# QUADRATIC ALGEBRAS OF TYPE AIII; II

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Abstract. This is the second article in a series of three where we study some quadratic algebras related to quantized matrix algebras. In here, we study the algebra $M_q(2)$ by elementary methods and construct the simple modules and the primitive ideals. We relate this to Artin's programme of non-commutative algebraic geometry ([1]), in particular, we determine the point and line modules of this algebra.

## 1. Introduction

In the first of this series of three articles, [2], we established that for an quantized algebra $U_q(\mathfrak{g})$ of type $A_{m+n-1}$ there is a decomposition

$$U_q(\mathfrak{g}) = A^-_{m,n} \cdot \mathcal{U}_q(\mathfrak{k}) \cdot A_{m,n}, \tag{1}$$

where $A^-_{m,n}$ and $A_{m,n}$ are quadratic algebras which moreover are $U_q(\mathfrak{k})$ modules. Here, $\mathfrak{k}$ is the quantized enveloping algebra of a maximal compact subalgebra of $\mathfrak{g}$ obtained by viewing $\mathfrak{g}$ as the complexification of the Lie algebra $\mathfrak{g}_0$ corresponding to a hermitian symmetric space and (1) then generalizes the decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{p}^-) \cdot \mathcal{U}(\mathfrak{k}) \cdot \mathcal{U}(\mathfrak{p}^+).$$

This result has been generalized in [3] to arbitrary hermitian symmetric spaces.

Actually, the algebras $A_{m,n}$ and $A^-_{m,n}$ in (1) are not uniquely determined. One of the candidates for the part of e.g. $A_{m,n}$ is actually the well-known quantized matrix algebra $M_q(n)$ ([4]). The latter has been the subject of many investigations, (see e.g. [5] and references therein)

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but it seems that one always has been taking a somewhat different point of view, namely that of getting hold of a quantum $GL(n)$ or $SL(n)$. Our point of view is really to take over Artins program [1]. According to this program one must determine a certain associated variety, determine the point, line, plane, etc. modules, the "fat points" and so on, and relate these to the variety. It has been a guiding principle for us that the variety should be "nice" and for this reason we choose a family of quadratic algebras that does not reduce to $M_q(n)$ in the case where $m = n$.

The previous discussion of the various families should mostly be taken as an advertisement for our family. In the present article we have chosen to compute all details in the special case $M_q(2)$ and here the families do agree.

The material is organized as follows: In Section 4 we construct the simple modules and the primitive ideals of $M_q(2)$. In Section 5 we find the point and line modules of the algebra $M_q(2)$.

After having finished this paper the articles by Vancliff [6], [7] were brought to our attention. In there, the point and line modules have been obtained for $M_q(2)$ by methods which are quite different from our quite elementary approach.

## 2. Simple representations and primitive ideals for $M_q(2)$.

In this section we consider the algebra $M_q(2)$, i.e. the $\mathbb{C}$-algebra generated by the elements $z_1, z_2, z_3$ and $z_4$ subject to the relations

$$\text{(2)}\qquad \begin{array}{lrcllrcl} (\mathrm{r}_1) & z_1z_2 & = & qz_2z_1 & (\mathrm{r}_2) & z_1z_3 & = & qz_3z_1 \\ (\mathrm{r}_3) & z_2z_3 & = & z_3z_2 & (\mathrm{r}_4) & z_2z_4 & = & qz_4z_2 \\ (\mathrm{r}_5) & z_3z_4 & = & qz_4z_3 & (\mathrm{r}_6) & z_1z_4 - z_4z_1 & = & (q-q^{-1})z_2z_3, \end{array}$$

where $q \in \mathbb{C} \setminus \{0, 1, -1\}$.

We find all simple representations as well as the primitive ideals of this algebra in case $q$ is a primitive $n$'th root of unity. Throughout, we assume that $q \neq \pm 1$. In case $q$ is not a root of unity the simple representations are already well studied in the literature (cf. [8], [9]). Later on we will find all point, line, and planes modules for the algebra.

It is well-known that $M_q(2)$ is an iterated Ore extension. From this we get that $M_q(2)$ is a left and right Noetherian domain and $M_q(2)$ is a free $\mathbb{C}$-module with bases

$$z_1^{k_1} z_2^{k_2} z_3^{k_3} z_4^{k_4} \qquad k_i \in \mathbb{N}_0 \quad i = 1 \cdots 4$$

We also need the following relations for later purposes

$$z_1z_4^j = z_4z_1z_4^{j-1} + (q-q^{-1})z_2z_3z_4^{j-1} =$$
$$z_4^rz_1z_4^{j-r} + (1+q^{-2}+q^{-2(r-1)})(q-q^{-1})z_2z_3z_4^{j-1} =$$
$$z_4^jz_1 + q(1-q^{-j})z_2z_3z_4^{j-1}$$

In case $q$ is an $n$-th root of unity it is readily checked that $z_j^n, j = 1,2,3$ and 4 are all central elements and thus $M_q(2)$ is a finite module over its center, hence in this case $M_q(2)$ is a P. I. algebra and all simple representations are finite dimensional, [10, Theorem 13.10,3].

Kaplansky's Theorem for P. I. algebras gives that any primitive ideal of $M_q(2)$ is maximal and the corresponding factoring is a central simple algebra of finite dimension over its center which is a field $K$.

Now the Artin-Tate lemma [10, 13.9.10] implies that $K$ is a finitely generated $\mathbb{C}$-algebra and $K = \mathbb{C}$ by the classical Nullstellensatz and the fact that $\mathbb{C}$ is algebraically closed. Thus we have shown that all primitive factoring of $M_q(2)$ are full matrix rings over $\mathbb{C}$.

Throughout this section we assume $q$ is a primitive $n$'th root of unity. For short we let $A$ denote $M_q(2)$. If $V$ is a representation of $A$, $z_i(v)$ denotes the representation of $z_i$ acting on $v$.

We will now determine all simple representations and all primitive ideals of $A$. Observe that $z_2$ ($z_3$) is a normal element and hence $\{v \in V \mid z_2(v) = 0\}$ is an invariant subspace. Thus, in a simple representation $z_2$ ($z_3$) is mapped either to zero or to an invertible operator.

Consider first those primitive ideals $P$ for which $A/P \simeq \mathbb{C}$. The relation ($\mathrm{r}_6$) in (2) shows that in this case either $z_2$ or $z_3$ is in $P$.

If $z_2, z_3 \in P$ then $A/P$ is a factor algebra of $\mathbb{C}[z_1, z_4]$, the polynomial ring in 2 indeterminates, and therefore by the classical Nullstellensatz, $P = (z_1-\alpha_1, z_2, z_3, z_4-\alpha_4)$ for suitable $\alpha_1, \alpha_4 \in \mathbb{C}$.

Next assume $z_2 \in P$ and $z_3 \notin P$. Then in $A/P$, $z_1z_3 = qz_3z_1$, hence $z_1 \in P$ and likewise $z_4 \in P$.

So we have the following complete list of kernels of 1-dimensional representations

- $(z_1-\alpha_1, z_2, z_3, z_4-\alpha_4)$
- $(z_1, z_2, z_3-\alpha_3, z_4)$ and • $(z_1, z_2-\alpha_2, z_3.z_4)$

$\alpha_2$ and $\alpha_3$ in $\mathbb{C}\setminus\{0\}$.

We will next describe the primitive ideals $P$ and simple representations $S_p$ for which $z_2 \in P$ and $\dim_{\mathbb{C}} S_p > 1$. Since $z_2 \in P$, the primitive ideals and simple representations are the ones for the algebra

$\mathbb{C}[z_1, z_3, z_4]$ with defining relations

$$\begin{array}{cccccc} (i) & z_1z_3 & = & qz_3z_1 & (ii) & z_3z_4 = qz_4z_3 \\ (iii) & z_1z_4 & = & z_4z_1. & & \end{array}$$

This algebra is in fact $A_1 = \mathbb{C}[z_1, z_4][z_3; \sigma]$ where $\sigma(z_1) = qz_1$ and $\sigma(z_4) = q^{-1}z_4$.

The representation theory for this algebra is treated in [11] in a more general setting, but it is in fact quite easy to list the primitive ideals and corresponding simple representations.

First notice that $z_3$ is a normal element in $A_1$, so $z_3$ is represented by an invertible matrix since it follows that $z_3 \notin P$ due to the assumptions on $\dim S_p$.

Let $V$ be a simple representation. Then $z_1$ and $z_4$ commute and we let $v \in V$ be a common eigenvector with eigenvalues $\lambda_1$ and $\lambda_4$, respectively. Since $z_3^n$ is central, we get that $z_3^n(v) = \lambda v, \lambda \neq 0$, and hence

$$\mathbb{C}v + \mathbb{C}z_3(v) + \cdots + \mathbb{C}z_3^{n-1}(v)$$

is an invariant subspace of $V$. For instance, $z_1(z_3^i(v)) = q^i z_3^i(z_1(v)) = q^i\lambda_1 z_3^i(v)$. So in case either $\lambda_1$ or $\lambda_4$ is non-zero, $\dim V = n$. If $\lambda_1 = 0$, $z_1 = 0$ by the above. But $\lambda_1$ and $\lambda_4$ cannot both be 0, since the representation is assumed to have dimension strictly larger than 1.

Assuming $\lambda_1 = 0$ and $\lambda_4 \neq 0$, the algebra $A/P$ is a factor of $\mathbb{C}[z_3, z_4]$, with defining relations

$$z_3z_4 = qz_4z_3 \quad z_3^n = \mu I \quad \text{and} \quad z_4^n = \lambda I,$$

$\lambda$ and $\mu$ both non-zero. This algebra is in fact simple and therefore isomorphic to $M_n(\mathbb{C})$: It is easily seen that every element is a $\mathbb{C}$-combination of monomials of the form $z_3^i z_4^j$, $0 \leq i, j \leq n-1$, and this representation is unique. Let $\mathfrak{a}$ be a non-zero two-sided ideal and pick $f \neq 0$ in $\mathfrak{a}$ such that $f = \sum_{v=0}^{m} p_v(z_3)z_4^v$, where $f$ is a polynomial of degree less than $n$ and $m$ is chosen minimal among all non-zero $f$'s in $\mathfrak{a}$. If $m \neq 0$, then

$$q^{-m}z_3f - fz_3 = \sum_{v=0}^{m-1}(q^{-m} - q^{-v})z_3p_v(z_3)z_4^v$$

and by minimality of $m$, $p_m(z_3)z_4^m \in \mathfrak{a}$ and hence $p_m(z_3)z_4^n = p_m(z_3)\lambda \in \mathfrak{a}$. Thus $m = 0$.

Next, pick $g = a_0 + \cdots + a_k z_3^k$ in $\mathfrak{a}$ with $a_k \neq 0$ and $k$ minimal. Then

$$\mathfrak{a} \ni gz_4 - q^k z_4 g = \left(\sum_{i=0}^{k-1} a_i(1 - q^{k-i})z_3^i\right) z_4.$$

Multiplying by $z_4^{n-1}$ gives that $\sum_{i=0}^{k-1} a_i(1-q^{k-i})z_3^i \in \mathfrak{a}$ and by the minimality of $k$ we get $a_i = 0$ for $i = 0, \dots, k-1$, so $z_3^k \in \mathfrak{a}$, hence $\mu I \in \mathfrak{a}$, hence $\mathfrak{a}$ is the whole ring, Thus, the algebra is simple as claimed.

Hence we have the following primitive ideals

- $(z_1, z_2, z_3^n - \alpha_3 1, z_4^n - \alpha_4 1) \quad \alpha_3, \alpha_4 \neq 0,$
- $(z_1^n - \alpha_1 1, z_2, z_3^n - \alpha_3 1, z_4) \quad \alpha_1, \alpha_3 \neq 0,$

and similar primitive ideals with $z_2$ and $z_3$ interchanged.

We are now left with the case $z_1(v) \neq 0$, $z_4(v) \neq 0$, $z_2 \in P$, and $z_3(v) \neq 0$, where $v$ is a common eigenvector for $z_1$ and $z_4$ with eigenvalues $\lambda_1$ and $\lambda_4$. Then

$$V = \mathbb{C}v + \cdots + \mathbb{C}z_3^{n-1}v.$$

Observe that for any $j = 0, \dots, n-1$,

$$\lambda_4 z_1^{n-1}(z_3^j(v)) = \lambda_4 q^{-j} z_3^j(\lambda_1^{n-1}(v)) \text{ and}$$

$$\lambda_1^{n-1} z_4(z_3^j(v)) = \lambda_1^{n-1} q^{-j} z_3^j(\lambda_4(v)).$$

Hence $z_1^{n-1} = dz_4$ for some non-zero constant $d$ and the algebra is thus generated by $z_1$ and $z_3$ with defining relations $z_1z_3 = qz_3z_1$ $z_3^n = \mu I$, and $z_1^n = \lambda I$. This algebra is simple by the previous argument.

Thus we have the following primitive ideals corresponding to some $n$ dimensional simple representations:

$$(z_1^n - \alpha_1, z_2, z_3^n - \alpha_3, z_4 - dz_1^{n-1}),$$

where $d, \alpha_3, \alpha_4$ are non-zero.

So far we have assumed that either $z_2$ or $z_3$ is in the primitive ideal corresponding to the simple representation.

Suppose now that neither $z_2$ nor $z_3$ is in $P$, a primitive ideal of $A$, and consider the corresponding simple representation $V$. Since $z_2$ and $z_3$ are normal elements they must act as isomorphisms on $V$. The simple representations of $A$ such that $z_2$ and $z_3$ act as isomorphisms are the ones for the algebra obtained by adjoining the inverses of $z_2$ and $z_3$ to $A$.

Let $v$ be a common eigenvector for the commuting maps $z_2, z_3$ and $z_1z_4$. Denote the corresponding eigenvalues by $\lambda_2, \lambda_3$, and $\mu$, where $\lambda_2$ and $\lambda_3$ both are non-zero.

1°. Let us first assume that $z_4$ is not nilpotent on $V$; then $v, z_4(v), \dots, z_4^{n-1}(v)$ are all non-zero and hence linearly independent being eigenvector for $z_2$ with different eigenvalues. Moreover, $V_0 = \mathrm{Span}(v, z_4(v), \dots, z_4^{n-1}(v))$ is invariant under the action of $z_2, z_3$ and $z_4$. It is also invariant under $z_1$ and so $V_0 = V$. To see this, observe that $z_4^n = \rho I$.

So $z_1z_4(v)) = \mu v$, $z_1z_4^k(v) = z_4z_1z_4^{k-1}(v) + (q-q^{-1})z_2z_3z_4^{k-1}(v)$, and $z_1z_4^n(v) = z_1(\rho v)$ for some non-zero $\rho$. Thus $V_0$ is an invariant subspace of $V$.

By $I_q$ we denote the matrix diag $(1, q, \ldots, q^{n-1})$;

$$\begin{pmatrix} 1 & & & \\ & q & & \\ & & \ddots & \\ & & & q^{n-1} \end{pmatrix}.$$

In the given base we have the following representation of $A$ on $V$:

$$z_2 \to \lambda_2 I_q \quad z_3 \to \lambda_3 I_q \quad \text{and thus} \quad z_3 = \lambda_3\lambda_2^{-1}z_2$$

$$z_4 \to \begin{pmatrix} 0 & & & \rho \\ 1 & 0 & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix} \text{ and } z_1 \to \begin{pmatrix} 0 & a_2 & & \\ & 0 & \ddots & \\ & & 0 & a_n \\ a_1 & & & 0 \end{pmatrix}. \tag{3}$$

Here, $\rho \in \mathbb{C} \setminus \{0\}$. All entries in the matrices above that are not explicitly given, are 0. The actual form of $z_1$ is a consequence of the relation $z_1z_2 = qz_2z_1$.

All defining relations except ($r_6$) for the algebra $M_q(2)$ hold for the matrices defined above.

From ($r_6$) we get

$$\begin{aligned} a_2 - \rho a_1 &= (q-q^{-1})\lambda_2\lambda_3 \\ a_3 - a_2 &= (q-q^{-1})q^2\lambda_2\lambda_3 \end{aligned}$$

and

$$\begin{aligned} a_n - a_{n-1} &= (q-q^{-1})q^{2n-4}\lambda_2\lambda_3 \\ a_1\rho - a_n &= (q-q^{-1})q^{2n-2}\lambda_2\lambda_3 \end{aligned}$$

Thus, we get that $a_2, \ldots, a_n$ are uniquely determined by $a_1, \rho$, and we have to verify the last condition $a_1\rho - a_n = (q-q^{-1})q^{2n-2}\lambda_2\lambda_3$. But

$$a_n = a_1\rho + (q-q^{-1})(\lambda_2\lambda_3)(1 + q^2 + \cdots + q^{2n-4}),$$

and since $q^2$ is an $n$'th root of unity we get the result.

This representation is in fact simple: Suppose $U$ is a non-zero invariant subspace and pick $0 \neq u = \sum_{i=0}^{n-1} x_i z_4^i(v)$ such that the number of non-zero $x_i$'s is smallest possible. Eventually after having applied $z_4$ a number of times we may assume $x_0 \neq 0$. Now $\lambda_2 u - z_2(u) \in U$ and has fewer non-zero coefficients thus $u = x_0 v$ and $U = V$.

To describe the corresponding primitive ideal we replace $z_2, z_3$ and $z_4$ by $\tilde{z}_2, \tilde{z}_3$ and $\tilde{z}_4$, where

$$\tilde{z}_2 = \lambda_2^{-1} z_2, \quad \tilde{z}_3 = \lambda_3^{-1} z_3, \text{ and } \tilde{z}_4 = \rho^{-\frac{1}{n}} z_4.$$

Then with these new generators we get $\tilde{z}_2 = \tilde{z}_3$, $\tilde{z}_4^n = 1, \tilde{z}_2^n = 1$, and $\tilde{z}_1 = c\tilde{z}_4^{-1} + \tilde{z}_2 \tilde{z}_4^{-1} \tilde{z}_2$. So the algebra is generated by $\tilde{z}_2$ and $\tilde{z}_4$ where $\tilde{z}_2^n = \tilde{z}_4^n = 1$ and $\tilde{z}_2 \tilde{z}_4 = q \tilde{z}_4 \tilde{z}_2$ and we have already shown that this algebra is simple.

In case $z_1$ is not nilpotent we get a similar result.

2°. We now assume that both $z_1$ and $z_4$ are nilpotent.

Let $V_i = \{v \in V \mid z_i(v) = 0\}$, $i = 1$ and 4. If $V_1 \cap V_4 \neq 0$, then by relation ($r_6$), $z_2$ or $z_3$ is not injective on this invariant subspace. Thus $V_1 \cap V_4 = 0$.

Let $U = \{v \in V \mid z_1(v) = 0\}$. Clearly $U$ is invariant under the action of $z_2, z_3$, and $z_1 z_4$ and we can pick a common eigenvector, $u \in U$, for these three endomorphisms with corresponding eigenvalues $\lambda_2, \lambda_3$, and $\mu$. Observe that $\mu \neq 0$ by previous argument.

As seen in the previous case, $\text{Span}(u, z_4(u), \ldots, z_4^{n-1}(u))$ is an invariant subspace of $V$ and hence equal to $V$.

If $z_4^j(u) \neq 0$, then $z_4^j(u)$ is an eigenvector for $z_2$ with eigenvalue $\lambda_2 q^j$. Thus $V$ has a base $(u, z_4(u), \ldots, z_4^{k-1}(u))$ where $z_4^k(u) = 0$.

In the corresponding matrix representation

$$z_2 \to \lambda_2 I_q \quad z_3 \to \lambda_3 I_q, \text{ and}$$

$$z_4 \to \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Since $z_1 z_2 = q z_2 z_1$, we get as in (3)

$$z_1 \to \begin{pmatrix} 0 & a_2 & & \\ & 0 & \ddots & \\ & & 0 & a_k \\ a_1 & & & 0 \end{pmatrix}.$$

From ($r_6$) we get

$$\begin{pmatrix} a_2 & & & \\ & \ddots & & \\ & & a_k & \\ & & & 0 \end{pmatrix} - \begin{pmatrix} 0 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_k \end{pmatrix} = (q - q^{-1}) \lambda_2 \lambda_3 I_{q^2}.$$

Let us for short write $c = (q - q^{-1})\lambda_2\lambda_3$ and observe that $a_1 = 0$ since $z_1$ is nilpotent. Hence

$$\begin{aligned} a_2 &= c, \quad a_3 = c + cq^2 = c(1+q^2) \\ a_4 &= (1+q^2) + cq^4 = c(1+q^2+q^4) \\ &\vdots \\ a_k &= c(1+\cdots+q^{2(k-2)}) \quad \text{and} \quad a_k = -cq^{2(k-1)}. \end{aligned}$$

Thus, ($\mathrm{r}_6$) is satisfied if and only if

$$c(1+\cdots+q^{2(k-2)}+q^{2(k-1)}) = 0,$$

hence $q^2$ must be a $k$'th root of unity. In conclusion, for $n$ odd we get for each $\lambda_2, \lambda_3$ a representation of degree $n$.

For $n$ even we get a representation of degree $n/2$ for each $\lambda_2, \lambda_3$ and one of degree $n$. But in the degree $n$ representation $a_{\frac{n}{2}+1} = 0$, hence this representation is not simple.

We claim that the remaining representations are simple.

Let $U$ be an invariant subspace of $V$. Then by standard arguments, $U$ contains eigenvectors for $z_2$ and $z_3$. But together, $z_1$ and $z_4$ permute *all* eigenvectors, hence $U = V$.

We conclude this section by determining the primitive ideals in case $z_1$ and $z_4$ acts a nilpotent matrices.

Consider first the case $n$ odd:

$$z_1^n, z_4^n, z_2^n - \lambda I, \text{ and } z_3 - \mu z_2 \text{ are in } P.$$

Let us for simplicity set $\lambda = \mu = 1$. In addition to the above, it is easy to see that the central element $z_1z_4 - qz_2z_3$ acts as $-q^{-1}$.

It follows that $z_2^2$ can be expressed via $z_1z_4$, and hence so can $z_2$ since it is equivalent to $z_2^{n+1}$. Furthermore, $z_1$ and $z_4$ satisfy (after having absorbed the factor $q - q^{-1}$ into one of them)

$$z_1z_4 - q^2z_4z_1 = 1. \tag{4}$$

So the algebra is generated by $z_1$ and $z_4$ subject to the relation (4) together with $z_1^n = z_4^n = 0$. We now prove that the latter algebra is simple, hence isomorphic to $A/P$.

First observe that the algebra clearly has dimension at most $n^2$. Next consider a simple representation of the algebra on a vector space $V$. Let $v \in V$ be a non-zero eigenvector of $z_1$ (of eigenvalue 0, of course). Then the subspace $V_0 = \mathrm{Span}\{v, z_4v, \ldots, z_4^{n-1}\}$ is clearly invariant and hence equal to $V$. Furthermore,

$$z_1(z_4^iv) = (1+q^2+\cdots+q^{2i-2})z_4^{i-1}.$$

It follows easily from this that $\{z_4^i\}_{i=0}^{n-1}$ is a basis of $V$ and hence $V$ is of dimension $n$.

Thus,

$$P = (z_1^n, z_4^n, z_2^n - 1, z_3 - z_2, z_1 z_4 - q z_2 z_3 + q^{-1}).$$

A similar but slightly more complicated argument shows that the ideal $\tilde{P}$ below (where we again have set $\mu = \lambda = 1$) is primitive in the case where $n$ is even. In fact, one can show that $z_2$ can be obtained from expressions of the form $z_1^i z_4^i$, $i = 0, \dots, n-1$. Thus, the primitive ideal is given as

$$\tilde{P} = (z_1^{n/2}, z_4^{n/2}, z_3 - z_2, z_2^n - 1, z_1 z_4 - q z_2 z_3 + q^{-1}).$$

**Remark 2.1.** *Before summarizing our findings, we make some observations about equivalence of representations and primitive ideals:*

- *Primitive ideals are by definition annihilators of simple modules, thus equivalent simple modules must define the* **same** *primitive ideal.*
- *As remarked earlier, if $P$ is a primitive ideal in $A$ then $A/P$ is a full matrix algebra. Thus,*

$$A/P = \underbrace{S \oplus \dots S,}_{k}$$

  *where $S$ is a simple $k$-dimensional module whose annihilator is $P$.*
- *One may define an $A$ equivalence of simple representations by saying that $\pi_1$ is equivalent to $\pi_2$ if there exists a graded automorphism $\alpha$ of $A$ such that*

$$\forall a \in A : \pi_2(a) = \pi_1(\alpha(a)).$$

  *The group of graded automorphisms of $A$ is generated by the maps $(z_1, z_2, z_3, z_4) \to (\lambda_1 z_1, \lambda_2 z_2, \lambda_3 z_3, \lambda_4 z_4)$ with $\lambda_1, \dots, \lambda_4 \in \mathbb{C}^*$ and $\lambda_1 \lambda_4 = \lambda_2 \lambda_3$ and the map $z_2 \leftrightarrow z_3$. The map $z_1 \leftrightarrow z_4$ may be viewed as a conjugate-linear automorphism. When we list our result below we do it modulo these graded linear or anti-linear automorphisms*

The result may then be given as follows:

**Theorem 2.2.** *Up to graded (possibly conjugate linear) isomorphism, the algebra $M_q(2)$ has the following primitive ideals*

| | *primitive ideal* | *dimension of rep.* |
|---|---|---|
| *1* | $(z_1-1, z_2, z_3, z_4-1)$ | *1* |
| *2* | $(z_1, z_2, z_3-1, z_4)$ | *1* |
| *3* | $(z_1^n-1, z_2, z_3^n-1, z_4)$ | $n$ |
| *4* | $(z_1^n-1, z_2, z_3^n-1, z_4-dz_1^{n-1})$ | $n$ |
| *5* | $(z_1^n-1, z_2^n-1, z_2-z_3, z_4-z_2z_1^{n-1}z_3-cz_1^{n-1})$ | $n$ |
| *6* | $(z_1^n, z_2^n-1, z_2-z_3, z_4^n, z_1z_4-qz_2z_3+q^{-1})$ | $n$ *($n$ odd)* |
| *7* | $(z_1^{n/2}, z_2^n-1, z_2-z_3, z_4^{n/2}, z_1z_4-qz_2z_3+q^{-1})$ | $n/2$ *($n$ even)* |

**Remark 2.3.** *Continuing the discussion in Remark 2.1 above, one may call two representations quasi-equivalent if they are connected by a non-graded automorphism. For instance the maps $(z_1, z_2, z_3, z_4) \to (z_1, z_2, z_3, z_4 + c_iz_2^iz_3^{n-i}z_1^{n-1})$ (i fixed) and $(z_1, z_2, z_3, z_4) \to (z_1, z_2 + qaz_3z_1^n, z_3, z_4 + cz_1^{n-1} + az_3^2z_1^{n-1})$ are examples of such. By means of these, it is easy to see that 3 is quasi-equivalent to 4 and that 5 is quasi-equivalent to $(z_1^n-1, z_2^n-1, z_2-z_3, z_4^n)$. Representations with different degree are of course inequivalent. Furthermore, it seems that 5 relations are needed for the primitive ideal in 6 and hence this would appear to be inequivalent from the others. Indeed, we believe that they are all inequivalent but have not been able to prove this. It also seems difficult to determine the group of automorphisms.*

In case $n$ is odd more can be said about $A_{2,2}$. First observe that for every primitive ideal $P$ such that $z_2 \notin P$ and $z_3 \notin P$ we have that $\dim_{\mathbb{C}} A/P = n^2$. Since both $z_2$ and $z_3$ are normal elements we can form $A[z_2^{-1}, z_3^{-1}] = B$ which is an Azumaya algebra by the Artin-Procesi Theorem [10, 7.14 Theorem].

Since $B$ is a finitely generated projective module over its center $C$, $\dim_{\mathbb{C}_m} B_m$ is a constant and equal $\dim_F B \otimes_c F = \dim_F Q(A)$ by [12, Chap. II, §5, $n^0 3$], where $F$ is the quotient of the center of $A$ and also $C$, and $Q(A)$ is the quotient division-ring of $A$. (Posner's Theorem).

Notice $\dim_{C_m} B_m = \dim_{(C/m)_m} (B/mB)_m = \dim_{C/m} B/mB$, $mB$ is a maximal ideal in $B$, since $B$ is Azumaya and $C/m$ being a field and affine over $\mathbb{C}$ is isomorphic to $\mathbb{C}$ by Nullstellensatz and Artin-Tate lemma.

In case $q$ is not a primitive root of unity the primitive ideals of the factor algebra $R_q[SL(2)]$ of $A$ (corresponding to $z_1z_4 - qz_2z_3 = 1$) were found in [13]. A complete list of the primitive ideals in this case is

$$\begin{array}{ll} (i) & (z_2, z_3, z_1 - \lambda, z_4 - \lambda^{-1}), \quad \lambda \neq 0 \\ (ii) & (z_2) \\ (ii)' & (z_3) \\ (iii) & (z_2 - \lambda z_3), \quad \lambda \neq 0. \end{array}$$

## 3. Point, Line, and Plane Modules.

In this section we determine the point, line, and plane modules for $M_q(2)$.

Let us first recall the definition from [14]:

**Definition 3.1.** *Let $M$ be a cyclic $\mathbb{Z}$ graded module. We say*

- *$M$ is a point module if $h_M(t) = (1-t)^{-1}$.*
- *$M$ is a line module if $h_M(t) = (1-t)^{-2}$.*
- *$M$ is a plane module if $h_M(t) = (1-t)^{-3}$.*

In the paper [14], results for these classes of modules were obtained for the four dimensional Sklyanin algebra. An examination of the arguments in that paper reveals that only some general properties of the Sklyanin algebra were in play. In fact, it was only used that the algebra is Auslander regular, i.e. Auslander–Gorenstein of finite global dimension. (For the definitions and simple properties of Cohen–Macaulay, Auslander–Gorenstein, and Auslander regular the reader is referred to [1], [15], and [14].)

That $M_q(2)$ satisfies these properties follows from the following result due to T. Levasseur, cf. [16] and to the fact that $gl\dim M_q(2) = 4$ (cf. Section 2).

**Proposition 3.2.** *Suppose a graded ring $B$ contains a regular normalizing sequence $z_2, z_3$ of homogeneous elements of positive degree. If $B/(z_2, z_3)$ is Auslander-Gorenstein of dimension $l$ and satisfies the Cohen-Macaulay property, then $B$ is Auslander-Gorenstein of dimension $l+2$ and satisfies the Cohen-Macaulay property.*

Since $M_q(2)$ has global dimension 4, $M_q(2)$ is Auslander-regular as well.

Thus, from [14, Corollary 1.11] we get

**Proposition 3.3.** *Let $M$ be a point, line or plane module, then $M$ is Cohen-Macaulay, $M$ is $n$-pure and since such an $M$ has $e(M) = 1$, $M$ is critical.*

We first list some general results following the lines of [14]; all modules are graded modules.

**Proposition 3.4.** *An $M_q(2)$ module $M$ is a plane module if and only if $M \simeq A/Aa, a \in A_1$. Here, $A_1$ denotes the homogeneous elements of degree 1.*

*Proof.* Suppose $M$ is a plane module. Then $\dim M_0 = 1$ and $\dim M_1 = 3$. Since $\dim A_1 = 4$ and $M$ is cyclic there exists a non-zero $a \in A_1$ such that $aM_0 = 0$. Thus, $M = AM_0$ is a quotient of $A/Aa$ and since $A$ is a domain, $M$ and $A/Aa$ have the same Hilbert series, hence they are isomorphic. □

A similar argument shows

**Proposition 3.5** ([14, Proposition 2.8])**.** *$M$ is a line module if and only if $M \simeq A/Au + Av$, where $u, v \in A_1$ are linearly independent elements such that $A_1u \cap A_1v \neq 0$.*

We first determine the point modules for $M_q(2)$. The main tool is the following result from [1] which establishes a one-to-one correspondence between point modules and the points on a certain variety related to the algebra. The result is valid for a large class of quadratic algebras and we shall state it in this generality.

**Theorem 3.6** ([1])**.** *Consider an affine quadratic algebra $A$, i.e.*

$$A = T(V)/(R),$$

*where $V$ is a finite-dimensional vector space over a field $k$ and $R \subseteq V \otimes V$ is the set of (quadratic) relations.*

*Let $\mathbb{P} = \mathbb{P}(V^*)$ and consider the associated variety $\Gamma = \mathcal{V}(R) \subseteq \mathbb{P} \otimes \mathbb{P}$. The canonical projections $\mathbb{P} \times \mathbb{P} \to \mathbb{P}$ are denoted $\pi_1$ and $\pi_2$. If $\pi_1(\Gamma_n) = \pi_2(\Gamma_n) = E$ and the $\pi_i|_{\Gamma_n}$ are injective then there is a bijection*

$$E \leftrightarrow \{\textit{Isomorphism classes of point modules}\}$$

*given by*

$$E \ni p \leftrightarrow M(p)$$

*where $M(p) = A/(Ax_1 + \cdots + Ax_n)$ and $x_1, \ldots, x_n \in V$ are determined by*

$$\mathcal{V}(x_1, \ldots, x_n) = \{p\}.$$

By Theorem 3.3 in [2] we have that

$$E = \{[p_1; p_2; p_3; p_4] \in \mathbb{P} \mid p_1p_4 - p_2p_3 = 0 \text{ or } p_2 = p_3 = 0\}$$

and

$$\Gamma_n = \{(p, p^\sigma) \in \mathbb{P} \times \mathbb{P} \mid p \in E\},$$

where $\sigma([p_1; p_2; p_3; p_4]) = [p_1; qp_2; qp_3; q^2p_4]$ for any point on the quadric and $\sigma$ acts as the identity on the line.

Thus the above result of [1] applies to give

**Theorem 3.7.** *The point modules of $A$ are (up to isomorphism) in one-to-one correspondence with the points of $E$ (determined above), i.e. "there is a quadric and a line of point modules". In particular, the variety of point modules is independent of $q$.*

Plane modules are characterized by Proposition 3.5 and are in 1-1 correspondence with hyperplanes in $\mathbb{P}^3$.

We now determine the line modules. By Proposition 3.5, we need to find all pairs of elements $u, v \in A_1$ that are a) linearly independent over $\mathbb{C}$ and for which b) $A_1u \cap A_1v \neq 0$. Equivalently, we will characterize the varieties in $\mathbb{P}^3$ defined by complex vector spaces $A_1u + A_1v$ where $u, v$ satisfy a) and b).

First observe that we may add a scalar multiple of $u$ to $v$ (or vice versa), and we may multiply either one by a non-zero complex number without changing the variety and without affecting the properties a) and b).

Now write

$$\begin{aligned} u &= a_1z_1 + a_2z_2 + a_3z_3 + a_4z_4 \\ v &= b_1z_1 + b_2z_2 + b_3z_3 + b_4z_4 \end{aligned}$$

We divide our investigation into several cases:
1°: $a_1 = b_1 = 0$.

We claim that $L \simeq A \otimes_B L_0$, where $B = \mathbb{C}[z_2, z_3][z_4, \sigma_4]$ and $L_0$ is a point module over $B$.

First of all, the point modules over $B$ are easy to find. To wit, an argument similar to the one in Proposition 3.4 shows that $L_0$ is a point module over $B$ if and only if $L_0 = B/Bu + Bv$ where $u, v$ are linearly independent elements in $B_1$ such that $B_1u \cap B_1v \neq 0$. But a straightforward computation shows that the latter condition is always satisfied for non-zero $u, v$ (and then of course we also have $A_1u \cap A_1v \neq 0$. Next notice that the PBW Theorem tells us that $A$ is a free $B$ module. Thus

$$A \otimes_B B/Bu + Bv \simeq A/Au + Av = L,$$

i.e. we have shown that in this case the line modules over $A$ are in 1-1 correspondence with the point modules over $B$ (quantum $\mathbb{P}^2$).
2°: $a_1 \neq 0$ or $b_1 \neq 0$.

We may without loss of generality assume that $a_1 = 1, b_1 = 0$.

Either $b_2 = b_3 = 0$ or we may assume $b_2 \neq 0$ in which case we may take $a_2 = 0$. Let us consider the last case first.

2° (i)

$$u = z_1 + a_3 z_3 + a_4 z_4 \text{ and } v = z_2 + b_3 z_3 + b_4 z_4.$$

Suppose we have a non-trivial relation

$$\begin{aligned} &(r_1 z_1 + r_2 z_2 + r_3 z_3 + r_4 z_4) \cdot (z_1 + a_3 z_3 + a_4 z_4) \\ = &(s_1 z_1 + s_2 z_2 + s_3 z_3 + s_4 z_4) \cdot (z_2 + b_3 z_3 + b_4 z_4) \ . \end{aligned}$$

We get $r_1 z_1^2 = 0$, $s_2 z_2^2 = 0$ and hence $r_1 = s_2 = 0$.

Moreover,

$$\begin{array}{rclcrcl} r_2 z_2 z_1 & = & s_1 z_1 z_2 & \text{so} & r_2 & = & s_1 q \ , \\ r_3 z_3 z_1 & = & s_1 b_3 z_1 z_3 & \text{so} & r_3 & = & s_1 b_3 q \ , \\ r_4 z_4 z_1 & = & s_1 b_4 z_1 z_4 & \text{so} & r_4 & = & s_1 b_4 \ , \end{array}$$

since the relation is non-trivial, $s_1 \neq 0$, and we may multiply the relation by $s_1^{-1}$ to get

$$\begin{gathered} (q z_2 + q b_3 z_3 + b_4 z_4)(z_1 + a_3 z_3 + a_4 z_4) = \\ (z_1 + s_3 z_3 + s_4 z_4)(z_2 + b_3 z_3 + b_4 z_4) \ . \end{gathered}$$

Consequently,

$$\begin{array}{rclcl} b_4 a_4 z_4^2 & = & s_4 b_4 z_4^2 & \text{so} & a_4 b_4 = s_4 b_4 \\ q a_4 z_2 z_4 & = & s_4 z_4 z_2 & \text{so} & q^2 a_4 = s_4 \ . \end{array}$$

If $a_4 = b_4 = 0$ we are back in case 1° with $z_1$ replaced by $z_4$.

2° (i) a) If $b_4 \neq 0$ then $a_4 = s_4 = q^{-2} s_4$ so $a_4 = s_4 = 0$ and we have the following relation

$$\begin{gathered} (q z_2 + q b_3 z_3 + b_4 z_4)(z_1 + a_3 z_3) = \\ (z_1 + s_3 z_3)(z_2 + b_3 z_3 + b_4 z_4) \end{gathered}$$

and we conclude

$$\begin{array}{llcll} (i) & q b_3 a_3 z_3^2 & = & s_3 b_3 z_3^2 & \text{so } (q a_3 - s_3) b_3 = 0; \\ (ii) & b_4 a_3 z_4 z_3 & = & s_3 b_4 z_3 z_4 & \text{so } a_3 = q s_3; \\ (iii) & q a_3 z_2 z_3 & = & s_3 z_2 z_3 + b_4 (q - q^{-1}) z_2 z_3, & \end{array}$$

and by (ii) and (iii), $q a_3 - q^{-1} a_3 = b_4 (q - q^{-1})$, so $a_3 = b_4$. If $b_3 \neq 0$ then $s_3 = q a_3 = q^2 s_3$ so $s_3 = 0$ and $a_3 = 0$ and hence $b_4 = 0$, a contradiction. Thus $b_3 = 0$ and the pair $(u, v) = (z_1 + a_3 z_3,\ z_2 + a_3 z_4)$ does determine a line module. The variety in $\mathbb{P}^3$ is $\{[-a_3 t_1; -a_3 t_2; t_1; t_2] \mid t_i \in \mathbb{C}\}$ which is a line on the quadric $E$.

Similarly, $z_1 + a_2 z_2$ and $z_3 + a_2 z_4$ determine line modules and the variety in $\mathbb{P}^3$ is

$$\{[-a_2 t_1; t_1; -a_2 t_2; t_2] \mid t_i \in \mathbb{C}\}.$$

2° (i) b) $b_4 = 0$ and $a_4 \neq 0$.

The relation in this case is

$$(qz_2 + qb_3z_3)(z_1 + a_3z_3 + a_4z_4) = \\ (z_1 + s_3z_3 + s_4z_4)(z_2 + b_3z_3)$$

Thus

$$\begin{array}{rclcrcl} qa_3z_2z_3 & = & s_3z_2z_3 & \text{so} & s_3 & = & qa_3 \\ qa_4z_2z_4 & = & s_4z_4z_2 & \text{so} & s_4 & = & q^2a_4 \\ qb_3a_4z_3z_4 & = & s_4b_3z_4z_3 & \text{so} & s_4b_3 & = & q^2b_3a_4 \\ qb_3z_3z_1 & = & b_3z_1z_3 & \text{so} & qb_3 & = & qb_3 \\ qb_3a_3z_3^2 & = & s_3b_3z_3^2 & \text{so} & b_3(qa_3 - s_3) & = & 0 \end{array}$$

In fact,

$$(qz_2 + qb_3z_3)(z_1 + a_3z_3 + a_4z_4) = \\ (z_1 + qa_3z_3 + q^2a_4z_4)(z_2 + b_3z_3),$$

and hence $(z_1 + a_3z_3 + a_4z_4, z_2 + b_3z_3)$ determines a line module. The variety in $\mathbb{P}^3$ is $\{[-a_3t_1 - a_4t_2; -b_3t_1; t_1; t_2] \mid t_i \in \mathbb{C}\}$ and this is not a line on the quadric since $a_4 \neq 0$.

2° ii) $a_1 = 1, b_1 = b_2 = b_3 = 0$.

We may take $u = z_1 + a_2z_2 + a_3z_3$ and $v = z_4$. But it is easily seen then that $A_1u \cap A_1v = 0$, so in this case we get no line modules.

Summing up, we have found the following types of line modules.

1°: Line modules coming from point modules over $\mathbb{C}[z_2, z_3][z_i, \sigma_i]$, $i = 1, 4$.

2° i)(a) Lines on the defining quadric corresponding to

$$u = z_1 + a_3z_3, \qquad v = z_2 + a_3z_4$$

or

$$u = z_1 + a_2z_2, \qquad v = z_3 + a_2z_4.$$

2° i)(b) Special lines where

$$u = z_1 + a_3z_3 + a_4z_4, \qquad v = z_2 + b_2z_3.$$

This is also the result of Vancliff ([7]).

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