TENSORING WITH SMALL QUANTIZED REPRESENTATIONS

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Abstract. We give an elementary proof of the associativity of the reduced tensor product that also works for primitive roots of -1. At the same time, we get a useful understanding of how representations "fuse" into each other.

1. Introduction

Following the fundamental paper of Reshetikin and Turaev [18], a number of investigations of invariants for 3-manifolds, e.g. [20], [21], and [7], uses implicitly or explicitly a "reduced tensor product" $\overline{\otimes}$ which is applied to a finite set I of representations of an algebra $\mathcal{A}$. For $\pi_1, \pi_2 \in I$, $\pi_1 \overline{\otimes} \pi_2$ is obtained from $\pi_1 \otimes \pi_2$ by removing, in a prescribed way, a maximal summand of "quantum dimension zero". The crucial requirements are that I must be closed under this tensor product and $\overline{\otimes}$ must be associative.

In the case where $\mathcal{A}$ is a quantum group at a primitive root of 1, the associativity, while apparently "well known to physicists", to our knowledge was first rigorously established in [20] for the case of A_n and the general case was obtained in [1].

Especially the investigation [7] requires explicitly the representations to be unitary, and for this reason one is forced to consider primitive roots of (-1). At the same time the investigation in [1] uses some deep results from algebra as well as Lusztig's canonical bases and hence a more elementary approach might give a useful perspective. The current paper is the result of the desire to meet these requirements.

Our basic tool is what we call a "small" representation of a quantum group. This concept is defined in terms of a number of, for our purposes, useful properties. It turns out that each quantum group has at least one such.

It may be said that our approach is in spirit related to that of [21]. Upon describing our program to Andersen we learned that he and Paradowski, in anticipation of certain results Lusztig's book [17], had launched a program which, among other things, would deal with other roots of 1. Their efforts have recently found a successful completion in [5].

Finally, we would like to thank H.H. Andersen as well as the members in our local q-group, B. Durhuus, A. Jensen, S. Jøndrup, and R. Nest, for helpful conversations.

2. Notation and background results

We consider the Cartan matrix $\{a_{ij}\}_{i,j=1}^n$ corresponding to a simple finite dimensional complex Lie algebra and let U_K denote the quantum group over the field $K = \mathbb{Q}(q)$ generated by the $4n$ generators $E_1, \dots, E_n, F_1, \dots, F_n, K_1^{\pm}, \dots, K_n^{\pm 1}$ with the usual "quantized Serre relations". For $i = 1, \dots, n$ we let $d_i \in \{1, 2, 3\}$ be chosen such that $\{d_i a_{ij}\}$ is symmetric, and more generally use the notation of [2].

$\mathbb{N}_0$ denotes the non-negative integers.

We recall the following definition:

Definition 2.1 (Lusztig [16]**).** *Let* $A = \mathbb{Z}[q, q^{-1}]$.

- U_A^+ *is the* A*-subalgebra of* U_K *generated by* $E_i^{(r)}, r \geq 0, i = 1, \dots, n$.
- U_A^- *is the* A*-subalgebra of* U_K *generated by* $F_i^{(r)}, r \geq 0, i = 1, \dots, n$.
- U_A^0 *is the* A*-subalgebra of* U_K *generated by* $K_i^{\pm}$, *and* $\begin{bmatrix} K_i; c \\ t \end{bmatrix}$, *where*
- $i = 1, \dots, n, c \in \mathbb{Z}, t \in N_0$.

Finally, U_A *is the* A*-subalgebra of* U_K *generated by* $E_i^{(r)}, F_i^{(r)}, K_i^{\pm 1}, r \geq 0, i = 1, \dots, n$, *and we introduce the notation* $U_A(\mathfrak{b}^-) = U_A^- U_A^0$.

For $\omega \in (\mathbb{Q}(q)^*)^n$ we denote by $L(\omega)$ the unique irreducible highest weight module for U_K. If $\Lambda = (q^{d_1 m_1}, \dots, q^{d_n m_n})$ for some $m_1, \dots, m_n \in \mathbb{N}_0$, we have in a natural way a U_A-submodule $L_A(\Lambda)$ of $L(\Lambda)$ generated by the primitive vector.

In the following our field is always $\mathbb{C}$. For a $\zeta \in \mathbb{C}^*$ we let $U_\zeta = U_A \otimes_A \mathbb{C}$ where $\mathbb{C}$ is made into an A -algebra by specialization at ζ. Similarly, for any A-module M we set $M_\zeta = M \otimes \mathbb{C}$. $\mathcal{C}_\zeta$ likewise denotes the specialization of the integrable modules. We will always take ζ to be a primitive ℓth root of ± 1 or, occasionally, we take $\zeta = 1$. We assume throughout that ℓ is prime to the non-zero entries in the Cartan matrix and that it is bigger than the Coxeter number.

Remark 2.2. *In many cases one is interested in yet another quantum algebra,* U_{lit}, *defined directly from the Serre relations by viewing the parameter* q *in the Serre relations as a complex number. In the case where* q *is not a root of unity there is not a great deal of difference, but in the root of unity case, which is the interesting one, this algebra has a big center. Indeed, it is finite dimensional over its center (*[6]*). The results about representations and tensor products that we obtain below carry over directly to* U_{lit} *basically because* U_{lit} *is a subalgebra of* U_q. *Observe that high powers of the elements* E_i^h *and* F_j^h *are mapped to zero in representations. This follows because they are of the form e.g.* $E_i^h = [h]_{d_i}! \cdot E_i^{(h)}$ *and* $[h]!$ *specializes to zero at a primitive* ℓ*th root of unity provided that* $h \geq \ell$.

Definition 2.3. *Let* M_ζ *be a finite-dimensional module and let* ϕ *be an endomorphism. The* ***quantum trace*** $tr_\zeta(\phi)$ *is given by the formula*

$$tr_\zeta(\phi) = tr(K_{2\rho}\phi), \tag{1}$$

*where $K_{2\rho} = \prod_{\beta\in\triangle^+} K_\beta$. In particular, the **quantum dimension** is given by*

$$dim_\zeta(M_\zeta) = tr(K_{2\rho}). \tag{2}$$

Let $X = \mathbb{Z}^n$ and $X^+ = \mathbb{N}_0^n \subseteq X$. If $\Lambda = (\lambda_1, \dots, \lambda_n) \in X$ we denote by ξ_Λ the following character of U_ζ^0:

$$\Lambda(K_i) = \zeta^{d_i\lambda_i}, \Lambda\left(\begin{bmatrix} K_i; c \\ t \end{bmatrix}\right) = \begin{bmatrix} \lambda_i + c \\ t \end{bmatrix}_{d_i}, \tag{3}$$

and we extend this to a character, also denoted ξ_Λ, on $U_\zeta(\mathfrak{b}^-)$ in the usual way.

The induced module corresponding to Λ (see Definition 3.1 below) is denoted by $H_\zeta^0(\Lambda)$, and the irreducible highest weight module is denoted by $L_\zeta(\Lambda)$.

The following formula is well known: ([11])

Proposition 2.4.

$$dim_q(\Lambda) = \prod_{\beta\in\triangle^+} \frac{q^{d_\beta(\Lambda+\rho,\beta^\vee)} - q^{-d_\beta(\Lambda+\rho,\beta^\vee)}}{q^{d_\beta(\rho,\beta^\vee)} - q^{-d_\beta(\rho,\beta^\vee)}}. \tag{4}$$

Finally we recall the Shapovalov determinant of the hermitian form in the quantum case as proved by de Concini and Kac ([6], see also [10]): If $\Lambda \in X$, the determinant of the contravariant hermitian form on the weight space $M_\zeta(\Lambda)^{-\eta}$ of the Verma module $M_\zeta(\Lambda)$ of highest weight Λ (and type $(+, +, \dots, +)$) is given by

$$det_{\Lambda,\zeta}(\eta) = \prod_{\beta\in\triangle^+} \prod_{m\in\mathbb{N}} \left([m]_{d_\beta} \frac{\zeta^{d_\beta(\Lambda+\rho-\frac{m}{2}\beta,\beta^\vee)} - \zeta^{-d_\beta(\Lambda+\rho-\frac{m}{2}\beta,\beta^\vee)}}{\zeta^{d_\beta} - \zeta^{-d_\beta}}\right)^{\mathrm{Par}(\eta-m\beta)}. \tag{5}$$

We denote by C the first dominant alcove,

$$C = \{\Lambda \in X^+ \mid (\Lambda+\rho, \alpha^\vee) < \ell \quad \text{for all } \alpha \in \triangle^+\}. \tag{6}$$

The following has been established in [2], but it is also an easy consequence of the formula (5) together with the generic irreducibility of the induced modules ([2]):

Proposition 2.5. *If $\Lambda \in \overline{C}$, then $H_\zeta^0(\Lambda) = L_\zeta(\Lambda)$.*

Finally, we let α_0 denote the highest short root.

Lemma 2.6.

$$(\Lambda + \rho, \alpha^\vee) \tag{7}$$

attains its maximum precisely for $\alpha = \alpha_0$.

Proof. We shall assume that there are two root lengths since the claim otherwise is trivially true. If α_h denotes the highest root, then this is long, and $\alpha_0 = \alpha_h - \alpha_s$ for some short root α_s. It follows that

$$(\Lambda + \rho, \alpha_h^\vee) = ((\Lambda + \rho, \alpha_s^\vee) + (\Lambda + \rho, \alpha_0^\vee)) \frac{(\alpha_0, \alpha_0)}{(\alpha_h, \alpha_h)} < (\Lambda + \rho, \alpha_0^\vee), \tag{8}$$

where the strict inequality follows because of the presence of ρ. □

3. Induction

We now briefly introduce the induced modules of H.H. Andersen et al ([2]): Let $\mathcal{C}_A$ denote the category of integrable U_A-modules and introduce the notation $\mathcal{C}_a(\mathfrak{b}^-)$ for the category of integrable $U_A(\mathfrak{b}^-)$-modules.

The induction functor

$$H_A^0(U_A/U_A(\mathfrak{b}^-), -) : \mathcal{C}_A(\mathfrak{b}^-) \to \mathcal{C}_A \tag{9}$$

is defined by

Definition 3.1. *Let M be a $U_A(\mathfrak{b}^-)$-module in $\mathcal{C}_A(\mathfrak{b}^-)$. Set*

(10)

$$\hom_{U_A(\mathfrak{b}^-)}(U_A, M) = \{f \in \hom(U_A, M) \mid f(u^2u^1) = u^2 f(u^1), u^1 \in U_A, u^2 \in U_A(\mathfrak{b}^-)\},$$

and consider this as a U_A-module via

$$uf(x) = f(xu). \tag{11}$$

Let $F(V)$ denote the integrable part of any $\mathfrak{g}$-module V and set

$$H_A^0(U_A/U_A(\mathfrak{b}^-), M) = F(\hom_{U_A(\mathfrak{b}^-)}(U_A, M)) \tag{12}$$

for any $U_A(\mathfrak{b}^-)$-module. We call this the induced module and set

$$H_A^0(\Lambda) = H_A^0(U_A/U_A(\mathfrak{b}^-), \xi_\Lambda) \tag{13}$$

for the one-dimensional representation ξ_Λ of $\mathfrak{b}^-$. (Λ dominant integral.)

Finally, we introduce in a similar way the induction functor H_K^0 and the modules $H_K^0(\Lambda)$.

In the following we wish to evaluate finite-dimensional U_K-modules. This we define as follows, where we use the fact ([2]) that such modules are completely reducible together with the following result:

Lemma 3.2. *Let Λ be dominant integral. Then*

$$H_K^0(\Lambda) = K \otimes_A H_A^0(\Lambda). \tag{14}$$

Proof. This follows because $A \to K$ is flat ([2]). □

Lemma 3.2 also tells what the U_A-invariant subspaces of $H_A^0(\Lambda)$ are, since $H_K^0(\Lambda)$ is known ([3]) to be an irreducible U_K-module.

Lemma 3.3. *Any non-zero U_A-invariant subspace of $H^0_A(\Lambda)$ has the same weight multiplicities as the full space.*

Proof. Specializing $H^0_A(\Lambda)$ at a generic ζ gives a U_ζ -module which is known to be irreducible (and the weight multiplicities in $H^0_A(\Lambda)$ are the same as e.g. in $H^0_1(\Lambda)$). The statement follows immediately from this. □

The following result has been established for primitive roots of 1 by Andersen [1], basically using Kempf vanishing as established in [2]. It has been extended to arbitrary roots of 1 in [19], by employing deep results about canonical bases.

Proposition 3.4.

$$H^0_\zeta(\Lambda) = \mathbb{C} \otimes_A H^0_A(\Lambda). \tag{15}$$

Proof. The weight multiplicities in U^+_ζ are equal to those in U^+_A. Furthermore, it is clear that a weight space in $H^0_A(\Lambda)$ cannot specialize to zero since the elements in $H^0_A(\Lambda)$ simply are homomorphisms that satisfy a certain finiteness condition. Hence, specializing to zero would mean that A specializes to zero, which is absurd. (c.f. Lemma 3.3).

Suppose now that specialization is not surjective. Consider the *right* U_A-module

$$R_A(\Lambda) = 1_\Lambda \otimes_{U_A(\mathfrak{b}^-)} U_A, \tag{16}$$

where 1_Λ denotes the 1-dimensional $U_A(\mathfrak{b}^-)$-module defined by the character ξ_Λ. In an appropriate sense, the left module $\hom_{U_A(\mathfrak{b}^-)}(U_A, 1_\Lambda)$ is the dual of this module. The Weyl module is equivalent to the quotient of $R_A(\Lambda)$ by the space P_A generated by the elements $E^{(s)} \otimes 1$, $s > \lambda_i$. Comparing with the situation at a generic ζ it follows (analogous to the proof of Lemma 3.3) that this is the dual of $H^0_A(\Lambda)$.

In a similar way,

$$R_\zeta(\Lambda) = \xi_\Lambda \otimes_{U_\zeta(\mathfrak{b}^-)} U_\zeta \tag{17}$$

is the dual of $\hom_{U_\zeta(\mathfrak{b}^-)}(U_\zeta, \xi_\Lambda)$. We know ([2]) that $H^0_\zeta(\Lambda)$ is finite-dimensional (and its weights are conjugate under the Weyl group).

The polar P_ζ (annihilator) of $H^0_\zeta(\Lambda)$ in $R_\zeta(\Lambda)$ is, naturally, invariant. Since the specialization $H^0_A(\Lambda) \to H^0_\zeta(\Lambda)$ is injective, $P_\zeta \subseteq (P_A)_\zeta$. On the other hand, P_ζ contains, by a simple computation, the specialization of the elements $E_i^{(s)} \otimes 1$, $s > \lambda_i$. Thus, $(P_A)_\zeta \subseteq P_\zeta$. □

By Lemma 3.2 and Proposition 3.4 we may say that $H^0_\zeta(\Lambda)$ has been obtained by specialization of the module $H^0_K(\Lambda)$ or that $H^0_\zeta(\Lambda)$ is $H^0_K(\Lambda)$ "evaluated at ζ". We will do that and also extend the terminology to e.g. tensor products of such modules.

4. Tensoring with the adjoint representation

We denote the adjoint representation by Ad and we denote the representation whose highest weight is that of α_0 by Ad^ℓ (the "little adjoint" representation). More precisely, the names of these representations ought perhaps to have the prefix "the quantum analogues of", but we omit this.

We first prove four lemmas for the "generic case" U_K. In fact, we prove them for the specializations to $\zeta = 1$. Since we have complete reducibility over the field K, there will be a perfect match-up between the primitive vectors in the K-modules considered (and their weights) and the primitive weight vectors in the specializations of the modules (as defined below).

Lemma 4.1. *Let*

$$x_{\Lambda_1+\Lambda_2-\omega} = \sum_{\omega_1\geq 0,\omega_2\geq 0,\omega_1+\omega_2=\omega} y_{\Lambda_1-\omega_1} \otimes z_{\Lambda_2-\omega_2} \tag{18}$$

be a non-zero highest weight vector in $L_1(\Lambda_1)\otimes L_1(\Lambda_2)$ of highest weight $\Lambda_1+\Lambda_2-\omega$. Then y_{Λ_1} and z_{Λ_2} occur in non-zero expressions.

Proof. If not, let, say, $y_{\Lambda_1-\tilde{\omega}}$ be a weight vector of a highest weight occurring in the sum. It follows that this must be a highest weight vector in $L_1(\Lambda_1)$. □

Lemma 4.2. *The natural map*

$$Ad_1 \otimes L_1(\Lambda) \ni x\otimes v \to x\cdot v \in L_1(\Lambda) \tag{19}$$

is a $\mathfrak{g}$-map. It is non-trivial exactly when $L_1(\Lambda)$ is non-trivial.

Proof. Obvious. □

Before stating the next lemma we need some notation: For each $\alpha \in \triangle$, choose $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha}\in\mathfrak{g}_{-\alpha}$ such that $B(e_\alpha, e_{-\alpha}) = 1$, where B denotes the Killing form. For each simple root $\alpha_i, i = 1,\dots,n$, choose $h_i, h^i \in \mathfrak{h}$ such that

$$h_i = [e_{\alpha_i}, e_{-\alpha_i}], \text{ and} \tag{20}$$

$$\forall i,j : [h^i, z_{\alpha_j}] = \delta_{i,j} z_{\alpha_j}. \tag{21}$$

Lemma 4.3. *Let $L_1(\Lambda)$ be a highest weight representation of highest weight Λ and highest weight vector v_Λ. The vector*

$$\sum_{\alpha\in\triangle} e_{-\alpha}\otimes e_\alpha\cdot v_\Lambda + \sum_{i=1}^{n} h^i\otimes h_i\cdot v_\Lambda \in Ad_1\otimes L_1(\Lambda) \tag{22}$$

is a highest weight vector of weight Λ if and only if $\Lambda\neq 0$.

Proof. This follows easily from [11, pages 19–20]. □

Lemma 4.4. *There can be at most as many copies of $L_1(\Lambda)$ in $Ad_1\otimes L_1(\Lambda)$ as there are $\Lambda_i \neq 0$.*

Proof. A highest weight vector of weight Λ in $Ad_1 \otimes L_1(\Lambda)$ must have the form

$$\tilde{v} = \sum_i c_i h_i \otimes v_\Lambda + \sum_{\mu\in\triangle^+} \sum_j x_\mu^{(j)} \otimes v_{\Lambda-\mu}^{(j)}, \tag{23}$$

where v_Λ denotes a non-zero highest weight vector in $L_1(\Lambda)$, $x_\mu \in \mathfrak{g}^\mu$, and $v_{\Lambda-\mu}^{(j)} \in L_1(\Lambda)$ has weight $\Lambda - \mu$. Suppose $\Lambda_j = 0$. Then

$$z_j^+ \tilde{v} = -c_j z_j^+ \otimes v_\Lambda + \text{ (rest) }, \tag{24}$$

where the "rest" term cannot contain anything proportional to $z_j^+ \otimes v_\Lambda$, since this would have to originate from $z_j^+(z_j^+ \otimes v_{\Lambda-\alpha_j}^{(k)})$ and by assumption, $L_1(\Lambda)^{\Lambda-\alpha_j} = 0$. Thus, $c_j = 0$. □

Proposition 4.5.

$$\hom_{U_A}(H_A^0(\Lambda), Ad \otimes H_A^0(\Lambda)) = \hom_{U_A(\mathfrak{b}^-)}(H_A^0(\Lambda), Ad \otimes \xi_\Lambda). \tag{25}$$

Proof. This follows directly from the Frobenious reciprocity law combined with the so-called tensor identity. □

Proposition 4.6. *The multiplicity of the representation $L(\Lambda)$ in $Ad \otimes L(\Lambda)$ is equal to the number of simple roots α_i for which $\Lambda_i \neq 0$.*

Proof. We may pass between H_K^0, H_A^0, and $L(\Lambda)$ at our convenience. We first consider a representation Λ_0 for which exactly one $\Lambda_{i_0} \neq 0$. As mentioned previously, we may use the results for $\zeta = 1$ in the generic case and so, by combining Lemma 4.2 with Lemma 4.4, it follows that $L(\Lambda_0)$ occurs exactly once in $Ad \otimes L(\Lambda_0)$. Thus, by Proposition 4.5 there is exactly one non-trivial $U(\mathfrak{b}^-)$-homomorphism $\phi_0 : L(\Lambda_0) \to Ad \otimes \xi_{\Lambda_0}$, and ϕ_0 satisfies in particular

$$\phi_0(v_{\Lambda_0}) = h_{i_0} \otimes c_{i_0}, \tag{26}$$

where c_{i_0} denotes a non-zero element in the 1-dimensional module ξ_{Λ_0}.

Let us now consider an arbitrary finite-dimensional highest weight module $L(\Lambda)$, and let V_Λ^- be the $U_A(\mathfrak{b}^-)$-invariant subspace consisting of all weight vectors of weight strictly smaller than Λ. The quotient map

$$\pi : L(\Lambda) \otimes Ad \otimes \xi_{\Lambda_0} \to L(\Lambda)/V_\Lambda^- \otimes Ad \otimes \xi_{\Lambda_0} \tag{27}$$

is then a $\mathfrak{b}^-$-module map. The target space is clearly isomorphic to $Ad \otimes \xi_\Lambda \otimes \xi_{\Lambda_0}$, and we view π as taking its values in this space. All in all, we now have a non-trivial $U(\mathfrak{b}^-)$-map

$$\pi \circ (1 \otimes \phi_0) : L(\Lambda) \otimes L(\Lambda_0) \to Ad \otimes \xi_{\Lambda+\Lambda_0}. \tag{28}$$

Evidently, this map is also non-zero when restricted to the U_A-invariant subspace $L(\Lambda + \Lambda_0) \subseteq L(\Lambda) \otimes L(\Lambda_0)$. In fact, we must clearly have

$$\pi \circ \phi_0(v_{\Lambda+\Lambda_0}) = h_{i_0} \otimes \tilde{c}_{i_0}, \tag{29}$$

where $\tilde{c}_{i_0}$ denotes a non-zero element in $\xi_{\Lambda+\Lambda_0}$.

It follows from this argument that there is a $U_A(\mathfrak{b}^-)$-map from any $L(\Lambda)$ with $\Lambda_i \neq 0$ to the corresponding module $Ad \otimes \xi_\Lambda$, and this map sends the highest weight vector v_Λ to an element of the form $h_i \otimes c$ with $c \neq 0$. □

There are analogous results for Ad^ℓ:

Lemma 4.7. *For the algebras B_n, G_2, and F_4 there can be at most as many copies of $L(\Lambda)$ in $Ad^\ell \otimes L(\Lambda)$ as there are $\Lambda_i \neq 0$.*

Proof. This follows by the same kind of argument as for Ad. □

Proposition 4.8. *For the algebras B_n, G_2, and F_4 the multiplicity of the representation $L(\Lambda)$ in $Ad^\ell \otimes L(\Lambda)$ is equal to the number of simple* short *roots α_i for which $\Lambda_i \neq 0$.*

Proof. This follows by arguments similar to those for Ad. One simply has to introduce the 0-weight space in Ad^ℓ in the left-hand tensors in the vector in (22). □

5. Filtrations

The following definition of a tilting module is not the usual definition, but it is a theorem [1] that we loose no generality by doing it. Let Γ be a commutative A-algebra and set $U_\Gamma = U_A \otimes_A \Gamma$.

Definition 5.1. *A* ***good*** *filtration of an integrable U_Γ-module M is a filtration*

$$0 = F_0 \subset F_1 \cdots \subset F_n = M, \tag{30}$$

with $F_i/F_{i-1} \equiv H^0_\Gamma(\Lambda_i)$ for some $\Lambda_1 \in X^+, i = 1, \ldots, n$. A ***tilting*** *module is an integrable module for which both M and M^* has a good filtration.*

The following follows from general results of Donkin and Ringel as proved by Paradowski and Andersen [5]:

Proposition 5.2. *For each $\Lambda \in X^+$ there is a unique indecomposable tilting module $d(\Lambda)$ satisfying*

- *Every weight μ of $D(\Lambda)$ satisfies $\mu \leq \Lambda$.*
- $\dim D(\Lambda)_\Lambda = 1$.

The crucial property we shall be using is the following ([1])

Proposition 5.3. *Let M be a tilting module. Then there exist uniquely determined non-negative integers $a_\Lambda(M), \Lambda \in X^+$, such that*

$$M = \oplus_{\Lambda \in X^+} D(\Lambda)^{a_\Lambda(M)}. \tag{31}$$

Proposition 5.4.

$$H^0_\zeta(\Lambda_1) \otimes H^0_\zeta(\Lambda_2) \tag{32}$$

has a good filtration.

Proof. Clearly we may view $H^0_A(\Lambda_1)\otimes H^0_A(\Lambda_2)$ as a space of homomorphisms defined on $U_A\otimes U_A$ and with values in $\xi_{\Lambda_1}\otimes\xi_{\Lambda_2}$. Analogous properties are held by the specialized objects.

We may then decompose

$$H^0_\zeta(\Lambda_1)\otimes H^0_\zeta(\Lambda_2) \tag{33}$$

directly by considering a filtration according to the "degree of vanishing on the diagonal" in the spirit of [9]. At the same time, we will pay attention to the decomposition of the tensor products of the corresponding U_A- and U_K-modules, especially because we have complete reducibility for the latter.

First consider the restriction homomorphism

$$\begin{aligned}\mathcal{R}^0 : H^0_\zeta(\Lambda_1)\otimes H^0_\zeta(\Lambda_2) &\longrightarrow H^0_\zeta(\Lambda_1+\Lambda_2),\\ (\mathcal{R}^0(\phi))(u) &= \phi(\Delta(u)).\end{aligned} \tag{34}$$

That $\mathcal{R}^0$ indeed takes values in the said space follows from the formulas ([16])

$$\Delta(E_i^{(r)}) = \sum_{t=0}^{r} q^{d_it(r-t)}E_i^{(r-t)}K_i^t\otimes E_i^{(t)}, \tag{35}$$

$$\Delta(F_i^{(r)}) = \sum_{t=0}^{r} q^{-d_it(r-t)}F_i^{(t)}\otimes K_i^{-t}F_i^{(r-t)}, \tag{36}$$

$$\Delta(K_i) = K_i\otimes K_i. \tag{37}$$

The surjectivity of $\mathcal{R}^0$ comes about as follows: It follows either from (35) or by using the counit that

$$\Delta_\zeta : U_\zeta\to U_\zeta\otimes U_\zeta \tag{38}$$

(restricted to U^+_ξ) is injective. On the level of homomorphisms, i.e. forgetting the finiteness criterion for being in a space H^0, we then obtain all homomorphisms.

Finally, the finiteness condition does not affect this at all. This is because we in the generic case (or in the U_A-case) get as image a module which contains all weights (Lemma 3.3). The claim then follows from Proposition 3.4. Put differently, the only way surjectivity could be ruined would be if Δ_ζ were to map some element which is not in the annihilator of $H^0_\zeta(\Lambda_1+\Lambda_2)$ into the annihilator of $H^0_\zeta(\Lambda_1)\otimes H^0_\zeta(\Lambda_2)$ in $U^+_\zeta\otimes U^+_\zeta$. But Δ_ζ is the localization of Δ_A hence the latter would have to share this property. But this clearly is in conflict with the fact that for generic ζ we have full reducibility (with one summand equivalent to $H^0_\zeta(\Lambda_1+\Lambda_2)$).

Now let $\mathcal{K}^0$ denote the kernel of the map — "the homomorphisms that vanish on the diagonal Δ". Then we proceed to analyze $\mathcal{K}^0$ according to the degree of vanishing on the diagonal:

Consider a (PBW) decomposition

$$U^0_AU^+_A\otimes U^+_A = S\cdot\Delta(U^+_A) \tag{39}$$

where $S\subseteq U^0_AU^+_A\otimes U^+_A$ is chosen such that the decomposition $u=s\Delta(u_1)$ of an element $u\in U^0_AU^+_A\otimes U^+_A$ is unique. To be specific, choose $S=U^0_AU^+_A\otimes 1$. It follows easily by induction on the hight of the weight of the term in the second place in the tensor product that this choice has the desired properties.

At the first level we then consider maps

$$\mathcal{R}^1 : \mathcal{K}^0 \longrightarrow H^0_\zeta(\Lambda_1 + \Lambda_2 - \alpha_i) \tag{40}$$

given by

$$(\mathcal{R}^1(\phi))(u) = \phi(s_{\alpha_i}\Delta(u)), \tag{41}$$

where $s_{\alpha_i} = E_i^{(1)} \otimes 1$. For similar reasons we again have surjectivity (if non-trivial). For each i we then get a kernel $\mathcal{K}_i^1$ and these kernels together span the space of homomorphism that vanish to the first order on the diagonal. This argument may clearly be continued to give the full filtration. □

Remark 5.5. *Notice that we actually only need this result in the case where the modules $H^0_\zeta(\Lambda_1)$ and $H^0_\zeta(\Lambda_1)$ are irreducible. In this case an even simpler proof of Proposition* 5.4 *may be given by elementary bookkeeping of weights.*

Corollary 5.6. *Let $\Lambda_1, \Lambda_2 \in X^+$. Suppose that the dual of $H^0_\zeta(\Lambda_i)$ is equal to $H^0_\zeta(\tilde{\Lambda}_i)$ for some $\tilde{\Lambda}_i \in X^+$ for $i = 1, 2$. Then*

$$H^0_\zeta(\Lambda_1) \otimes H^0_\zeta(\Lambda_2) \tag{42}$$

is a tilting module.

Proof. This follows directly from the definition of tilting module together with Proposition 5.4. □

6. Small Representations

Lemma 6.1. *Suppose*

$$(\Lambda_1 + \rho, \alpha_0^\vee) = x \text{ and } (\Lambda_2, \alpha_0^\vee) = y, \tag{43}$$

and suppose $\tilde{\Lambda}$ is a highest weight representation that occurs in $\Lambda_1 \otimes \Lambda_2$. Then

$$(\tilde{\Lambda} + \rho, \alpha_0^\vee) \le x + y \tag{44}$$

with equality especially when $\tilde{\Lambda} = \Lambda_1 + \Lambda_2$.

Proof. Trivial. □

We now introduce some terminology:

Definition 6.2. *A representation Λ_{gd} is called **good** if*

$$(\Lambda_{gd}, \alpha_0^\vee) \leq 2, \tag{45}$$

and if, furthermore, for any representation Λ, the only representation $\tilde{\Lambda}$ in $L(\Lambda) \otimes L(\Lambda_{gd})$ for which

$$(\tilde{\Lambda} + \rho, \alpha_0^\vee) = (\Lambda + \rho, \alpha_0^\vee) + 2 \tag{46}$$

is $\tilde{\Lambda} = \Lambda + \Lambda_{gd}$.

*The representation Λ_e is called **excellent** if*

$$(\Lambda_e, \alpha_0^\vee) \leq 1. \tag{47}$$

*The representation Λ_s is called **small** if it is either good or excellent. Finally, Λ_g is called **generating** if any Λ occurs is the nth fold tensor product of Λ_g with itself.*

The following is easily verified:

Lemma 6.3. *For any $\mathfrak{g}$ there is a small representation. Moreover, with the exception of B_n it can be chosen to be generating. In the case of B_n there is exactly one representation which is not generated, namely $\Lambda = (0, \dots, 0, 1)$.*

Remark 6.4. *Since the mentioned representation for B_n in particular satisfies*

$$\forall \tilde{\Lambda} \in D : (\tilde{\Lambda}, \alpha_0^\vee) \ll \ell, \tag{48}$$

we do not have to take it into consideration, c.f. the proof of Proposition 8.2.

Definition 6.5. *We denote the representation in Lemma* 6.3 *by Λ_S.*

Definition 6.6. *Suppose that $H^0_\zeta(\Lambda_1)$ and $H^0_\zeta(\Lambda_2)$ are irreducible. Let $H^0_\zeta(\overline{\Lambda}_1)$ and $H^0_\zeta(\overline{\Lambda}_2)$ denote the dual representations. We then denote by $\mathcal{K}^0(\Lambda_1, \Lambda_2)$ the kernel in $H^0_\zeta(\Lambda_1) \otimes H^0_\zeta(\Lambda_2)$ of the restriction homomorphism $\mathcal{R}^0$ (*34*) and we denote by $\mathcal{P}^0$ the annihilator in $H^0_\zeta(\Lambda_1) \otimes H^0_\zeta(\Lambda_2)$ of $\mathcal{K}^0(\overline{\Lambda}_1, \overline{\Lambda}_2)$.*

Proposition 6.7. *Suppose that Λ_{gd} is good and that*

$$(\Lambda + \rho, \alpha_0^\vee) = \ell - 1. \tag{49}$$

Then $\mathcal{K}^0(\Lambda_1, \Lambda_2)$ is semi-simple and $\mathcal{P}^0$ is equivalent to $H^0_\zeta(\Lambda + \Lambda_{gd})$. Let

$$P_1 = \mathcal{K}^0 \cap \mathcal{P}^0. \tag{50}$$

Then P_1 is equivalent to $H^0_\zeta(\Lambda)$ and is non-complemented in $\mathcal{P}^0$. Let $W_1 = \mathcal{K}^0 \ominus P_1$. Then there is an invariant subspace W_2 such that

$$H^0_\zeta(\Lambda) \otimes H^0_\zeta(\Lambda_{gd}) = W_1 \oplus W_2. \tag{51}$$

Finally, there is a non-split exact sequence

$$0 \to H^0_\zeta(\Lambda + \Lambda_{gd}) \to W_2 \to H^0_\zeta(\Lambda) \to 0. \tag{52}$$

Proof. The most important observation is that according to Definition 6.2 and formula (5), the Verma module of highest weight $\Lambda+\Lambda_{gd}$ contains a primitive vector of weight Λ and multiplicity 1. Clearly, $\mathcal{P}^0$ as an A-module must be the Weyl module, hence also the specialized module and the primitive vector of weight Λ is contained in this space. It is clear that $\mathcal{K}^0$ is always completely reducible since its weights are all below the critical height. Hence all primitive weight vectors of weight Λ are contained in $\mathcal{K}^0$. Hence P_1 is non-trivial and can be reached from the highest weight space. Consider the space

$$H^0_\zeta(\Lambda)\otimes H^0_\zeta(\Lambda_{gd})/\mathcal{P}^0. \tag{53}$$

For similar reasons, this is completely reducible and there will be the same number of summands of highest weight Λ in (53), say r, as there are in $\mathcal{K}^0$ or “generically”. The reason behind this fact is that

$$(H^0_\zeta(\Lambda)\otimes H^0_\zeta(\Lambda_{gd})/\mathcal{K}^0)\equiv H_\zeta(\Lambda+\Lambda_{gd})\quad(\equiv\mathcal{P}^0), \tag{54}$$

hence, because $\mathcal{K}^0\cap\mathcal{P}^0\equiv H^0_\zeta(\Lambda)$, $\mathcal{P}^0+\mathcal{K}^0$ is not the full space.

In the space of primitive vectors of weight Λ we now choose a basis containing the one from $\mathcal{P}^0$. We can use the remaining $r-1$ in the decomposition (53). Let W_Λ denote the remaining summand in (53). Then we may take W_2 as the inverse image of this space in the full tensor product. The other claims follow immediately. □

Remark 6.8. *There are of course identical results for the case where the two factors in the $\otimes$-product are interchanged. We shall see later (Corollary 7.4) that $dim_qW_2=0$.*

7. Technical matters

To substantiate some of our previous and coming claims, we present here some facts about simple Lie algebras and selected representations of these.

For the classical groups, the defining representation is λ_1. We label the simple roots as in [8].

$\mathfrak{g}$	Highest short root $=\alpha_0$	α_0	Ad	$\Lambda\ ((\Lambda,\alpha_0^\vee))$
A_n	$\alpha_1+\alpha_2+\cdots+\alpha_n$	$\lambda_1+\lambda_n$	α_0	$\lambda_1\ (1)\quad(\lambda_i\ (1))$
B_n	$\alpha_1+\alpha_2+\cdots+\alpha_n$	λ_1	λ_2	$\lambda_n\ (1)\quad(\lambda_i\ (2), i<n)$
C_n	$\alpha_1+2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n$	λ_2	$2\lambda_1$	$\lambda_1\ (1)\quad(\lambda_i\ (2), i>1)$
D_n	$\alpha_1+2\alpha_2+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$	λ_{n-2}	α_0	$\lambda_1\ (1), \lambda_{n-1}\ (1), \lambda_n\ (1)$
E_6	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	λ_2	α_0	$\lambda_1\ (1), \lambda_6\ (1)$
E_7	$2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$	λ_1	α_0	$\lambda_7\ (1)$
E_8	$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8$	λ_8	α_0	$\lambda_8\ (2)$
G_2	$2\alpha_1+\alpha_2$	λ_1	λ_2	$\lambda_1\ (2)$
F_4	$\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$	λ_4	λ_1	$\lambda_4\ (2)$

$\mathfrak{g}$	Λ_S $((\Lambda_S, \alpha_0^\vee))$	Comment	Also small
A_n	λ_1 (1)	Defining	λ_i (1) and $(\lambda_1+\lambda_n)$ (2) (Ad)
B_n	λ_1 (2)	Defining and Ad^ℓ	λ_n (1) (not really needed)
C_n	λ_1 (1)	Defining	λ_2 (2) (Ad^ℓ)
D_n	λ_1 (1)	Defining	λ_{n-1} (1), λ_n (1), and λ_{n-2} (2) (Ad^ℓ)
E_6	λ_1 (1)	–	λ_6 (1) and λ_2 (2) (Ad)
E_7	λ_7 (1)	–	λ_1 (2) (Ad)
E_8	λ_8 (2)	Ad = Ad^ℓ	–
G_2	λ_1 (2)	Ad^ℓ	–
F_4	$\lambda_4(2)$	Ad^ℓ	–

Lemma 7.1. *Let $m \in \mathbb{N}$, that q is an lth root of -1, and $\alpha \in \triangle^+$. Suppose that*

$$(\Lambda+\rho, \alpha^\vee) = l + m. \tag{55}$$

Then

$$dim_q(\Lambda - m\alpha) = (-1)^{\ell(S_\alpha)} dim_q(\Lambda), \tag{56}$$

where $\ell(S_\alpha)$ denotes the length of the element S_α of the Weyl group.

Proof. We have that

$$(\Lambda+\rho-m\alpha, \beta^\vee) = (S_\alpha(\Lambda+\rho)+l\alpha, \beta^\vee), \tag{57}$$

and

$$q^{d_\beta(l\alpha,\beta^\vee)} = q^{-d_\beta(l\alpha,\beta^\vee)} = (-1)^{(\alpha,\beta)}. \tag{58}$$

Thus,

$$\begin{aligned} dim_q(\Lambda - m\alpha) &= (-1)^{\sum_{\beta\in\triangle^+}(\alpha,\beta)} \prod_{\beta\in S_\alpha(\triangle^+)} \frac{q^{d_\beta(\Lambda+\rho,\beta^\vee)} - q^{-d_\beta(\Lambda+\rho,\beta^\vee)}}{q^{d_\beta(\rho,\beta^\vee)} - q^{-d_\beta(\rho,\beta^\vee)}} && (59)\\ &= (-1)^{2(\alpha,\rho)}(-1)^{\ell(S_\alpha)} dim_q(\Lambda), && (60)\end{aligned}$$

as follows from e.g. Humphreys Lemma 10.3.A. □

Lemma 7.2. *For the algebras G_2, F_4; and E_8 the lengths of the reflexions corresponding to the highest short root are 5, 15, and 57, respectively.*

Proof. This follows in an elementary way from a representation of the relevant root systems, see e.g. Humphreys p. 65. □

Lemma 7.3. *For G_2 the representation $\lambda_1 \equiv 2\alpha_1 + \alpha_2$ has dimension 7. We have*

$$((n_1+1, n_2+1), (2\alpha_1+\alpha_2)) = 2(n_1+1) + 3(n_2+1). \tag{61}$$

If $n_1 = 0$ there may be a problem with $L(\Lambda) \subset L(\lambda_1) \otimes L(\Lambda)$. However, if we demand that l is not divisible by 3 then this cannot occur.

For F_4 the dimension of $\lambda_4 \equiv \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ is 26. The 0-weight space is 2-dimensional and

$$((n_1+1, n_2+1, n_3+1, n_4+1)), (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)) = (n_1+1) + 4(n_2+1) + 3(n_3+1) + 2(n_4+2). \tag{62}$$

Clearly, $n_3 = 0$ is impossible. In fact, n_3 must be odd if the expression in (62) is to equal $l-1$.

Finally, by combining Lemma 7.1, Lemma 7.2, and Lemma 7.3 with Proposition 6.7, we get

Corollary 7.4. *The representation W_2 in (52) has q-dimension 0.*

8. Tensoring versus q-dimension

The following is a fundamental result:

Proposition 8.1 (Andersen, [1]). *Let $E, M \in \mathcal{C}$ be finite-dimensional and suppose that $tr_q(f) = 0$ for all $f \in \mathrm{End}_{U_K}(M)$. Then $tr_q(\phi) = 0$ for all $\phi \in \mathrm{End}_{U_K}(E \otimes M)$. In particular, $dim_q(Q) = 0$ for all summands of $E \otimes M$.*

We can now state our main result:

Proposition 8.2. *Let Λ_1 and Λ_2 be in the first dominant alcove C. Then $L_\zeta(\Lambda_1) \otimes L_\zeta(\Lambda_2)$ is a direct sum*

$$L_\zeta(\Lambda_1) \otimes L_\zeta(\Lambda_2) = L_\zeta(\Lambda_{i(1)}) \oplus \cdots \oplus L_\zeta(\Lambda_{i(n(1,2))}) \oplus S, \tag{63}$$

where $\Lambda_{i(1)}, \dots, \Lambda_{i(n(1,2))} \in C$ and S is a direct sum

$$S = \oplus D(\Lambda_k), \tag{64}$$

where each $D(\Lambda_k)$ has q-dimension 0.

Proof. Let $s = (\Lambda_1 + \Lambda_2 + \rho, \alpha_0^\vee)$. For $s < \ell$ everything is completely reducible and hence all right. In fact, even for $s = \ell$ we have the result since the induced modules stay irreducible in the closure $\overline{C}$ of the fundamental alcove. Next we observe that the result is true if one of the factors is Λ_S (Proposition 6.7 and Corollary 7.4). Next observe that for any representation $\Lambda \in C$, $L_\zeta(\Lambda) = H^0_\zeta(\Lambda)$ and C is closed under taking dual modules. Thus, we can use the results of Section 5, in particular the important results, Proposition 5.3 and Corollary 5.6.

Suppose now that the formula (63) holds for a given pair Λ_1, Λ_2. It then follows that the module S is a direct sum of $D(\Lambda_j)$'s, each of q-dimension 0. By tensoring both sides by Λ_S and using Proposition 8.1 we clearly get a right-hand-side which is a direct sum of simple modules $L_\zeta(\Lambda_j)$ with $\Lambda_j \in C$ and a module $\tilde{S}$ for which $tr_q(\phi) = 0$ for all $\phi \in \mathrm{End}_U(\tilde{S})$.

Concerning the left-hand-side, we can consider the tensor product $\Lambda_S \otimes \Lambda_1$ which we can assume to decompose into a direct sum $\oplus_i \Lambda_i$ of representations from C and thus the left-hand-side is a sum of the form $\oplus_i \Lambda_i \otimes \Lambda_2$. (We might, of course, have chosen to tensor Λ_S onto Λ_2 first instead.) It follows by the uniqueness of the decomposition of tilting modules that each of the summands $\Lambda_i \otimes \Lambda_2$ satisfies a formula like (63). Moreover, the left-hand-side can be written as a direct sum of tilting modules, hence so can the right-hand-side. Moreover, by the uniqueness of the decomposition into tilting modules, we can keep track of all the tilting modules with q-dimension 0 on the right-hand-side and hence on the left-hand-side.

Thus, we can extend the set of $\Lambda_1, \Lambda_2 \in C \times C$ for which (63) holds. Eventually, by the property of Λ_S, we get it to hold for all pairs in $C \times C$. □

We can now introduce the reduced tensor product $\overline{\otimes}$:

Definition 8.3. *In the notation of Proposition* 8.2 *we set*

$$L_\zeta(\Lambda_1) \,\overline{\otimes}\, L_\zeta(\Lambda_2) = L_\zeta(\Lambda_{i(1)}) \oplus \cdots \oplus L_\zeta(\Lambda_{i(n(1,2))}). \tag{65}$$

Theorem 8.4. *The reduced tensor product is associative.*

Proof. Consider

$$L_\zeta(\Lambda_1) \otimes L_\zeta(\Lambda_2) \otimes L_\zeta(\Lambda_3). \tag{66}$$

If we first decompose the tensor product involving Λ_1, Λ_2 and then tensor onto the result with $L_\zeta(\Lambda_3)$ we get a result, call it (1–2)–3, which by Proposition 8.2 is a sum of simple modules corresponding to certain $\Lambda_i \in C$ together with a module S_{12} such that each summand of $S_{12,3}$ has q-dimension 0.

$$(1\text{–}2)\text{–}3 = \oplus_i L_\zeta(\Lambda_i^{12,3}) \oplus S_{12,3}. \tag{67}$$

In the same way we get a result 1–(2–3) for the other way of grouping together in the tensor product,

$$1\text{–}(2\text{–}3) = \oplus_j L_\zeta(\Lambda_j^{1,23}) \oplus S_{1,23}. \tag{68}$$

By the associativity of the usual tensor product, (1–2)–3=1–(2–3), and cutting down by central character to one of the simple summands we get

$$N^i_{12,3} L_\zeta(\Lambda_i) \oplus S^i_{12,3} = N^i_{1,23} L_\zeta(\Lambda_i) \oplus S^i_{12,3}. \tag{69}$$

Taking q-dimension on both sides we get that $N^i_{12,3} = N^i_{1,23}$. □

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