THE CENTER OF THE QUANTIZED MATRIX ALGEBRA

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ABSTRACT. We compute the center in the case where q is a root of unity. Main steps are to compute the degree of an associated quasipolynomial algebra and to compute the dimensions of some interesting irreducible modules.

1. INTRODUCTION

Quantum groups arose from a quantum mechanical problem in statistical mechanics, the quantum Yang-Baxter equation (QYBE). Following Drinfeld [5], a quantum group is the dual of a Hopf algebra and a quantum affine space is the dual of an associative algebra. This algebra is called the coordinate algebra of the quantum affine space. In [11] and [12], a quantum matrix space $M_q(n)$ was defined which can be viewed as a deformation of the full matrix algebra $M_n(\mathbb{C})$. The coordinate algebra of $M_q(n)$ is an associative algebra $\mathcal{A}_n$ (see [1] and [12] for more details), generated by elements $Z_{i,j}, i, j = 1, 2, \cdots, n$, subject to the following relations:

$$\begin{aligned} Z_{i,j}Z_{i,k} &= qZ_{i,k}Z_{i,j} \text{ if } j < k, \\ Z_{i,j}Z_{k,j} &= qZ_{k,j}Z_{i,j} \text{ if } i < k, \\ Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} \text{ if } i < s, t < j, \\ Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} + (q - q^{-1})Z_{i,t}Z_{s,j} \text{ if } i < s, j < t, \end{aligned} \tag{1}$$

where $i, j, k, s, t = 1, 2, \cdots, n$, and $q \in \mathbb{C}^*$ is the quantum parameter.

Our point of view is related to the quantization of the hermitian symmetric space that corresponds to a Lie algebra of type A_n as introduced in [8]. This seems to be different from most others which are more directed towards a quantization of $Gl(n, \mathbb{R})$. Actually, for each hermitian symmetric space there is a quantization in the following sense as explained in [9]: Let $\mathfrak{g}$ be a finite dimensional Lie algebra which corresponds to a non-compact hermitian symmetric space and consider the corresponding quantum group $U_q(\mathfrak{g})$ defined by generators $E_i, F_i, K_i, K_i^{-1}, 1 \leq i \leq n$ and quantized Serre relations (see [10]). Analogous to the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{p}^-)U(\mathfrak{k})U(\mathfrak{p}^+), \tag{2}$$

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there is a decomposition

$$U_q(\mathfrak{g}) = A^- U_q(\mathfrak{k}) A^+, \tag{3}$$

where A^- and A^+ are quadratic algebras.

Specializing to the case of $su(n,n)$ the construction is as follows: For a simple compact root vector E_μ and E_α an arbitrary element of $U_q(\mathfrak{su}(n,n))$ of weigh α, set

$$(adE_\mu)(E_\alpha) = E_\mu E_\alpha - q^{\langle\alpha,\mu\rangle} E_\alpha E_\mu, \tag{4}$$

where, as usual, $\langle\alpha,\mu\rangle = \frac{2(\alpha,\mu)}{(\mu,\mu)}$. This is the so-called quantum commutator.

The set of simple compact roots breaks up into two orthogonal sets:

$$\Sigma_c = \{\nu_1, \nu_2, \cdots, \nu_{n-1}\} \cup \{\mu_1, \mu_2, \cdots, \mu_{n-1}\}. \tag{5}$$

Thus

$$E_{\mu_i} E_{\nu_j} = E_{\nu_j} E_{\mu_i} \tag{6}$$

for all i, j.

Assume that these roots have been labeled in such a way that

$$\langle\beta,\mu_1\rangle = \langle\beta,\nu_1\rangle = \langle\mu_i,\mu_{i+1}\rangle = \langle\nu_i,\nu_{i+1}\rangle = -1, \tag{7}$$

for all i, j, and where β is the unique non-compact simple root.

We then define $Z_{0,0} = E_\beta$ and

$$Z_{i,j} = (adE_{\mu_i}) \cdots (adE_{\mu_1})(adE_{\nu_j}) \cdots (adE_{\nu_1})(E_\beta). \tag{8}$$

for $i, j = 0, 1, \cdots, n-1$.

It is then straightforward to prove that the elements $Z_{i,j}, i, j = 0, 1, \cdots, n-1$, generate a subalgebra A^+ of $U_q(\mathfrak{g})$ which is isomorphic to $\mathcal{A}_n$ (see [8]).

For any matrix $A = (a_{i,j})_{i,j=1}^n \in M_n(\mathbb{Z}_+)$ we define a monomial Z^A in $\mathcal{A}_n$ by

$$Z^A = \Pi_{i,j=1}^n Z_{i,j}^{a_{i,j}}, \tag{9}$$

where the factors are arranged in the lexicographic order on $I(n) = \{(i,j) \mid i,j = 1, \ldots, n\}$.

In [11] it was proved that $\mathcal{A}_n$ has a basis

$$\{Z^A \mid A \in M_n(\mathbb{Z}_+)\}, \tag{10}$$

by using the so-called diamond lemma. The above ordering then induces a total ordering on this basis. Let N_- be the subalgebra of $\mathcal{A}_n$ generated by the elements $Z_{i,j}$ for $i, j = 1, 2, \cdots, n$ and $i + j \leq n$. Let N_+ be the subalgebra of $\mathcal{A}_n$ generated by the $Z_{i,j}$ for $i, j = 1, 2, \cdots, n$ and $i + j \geq n + 2$, and let H be the subalgebra of $\mathcal{A}_n$ generated by the $Z_{i,n-i+1}$ for $i = 1, 2, \cdots, n$. Then

$$\mathcal{A}_n = N_- H N_+, \tag{11}$$

which is called the triangular decomposition (or Weyl decomposition) of $\mathcal{A}_n$.

It is well known that $\mathcal{A}_n$ is a bialgebra (Cf. [12] for more details) for which the coproduct Δ and counit ϵ are defined as follows:

$$\Delta : \mathcal{A}_n \longrightarrow \mathcal{A}_n \otimes \mathcal{A}_n, \tag{12}$$

$$\Delta(Z_{i,j}) = \sum_k Z_{i,k} \otimes Z_{k,j}, \tag{13}$$

and

$$\epsilon : \mathcal{A}_n \longrightarrow \mathbb{C}, \tag{14}$$

$$\epsilon(Z_{i,j}) = \delta_{i,j}. \tag{15}$$

The quantum determinant $\det_q$ is defined as follows (motivated by the bialgebra structure of $\mathcal{A}_n$):

$$\det_q = \Sigma_{\sigma \in S_n} (-q)^{l(\sigma)} Z_{1,\sigma(1)} Z_{2,\sigma(2)} \cdots Z_{n,\sigma(n)}. \tag{16}$$

For later reference we now introduce some terminology: Let $m \leq n$ be a positive integer. Given any two subsets $I = \{i_1, i_2, \cdots, i_m\}$ and $J = \{j_1, j_2, \cdots, j_m\}$ of $\{1, 2, \cdots, n\}$, each having cardinality m, it is clear that the subalgebra of $\mathcal{A}_n$ generated by the elements Z_{i_r, j_s} with $r, s = 1, 2, \cdots, m$, is isomorphic to $\mathcal{A}_m$, so we can talk about its determinant. Such a determinant is called a subdeterminant of $\det_q$, and will be denoted by $\det_q(I, J)$. If $I = \{1, 2, \cdots, n\} \setminus \{i\}$, $J = \{1, 2, \cdots, n\} \setminus \{j\}$, $\det_q(I, J)$ will be denoted by $A(i, j)$.

In [12], Parshall and Wang proved that $\det_q$ is a central element of $\mathcal{A}_n$. Moreover, in [11], Noumi, Yamada, and Mimachi proved that the center of $\mathcal{A}_n$ is generated by $\det_q$ if q is generic.

However, when q is a root of unity the situation changes dramatically. For example, $\mathcal{A}_n$ is finite over its center and so every irreducible module over $\mathcal{A}_n$ is finite dimensional. In this paper we compute the center of $\mathcal{A}_n$ when q is a root of unity by calculating the degree of $\mathcal{A}_n$.

After our research was completed we learned of the interesting results of De Concini and Lyubashenko [2], De Concini and Procesi [4], and Gaiffi [7]. In these articles, among other things, analogous results are obtained, by different methods, for the quantum function algebras.

The arrangement of the paper is as follows: In Section 2 we review some of the results obtained by De Concini and Procesi in [3] about the degree of a prime algebra, especially for certain quasipolynomial algebras. We introduce some Verma modules in Section 3. They are mainly used here because we can compute their dimensions, but we believe they will be of further interest. In Section 4 we find some covariant elements of $\mathcal{A}_n$ by using the subdeterminants above. In Section 5 we compute the canonical form of a certain $n^2 \times n^2$ skew-symmetric matrix and this enables us to compute the degree of $\overline{\mathcal{A}_n}$, the associated quasipolynomial algebra of $\mathcal{A}_n$. It is a special case of a fundamental theorem of [3] that this degree equals the degree of $\mathcal{A}_n$. Finally, in Section 6 we calculate the center $C(\mathcal{A}_n)$ of $\mathcal{A}_n$ as well as the center $C(\overline{\mathcal{A}_n})$ of $\overline{\mathcal{A}_n}$.

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2. The degree of a prime algebra

The main tool used to compute the degree of $\mathcal{A}_n$ is the theory developed in [3] by De Concini and Procesi. So we shall first recall some results from there specialized to our situation.

Let A be a prime algebra (i.e. $aAb = 0$ implies $a = 0$ or $b = 0$) over the complex numbers $\mathbb{C}$ and Z be the center of A. Then Z is a domain and A is a torsion free module over Z. Assume that A is a finite module over Z. Then A embeds in a finite dimensional central simple algebra $Q(A) = Q(Z) \otimes_Z A$, where $Q(Z)$ is the fraction field of Z. If $\overline{Q(Z)}$ denotes the algebraic closure of $Q(Z)$, we have that $\overline{Q(Z)} \otimes_Z A$ is the full algebra $M_d(\overline{Q(Z)})$ of $d \times d$ matrices over $\overline{Q(Z)}$. Then d is called the degree of A.

Given an $n \times n$ skew-symmetric matrix $H = (h_{i,j})$ over $\mathbb{Z}$ we construct the quasipolynomial algebra $\mathbb{C}_H[x_1, x_2, \cdots, x_n]$ as follows: It is the algebra generated by elements $x_1, x_2, \cdots, x_n$ with the following defining relations:

$$x_i x_j = q^{h_{i,j}} x_j x_i \text{ for } i, j = 1, 2, \cdots, n. \tag{17}$$

It can be viewed as an iterated Ore extension with respect to any ordering of the indeterminates x_i. Given $a = (a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n$ we write $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$.

Let q be a primitive mth-root of unity. We consider the matrix H as a matrix of the homomorphism $H : \mathbb{Z}^n \longrightarrow (\mathbb{Z}/m\mathbb{Z})^n$ and we denote by K the kernel of H and by h the cardinality of the image of H. The following was proved in [3]:

Theorem 2.1. *(a) The monomials x^a with $a \in K \cap \mathbb{Z}_+^n$ form a basis of the center of $\mathbb{C}_H[x_1, x_2, \cdots, x_n]$.*

(b) degree $\mathbb{C}_H[x_1, x_2, \cdots, x_n] = \sqrt{h}$.

It is well known that a skew-symmetric matrix over $\mathbb{Z}$ such as our matrix H can be brought into a 2×2 block diagonal form by an element $W \in SL(\mathbb{Z})$. Specifically, there is a $W \in SL(\mathbb{Z})$ and a sequence of 2×2 matrices $S(m_i) = \begin{pmatrix} 0 & -m_i \\ m_i & 0 \end{pmatrix}, i = 1, \ldots, N$, with $m_i \in \mathbb{Z}$ for each $i = 1, \ldots, N$, such that

$$W \cdot H \cdot W^t = \begin{cases} Diag\,(S(m_1), \ldots, S(m_N), 0) \text{ with } N = \frac{n-1}{2}, \text{ if } n \text{ is odd} \\ Diag\,(S_1(m_1), \ldots, S(m_N)) \text{ with } N = \frac{n}{2}, \text{ if } n \text{ is even} \end{cases}. \tag{18}$$

Definition 2.2. *Any matrix of the form of the right-hand-side in (18) will be called a canonical form of H and will occasionally be denoted by J_H.*

Thus, a canonical form of H reduces the algebra to the tensor product of Laurent quasipolynomial algebras in two variables with commutation relation $xy = q^r yx$. By Theorem 2.1 it follows in particular that the degree of a Laurent quasipolynomial algebra in two variables is equal to $m/(m,r)$, where (m,r) is the greatest common divisor of m and r.

3. Some irreducible modules

We now introduce some modules which turn out to be very useful.

Definition 3.1. *For an integer m we set*

$$m' = \begin{cases} m \text{ if } m \text{ is odd} \\ \frac{m}{2} \text{ if } m \text{ is even} \end{cases} . \tag{19}$$

Definition 3.2. *Let $\Lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and let $I(\Lambda)$ be the left ideal in $\mathcal{A}_n$ generated by the elements $Z_{i,j}$ with $i+j \geq n+2$ together with the elements $Z_{k,n+1-k} - \lambda_k$ for $k = 1, \dots, n$. The Verma module $M(\Lambda)$ is defined as*

$$M(\Lambda) = \mathcal{A}_n / I(\Lambda). \tag{20}$$

We denote by v_Λ the image of 1 in the quotient.

Before introducing one more type of modules we need

Lemma 3.3. *If $i < k$ and $j < l$ then $Z_{i,j}^s Z_{k,l} = Z_{k,l} Z_{i,j}^s + (q - q^{1-2s}) Z_{i,j}^{s-1} Z_{i,l} Z_{k,j}$.*

Proof. We use induction on s. The case $s = 1$ is trivial. We have

$$\begin{aligned} Z_{i,j}^s Z_{k,l} &= Z_{i,j}^{s-1}[Z_{k,l} Z_{i,j} + (q - q^{-1}) Z_{i,l} Z_{k,j}] \\ &= [Z_{k,l} Z_{i,j}^{s-1} + (q - q^{3-2s}) Z_{i,j}^{s-2} Z_{i,l} Z_{k,j}] Z_{i,j} + (q - q^{-1}) Z_{i,j}^{s-1} Z_{i,l} Z_{k,j} \\ &= Z_{k,l} Z_{i,j}^s + (q - q^{1-2s}) Z_{i,j}^{s-1} Z_{i,l} Z_{k,j}. \end{aligned} \tag{21}$$

This completes the proof. □

Corollary 3.4. *If q is a primitive mth-root of unity then $Z_{i,j}^m$ is a central element for all $i, j = 1, 2, \cdots, n$.*

Remark 3.5. *Notice that in (21) the very last term vanishes for $s = m'$. However, for other commutators we only have covariance (cf. Definition 4.1), e.g. $Z_{i,1} Z_{i,j}^{m'} = q^{m'} Z_{i,j}^{m'} Z_{i,1}$.*

Definition 3.6. *Let $\overline{I(\Lambda)}$ denote the left ideal in $\mathcal{A}_n$ generated by $I(\Lambda)$ together with the elements $Z_{i,j}^{m'}$ for $i + j \leq n$. The restricted Verma module $\overline{M(\Lambda)}$ (sometimes called the "Baby Verma module") is given as*

$$\overline{M(\Lambda)} = \mathcal{A}_n / \overline{I(\Lambda)}. \tag{22}$$

Theorem 3.7. *$\overline{M(\Lambda)}$ is irreducible if and only if $\forall i = 1, \dots, n : \lambda_i \neq 0$.*

Proof. Observe first that the case $m = 2$ is trivial. So let $m \geq 3$ and assume that we have a proper invariant subspace U of the Verma module. It follows that there must be a non-zero vector v_1 in U which is annihilated by all elements strictly above the diagonal $Z_{1,n}, Z_{2,n-1}, \cdots, Z_{n,1}$. Moreover, we may assume that v_1 is a weight vector for the elements in the diagonal. We proceed to prove the claim by induction. The case $n = 2$ follows easily from Lemma 3.3. Now Let

$$Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{n-2,1}^{i_{n-2}} Z_{n-1,1}^{i_{n-1}} \cdot p_{i_1,\cdots,i_{n-1}} v_\Lambda \tag{23}$$

be a term in v_1 constructed as follows: Locate the terms in v_1 with the highest $Z_{1,1}$ degree. Among these, choose those with the biggest $Z_{2,1}$ degree, and so on, yielding a unique monomial multiplied onto a term $p_{i_1,\cdots,i_{n-1}} v_\Lambda$ which only contains elements of the form $Z_{i,j}$ with $j \geq 2$ and $i + j \leq n - 1$. We may assume that at least one $i_j > 0$. By construction and by the action of elements of the form $Z_{i,j}$ with $i \leq n-1$, $j \geq 2$, and $i = j \geq n + 2$, it follows that $p_{i_1,\cdots,i_{n-1}} v_\Lambda$ is a highest weight vector for these elements, and hence is a (non-zero) constant. Furthermore, in view of the action of $Z_{n,2}$ on $Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{n-2,1}^{i_{n-2}} Z_{n-1,1}^{i_{n-1}}$, it is impossible to have a positive degree i_{n-1} on $Z_{n-1,1}$ if v_1 is to be annihilated. The conclusion of this analysis is that we may assume that

$$v_1 = Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k} v_\Lambda + \ldots \tag{24}$$

for some $k \leq n - 2$.

Next we again consider the action of $Z_{n,2}$ and conclude that there must be a term proportional to $Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k - 1} Z_{n-1,1} Z_{k,2} v_\Lambda$. Thus,

$$v_1 = (Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k} + \beta \cdot Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k - 1} Z_{n-1,1} Z_{k,2}) v_\Lambda + \ldots \tag{25}$$

for some non-zero constant β.

Let us for simplicity write the formula (21) as $Z_{k,l} Z_{i,j}^s = Z_{i,j}^s Z_{k,l} + \psi_s Z_{i,j}^{s-1} Z_{i,l} Z_{k,j}$. Applying $Z_{n,n+1-k}$ to v_1 we now get

$$(Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k - 1} \psi_{i_k} \lambda_n \lambda_k + \beta \cdot Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k - 1} \psi_1^2 \lambda_n \lambda_{n-1} \lambda_k) v_\Lambda + \ldots, \tag{26}$$

where the dots indicate terms which cannot affect the one given, and vice versa, and hence must in fact be zero. Thus,

$$\beta = -\frac{\psi_{i_k}}{\lambda_{n-1} \psi_1^2}. \tag{27}$$

However, we may assume that v_1 is a highest weight vector for $Z_{n-1,2}$ (of weight λ_{n-1}, naturally) and hence we get that

$$\begin{aligned} & \beta \lambda_{n-1} q^{-2} Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k} Z_{n-1,1} Z_{k,2} + \psi_{i_k} Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k} Z_{n-1,1} Z_{k,2} \\ = \; & \lambda_{n-1} \beta \cdot Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k} Z_{n-1,1} Z_{k,2}, \end{aligned} \tag{28}$$

and hence

$$\beta = -\frac{\psi_{i_k}}{\lambda_{n-1}(1 - q^{-2})}. \tag{29}$$

This is a contradiction since

$$\psi_1^2 = (q - q^{-1})^2 = q^2(1 - q^{-2})^2 \neq 1 - q^{-2}. \tag{30}$$

Conversely, if some $\lambda_{i_0} = 0$ it is easy to see that there is a proper invariant subspace generated by the elements $Z_{i_0,n-i_0}v_\Lambda$ and $Z_{i_0+1,n-i_0-1}v_\Lambda$. □

Corollary 3.8. *If $\overline{M(\Lambda)}$ is irreducible, then*

$$\dim \overline{M(\Lambda)} = (m')^{\frac{n^2-n}{2}}. \tag{31}$$

Proof. This follows from the PBW theorem ([11]). □

To deal with the situation where m is even, we now introduce some "generalized Verma modules" and some "restricted generalized Verma modules":

Definition 3.9. *Let $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, let $\Phi = (\phi_1, \ldots, \phi_{n-1}) \in \mathbb{C}^{n-1}$ and let $I^0(\Lambda, \Phi)$ be the left ideal in $\mathcal{A}_n$ generated by the elements $Z_{i,j}$ with $i + j \geq n + 2$ together with the elements $Z_{k,n+1-k} - \lambda_k$ for $k = 1, \ldots, n$ and the elements $Z^m_{k,n-k} - \phi_k$ for $k = 1, \ldots, n-1$. The generalized Verma module $M^0(\Lambda, \Phi)$ is defined as*

$$M^0(\Lambda, \Phi) = \mathcal{A}_n / I^0(\Lambda, \Phi). \tag{32}$$

Definition 3.10. *Let $\overline{I^0(\Lambda, \Phi)}$ denote the left ideal in $\mathcal{A}_n$ generated by $I^0(\Lambda, \Phi)$ together with the elements $Z^{m'}_{i,j}$ for $i + j \leq n - 1$. The restricted generalized Verma module $\overline{M^0(\Lambda, \Phi)}$ is given as*

$$\overline{M^0(\Lambda, \Phi)} = \mathcal{A}_n / \overline{I^0(\Lambda, \Phi)}. \tag{33}$$

We denote by $v^0_{\Lambda,\Phi}$ the image of 1 in the quotient.

Theorem 3.11. *The module $\overline{M^0(\Lambda, \Phi)}$ is irreducible for $\Psi = (1, \ldots, 1)$ and $\Lambda = (1, \ldots, 1)$. It has dimension $m^{n-1}(m')^{(n-1)(n-2)/2}$.*

Proof. As we shall only need this in the case m even, we only give the details in this case. The proof for the case m odd is similar. We may then use induction as in the case of Theorem 3.7. However, there are some new features that must be taken into account. First of all, $Z^{m'}_{n-1,1} \cdot v^0_{\Lambda,\Phi}$ is a primitive vector which is different from $v^0_{\Lambda,\Phi}$. However, it does not generate a non-trivial invariant subspace since we can multiply it with $Z^{m'}_{n-1,1}$ and thus get back to the highest weight vector. Also observe that we can separate it from the highest weight vector since $Z_{n,1}Z^{m'}_{n-1,1} = -Z^{m'}_{n-1,1}Z_{n,1}$. Besides this we may proceed to investigate hypothetical invariant subspaces U. It is again clear that any invariant subspace must contain a primitive vector, v_p. It follows again that

$$v_p = Z^{i_1}_{1,1} Z^{i_2}_{2,1} \cdots Z^{i_k}_{k,1} Z^{\alpha}_{n-1,1} \cdot v^0_{\Lambda,\Psi} + \ldots, \tag{34}$$

where the power α may be 0 or m'. Again we look at the action of $Z_{n,2}$, but now we must allow for one more term, at least when $k \leq n-3$, namely

$$v_p = (Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k} + \beta \cdot Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k - 1} Z_{n-1,1} Z_{k,2} + \gamma \cdot Z_{1,1}^{i_1} Z_{2,1}^{i_2} \cdots Z_{k,1}^{i_k-1} Z_{k,2} Z_{n-2,1} Z_{n-2,2}^{m-1}) \cdot Z_{n-1,1}^{\alpha} \cdot v_{\Lambda,\Psi} + \dots. \tag{35}$$

Looking at the action of $Z_{k+1,n-k}$ it follows immediately that there can be no primitive vector when $k \leq n-4$. In the case $k = n-3$ there can be no β and if $k = n-2$ there can be no γ. The case $k = n-1$ is included in the previous due to the factor $Z_{n-1,1}^{\alpha}$ unless the leading term actually has the form $Z_{n-1,1}^{i_{n-1}}$ for some positive power i_{n-1}. In the first two cases we proceed as in the proof of Theorem 3.7 and get two equations for β as well as for γ. They both lead to absurdities. The β equations are the same as previously. The γ equations lead to

$$\psi_1^2 = \psi_1 - 1 \tag{36}$$

which is impossible. Finally, if the leading term is $Z_{n-1,1}^{i_{n-1}}$ it follows by looking at the action of $Z_{n,1}$ that the primitive vector must be $Z_{n-1,1}^{m'}$ and this case has already been disposed of. $\square$

Remark 3.12. *The modules $\overline{M^0(\Lambda,\Phi)}$ are generically irreducible.*

Remark 3.13. *One could try to generalize the Verma modules even further by imposing the condition $Z_{i,j}^m = d_{i,j}$ on all elements for which $i + j \leq n$. However, this will in general not lead to irreducible modules. There will be more primitive vectors, and some of these will generate proper invariant subspaces. Indeed, if these modules were irreducible, the degree of the corresponding algebra would be bigger than what has been established.*

4. Some covariant elements

Definition 4.1. *An element $x \in \mathcal{A}_n$ is called covariant if for any $Z_{i,j}$ there exists an integer $n_{i,j}$ such that*

$$xZ_{i,j} = q^{n_{i,j}} Z_{i,j} x. \tag{37}$$

Clearly, $Z_{1,n}$ and $Z_{n,1}$ are covariant.

The following proposition was proved by Parshall and Wang in [12]. OBS $q \to q^{-1}$!!!!!

Proposition 4.2. *Let $i, k \leq n$ be fixed integers. Then*

$$\delta_{i,k} \mathrm{det}_q = \sum_{j=1}^{n} (-q)^{k-j} Z_{i,j} A(k,j) = \sum_j (-q)^{j-i} A(i,j) Z_{k,j} \tag{38}$$

$$= \sum_j (-q)^{k-j} Z_{j,i} A(j,k) = \sum_j (-q)^{j-i} A(j,i) Z_{j,k}. \tag{39}$$

The above formula is called the quantum Laplace expansion.

Let $\det_q(t) = \det_q(\{1, \cdots, t\}, \{n-t+1, \cdots, n\})$, for $t = 1, 2, \cdots, n$. Then we have

Theorem 4.3. $\det_q(t)$ *is covariant for all* t.

Proof. Let $M_t^- = \{(i,j) \in \mathbb{N}^2 \mid 1 \le i \le t \text{ and } 1 \le j \le n-t\}$, $M_t^+ = \{(i,j) \in \mathbb{N}^2 \mid t+1 \le i \le n \text{ and } n-t+1 \le j \le n\}$, $M_t^l = \{(i,j) \in \mathbb{N}^2 \mid t+1 \le i \le n \text{ and } 1 \le j \le n-t\}$, and $M_t^r = \{(i,j) \in \mathbb{N}^2 \mid 1 \le i \le n \text{ and } n-t+1 \le j \le n\}$. We use induction on t to prove that

$$\begin{aligned} Z_{i,j}\det_q(t) &= \det_q(t)Z_{i,j} \text{ if } (i,j) \in M_t^l \cup M_t^r, \\ Z_{i,j}\det_q(t) &= q\det_q(t)Z_{i,j} \text{ if } (i,j) \in M_t^-, \text{ and} \\ Z_{i,j}\det_q(t) &= q^{-1}\det_q(t)Z_{i,j} \text{ if } (i,j) \in M_t^+. \end{aligned} \tag{40}$$

If $t = 1$, $\det_q(1) = Z_{1,n}$ and the above formulas hold for $Z_{1,n}$. Now we assume that these formulas hold for $\det_q(t-1)$ for any n.

If $(i,j) \in M_t^l$, by the defining relations of $\mathcal{A}_n$ we have

$$Z_{i,j}\det_q(t) = \det_q(t)Z_{i,j}. \tag{41}$$

For $(i,j) \in M_t^-$ we claim that

$$Z_{i,j}\det_q(t) = q\det_q(t)Z_{i,j}. \tag{42}$$

Clearly (42) holds if $i = t$. Now we assume that $i < t$. By the quantum Laplace expansion we have

$$Z_{i,j}\det_q(t) = \sum_{s=1}^{t} (-q)^{s-t} Z_{i,j} Z_{t,n-s+1} A_t(t, n-s+1), \tag{43}$$

where $A_t(k,l)$ is a subdeterminant of $\det_q(t)$.

By the induction hypothesis we have

$$Z_{i,j} A_t(t, n-s+1) = q A_t(t, n-s+1) Z_{i,j}. \tag{44}$$

Hence,

$$\begin{aligned} Z_{i,j}\det_q(t) &= \sum_s (-q)^{s-t} Z_{i,j} Z_{t,n-s+1} A_t(t, n-s+1) \\ &= \sum_s (-q)^{s-t} \left(Z_{t,n-s+1} Z_{i,j} - (q - q^{-1}) Z_{i,n-s+1} Z_{t,j} \right) A_t(t, n-s+1) \\ &= \sum_s (-q)^{s-t} q Z_{t,n-s+1} A_t(t, n-s+1) Z_{i,j} - \sum_s (-q)^{s-t} Z_{t,j} Z_{i,n-s+1} A_t(t, n-s+1). \end{aligned} \tag{45}$$

By the quantum Laplace expansion we know that

$$\sum_s (-q)^{s-t} Z_{i,n-s+1} A_t(t, n-s+1) = 0, \tag{46}$$

and this proves that

$$Z_{i,j}\det_q(t) = q\det_q(t)Z_{i,j}. \tag{47}$$

In a similar way it follows that if $t+1 \le i \le n$ and $n-t+1 \le j \le n$ then

$$Z_{i,j}\det_q(t) = q^{-1}\det_q(t)Z_{i,j}. \tag{48}$$

Note that $\det_q(t)$ is a central element of the subalgebra generated by the $Z_{i,j}$ for $i = 1, 2, \cdots, t$ and $j = n-t+1, \cdots, n$, so

$$Z_{i,j}\det_q(t) = \det_q(t)Z_{i,j} \text{ for } (i,j) \in M_t^r. \tag{49}$$

This completes the proof. □

The following proposition is essentially Proposition 3.7.1 in [12].

Proposition 4.4. *The mapping τ sending $Z_{i,j}$ to $Z_{j,i}$ extends to an algebra automorphism of $\mathcal{A}_n$.*

By the proof of Theorem 4.3, specifically (40), we have

Corollary 4.5. *If q is a primitive mth-root of unity then*

$$\det_q(t)^{m-r} \cdot \tau((\det_q(n-t))^r) \tag{50}$$

is a central element for all t and r.

5. The degree of $\mathcal{A}_n$

Let $I(n) = \{(i,j) \mid i,j = 1, 2, \cdots, n\}$. Let $\le$ be the lexicographic ordering on $I(n)$ and let $J(n) = (a_{(i,j),(k,l)})_{i,j,k,l=1}^n$ be the $n^2 \times n^2$ skew-symmetric matrix where $a_{(i,j),(i,k)} = 1 = -a_{(i,k),(i,j)}$ if $j < k$, $a_{(i,j),(k,j)} = 1 = -a_{(k,j),(i,j)}$ if $i < k$, and $a_{(i,j),(k,l)} = 0$ otherwise.

Let $\overline{\mathcal{A}_n}$ be the associative algebra generated by elements $X_{i,j}$ for $i,j = 1, 2, \cdots, n$ with the defining relations:

$$X_{i,j}X_{k,l} = q^{a_{(i,j),(k,l)}}X_{k,l}X_{i,j}, i,j,k,l = 1, 2, \cdots, n. \tag{51}$$

$\overline{\mathcal{A}_n}$ is the associated quasipolynomial algebra of $\mathcal{A}_n$ ([3]) and it is the quasipolynomial algebra in n^2 variables associated to the skew-symmetric matrix $J(n)$ as in Section 2.

It is well known that $\mathcal{A}_n$ is an iterated twisted Ore extension ([6]) that falls into the class of algebras considered by De Concini and Procesi in [3]. The following is a special case of the theory developed there:

Theorem 5.1. *The degree of $\mathcal{A}_n$ is equal to the degree of the associated quasipolynomial algebra $\overline{\mathcal{A}_n}$.*

This, together with Theorem 2.1 will be our main tools.

Theorem 5.2. rank $J(n) = n^2 - n$.

Proof. Let $M_n(\mathbb{C})$ be the full matrix algebra of $n\times n$ matrices. Let $E_{i,j}$ be the matrix unit with (i,j)th entry 1, and 0 elsewhere. Let $H=\sum_{i>j}(E_{i,j}-E_{j,i})$. We define a linear map

$$J : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C}) \tag{52}$$

by

$$N \stackrel{J}{\longrightarrow} HN - NH. \tag{53}$$

A direct calculation shows that $J(n)$ is the matrix of J with respect to the basis $\{E_{i,j} \mid i,j=1,2,\cdots,n\}$.

Let $N\in kerJ$ i.e. $HN-NH=0$. Note that H is diagonalizable so we can find a non-degenerate matrix $P\in M_n(\mathbb{C})$ such that $PHP^{-1}=\sum_i d_iE_{i,i}$. Let $PNP^{-1}=(x_{i,j})_{i,j=1}^n$. Then we have

$$(d_i-d_j)x_{i,j}=0 \text{ for all } i,j=1,2,\cdots,n. \tag{54}$$

If $d_i\neq d_j$ for each pair i,j with $i\neq j$, then $x_{i,j}=0$ unless $i=j$. Hence $\dim ker\, J=n$ and thus $rank\, J(n)=n^2-n$, provided that the eigenvalues of H are all distinct.

Let $S=\sum_{i=2}^n E_{i,i-1}-E_{1,n}$. Then we have

$$H=S+S^2+S^3+\cdots S^{n-1}. \tag{55}$$

Observe that $S^n=-1$, hence S is diagonalizable and its eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$ are the n distinct nth roots of -1. Consider H and S as linear transformations of the vector space $\mathbb{C}^n$. Let v_i be an eigenvector of S belonging to the eigenvalue λ_i. Then $v_1,v_2,\cdots,v_n$ is a basis of $\mathbb{C}^n$. It is easy to see that $HS+S=H-1$, so $(1-\lambda_i)Hv_i=(1+\lambda_i)v_i$. The eigenvalues of H are then $\frac{1+\lambda_i}{1-\lambda_i}$, $i=1,2,\cdots,n$, where λ_i is an nth-root of -1. Thus, the n eigenvalues are indeed different. This completes the proof. □

For later use we now study the special case of $q=-1$.

Definition 5.3. *Let B_n be the associative algebra generated by elements $Y_{i,j}$ for $i,j=1,2,\cdots,n$, subject to the following relations:*

$$\begin{aligned} Y_{i,j}Y_{s,j} &= -Y_{s,j}Y_{i,j} \text{ if } i\neq s,\\ Y_{i,j}Y_{i,t} &= -Y_{i,t}Y_{i,j} \text{ if } j\neq t, \text{ and}\\ Y_{i,j}Y_{s,t} &= Y_{s,t}Y_{i,j} \text{ if } i\neq s \text{ and } j\neq t, \end{aligned} \tag{56}$$

for all $i,j,s,t=1,2,\cdots,n$.

Lemma 5.4. *$Y_{s,i}Y_{s,j}Y_{t,i}Y_{t,j}$ is in the center $\mathcal{Z}$ of B_n for any $i,j,s,t=1,2,\cdots,n$.*

Proof. The proof is straightforward, we leave it to the reader. □

Proposition 5.5. *$\mathcal{Z}$ is generated by the following central elements:*

$$Y_{i,j}^2, Y_{s,i}Y_{s,j}Y_{t,i}Y_{t,j}, Y_{k,1}Y_{k+1,2}\cdots Y_{n,n-k+1}Y_{1,n-k+2}Y_{2,n-k+3}\cdots Y_{k-1,n}, \tag{57}$$

for $i,j,s,t,k = 1,2,\cdots,n$.

Proof. It is clear that the elements stated above are central. Let $\mathcal{Z}'$ be the central subalgebra generated by these elements. By

$$Y_{1,s}Y_{1,j}Y_{s,s}Y_{s,j}Y_{s,j} = Y_{s,j}^2 Y_{1,s}Y_{1,j}Y_{s,s} \tag{58}$$

and

$$Y_{1,1}Y_{2,2}\cdots Y_{n,n}Y_{1,1} = Y_{1,1}^2 Y_{2,2}\cdots Y_{n,n} \tag{59}$$

for all $s = 2,3,\cdots,n$ and $j = 1,2,\cdots,n$, it is easy to see that $B_n \otimes Q(\mathcal{Z}')$ is spanned by

$$\{Y_{2,2}^{c_{2,2}}Y_{3,3}^{c_{3,3}}\cdots Y_{n,n}^{c_{n,n}}Y_{1,2}^{c_{1,2}}\cdots Y_{1,n}^{c_{1,n}} \mid c_{i,i}, c_{1,j} \in \{0,1\} \text{ for } i,j = 2,3,\cdots,n\} \tag{60}$$

over $Q(\mathcal{Z}')$.

For any central element X which is a monomial there exists an element Y as in (60) and $W, V \in \mathcal{Z}'$ such that $WX = VY$. So Y is also a central element. By direct calculation it is easy to see that $Y = 1$. Therefore $WX = V$. Comparing the two sides, it follows that $X \in \mathcal{Z}'$. This completes the proof since it is enough to consider monomials. □

Corollary 5.6.

$$\text{degree } B_n = 2^{n-1} \tag{61}$$

Proof. It follows by the above that the degree is less than or equal to 2^{n-1}. But by Theorem 3.11 ($m = 2$) it must also be of at least that magnitude. □

Theorem 5.7. *Let $S_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $i = 1,2,\cdots,n-1$ and $S_j = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ for $j = n,\cdots,\frac{n^2-n}{2}$. A canonical form of $J(n)$ is* Diag $(S_1, S_2, \cdots, S_{\frac{n^2-n}{2}}, 0, \cdots, 0)$*. In particular, the degree is given by*

$$\deg \mathcal{A}_n = m^{n-1}(m')^{(n-1)(n-2)/2}. \tag{62}$$

Proof. By Theorem 5.2, Theorem 5.1, and Corollary 3.8 the canonical form of $J(n)$ must be of the form

$$\mathit{Diag}\,(S_1, \ldots, S_{\frac{n^2-n}{2}}, 0, \ldots, 0), \tag{63}$$

where $S_i = \begin{pmatrix} 0 & 2^{r_i} \\ -2^{r_i} & 0 \end{pmatrix}$. By Corollary 5.6 it follows that $r_i = 0$ for $i = 1,\ldots,n-1$ and Theorem 3.11 gives that the remaining S_i's must be equal to $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$. □

6. The centers of $\overline{\mathcal{A}_n}$ and $\mathcal{A}_n$.

Let $C(\mathcal{A}_n)$ be the center of $\mathcal{A}_n$ and $C(\overline{\mathcal{A}_n})$ the center of $\overline{\mathcal{A}_n}$.

For any $P \in C(\mathcal{A}_n)$ the leading term of P must be of the form cZ^A, $c \in \mathbb{C}^*$, for some $A = (a_{i,j})_{i,j=1}^n \in M_n(\mathbb{Z}_+)$. For any $Z_{k,l}$, the leading term of $P \cdot Z_{k,l}$ is $cq^{r_{k,l}} Z^{A+E_{k,l}}$, where $r_{k,l} = \sum_{j>l} a_{k,j} + \sum_{i>k} a_{i,l}$. The leading term of $Z_{k,l} \cdot P$ is $c \cdot q^{l_{k,l}} Z^{A+E_{k,l}}$, where $l_{k,l} = \sum_{j<l} a_{k,j} + \sum_{i<k} a_{i,l}$. By $ZZ_{k,l} = Z_{k,l}Z$ we get $q^{l_{k,l}} = q^{r_{k,l}}$ and this implies that cX^A is a central element of the quasipolynomial algebra $\overline{\mathcal{A}_n}$. Hence we can define a map

$$T : C(\mathcal{A}_n) \longrightarrow C(\overline{\mathcal{A}_n}) \tag{64}$$

by

$$P \xrightarrow{T} cX^A, \tag{65}$$

if the leading term of P is cZ^A, $c \in \mathbb{C}^*$.

Observe that $T(P) = 0$ implies $P = 0$. But, of course, T is not linear.

Theorem 6.1. *If q is a primitive mth-root of unity and m is an odd integer then $C(\overline{\mathcal{A}_n})$ is generated by the elements*

$$X_{i,j}^m, X_{1,1}X_{2,2}\cdots X_{n,n}, X_{i,1}^r X_{i+1,2}^r \cdots X_{n,n-i+1}^r X_{1,n-i+2}^{m-r} X_{2,n-i+3}^{m-r} \cdots X_{i-1,n}^{m-r} \tag{66}$$

for $i, j = 1, 2, \cdots, n$ and $r = 0, 1, \cdots, m$.

Proof. By Theorem 5.7 we now know that *degree* $\overline{\mathcal{A}_n} = m^{\frac{n^2-n}{2}}$ if m is an odd integer.

By using the map T in (64) together with Corollary 4.5 it is easy to prove that all elements mentioned in the theorem are central. Denote by Z the subalgebra generated by these central elements. By the observation

$$\begin{aligned} & X_{i,1}(X_{i,1}^{m-1} X_{i+1,2}^{m-1} \cdots X_{n,n-i+1}^{m-1} X_{1,n-i+2} X_{2,n-i+3} \cdots X_{i-1,n}) \\ &= X_{i,1}^m (X_{i+1,2}^{m-1} \cdots X_{n,n-i+1}^{m-1} X_{1,n-i+2} X_{2,n-i+3} \cdots X_{i-1,n}), \end{aligned} \tag{67}$$

it follows that $Q(Z) \otimes_Z \overline{\mathcal{A}_n}$ is spanned by

$$\{\Pi_{i=1,j=2}^n X_{i,j}^{s_{i,j}} \mid 0 \le s_{i,j} \le m-1\} \tag{68}$$

over $Q(Z)$. Hence

$$\dim(Q(Z) \otimes \overline{\mathcal{A}_n}) \le m^{n^2-n}. \tag{69}$$

Since $\dim(Q(C(\overline{\mathcal{A}_n})) \otimes \overline{\mathcal{A}_n} = m^{n^2-n}$, we get $Q(Z) = Q(C(\overline{\mathcal{A}_n}))$. This proves that as a field, $Q(C(\overline{\mathcal{A}_n}))$ is generated by the central elements mentioned in the theorem. Let K be the subgroup of $(\mathbb{Z})^{n^2}$ which, when the latter is viewed as the $n \times n$ matrix algebra over $\mathbb{Z}$, is generated by $m \cdot E_{i,j}$, $\sum_{l=1}^n E_{l,l}$, $s \cdot \sum_{i=1}^{n-t+1} E_{t,i} + (m-s) \sum_{j=1}^{t-1} E_{j,n-t+j+1}$ for $i, j, t = 1, 2, \cdots, n$. It is easy to see that $K \cap (\mathbb{Z}_+)^{n^2}$ is a sub-semigroup of K generated by the above elements. Hence, $C(\overline{\mathcal{A}_n}) = Z$. This completes the proof. □

The above discussion now enables us to calculate the center of $\mathcal{A}_n$.

Theorem 6.2. *If q is a primitive mth-root of unity, m an odd positive integer, then the center $C(\mathcal{A}_n)$ of $\mathcal{A}_n$ is generated by*

(70)
$$\{Z_{i,j}^m, \det_q, (\det_q(t))^r \tau((\det_q(n-t))^{m-r}) \mid i,j,t = 1, \cdots, n \text{ and } r = 0, 1, \cdots, m\}.$$

Proof. Let C_0 be the subalgebra generated by the elements in (70). We use induction on the order of the leading term to prove that any $Y \in C(\mathcal{A}_n)$ belongs to C_0: we have that $T(Y) \in C(\overline{\mathcal{A}_n})$. Hence there is an element $Y' \in C_0$ with the same leading term as Y. Thus, the leading term of $Y - Y'$ is smaller than that of Y and hence $Y - Y' \in C_0$ and then also $Y \in C_0$. □

In a similar way one obtains

Theorem 6.3. *If m is even, $m = 2m'$, then the center $C(\mathcal{A}_n)$ is generated by $Z_{i,j}^m$, $\det_q$, $\det_q(t)^s(\tau\det_q(n-t))^{m-s}$, $Z_{k,i}^{m'}Z_{k,j}^{m'}Z_{l,i}^{m'}Z_{l,j}^{m'}$, and $\Pi_{i=1}^n Z_{i,i}^{m'}$, where $i,j,k,l,t = 1, 2, \cdots, n$ and $s = 0, 1, \cdots, m$.*

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