad a\, e_\alpha v = 0 ,$$

and we may write $v = u_0 . v_\varphi$ for a unique $u_0 \in \mathcal{U}(\dot{\mathfrak{g}}_{-1} \otimes R)$. Let $\alpha_a \in \Pi \setminus Y$.

Choose an ordering of $\Delta^- \setminus \Delta^-_Y$ such that the smallest elements are those α for which $[e_{\alpha_a}, e_\alpha]=0$. In a Poincaré-Birkhoff-Witt basis we write those terms to the left. Further we shall assume, as we may, that at this level, $\alpha_1+\alpha_2$ is bigger than α_1 and α_2. We write monomials in $\mathcal{U}(\mathring{\mathfrak{g}}_{-1}\otimes R)$ made up entirely of elements whose $\mathring{\mathfrak{g}}_{-1}$-part e_α commutes with e_{α_a} as $u(0)$. Again, inside a $u(0)$, the smallest elements go to the left. Next in the ordering come those $-\alpha$'s for which $[e_{\alpha_a}, e_{-\alpha}] \neq 0$ and $\alpha = \alpha_a + \sum_{\alpha_i \in \Pi\setminus\{\alpha_a\}} n_i\alpha_i$ has α_a -coefficient 1. These we order in an arbitrary fixed way. Monomials in $\mathcal{U}(\mathring{\mathfrak{g}}_{-1}\otimes R)$ corresponding entirely to such roots are denoted by $u(1)$. Then we define $u(2),\ldots u(j)$ analogously. For convenience we also allow the $u(j)$'s to be constants. It follows from Lemma 6.1 that there exists at least one $\alpha_a \in \Pi\setminus Y$ which satisfies that when we write u_0 as a sum of expressions $u(0)u(1)\ldots u(j)$, then at least one $u(1)$ contains a factor $a\,e_{-\alpha_a}$ for some $a \in R$. We proceed with a fixed α_a satisfying this and now claim that any of the $u(1)$'s occuring is made up entirely of products of $r_i\,e_{-\alpha_a}$'s and that, furthermore, the $u(j)$'s for j> 1 must be constants. Again we argue by contradiction : Suppose that some $u(1)$ contains a factor $b\,e_{-\alpha}$ with $\alpha \neq \alpha_a$. Let us describe this by saying that α satisfies (*). Let $\bar{\alpha}$ be that element of the set of roots satisfying (*) for which the root $\alpha' = \alpha_a - \alpha$ is smallest. Consider a monomial $u(0)u(1)\ldots u(\ell)$ where the full factor in $u(1)$ containing $e_{\bar{\alpha}}$'s is, say, $(r_1 e_{-\bar{\alpha}})\ldots(r_n e_{-\bar{\alpha}})$.

Assume for simplicity that the r_i''s are linearly independent. When we compute $[c\,e_{\alpha_a}, u_0]$ we then get a monomial where, say, the $u(0)$ part contains a $c\,r_1\,e_{-\alpha'}$ and where the $u(1)$ part contains $(r_2 e_{-\bar{\alpha}})\ldots(r_n e_{-\bar{\alpha}})$. On the other hand, by (6.6) we must get zero and hence we must remove this expression again. But this can only be done through the adjoint action of $c\,e_{\alpha_a}$ on some $u(j)$ with j > 1 or on portions of $u(1)$ not of the form $r\,e_{-\bar{\alpha}}$, and this does not change the coefficients of the terms in $u(0)$ involving $r\,e_{-\alpha'}$'s. Hence, for all $c \in R$, $c\cdot r_1$ must be in the span S of the elements s for which $s\,e_{-\alpha'}$ occurs in some $u(0)$ in the decomposition of u_0. In other words, $R\cdot r_1 \subset S$ and S is finite-dimensional This contradicts (6.3b). Further, it now follows analogously that the $u(j)$'s are constants for j > 1. Thus u_0 is a sum of terms of the form $u(0)\left(\sum_i (a_i f_{-\alpha_a})^N\right)$ for some fixed N, and where the $u(0)$'s are linearly independent. It then follows from (6.6) that there exists a non-zero element $a \in R$, and some i such that $\varphi_i(ac)=0$ for all $c \in R$. Thus, φ_i violates (6.3a), and this is a contradiction. □

Remark. Let X be a compact pathwise connected Hausdorff space and let R be a non-trivial (i.e. $R \neq 0$, $R \neq \mathbb{C}$) subalgebra of C(X). Then R contains no non-trivial finite-dimensional ideals. □

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Added in proof : The exceptional representations of $sl_n(\mathbb{C}[z, z^{-1}])$ can be integrated to (projective) unitary representations of the group of polynomial loops on SU(n,1).