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Angular Momentum of the Cosmic Background Radiation

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The effect of a nontrivial conserved isotropic total angular momentum $M^2 = m_x^2 + m_y^2 + m_z^2$ on the equilibrium distribution of energy in a photon gas is examined. It is shown that the correspondingly modified Planck law takes the form

$$F(\nu) = \text{const}\,\nu\gamma^{-1}\ln[(1-e^{-\beta\nu-\gamma\nu^2})/(1-e^{-\beta\nu})],$$

where β and γ are parameters determined by the energy and angular momentum. This law provides a good fit to the spectrum of the cosmic background radiation observed by Woody and Richards.

Woody and Richards[1] have reported new measurements of the cosmic background radiation (CBR) out to 40 cm^{-1}, and noted deviations from a simple blackbody law which are significant at the 5σ level. The possibility of deviations arising from a nontrivial isotropic angular momentum for the CBR had been noted earlier in the chronometric cosmology (for which see Segal[2]), and it is natural, therefore, to explore this possibility and to compare its implications with the recent observations.

More specifically, in the chronometric cosmology, the CBR prediction arises from the temporal homogeneity of the theory, which implies energy conservation, together with the maximization of randomness as measured by the entropy. In accordance with general statistical mechanics, this yields the Planck law (cf. Mayer[3]), but the possibility remains open *a priori* that other conserved quantities may exist, and modify the result correspondingly, which would then not appear as a simple blackbody law. Thus, if in addition to the energy H, the quantities $K_1, K_2, \ldots$ are conserved, then maximization of entropy with these constraints leads to the density matrix $D = C \times \exp[-(\beta H + \gamma_1 K_1 + \gamma_2 K_2 + \ldots)]$, where C is a normalizing constant, and β and the γ_1 are parameters determined by the constraints.

The major possibility for a conserved quantity of substantial expectation value would appear to be the total angular momentum $M^2 = m_x{}^2 + m_y{}^2 + m_z{}^2$. Isotropy requires the vanishing of the expectation values of m_z and the other components; but not that of M^2 (which indeed is strictly nonvanishing in every photon state, though possibly unobservably small). There is no *a priori* reason why the angular momentum should not be large enough to product an observable effect on the CBR.[4]

On the basis of general principles, this note determines the modified Planck law when the total angular momentum is significant, and compares the result with the observations of Woody and Richards.

(1) *Maximal entropy states of a quantized boson field.*—The preliminary analysis is the same for any boson. Suppose it inhabits a single-particle Hilbert space $\mathcal{K}$ (i.e., for photons, the space of all normalizable solutions of Maxwell's equations in empty space). Let A denote the single-particle energy operator, and let $B_1, B_2, \ldots$ denote other self-adjoint operators in $\mathcal{K}$ which, together with A, provide a complete set of quantum numbers for the boson (i.e., generate a maximal commuting set of operators on $\mathcal{K}$). Then there exists a complete eigenbasis for $\mathcal{K}$, which will be written in discrete terms for simplicity (but note in passing that this discreteness is strictly satisfied in the application to the chronometric cosmology): $e_{a, b_1, b_2, \ldots}$, on which A has the eigenvalue a, and B_j the eigenvalue b_j.

If $\mathcal{K}$ denotes the Hilbert space for the corresponding quantized field, the occupation numbers $n_{a, b_1, b_2, \ldots}$ generate a maximal commuting set of operators on $\mathcal{K}$. The total field energy is $\tilde{A} = \sum a n_{a, b_1, b_2, \ldots}$, where the superposed tilde denotes the quantum field operator corresponding to the indicated single-particle operator[5]; and $\tilde{B}_j = \sum b_j n_{a, b_1, b_2, \ldots}$, $\sum$ denoting summation over all mutually compatible $a, b_1, b_2, \ldots$. The maximal-entropy state when $\langle \tilde{A} \rangle$ and the $\langle \tilde{B} \rangle$ are given has density matrix

$$D = C \exp[-\sum (\beta a + \gamma_1 b_1 + \gamma_2 b_2 + \ldots) n_{a, b_1, b_2 \ldots}],$$

where C is a constant chosen so that $\mathrm{tr} D = 1$, and β and the γ_i are numerical parameters to be deduced from the constraints on the expectation values.

The joint probability distribution of the occupation numbers $n_{a, b_1, b_2, \ldots}$ (forming an infinite collection as $a, b_1, b_2, \ldots$ vary) is then determined (cf. Kon[6]) and the expected energy in the range from ν to ν' takes the form

$$C \overset{*}{\sum}{}' a n_{a, b_1, b_2, \ldots} \times \exp[-(\beta a + \gamma_1 b_1 + \gamma_2 b_2 + \ldots) n_{a, b_1, b_2, \ldots}],$$

where $\sum'$ is the same as $\sum$ except that values of a are required to lie in the range $\nu \leq a \leq \nu'$, and the $*$ over $\sum$ indicates that summation is also performed over all possible occupation numbers (eigenvalues of the $n_{a, b_1, b_2, \ldots}$). Evaluating C, summing over all occupation numbers (a geometric series), and assuming, partly for simplicity and partly for applicability, that the eigenvalues of A are positive integers when scaled appropriately: $1, 2, \ldots$, the expected total field energy is

$$F(\nu) = \nu \sum{}' (\zeta_{a, b_1, b_2, \ldots}{}^{-1})^{-1},$$

where

$$\zeta_{a, b_1, b_2, \ldots} = \exp(\beta a + \gamma_1 b_1 + \gamma_2 b_2 + \ldots)$$

and where the sum $\sum'$ is over all values of b_1, $b_2, \ldots$ compatible with the value ν for the energy.

(2) *The equilibrium energy spectrum of an isotropic photon gas of given angular momentum.*—A complete set of quantum numbers for the photon is formed by the energy, the total angular momentum $M^2 = m_x{}^2 + m_y{}^2 + m_z{}^2$, the z component m_z of the angular momentum, and the helicity λ. The chronometric energy operator differs from the conventional energy operator by a term of order $1/R$, where R is the "radius of the universe"; it will follow that the same law deduced using the chronometric energy applies also in the case of the conventional energy, this approximation providing an alternative to the approximation of Euclidean space by a box in order to achieve a discrete spectrum.

The chronometric energy ν has the eigenvalues $2, 3, \ldots$ in the natural units $\hbar = c = R = 1$. For given ν, M^2 has as possible eigenvalues the numbers $l(l+1)$, where l is an arbitrary integer in the range $1 \leq l < \nu$. For given l, m_z has as possible eigenvalues all integers between $-l$ and l. The helicity λ takes on the values ± 1. For the chronometric analysis of the Maxwell equations, cf. Jakobsen and Vergne.[7]

For microwave frequencies, the sum defining $F(\nu)$ in the preceding section has $\gtrsim 10^{20}$ terms, and may appropriately be approximated by the corresponding integral, the difference being of order $1/R$ and hence unobservably small. The same approximation permits $l(l+1)$ to be replaced by l^2. The resulting expression is, with use of the same approximation to replace the lower limit on l by 0,

$$F(\nu)=\nu\int_0^{\nu}\int_{-l}^{l}\sum_{\lambda}[\exp(\beta\nu+\gamma l^2+\gamma_2 m+\gamma_3\lambda)-1]^{-1}dm\,dl.$$

If we assume that the total m_z and λ are negligible, the corresponding coefficients γ_2 and γ_3 are likewise negligible, and will be set equal to zero, and γ_1 denoted as γ. The expression then takes the form[8]

$$F(\nu)=4\nu\int_0^{\nu}(e^{\beta\nu+\gamma l^2}-1)^{-1}l\,dl$$
$$=2\nu\gamma^{-1}\ln\frac{1-e^{-\beta\nu-\gamma\nu^2}}{1-e^{-\beta\nu}}.$$

To first order in γ, this may be expressed as

$$F(\nu)=\frac{2\nu^3}{e^{\beta\nu}-1}\left(1-\frac{\gamma}{2}\frac{\nu^2}{e^{\beta\nu}-1}\right).$$

To fit a pure ($\gamma=0$) blackbody law to one with $\gamma\neq 0$, one may equate total energies and the peak energies. With $\beta=2.68°K$ and $\gamma/\beta^2=0.1$, the result is shown superimposed on the graphical summary of Woody and Richards; the fit is good. (See Fig. 1.) Values of γ/β^2 in the range 0.075 to 0.175 also provide good fits when paired with suitable values for β.

The quantum numbers used in the preceding section naturally refer to the point of observation, and to a specific choice of Lorentz frame at this point; but the point is itself perfectly general, and the observable implications for measurements at the point are the same throughout space. There is consequently no preferred base point or rotational axis, the indicated spatial homogeneity being implemented analytically by the transformation of the density matrix D for a given base point by an arbitrary rotation of S^3, rather than by simple commutativity of the density matrix with space translations. In particular, the occupation numbers used have the same joint distribution at all points of S^3; but at two different points of S^3 have no joint distribution since they do not (in general) commute.

The deviations from a pure blackbody law noted by Woody and Richards are thus satisfactorily explained by a nontrivial isotropic angular momentum in the CBR. Such angular momentum would impart angular momentum to galaxies on absorption, and could contribute an amount significant for their evolution. The angular-momentum economy of dust, stars, etc., would be affected, and ultimately also small-scale features such as the solar system, whose angular-momentum anomalies could originate in the omission of angular momentum derived from the CBR.

Consequently, direct measurements of the angular momentum of the CBR should be of interest, as well as additional observations concerning the deviations reported.

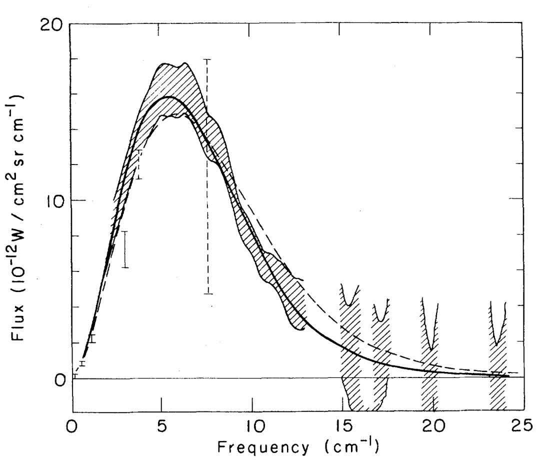

FIG. 1. Comparison of observation with the theoretical pure blackbody curve and this modified by angular momentum. The solid curve gives the latter curve, the dashed curve the former, according to the measurements of Woody and Richards (Ref. 1), from whom this figure is taken, apart from the present theoretical curve.

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[5]This canonical correspondence between single-particle and field quantities yields in the case of the total angular momentum the total (square sum) of the angu-

lar momenta of all field quanta. This differs from the square of the total angular momentum of the field, which is not at issue here. However, the two operators should have equal expectation values because of the consideration cited in Ref. 4.

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