

QUADRATIC ALGEBRAS OF TYPE AIII; I

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ABSTRACT. This is the first article in a series of three where we study some quadratic algebras related to quantized matrix algebras. In the first part we construct our algebras and relate them to quantized hermitian symmetric spaces. Our point of view is that of non-commutative algebraic geometry [1] and for this reason we determine the associated variety of a certain quantized matrix algebra which we feel should play a more prominent rôle. We hope the result will justify this.

1. INTRODUCTION

Let $\mathfrak{g} = su(m, n)^{\mathbb{C}}$ be the simple complex Lie algebra of type A_r ; $r + 1 = m + n$. This is the Lie algebra corresponding to the hermitian symmetric space of type AIII. Let \mathfrak{k} denote a maximal compact subalgebra of \mathfrak{g} , i.e. $\mathfrak{k} \equiv s(u(m) \times u(n))^{\mathbb{C}}$. Then it is a crucial fact in relation to analysis on the hermitian symmetric space that there exists a decomposition

$$(1) \quad \mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$$

of the Lie algebra into a sum of subalgebras and where, furthermore, \mathfrak{p}^{\pm} are abelian \mathfrak{k} -modules. Indeed, (1) leads to a decomposition on the level of enveloping algebras,

$$(2) \quad \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{p}^-) \cdot \mathcal{U}(\mathfrak{k}) \cdot \mathcal{U}(\mathfrak{p}^+),$$

and a good deal of the analysis, e.g. holomorphic representations, covariant differential operators, etc. can be transferred to algebraic questions via this so-called triangular, or Weyl, decomposition.

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²The last author is partially supported by NSF of China

Let \mathcal{U}_q be a quantum group defined by a Cartan matrix of type A_r together with the usual quantized Serre relations. We will treat the quantum parameter q as a non-zero constant. Both the “generic case” where q is not a primitive root of unity and the complementary case will be considered.

We will now investigate in what sense a hermitian symmetric space can be deformed or “quantized”. We choose to put emphasis on the decomposition (2). It is then quite natural just to replace the compact piece $\mathcal{U}(\mathfrak{k})$ by its quantum analogue. We do that.

In the classical case, the algebras $\mathcal{U}(\mathfrak{p}^-)$ and $\mathcal{U}(\mathfrak{p}^+)$ are polynomial algebras. Moreover, for holomorphically induced representations or, more or less equivalently, generalized Verma modules, the space of \mathfrak{k} -types is either $\mathcal{U}(\mathfrak{p}^-) \otimes V_\tau$ or a subset thereof, where V_τ denotes the \mathfrak{k} -type we induce up from.

As a start one could then simply choose to consider (generalized) highest weight modules of the quantum algebra \mathcal{U}_q corresponding to a 1-dimensional representation V_τ of the compact piece. Inspired by the bases and automorphisms used by Lusztig ([2]), the above guidelines have lead us to introduce quadratic algebras $A_{m,n}$ and $A_{m,n}^-$ that fulfill the following (Proposition 2.6)

$$(3) \quad U_q = A_{m,n}^- \cdot \mathcal{U}_q(\mathfrak{k}) \cdot A_{m,n}.$$

This result has been generalized in [3] to arbitrary hermitian symmetric spaces.

Actually, the algebras $A_{m,n}$ and $A_{m,n}^-$ in (3) are not uniquely determined. One of the candidates for the part of e.g. $A_{m,n}$ is actually the well-known quantized matrix algebra $M_q(n)$ ([4]). The latter has been the subject of many investigations, (see e.g. [5] and references therein) but it seems that one always has been taking a somewhat different point of view, namely that of getting hold of a quantum $GL(n)$ or $SL(n)$. Our point of view is really to take over Artins program [1]. According to this program one must determine a certain associated variety, determine the point, line, plane, etc. modules, the “fat points” and so on, and relate these to the variety. It has been a guiding principle for us that the variety should be “nice” and for this reason we choose a family of quadratic algebras that does not reduce to $M_q(n)$ in the case where $m = n$.

In the present article we introduce the algebras in Section 2 and in Section 3 we determine the so-called associated variety.

2. THE ALGEBRAS

We continue to work with $\mathfrak{g} = su(m, n)^\mathbb{C}$. We have the following critical Serre relation

$$(4) \quad E_1^2 E_2 - (q + q^{-1}) E_1 E_2 E_1 + E_2 E_1^2 = 0.$$

Lemma 2.1. *Let E_x, E_y satisfy (4). Set $E_{x,qy} = E_y E_x - q E_x E_y$. Then*

$$E_x E_{x,qy} = q E_{x,qy} E_x \text{ and } E_y E_{x,qy} - q^{-1} E_{x,qy} E_y = 0.$$

Proof. The first equation is directly equivalent to the Serre equation (4) and the other is directly equivalent to the Serre equation with E_x and E_y interchanged. \square

Lemma 2.2. *Let E_x, E_y be as above. Assume that E_z commutes with E_x and that $a_{yz} = -1$ (the entry in the Cartan matrix corresponding to x, y). Then $E_{x,qy}, E_z$ satisfy the Serre relation. In particular, if $E_{x,qy,qz} = E_z E_{x,qy} - E_{x,qy} E_z$ then $E_{x,qy} E_{x,qy,qz} = q E_{x,qy,qz} E_{x,qy}$.*

Proof. Let $E_2 = E_{x,qy}$. We must then prove that

$$(5) \quad E_2^2 E_z - (q + q^{-1}) E_2 E_z E_2 + E_z E_2^2 = 0$$

We now use the fact that there is a unique algebra homomorphism corresponding to E_y (Lusztig, [2]). It is given (in part) by

$$T_y(E_w) = \begin{cases} -F_w K_w & \text{if } z = w \\ E_w & \text{if } a_{zw} = 0 \\ -E_y E_w + \nu^{-1} E_w E_y & \text{if } a_{zw} = -1. \end{cases}$$

The “ ν ” is Lusztig’s quantum parameter. It may here be taken to be q^{-1} . We then can write $E_{y,qx} = -T_y(E_x)$ and equation (5) is equivalent to

$$E_1^2 T_y(E_z) - (q + q^{-1}) E_1 T_y(E_z) E_1 + T_y(E_z) E_1^2 = 0,$$

which is easily seen to be true, using the commutativity of E_x, E_z and the Serre relation for E_x, E_y .

The rest then follows from Lemma 2.1. \square

The following definition is a special case of the generally defined adjoint representation in a Hopf algebra (c.f. De Concini and Kac, [6]).

Definition 2.3. *For a simple compact root vector E_μ , and E_α an arbitrary element of the quantum algebra of weight α , set*

$$(ad E_\mu)(E_\alpha) = E_\mu E_\alpha - q^{\langle \alpha, \mu \rangle} E_\alpha E_\mu,$$

where, as usual, $\langle \alpha, \mu \rangle = \frac{2(\alpha, \mu)}{(\mu, \mu)}$.

In the current case of $su(m, n)$ the set of simple compact roots break up into two orthogonal sets: $\sum_c = \{\nu_1, \dots, \nu_{m-1}\} \cup \{\mu_1, \dots, \mu_{n-1}\}$. Thus,

$$\forall i, j : E_{\mu_i} E_{\nu_j} = E_{\nu_j} E_{\mu_i}.$$

Assume moreover that these roots have been labeled in such a way that

$$\langle \beta, \mu_1 \rangle = \langle \beta, \nu_1 \rangle = \langle \mu_i, \mu_{i+1} \rangle = \langle \nu_j, \nu_{j+1} \rangle = -1,$$

for all $i = 1, \dots, m-2$ and for all $j = 1, \dots, n-2$.

We can then introduce

Definition 2.4. Set $\tilde{Z}_{0,0} = E_\beta$, and

$$\tilde{Z}_{i,j} = (adE_{\mu_i}) \dots (adE_{\mu_0})(adE_{\nu_j}) \dots (adE_{\nu_0})(E_\beta).$$

for $i = 0, \dots, m-1$ and $j = 0, \dots, n-1$, where adE_{μ_0} and adE_{ν_0} are defined to be the identity operator.

Lemma 2.5. Let $\tilde{Z}_1 = \tilde{Z}_{i,j}$, $\tilde{Z}_2 = \tilde{Z}_{i,j+y}$, $\tilde{Z}_3 = \tilde{Z}_{i+x,j}$, and $\tilde{Z}_4 = \tilde{Z}_{i+x,j+y}$. Then

$$(6) \quad \begin{aligned} \tilde{Z}_1 \tilde{Z}_3 &= q \tilde{Z}_3 \tilde{Z}_1 \\ \tilde{Z}_1 \tilde{Z}_2 &= q \tilde{Z}_2 \tilde{Z}_1 \\ \tilde{Z}_2 \tilde{Z}_4 &= q \tilde{Z}_4 \tilde{Z}_2 \\ \tilde{Z}_3 \tilde{Z}_4 &= q \tilde{Z}_4 \tilde{Z}_3 \\ \tilde{Z}_2 \tilde{Z}_3 &= \tilde{Z}_3 \tilde{Z}_2 \\ \tilde{Z}_1 \tilde{Z}_4 - \tilde{Z}_4 \tilde{Z}_1 &= (q - q^{-1}) \tilde{Z}_2 \tilde{Z}_3. \end{aligned}$$

There are no further defining relations.

Proof. The relations follow directly from the previous lemmas. That there can be no further relations follows by a weight count in U_q or, equivalently, by observing that the Hilbert series must be the same as for the polynomial algebra in $m \cdot n$ variables. \square

Proposition 2.6. There exist quadratic algebras \mathcal{A}^+ and \mathcal{A}^- such that

$$(7) \quad U_q = \mathcal{A}^- \cdot \mathcal{U}_q(\mathfrak{k}) \cdot \mathcal{A}^+.$$

Proof. Define \mathcal{A}^+ to be the algebra generated by the elements $\tilde{Z}_{i,j}, i = 0, 1, \dots, m, j = 0, 1, \dots, n$, and with the relations (6) and define \mathcal{A}^- analogously by means of the negative roots. By construction, it is clear that they satisfy (7). \square

Remark 2.7. *The algebra \mathcal{A}^+ in the case $m = n$ is very well known. It is the so-called quantized matrix algebra $M_q(n)$.*

From our point of view these algebras are not quite satisfactory. In fact, the associated varieties are too fragmentary for our taste. But the procedure that was used to derive them was not in any way unique. In particular, it ignored completely the commutative algebra generated by the $K_{\mu_i}^\pm$'s and the $K_{\nu_j}^\pm$'s.

We have the following result:

Lemma 2.8. *It is possible to multiply the elements $\tilde{Z}_{i,j}$ from the right by certain (square roots) of elements from the algebra U_K generated by the $K_{\mu_i}^\pm$'s and the $K_{\nu_j}^\pm$'s in such a way that the resulting elements $Z_{i,j}$ satisfy:*

Let $Z_1 = Z_{i,j}, Z_2 = Z_{i,j+a}, Z_3 = Z_{i+b,j}$ and $Z_4 = Z_{i+b,j+a}$. Then

$$(8) \quad \begin{aligned} Z_1 Z_3 &= q^b Z_3 Z_1 \\ Z_1 Z_2 &= q^a Z_2 Z_1 \\ Z_2 Z_4 &= q^b Z_4 Z_2 \\ Z_3 Z_4 &= q^a Z_4 Z_3 \\ Z_2 Z_3 &= q^{b-a} Z_3 Z_2 \\ q^{1-a} Z_1 Z_4 - q^{b-1} Z_4 Z_1 &= (q - q^{-1}) Z_2 Z_3. \end{aligned}$$

Proof. Let ξ_β, ξ_{μ_i} , and ξ_{ν_j} denote the fundamental dominant weights. Then

$$\tilde{Z}_{i,j} K^{-(i+j+1)\xi_\beta - \xi_{\mu_i} - \xi_{\nu_j}}$$

satisfies the requirements. \square

Remark 2.9. *The above algebra is a twisted Ore extension but does not fit into Artins multiparameter family. On the other hand, $M_q(n)$ is an (iterated) Ore extension and does belong to Artins family.*

Definition 2.10. *By the quadratic algebra of type AIII(m, n) (or just $A_{m,n}$) we understand the algebra generated by the elements $Z_{i,j}, i = 0, 1, \dots, m-1, j = 0, 1, \dots, n-1$, and with relations (8). We let $J_q(n)$ denote the algebra $A_{n,n}$.*

We can of course also define an analogous algebra, $A_{m,n}^-$ based on the negative (non-compact) roots and it is clear that we also have

Proposition 2.11. *The quadratic algebras $A_{m,n}$ and $A_{m,n}^-$ satisfy*

$$U_q = A_{m,n}^- \cdot \mathcal{U}_q(\mathfrak{k}) \cdot A_{m,n}.$$

3. THE VARIETIES OF THE ALGEBRAS $J_q(n)$

In this section, we consider the associated variety of the modified algebra $J_q(n)$.

Following [1], [7], let V be a complex linear space and let $T(V)$ be the tensor algebra on V . Let $R \subset V \otimes V$ be a subspace and let (R) be the ideal of $T(V)$ generated by R . Set $A = T(V)/(R)$. This is a quadratic algebra. The *associated variety* $\Gamma = \Gamma(R) \subset \mathbb{P}(V^*) \times \mathbb{P}(V)^*$ is the variety defined by R .

In [8] the associated variety of the standard quantized matrix algebra was determined. Among other things it turned out to be independent of the quantum parameter q . In this section we (again) assume that the $q^2 \neq 1$ and we consider the associated variety Γ_n of the algebra $J_q(n)$. In some sense, this is the nicest.

A point $((a_{i,j}), (b_{i,j})) \in \Gamma_n$, where $(a_{i,j})$ and $(b_{i,j})$ are two $n \times n$ complex matrices, if and only if the following equations are satisfied for all 4-tuples $[a_1; a_2; a_3; a_4]$ and $[b_1; b_2; b_3; b_4]$ corresponding to the indices (i, j) , $(i + a, j)$, $(i, j + b)$, and $(i + a, j + b)$:

$$\begin{aligned} a_1 b_3 &= q^b a_3 b_1 \\ a_1 b_2 &= q^a a_2 b_1 \\ a_2 b_4 &= q^b a_4 b_2 \\ a_3 b_4 &= q^a a_4 b_3 \\ a_2 b_3 &= q^{b-a} a_3 b_2 \\ q^{1-a} a_1 b_4 - q^{b-1} a_4 b_1 &= (q - q^{-1}) a_2 b_3. \end{aligned}$$

Lemma 3.1. *Let the notations be as above. Then $a_{i,j} = 0$ if and only if $b_{i,j} = 0$.*

Proof. We assume that $a_{s,t} \neq 0$ for some (s, t) and $a_{i,j} = 0$ but $b_{i,j} \neq 0$. By

$$a_{i,j} b_{i,k} = q^{k-j} a_{i,k} b_{i,j},$$

we have $a_{i,k} = 0$ for all $k = 1, 2, \dots, n$.

If $(s - i)(t - j) \leq 0$ we have

$$a_{i,j}b_{s,t} = q^{s+t-i-j}a_{s,t}b_{i,j}$$

which implies that $b_{i,j} = 0$. Contradiction.

If $s > i$ and $t > j$ we have

$$q^{1-t+j}a_{i,j}b_{s,t} = q^{s-i-1}a_{s,t}b_{i,j} + (q - q^{-1})a_{i,t}b_{s,j}$$

which together with $a_{i,t} = 0$ imply $b_{i,j} = 0$ which again is a contradiction.

Similarly one can prove that if $s < i$ and $t < j$ we also get $b_{i,j} = 0$. This completes the proof. \square

Lemma 3.2. *Let $(a_{i,j})$ be an $n \times n$ complex matrix. Then $((a_{i,j}), (q^{i+j}a_{i,j})) \in \Gamma_n$ if and only if $(a_{i,j})$ is a rank one matrix.*

Proof. If the rank of $(a_{i,j})$ is equal to 1, the claim follows by an elementary computation. For the opposite direction, assume that $a_{i,j} \neq 0$ for some (i, j) . Then $b_{i,j} \neq 0$ and by multiplying through by some non-zero complex number we can assume that $b_{i,j} = q^{i+j}a_{i,j}$.

For any (s, t) , if $(s - i)(t - j) \leq 0$ we have

$$a_{i,j}b_{s,t} = q^{s+t-i-j}a_{s,t}b_{i,j},$$

so $b_{s,t} = q^{s+t}a_{s,t}$.

If $s > i$ and $t > j$, we have

$$q^{1-t+j}a_{i,j}b_{s,t} = q^{s-i-1}a_{s,t}b_{i,j} + (q - q^{-1})a_{i,t}b_{s,j}.$$

Since $b_{s,j} = q^{s+j}a_{s,j}$ and since rank 1 of the matrix $(a_{i,j})$ implies that $a_{i,t}a_{s,j} = a_{i,j}a_{s,t}$, we get $b_{s,t} = q^{s+t}a_{s,t}$.

Similarly, one can prove that if $s < i$ and $t < j$ then $b_{s,t} = q^{s+t}a_{s,t}$. This completes the proof. \square

Now we assume that the matrix $(a_{i,j})$ is indecomposable. Let $a_{i,j}$ be the first non-zero entry in the matrix $(a_{i,j})$ according to the lexicographic ordering and assume $b_{i,j} = q^{i+j}a_{i,j}$. Let

$$I_1 = \{(i, j) \mid b_{i,j} = q^{i+j}a_{i,j}\},$$

$$I_2 = \{(i, j) \mid b_{i,j} \neq q^{i+j}a_{i,j}\}.$$

Then it is easy to see that if $a_{i,k} \neq 0$ and $(i, k) \in I_1$ for some k , then $(i, j) \in I_1$ for all j . Similarly, if $a_{k,j} \neq 0$ and $(k, j) \in I_1$ for some k , then $(i, j) \in I_1$ for all i . Indeed, if $(k, l) \in I_1$ and if $(s - k)(t - l) \leq 0$ then $(s, t) \in I_1$. Since $I_1 \neq \emptyset$ and $(a_{i,j})$ is indecomposable, it follows easily

that $I_1 = I(n)$, i.e. $b_{i,j} = q^{i+j}a_{i,j}$ for all i, j . For any $i < s$ and $j < t$ we have

$$q^{1-t+j}a_{i,j}b_{s,t} = q^{s-i-1}a_{s,t}b_{i,j} + (q - q^{-1})a_{i,t}b_{s,j}.$$

Hence $a_{i,j}a_{s,t} = a_{i,t}a_{s,j}$. This proves that $\text{rank}(a_{i,j}) = 1$.

Now we assume that the matrix $(a_{i,j})$ is decomposable. Then $\text{rank}(a_{i,j}) \geq 2$. Let $a_{i,j}a_{s,t} \neq 0$, $(i, j) \in I_1$, $(s, t) \in I_2$. As above, $(s - i)(t - j) \leq 0$ is impossible. So without losing generality we assume that $s > i$ and $t > j$. We then must have $a_{i,t} = a_{s,j} = 0$. By

$$q^{1-t+j}a_{i,j}b_{s,t} = q^{s-i-1}a_{s,t}b_{i,j} + (q - q^{-1})a_{i,t}b_{s,j}$$

we get that $b_{s,t} = q^{s+t-2}a_{s,t}$ for $(s, t) \in I_2$. More generally this proves that the matrix $(a_{i,j})$ is in fact a direct sum of indecomposable matrices,

$$(a_{i,j}) = \text{diag}(D_1, D_2, \dots, D_r),$$

where each D_i 's is either zero or of rank one. Furthermore, the above analysis of how the relation between $a_{s,t}$ and $b_{s,t}$ follows from I_1 clearly implies (since $q^2 \neq 1$), that at most two of them are non-zero. Summarizing, we have proved

Theorem 3.3. *Let q be generic or q is an m th root of unity ($m \neq 2$). Let $((a_{i,j}), (b_{i,j})) \in \Gamma_n$. Then the matrix $(a_{i,j})$ is either of rank one and $b_{i,j} = q^{i+j}a_{i,j}$ for all i, j or $(a_{i,j})$ is of the following form:*

$$(a_{i,j}) = \text{diag}(0, 0, \dots, 0, D_1, 0, \dots, 0, D_2, 0, \dots, 0)$$

where D_i are rank one matrices. In this case,

$$(b_{i,j}) = \text{diag}(0, 0, \dots, T_1, 0, \dots, 0, T_2, 0, \dots, 0)$$

where $T_1 = (q^{i+j}a_{i,j}), T_2 = (q^{i+j-2}a_{i,j})$.

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