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Covariant chronogeometry and extreme distances: Elementary particles

(particle models/chronometric theory/space–time models)

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ABSTRACT We study a variant of elementary particle theory in which Minkowski space, M_0, is replaced by a natural alternative, the unique four-dimensional manifold $\tilde{M}$ with comparable properties of causality and symmetry. Free particles are considered to be associated (*i*) with positive-energy representations in bundles of prescribed spin over $\tilde{M}$ of the group of causality-preserving transformations on $\tilde{M}$ (or its mass-conserving subgroup) and (*ii*) with corresponding wave equations. In this study these bundles, representations, and equations are detailed, and some of their basic features are developed in the cases of spins 0 and ½. Preliminaries to a general study are included; issues of covariance, unitarity, and positivity of the energy are treated; appropriate quantum numbers are indicated; and possible physical applications are discussed.

There is only one four-dimensional space–time manifold, $\tilde{M}$, other than Minkowski space–time M_0 that enjoys comparable properties of symmetry, causality, and separability into time and space components (1). $\tilde{M}$ and its causal symmetry group $\widetilde{SU}(2,2)$ have been applied to cosmology, providing a basis for the chronometric redshift theory (2) primarily through the consideration of Maxwell's equations (3). These extend naturally to $\tilde{M}$, and the energy of a photon in $\tilde{M}$ splits Lorentz-covariantly into a local and delocalized part. The local part is represented by the conventional energy operator in M_0, which can be regarded as a submanifold of $\tilde{M}$; the delocalized part drives, essentially as an interaction hamiltonian, a redshift in very good agreement with objective observations on galaxies and quasars. Here we specify some of the formalism involved in the development of similar applications to particle theory, in the scalar and spinor cases, within the general framework indicated in ref. 4.

Although principles originating in the chronometric redshift theory are required to fix the physical interpretation, the present direction also may in part be construed independently of chronometric principles as towards the adaptation of conventional particle theory to the naively most simple alternative. This involves the replacement of the euclidean spatial part E^3 of M_0 by a sphere S^3 of small but nonvanishing curvature, paralleling Minkowski's treatment of the relation between the Lorentz and Galilean groups. Consequently, we refer to the local deformation of $\tilde{M}$ into M_0, and of $\widetilde{SU}(2,2)$ into the Poincaré group as the formation of the "flat" limit of the "curved" picture.

Chronometric geometry

The chronometric cosmos $\tilde{M}$ may be invariantly described in relation to M_0 as the universal cover of the causal (or, less physically, conformal) compactification $\overline{M}$. We refer to ref. 5 for a detailed description and here only review the most basic aspects and treat some new points.

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A causal manifold is one with an assignment to each of its points of a convex cone in the tangent space, representing physically the future directions at the point. The usual causality in M_0 extends to a causal structure in $\overline{M}$. Another causal structure is obtained by identifying the tangent space at a point in $U(2)$ with the space of 2×2 hermitian matrices as usual and then assigning the cone of all positive semidefinite matrices. It is basic in our treatment that $\overline{M}$ and $U(2)$ are equivalent as causal manifolds; that the connected causal invariance group of $\overline{M}$ is $SO_0(2,4)/\{\pm 1\}$; and that the same for $U(2)$ is $SU(2,2)/\{\pm i, \pm 1\}$. $\tilde{M}$ may be formulated as $\tilde{U}(2)$ and is then naturally a causal manifold, equivalent to $R^1 \times SU(2)$ with its natural causal structure when $SU(2)$ is identified with S^3 in the usual way.

We denote as G the group $SU(2,2)$; if g is the element of G given by the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in which A, B, C, D are 2×2 complex matrices, g being represented naturally relative to the hermitian form $z_1\bar{z}_1 + z_2\bar{z}_2 - z_3\bar{z}_3 - z_4\bar{z}_4$ that defines $SU(2,2)$, and if Z is arbitrary in $U(2)$, then g carries Z into $(AZ + B)(CZ + D)^{-1}$, which we denote as gZ. This action determines a corresponding action of $\tilde{G}$ on $\tilde{U}(2) = \tilde{M}$; the subgroup of $\tilde{G}$ that leaves fixed a given point in $\tilde{M}$ is $\tilde{P}^e$, where P^e is the Poincaré group extended by scale transformations.

$U(1) \times SU(2)$ covers $U(2)$ by the map $e^{it} \times V \dashrightarrow e^{it}V$. We denote the former space as $\overline{\overline{M}}$ when it is endowed with the corresponding causal structure and parametrize it as the totality of points $(u_{-1},u_0) \times (u_1,u_2,u_3,u_4)$ in $E^2 \times E^4$ so that $u_{-1}^2 + u_0^2 = 1 = u_1^2 + u_2^2 + u_3^2 + u_4^2$. In these terms the corresponding element of $U(2)$ is $(u_{-1} + iu_0)(u_1 i\sigma_1 + u_2 i\sigma_2 + u_3 i\sigma_3 + u_4)$, where the σ_j are the usual Pauli matrices. We imbed M_0 (causally) in $U(2)$ by the map $(x_0,x_1,x_2,x_3) \dashrightarrow U = (1 + (iH/2))(1 - (iH/2))^{-1}$, where $H = x_0\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$; near the origin in M_0, the coordinates u_j $(j = 0,1,2,3)$ of U agree with the x_j within terms of second order in the x_j.

Chronometric physics takes as its fundamental temporal evolution group the transformations $t \times W \dashrightarrow (t + s) \times W$ [$t \times W$ denoting an arbitrary point of $R^1 \times SU(2)$] instead of the infinitesimally equivalent conventional group, $(x_0,x_1,x_2,x_3) \dashrightarrow (x_0 + s,x_1,x_2,x_3)$. Similarly, as its fundamental spatial motion group, it takes the isometry group of $SU(2)$ in place of the euclidean group on E^3. These chronometric transformations generate the subgroup $\tilde{K}$ of $\tilde{G}$, parametrizable as $R^1 \times SU(2) \times SU(2)$, with the following action on $\tilde{M}$: $s \times U \times V$ sends $t \times W$ into $(t + s) \times UWV^{-1}$. The center of $\tilde{G}$ is generated by the two transformations in $\tilde{K}$, $\zeta = \pi \times -I \times I$ and $\eta = 0 \times -I \times -I$. The physical (usual) scale-extended Poincaré group is carried by the given imbedding of M_0 into $U(2)$ into the subgroup of G modulo its center leaving fixed the matrix $-I$. The corresponding subgroup of $\tilde{G}$ consists of those elements leaving invariant the point $\pi \times I$ of $\tilde{M}$ or any image of this point by an element of the center D of $\tilde{G}$, such as the point $0 \times -I$. Note that although from a Minkowskian standpoint $\pi \times I$ appears infinitely distant in time and $0 \times -I$ appears infinitely distant in space, from the point of observation $0 \times I$ corresponding to the origin in M_0, they both cover the same point $-I$ of $U(2)$; the space–time separation in $\tilde{M}$ is only infinitesimally the same as that in M_0.

Equivalences of causal symmetries

The infinitesimal causal symmetries may be given a variety of presentations, of which four of the most useful are given in Table 1. The real projective quadric $\Sigma_{i=-1}^{4} e_i q_i^2 = q_{-1}^2 + q_0^2 - q_1^2 - q_2^2 - q_3^2 - q_4^2 = 0$, where $q_{-1}, q_0, q_1, q_2, q_3$ parametrizes an arbitrary point of projective 5-space, has a unique causal structure invariant under the group [locally $SO(2,4)$] of projectivities that leave it fixed (within reversal), and is then equivalent to $\overline{M}$. The vector fields on $\overline{M}$ corresponding to the operators $e_i q_i (\partial/\partial q_j) - e_j q_j (\partial/\partial q_i)$ are consequently infinitesimal causal symmetries and, when lifted up to $\overline{\overline{M}}$, are denoted as L_{ij}, forming the entries of Table 1, column 1.

A basis for the vector fields on $\overline{\overline{M}}$ is formed by the infinitesimal generators X_j of the one-parameter causal groups of transformations lifted up from the action $U \dashrightarrow Ue^{it\sigma_j}$ on $\overline{M}$, identified with $U(2)$. Column 2 (Table 1) gives the expressions for the L_{ij} as linear combinations of the X_k. Expressions for the X_k in terms of the L_{ij} are: $X_0 = L_{-10}$, $X_1 = L_{14}-L_{23}$, $X_2 = L_{24}-L_{31}$, $X_3 = L_{34}-L_{12}$. Column 3 gives the matrix in $su(2,2)$, where lower-case letters denote the Lie algebra of the group designated by the corresponding capitals, corresponding to L_{ij} in the earlier-indicated local isomorphism of $SO(2,4)$ into $\tilde{S}U(2,2)$; b_j denotes $i\sigma_j$ ($j = 0,1,2,3$). Column 4 is the expression for the restrictions of the L_{ij} to M_0 as imbedded in $\overline{\overline{M}}$ [lifted up from the imbedding in $\overline{M}$ given earlier, forming two copies of M_0 of which the one containing the base point $(1,0) \times (0,0,0,1)$ is chosen] as linear combinations of the Minkowskian $\partial/\partial x_j$. The flat limit of the L_{ij} is seen from column 4 by replacing x_j by x_j/R in which R is the "radius of the universe" S^3 in laboratory units, then rescaling L_{ij} appropriately, and finally forming the limit as $R \dashrightarrow \infty$.

Transformation properties of chronometric fields

The transformation properties of fields on $\tilde{M}$ are determined by those for the subgroup of $\tilde{G}$ fixing one point; these form a linear representation R of this subgroup $\tilde{P}^e$. Conversely, every such representation R of $\tilde{P}^e$ determines a species of field on $\tilde{M}$ that form the R-bundle over M or homogeneous vector bundle on $\tilde{M}$ induced from R. In particular, the standard relativistic fields extend directly to $\tilde{M}$ and are locally identical to these extensions through the imbedding of M_0 in $\tilde{M}$. We consider only smooth, say C^∞, fields.

Free particles in $\tilde{M}$ are naturally described by $\tilde{K}$-invariant wave equations (*i*) incorporating finite propagation velocity (relative to the given causal structure), positive energy (suitably interpreted for antifermions, as usual), and $\tilde{G}$-covariance and (*ii*) having flat limits in agreement with the conventional ones in M_0 for fields of the same species. The explicit treatment of such questions is facilitated by the parallelization of bundles on $\tilde{M}$, which is possible because of its group structure. Parallelizing by left translations on $\tilde{M}$ as $\tilde{U}(2)$ and identifying $\tilde{M}$ with a subgroup of $\tilde{G}$ by mapping the element $t \times W$ of $\tilde{M}$ into the element $t \times W \times I$ of $\tilde{K}$, we obtain:

Table 1. Presentations of infinitesimal causal symmetries of $\tilde{M}$

$SO(2,4)$ generator	Vector field on $\overline{\overline{M}}$ as linear combination of the X_j	$SU(2,2)$ generator	Vector field on M_0 as linear combination of $\partial_j(=\partial/\partial x_j)$*
$L_{-1,0}$	X_0	$\frac{1}{2}\begin{pmatrix} b_0 & 0 \\ 0 & b_0 \end{pmatrix}$	$(1 - \frac{1}{4}x^2)\partial_0 + (x_0/2)S$
$L_{-1,1}$	$-u_0u_1X_0 + u_{-1}u_4X_1 + u_{-1}u_3X_2 - u_{-1}u_2X_3$	$\frac{1}{2}\begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix}$	$(1 - \frac{1}{4}x^2)\partial_1 - (x_1/2)S$
$L_{-1,2}$	$-u_0u_2X_0 - u_{-1}u_3X_1 + u_{-1}u_4X_2 + u_{-1}u_1X_3$	$\frac{1}{2}\begin{pmatrix} 0 & b_2 \\ -b_2 & 0 \end{pmatrix}$	$(1 - \frac{1}{4}x^2)\partial_2 - (x_2/2)S$
$L_{-1,3}$	$-u_0u_3X_0 + u_{-1}u_2X_1 - u_{-1}u_1X_2 + u_{-1}u_4X_3$	$\frac{1}{2}\begin{pmatrix} 0 & b_3 \\ -b_3 & 0 \end{pmatrix}$	$(1 - \frac{1}{4}x^2)\partial_3 - (x_3/2)S$
$L_{-1,4}$	$-u_0u_4X_0 - u_{-1}u_1X_1 - u_{-1}u_2X_2 - u_{-1}u_3X_8$	$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$-S$
$L_{0,1}$	$u_{-1}u_1X_0 + u_0u_4X_1 + u_0u_3X_2 - u_0u_2X_3$	$\frac{i}{2}\begin{pmatrix} 0 & b_1 \\ b_1 & 0 \end{pmatrix}$	$x_1\partial_0 + x_0\partial_1$
$L_{0,2}$	$u_{-1}u_2X_0 - u_0u_3X_1 + u_0u_4X_2 + u_0u_1X_3$	$\frac{i}{2}\begin{pmatrix} 0 & b_2 \\ b_2 & 0 \end{pmatrix}$	$x_2\partial_0 + x_0\partial_2$
$L_{0,3}$	$u_{-1}u_3X_0 + u_0u_2X_1 - u_0u_1X_2 + u_0u_4X_3$	$\frac{i}{2}\begin{pmatrix} 0 & b_3 \\ b_3 & 0 \end{pmatrix}$	$x_3\partial_0 + x_0\partial_3$
$L_{1,2}$	$(u_1u_3 + u_2u_4)X_1 + (u_2u_3 - u_1u_4)X_2 - (u_1^2 + u_2^2)X_3$	$\frac{1}{2}\begin{pmatrix} b_3 & 0 \\ 0 & b_3 \end{pmatrix}$	$x_2\partial_1 - x_1\partial_2$
$L_{2,3}$	$-(u_2^2 + u_3^2)X_1 + (u_1u_2 + u_3u_4)X_2 + (u_1u_3 - u_2u_4)X_3$	$\frac{1}{2}\begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix}$	$x_3\partial_2 - x_2\partial_3$
$L_{3,1}$	$(u_1u_2 - u_3u_4)X_1 - (u_1^2 + u_3^2)X_2 + (u_1u_4 + u_2u_3)X_3$	$\frac{1}{2}\begin{pmatrix} b_2 & 0 \\ 0 & b_2 \end{pmatrix}$	$x_1\partial_3 - x_3\partial_1$
$L_{0,4}$	$u_{-1}u_4X_0 - u_0u_1X_1 - u_0u_2X_2 - u_0u_3X_3$	$\frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$(1 + \frac{1}{4}x^2)\partial_0 - (x_0/2)S$
$L_{1,4}$	$(u_1^2 + u_4^2)X_1 + (u_1u_2 + u_3u_4)X_2 + (u_1u_3 - u_2u_4)X_3$	$\frac{1}{2}\begin{pmatrix} b_1 & 0 \\ 0 & -b_1 \end{pmatrix}$	$(1 + \frac{1}{4}x^2)\partial_1 + (x_1/2)S$
$L_{2,4}$	$(u_1u_2 - u_3u_4)X_1 + (u_2^2 + u_4^2)X_2 + (u_1u_4 + u_2u_3)X_3$	$\frac{1}{2}\begin{pmatrix} b_2 & 0 \\ 0 & -b_2 \end{pmatrix}$	$(1 + \frac{1}{4}x^2)\partial_2 + (x_2/2)S$
$L_{3,4}$	$(u_1u_3 + u_2u_4)X_1 + (u_2u_3 - u_1u_4)X_2 + (u_3^2 + u_4^2)X_3$	$\frac{1}{2}\begin{pmatrix} b_3 & 0 \\ 0 & -b_3 \end{pmatrix}$	$(1 + \frac{1}{4}x^2)\partial_3 + (x_3/2)S$

* S denotes the vector field $x_0\partial_0 + x_1\partial_1 + x_2\partial_2 + x_3\partial_3$, and $x^2 = x_0^2 - x_1^2 - x_3^2 - x_4^2$.

THEOREM 1. *Let* V_0 *denote the representation of* $\tilde{G}$ *in the bundle over* $\tilde{M}$ *induced from the representation* R *of the subgroup leaving fixed the point* x_0. *If* L *denotes the left parallelization map, the parallelized action* $V(g) = LV_0(g)L^{-1}$ ($g \in G$) *takes the form*

$$V(g): \Phi(x) \dashrightarrow R(x_0x^{-1}g{\cdot}g^{-1}(x)x_0^{-1})\Phi(g^{-1}x).$$

Φ is here a function on $\tilde{M}$ with values in the representation space of R, of "spin space" at the group unit; $g^{-1}(x)$ refers to the action of G as a transformation group on M. A proof of Theorem 1 is given in ref. 6. In general, $g^{-1}(x)$ can not be given explicitly (except locally) or globally on $\overline{M} = U(2)$. The map $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \dashrightarrow AD^{-1}$ is a local isomorphism of the subgroup of $SU(2,2)$ for which $B = C = 0$ into $U(2)$ that is locally equivalent to the earlier given map of $\tilde{M}$ into $\tilde{G}$. Its inverse carries the element Z of $U(2)$ into the element $(\det Z)^{-1/4}\begin{pmatrix} Z & 0 \\ 0 & I \end{pmatrix}$ of G. With this identification of $\overline{M}$ with a subgroup of G, the element $x_0x^{-1}g{\cdot}g^{-1}(x)x_0^{-1} = g^*$ (say, with x_0 chosen as $-I$ to facilitate induction from given representations of the physical Poincaré group) takes the form

$$g^* = (\det ZW^{-1})^{1/4}\begin{pmatrix} Z^{-1}AW & -Z^{-1}B \\ -CW & D \end{pmatrix};\ W = (A'Z + B')(C'Z + D')^{-1},\ g^{-1} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}.$$

In infinitesimal form, the effect of the "multiplier" $R(g^*)$ is additive, contributing an "internal" component to the total group action, which includes in addition the "external" component that consists of the negative of the vector field corresponding to the infinitesimal group element. Specifically, if $X \in \mathcal{G}$, where $\mathcal{G}$ denotes the Lie algebra of G, the internal component of the action of X on the R-bundle is (in its presently parallelized form) $r(Y)$, in which r is the infinitesimal representation corresponding to R and

$$Y = (1/4)\,tr\,(a + bZ^{-1} - cZ - d) + \begin{pmatrix} -Z^{-1}b + cZ + d & -Z^{-1}b \\ -cZ & d \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the representative for X in $su(2,2)$.

Scalar fields

These are defined by real one-dimensional representations R of $\tilde{P}^e$ that are trivial on $\tilde{P}$ itself and have the form $R(S_\lambda) = \lambda^\nu$ for some real ν, where S_λ is the transformation $x \dashrightarrow \lambda x$ on M_0. Substitution in the expression just given for the internal action of the generator of the one-parameter group $S_{\exp(t)}$ shows that this takes the form $-(\nu/4)tr(Z + Z^{-1}) = -\nu u_{-1}u_4$. Only the case $\nu = 1$ admits a "massless" invariant subspace or equivalently admits a covariant second-order wave operator. The salient facts concerning scalar fields in this case are summarized in:

THEOREM 2. *Let* V *denote the parallelized action of* G *on the ensemble* $\mathcal{S}$ *of smooth scalar fields over* $\overline{\overline{M}}$ *of the type indicated. Let* $\square$ *denote the differential operator* $X_0^2 - X_1^2 - X_2^2 - X_3^2 + 1$ *on* S. *Then* (i) $\square$ *is invariant under the* V(a), *with* a *in the maximal compact subgroup* K *of* G, *and covariant with respect to* G:

$$[\square, v(W)] = m(W)\square, \qquad [1]$$

where v *denotes the infinitesimal form of* V, $W \in \mathcal{G}$, *and* m(W) *is a function on* $\overline{\overline{M}}$.

(ii) *The space N of smooth complex solutions to the equation* $\square\Phi = 0$ *(the "massless subspace") admits a* G*-invariant decomposition into subspaces* N^+ *and* N^-, *on which* $-iv(X_0)$ *is respectively nonnegative and nonpositive; subspaces* $N^\pm$ *admit* G*-invariant positive-definite sesquilinear forms. On* N^+ *this form may be expressed as* $\langle\Phi,\Psi\rangle = -i\int_{SU(2)}(\partial\Phi/\partial t)\bar{\Psi}\,d_3u$, *where* d_3u *denotes the element of invariant volume on* SU(2).

(iii) $\mathcal{S}$ *admits the* G*-invariant sesquilinear form* $\langle\langle\Phi,\Psi\rangle\rangle = \int_{\overline{M}}(\square\Phi)\bar{\Psi}\,d_4u$, *where* d_4u *denotes the element of* K*-invariant volume on* $\overline{\overline{M}}$.

(iv) *There is a maximal* G*-invariant subspace (the "positive-energy" subspace) on which* $\langle\langle\cdot,\cdot\rangle\rangle$ *is positive semidefinite and on which* $-iv(X_0)$ *is nonnegative, spanned by fields* Φ *that are eigenvectors for* $v(X_0)$ *and for which* $\square\Phi = \lambda\Phi$ *for some* $\lambda \le 0$.

(v) *There is no complementary* G*-invariant subspace in the positive-energy subspace to the massless subspace: repeated application of* v(S), S *being the infinitesimal generator of scale transformations, carries every* K*-invariant subspace that is disjoint from the massless subspace into one having nontrivial intersection with it.*

Certain of the results and methods involved in this theorem are contained in work of E. G. Lee (7), B. Ørsted (8), H. P. Jakobsen and M. Vergne (9), and B. Speh (10) (cf. also ref. 4). Our proof proceeds in outline as follows. The invariance of $\square$ under the action of K follows from *Theorem 1*. Since K together with S generate G, it suffices for the proof of G-covariance, to establish this in the case of S. By using Table 1 to express S as a linear combination of the X_j, a computation shows that Eq. 1 holds with $m(S) = -2u_{-1}u_4$. Transformation by a general element of G shows that $m(W)$ is a scalar function on $\overline{M}$ for all W in the Lie algebra of G. For the treatment of *i*, see ref. 8. For *iii*, it suffices (as in the proof of *i*) to show invariance under S. By evaluating $(\partial/\partial t)J(e^{-tS})|t = 0$ as

$$-4 + [(x^2)^2 - 4x^2 + 8x_0^2]\,[2(1 - \tfrac{1}{4}x^2)^2 + 2x_0^2]^{-1} = -4u_{-1}u_4,$$

where $J(T)$ denotes the Jacobian of the transformation T with respect to the K-invariant measure on $\overline{\overline{M}}$, and by using the same presentation of S as earlier, this invariance follows.

To treat the positive-energy subspace (and for later purposes, such as diagonalization of quantum numbers), the following basis for S is useful:

$$\beta_{klmn} = e^{itn}\sin^l\rho\, C_k^{l+1}(\cos\rho)Y_{lm}(\theta,\phi),$$

where k,l,m,n are integral; $k,l \ge 0$, $-1 \le m \le l$. Here ρ, θ, and ϕ are the polar coordinates defined by the equations $u_1 = \sin\rho\sin\theta\cos\phi$, $u_2 = \sin\rho\sin\theta\sin\phi$, $u_3 = \sin\rho\cos\theta$, and $u_4 = \cos\rho$, where $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$, and $0 \le \rho \le \pi$; C_m^{l+1} and Y_{lm} denote the usual Gegenbauer polynomials and spherical harmonics. The β_{klmn} are eigenfunctions of $\square$ with eigenvalue $(k + l + 1)^2 - n^2$. Distinct β_{klmn} are orthogonal relative to the given inner product; under K they are transformed into other eigenfunctions of the same eigenvalue. The K-invariance and nonnegativity of the inner product on the positive-energy subspace follow.

To show G-invariance, it suffices as earlier to show invariance under S. The action of S is given explicitly in the present representation as follows:

$$\begin{aligned} v(S)\beta_{klmn} = {} & [(2l + k + 1)(-n + l + k + 1)/4(l + k + 1)]\,\beta_{k-1,lm,n+1} + [(k + 1)(-n - l - k - 1)/4(l + k + 1)]\,\beta_{k+1,lm,n+1} \\ & + [(2l + k + 1)(n + l + k + 1)/4(l + k + 1)]\,\beta_{k-1,lm,n-1} + [(k + 1)(n - l - k - 1)/4(l + k + 1)]\,\beta_{k+1,lm,n-1}. \end{aligned}$$

From this the invariance and maximality of the positive-energy subspace follows as does the fact that repeated application of S to any of the β_{klmn} in the positive-energy subspace, one of which must be contained in any nontrivial K-invariant subspace, eventually carries it into the massless subspace.

Spinor fields

In this case the representation R takes the form $R(g) = \begin{pmatrix} T & 0 \\ 0 & T^{*-1} \end{pmatrix}$, with T in $SL(2,C)$ if g is a transformation that (*i*) carries the hermitian matrix H, corresponding to an arbitrary point of M_0, into THT^*; (*ii*) carries the scale transformation $H \dashrightarrow \lambda H$ into $\lambda^{3/2} I_4$; and (*iii*) is trivial on the translation subgroup. In terms of the $SU(2,2)$ representation of the physical Poincaré group G_0 with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, R takes the form $R(g) = \begin{pmatrix} (A-B)\det(A-B) & 0 \\ 0 & (A+C)\det(A+C)^{-2} \end{pmatrix}$. G_0 may be defined by the condition that it leave $-I$ invariant, which is equivalent to the condition $A - B + C - D = 0$, implying that $\det(A + C) = \det(A - B)^{-1}$ for g in G_0.

Defining $\gamma_0 = -i\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$, $\gamma_j = \begin{pmatrix} 0 & -b_j \\ b_j & 0 \end{pmatrix}$, the canonical Dirac operator after parallelization takes the form

$$\mathfrak{D} = \gamma_0 X_0 + \gamma_1 X_1 + \gamma_2 X_2 + \gamma_3 X_3 - (3/2)\, \gamma_4 \gamma_5,$$

acing in the space of all functions from M to complex 4-space C^4.

THEOREM 3. *Let* V *denote the parallelized action of* G *on the spin bundle over* $\bar{\bar{M}}$. *Then*

(i) $\mathfrak{D}$ *is* K-*invariant and covariant with respect to* G: $[\mathfrak{D}, \mathrm{v(W)}] = \mathrm{n(W)}\,\mathfrak{D}$, *where* n(W) *is a matrix-valued function on* $\bar{\bar{M}}$, W *being arbitrary in* $\mathscr{G}$.

(ii) *The space of smooth solutions* Ψ *to the equation* $\mathfrak{D}\,\Psi = 0$ *admits a positive-definite* G-*invariant form*: $<\Phi,\Psi> = \int_{\bar{\bar{M}}} <<<\Phi(x),\Psi(x)>>>\, d_4u$, *where* $<<<\cdot,\cdot>>>$ *denotes the usual positive definite inner product in* C^4.

(iii) *The entire spin bundle admits the* G-*invariant sesquilinear form* $<\Phi,\Psi> = \int_{\bar{\bar{M}}} <\mathfrak{D}\,\Phi(x),\Psi(x)>'\, d_4u$, *where* $<\cdot,\cdot>'$ *denotes the Lorentz-invariant (indefinite) hermitian inner product (in spin space).*

The proof resembles that for *Theorem 2*, but is more complicated. Similar results regarding the Dirac operator in M_0 are given in the massless case by Gross (11) and in the massive case as well in ref 9. General aspects of covariance of operators have been treated by Kosmann (12).

Quantum numbers and physical discussion

A basic physical problem in the chronometric treatment of particles is the identification of quantum numbers that will agree in the flat local limit with conventional relativistic ones and at the same time be built up from the action of the mass-conserving subgroup $\tilde{K}$ of $\tilde{G}$. This identification is essential for the correlation of theory with realistic experiment, which can directly measure only the flat (Minkowkian) quantum numbers because only these are determined by the local structure of the wave functions and experiments must be conducted within a local region.

An appropriate system of quantum numbers constructed from the generators of $\tilde{K}$ consists in the following first- and second-order expressions in the generators. For simplicity, $v(L_{ij})$, where v is the infinitesimal form of the representation V giving the action of K, is denoted simply as L_{ij}.

As earlier, natural chronometric units are used in which $\hbar = c = R = 1$, where R is the radius of the universe $SU(2)\ S^3$.

Table 2. Chronometric Quantum numbers

Expression	Physical interpretation
$-iL_{-1,0}$	Chronometric energy
$-iL_{12}$	z-Component of angular momentum
$-(L_{12}{}^2 + L_{23}{}^2 + L_{31}{}^2)$	Total angular momentum (squared)
$-(L_{14}{}^2 + L_{24}{}^2 + L_{34}{}^2)^{1/2}$	Total chronometric linear momentum, denoted p
$(L_{12}L_{34} + L_{23}L_{14} + L_{31}L_{24})p^{-1}$	Chronometric helicity

The chronometric energy always exceeds the flat energy $-i(\partial/\partial x_0)$ in principle, but in sufficiently localized states there will be no observable difference. The "superrelativistic" excess of the chronometric over the flat energy is interpreted physically as manifested in large-scale diffuse processes, such as cosmological redshifting and, possibly, gravitation. The linear momenta L_{j4} ($j = 1,2,3$) have similar decompositions into flat and superrelativistic (or local and delocalized) components, and p will typically exceed the total flat linear momentum.

The angles ϕ and θ introduced earlier are identical with the corresponding angles in M_0, and the angular momentum quantum numbers are identical with those for the flat case. The other quantum numbers differ from their flat counterparts by terms of order R^{-1} in laboratory units, but the helicity as a quotient may be unstable in the flat limit for states of small total linear momentum. In general, therefore, the chronometric and Minkowskian theoretical predictions may be expected to be quite close energetically; however, they may differ in selection rules and even energetically for phenomena involving sufficient delocalization. The large masses of the elementary particles in chronometric units, $m_p \approx 10^{40}$, in part compensate for the smallness of an effect of order R^{-1}. Specific directions of possible applications may be indicated as follows.

A chronometric electron state with an approximately exact *flat* mass M, where $M^2 = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2 + (\partial/\partial x_3)^2 - (\partial/\partial x_0)^2$, would appear observationally as a well-defined elementary particle of spin 1/2 and mass M. In a sufficiently delocalized state, the flat mass M could be much greater than the chronometric mass m given by the equation $m^2 + \square = 0$, where $\square$ is the covariant wave operator earlier defined. Because the flat and curved wave operators do not at all commute, such states would be quite exceptional and therefore, expected to have an extremely limited flat (observed) mass spectrum. Thus the muon may be such a state.

The action of parity on $\tilde{K}$ is to carry $t \times U \times V$ into $t \times V \times U$. It follows that it does not leave ζ invariant but carries it into $\zeta\eta$. As central elements of $\tilde{G}$, both ζ and η will be absolute invariants for a given particle species, and η, which occurs in conventional theory as the nontrivial center of $\tilde{P}$, takes the value -1 on spinor fields. Therefore, parity and the quantum number ζ, which has no counterpart in M_0, can not simultaneously be conserved within an irreducible particle species. Besides the nonconservation of parity, a possible interpretation is a need for parity doublets, which thus arise naturally from the group-theoretic analysis of fields on $\tilde{M}$ of half-integral spin; on M_0, they have been considered for phenomenological reasons in refs. 13 and 14. With the latter interpretation, maximal parity violation as represented by the usual chiral-invariant four-fermion interaction appears as the local form of a parity-conserving interaction in $\tilde{M}$. This would not imply apparent parity violation for other interactions. In the case of quantum electrodynamics, for example, photon wave functions on $\tilde{M}$ are invariant under ζ, producing a cancellation in the integrated chronometric interaction hamiltonian due to antisymmetry under ζ of the current between doublet members of opposite parity.

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1. Segal, I. E. (1980) in *Symmetries in Science*, eds. Gruber, B. & Millman, R. S. (Plenum, New York), pp. 385–396.
2. Segal, I. E. (1972) *Astron. Astrophys.* **18**, 143–148.
3. Segal, I. E. (1976) *Proc. Natl. Acad. Sci. USA* **73**, 669–673.
4. Jakobsen, H. P., Ørsted, B., Segal, I. E., Speh, B. & Vergne, M. (1978) *Proc. Natl. Acad. Sci. USA* **75**, 1609–1611.
5. Segal, I. E. (1976) *Mathematical Cosmology and Extragalactic Astronomy* (Academic, New York).
6. Segal, I. E. (1980) in *Proceedings of the Conference on Differential Geometric Methods in Mathematical Physics*, ed. Doebner, H. D. (Clausthal, Federal Republic of Germany).
7. Lee, E. G. (1975) Dissertation (MIT, Cambridge, MA).
8. Ørsted, B. (1977) *Lett. Math. Phys.* **1**, 183.
9. Jakobsen, H. P. & Vergne, M. (1977) *J. Funct. Anal.* **24**, 52–106.
10. Speh, B. (1979) *J. Funct. Anal.* **33**, 95–118.
11. Gross, L. (1964) *J. Math. Phys.* **5**, 687–695.
12. Kosmann, Y. (1976) in *Differential Geometry and Relativity*, eds. Cahen, M. & Flato, M. (Reidel, Dordrecht, The Netherlands).
13. Lee, T. D. & Yang, C. N. (1955) *Phys. Rev.* **102**, 290.
14. Pati, J. C. (1977) *Unification of Elementary Forces and Gauge Theories*, Report of B. W. Lee Memorial Conference, Batavia, NY, eds. Cline, D. B. & Mills, F. E. (Harwood, London).