

THE CENTER OF $\mathcal{U}_q(\mathfrak{n}_\omega)$.

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ABSTRACT. We determine the center of a localization of $\mathcal{U}_q(\mathfrak{n}_\omega) \subseteq \mathcal{U}_q^+(\mathfrak{g})$ by the covariant elements (non-mutable elements). Here, \mathfrak{g} is any finite-dimensional complex Lie algebra and ω is any element in the Weyl group W . The non-zero complex parameter q is assumed not to be a root of unity. The center is determined by the null space of $1 + \omega$. Another family of quadratic algebras is also considered.

1. INTRODUCTION

The topics of quantum groups, quantized function algebras, quantized matrix algebras, and quantum cluster algebras have since long been seen to be intrinsically interwoven.

The groundbreaking research of Drinfeld ([10],[11]) and Jimbo ([26],[27]) was followed by deep results of Lusztig ([36],[38]), Kashiwara ([28],[29]). Then Levendorskii and Soibelman ([35], [33]) and later de Concini and Procesi ([8]) added the quadratic algebra side to this distinguished family. With the advent of the cluster algebras of Fomin and Zelevinsky ([14]) and Berenstein-Zelevinsky quantized cluster algebras ([3]) many new dimensions were added to the function algebra side.

Through many years, quantized function algebras have attracted a lot of attention ([8], [7], [9], [12], [16], [19], [21] [34], [35], [36], [39], [42], and many others). Many special examples were considered in the beginning, but also general families have more recently been considered ([15]).

The current research has its focus on the quadratic algebra side. It utilizes fundamental results in ([3]) and ([15]).

In certain families of examples ([22],[23]), it was seen that certain (signed) permutation matrices contained much information about the quantized matrix algebras. The topic of this article is to explain exactly the reason for that, while at the same time giving the full description of the centers. That we thereby also obtain an insight into very algebraic properties of quantized function algebras, even specializing these to roots of unity, is clear, but will not be pursued in this article. Here, we assume throughout that q is a not roots of unity.

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An important tool in our investigation is a family of quantized minors introduced by Berenstein and Zelevinsky ([3]). Later C. Geiss, B. Leclerc, and J. Schröer ([15]) have modified these in a way that turns out to be exactly suitable for our needs.

Given an element w in the Weyl group W one may construct, using Lusztig ([37]), a family of elements $Z_1, \dots, Z_{\ell(w)} \subset \mathcal{U}_q^+$. It is a key result of Levendorskii and Soibelman's ([35], [33]) that, for $1 \leq i < j \leq \ell(w)$,

$$Z_i Z_j = q^{(\gamma_i, \gamma_j)} Z_j Z_i + \text{terms involving only elements } Z_k \text{ with } i \leq k \leq j.$$

These relations can be taken as the defining relations of $\mathcal{U}_q(\mathfrak{n}_w)$.

Procesi and de Concini later reproved this result and introduced the associated quasi-polynomial algebra $\overline{\mathcal{U}}_q(\mathfrak{n}_w)$ with generators $z_1, \dots, z_{\ell(w)}$, and relations

$$z_i z_j = q^{(\gamma_i, \gamma_j)} z_j z_i.$$

They proved that eg. the P.I. degree of $\mathcal{U}_q(\mathfrak{n}_w)$ could be determined from this much simpler algebra.

The success of the present endeavor rests on the choice of a good basis of the associated quasi Laurent algebra $\overline{\mathcal{L}}_q(\mathfrak{n}_w)$. While looking at specific cases ([23]) such a basis was found essentially as the “diagonals” of the quantized minors of Berenstein-Zelevinsky.

Consider the symplectic form $\overline{\mathbb{L}}$ defined by the relations above, that is, the skew symmetric form defined by

$$i < j : \overline{\mathbb{L}}_{i,j} = (\gamma_i, \gamma_j).$$

Recall that a symplectic form may be brought to a block diagonal form by using matrices with integer coefficients and determinant 1.

The center of $\overline{\mathcal{L}}_q(\mathfrak{n}_w)$ is given by the null space of $\overline{\mathbb{L}}$.

The second major step forward comes with the construction, quite explicitly, and while again using ideas from ([22],[23]), of a partial inverse $\overline{\mathbb{B}}$ to $\overline{\mathbb{L}}$.

As an aside, we mention that we actually construct a quantum seed.

With these steps taken, the center of $\overline{\mathcal{L}}_q(\mathfrak{n}_w)$ is easily determined.

The final step towards determining the center of $\mathcal{L}_q(\mathfrak{n}_w)$ comes when one realizes that $\overline{\mathbb{L}}$ actually is also the symplectic form \mathbb{L} for a certain family of q commuting quantized minors and likewise $\overline{\mathbb{B}}$ is expressible in terms of these minors and their inverses and hence we get a compatible pair \mathbb{L}, \mathbb{B} .

As mentioned in the abstract, the center is given by the null space of

$$1 + \omega.$$

More precisely, to each fundamental weight Λ_s there is a covariant element $C_s(\omega)$ and the center is given by those $\prod_s C_s^{n_s}$ for which

$$(1 + \omega)\left(\sum_s n_s \Lambda_s\right) = 0.$$

We discuss some special examples of this in Subsection 5.3, Section 6, and Section 8.

Here is a table of contents:

Sections 2: Background; saturated sets of positive roots, Section 3: The quadratic algebra structure of $\mathcal{U}_q(\mathfrak{n}_\omega)$. Section 4: Basics; a diagrammatic way of representing \mathfrak{n}_ω and $\Delta^+(\mathfrak{n}_\omega)$ is introduced. Section 5: The associated quasi-polynomial algebra and an example. Section 6 is a Diophantine interlude in which the centers are computed for some specific elements $w \in W$ in type A_n . Then in Section 7 the quantum minors of Berenstein-Zelevinsky are introduced, the twist by ([15]) is given and two series of what we call Levendorskii-Soibelman quadratic algebras are introduced. Finally, the way is paved for Section 8 in which the previous results are extended to the general setting for these series of quadratic algebras.

2. ON PARABOLICS

The origin of the following lies in A. Borel [4], and B. Kostant [32]. Other main contributors are [2] and [44]. See also [6]. We have also found ([43]) useful.

We consider a simple Lie algebra \mathfrak{g} . \mathbf{A} is the Cartan matrix of \mathfrak{g} and is assumed to be of finite type. Π denotes a fixed choice of simple roots, and \mathbf{E}_Π denotes the euclidean space spanned by the simple roots. The fundamental weights corresponding to the simple roots are denoted by Λ_i .

Definition 2.1. *Let $w \in W$. Set*

$$\Phi_w = \{\alpha \in \Delta^+ \mid w^{-1}\alpha \in \Delta^-\} = w(\Delta^-) \cap \Delta^+.$$

We have that $\ell(w) = \ell(w^{-1}) = |\Phi_w|$.

Definition 2.2. *A subset S of Δ^+ is saturated if whenever $\alpha, \beta \in S$ and $\alpha + \beta$ is a root, then $\alpha + \beta \in S$.*

Theorem 2.3 ([32]). *The map*

$$w \mapsto \Phi_w$$

defines a bijection between W and the set of all subsets $\Phi \subseteq \Delta^+$ for which both Φ and $\Delta^+ \setminus \Phi$ are saturated.

In passing we observe that, trivially, for a saturated set Φ , both Φ and $\Delta^+ \setminus \Phi$ correspond to nilpotent subalgebras.

We will from now on set $\Phi_w = \Delta^+(w)$.

We will consider nilpotent quantized enveloping algebras of the form

$$\mathcal{U}_q(\mathfrak{n}_\omega),$$

where ω is an arbitrary element in the Weyl group W , and \mathfrak{n}_ω is the quantized nilpotent defined by the roots $\alpha \in \Phi_w$ (This is $\mathfrak{n}_{\omega^{-1}}$ of ([15]). It is convenient for us, also with an eye to forthcoming investigations, to assume that we are working with a fixed parabolic \mathfrak{p} with a Levi decomposition

$$\mathfrak{p} = \mathfrak{l} + \mathfrak{u}, \tag{1}$$

where \mathfrak{l} is the Levi subalgebra, and such that, on the classical level, $\mathfrak{n}_\omega \subseteq \mathfrak{u}$. There is no loss of generality in that.

Finally set

Definition 2.4.

$$\begin{aligned} W_p &= \{w \in W \mid \Phi_w \subseteq \Delta^+(\mathfrak{l})\} \\ W^p &= \{w \in W \mid \Phi_w \subseteq \Delta^+(\mathfrak{u})\}. \end{aligned}$$

W^p is a set of distinguished representatives of the right coset space $W_p \backslash W$.

It is well known (see eg ([43])) that any $w \in W$ can be written uniquely as $w = w_p w^p$ with $w_p \in W_p$ and $w^p \in W^p$.

3. THE QUADRATIC ALGEBRAS

Let $\omega = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_t}$ be an element of the Weyl group written in reduced form. Using his braid operators, given in a special case as

$$T_i(e_j) = \sum_{a+b=r} (-q_i)^{-b} e_i^{(a)} e_j e_i^{(b)},$$

where $r = -\langle h_i, \alpha_j \rangle$, Lusztig in ([37]) construct a sequence of elements $Z_1, \dots, Z_t \subseteq \mathcal{U}_q^+(\mathfrak{g})$. Specifically,

$$Z_i = T_{\omega_{i-1}}(E_{\alpha_i}), \quad i = 2, \dots, t, \quad \text{and} \quad Z_1 = E_{\alpha_1},$$

where, for each $i = 1, \dots, t$, $\omega_i = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_i}$. In particular, $\omega = \omega_t$. The weight of Z_i is given by $\gamma_i = \omega_{i-1}(\alpha_i)$. Here, and throughout, we use the notation of ([24]).

The following result is well known

Theorem 3.1 ([34],[33]). *Suppose that $1 \leq i < j \leq t$. Then*

$$Z_i Z_j = q^{(\gamma_i, \gamma_j)} Z_j Z_i + \text{terms involving only elements } Z_k \text{ with } i \leq k \leq j.$$

Our statement follows [24],[25]. Other authors, eg. [34], [15] have used the other Lusztig braid operators. The result is just a difference between q and q^{-1} . Proofs of this theorem which are more accessible are available ([8],[25]).

Proposition 3.2. $\mathcal{U}_q(\mathfrak{n}_\omega)$ is a quadratic algebra.

It is known that this algebra is isomorphic to the algebra of functions on $\mathcal{U}_q(\mathfrak{n}_\omega)$ satisfying the usual finiteness condition. It is analogously equivalent to the algebra of functions on $\mathcal{U}_q^-(\mathfrak{n}_\omega)$ satisfying a similar finiteness condition. See eg ([15]) and ([24]). We will not distinguish between these algebras.

4. BASIC STRUCTURE

Consider a fixed basis $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_R\}$ of Φ . Let us agree to write σ_{α_i} in W just as σ_i for $i = 1, \dots, R$.

We consider a fixed parabolic \mathfrak{p} with a Levi decomposition

$$\mathfrak{p} = \mathfrak{l} + \mathfrak{u} \tag{2}$$

with $\mathfrak{u} \neq \{0\}$.

Adapting to the language of [14], [3], and others, we will often label structures derived from ω_τ by the reduced word τ . The full structure with double words will not be required here.

Let $\omega^{\mathfrak{p}}$ be the maximal element in $W^{\mathfrak{p}}$. It is the one which maps all roots in $\Delta^+(\mathfrak{u})$ to Δ^- . (Indeed: To $\Delta^-(\mathfrak{u})$.) Let w_0 be the longest element in W and w_L the longest in the Weyl group of \mathfrak{l} , Then

$$w^{\mathfrak{p}} w_L = w_0. \tag{3}$$

Let $\omega_\tau = \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_r} \in W^{\mathfrak{p}}$ be fixed and written in a fixed reduced form. Then $\ell(\omega_\tau) = r$.

Set

$$\Delta^+(\omega_\tau) = \{\beta_{i_1}, \dots, \beta_{i_r}\}. \tag{4}$$

Consider $\alpha_{i_{r+1}} \in \Pi$ and set $w_1 = \omega_\tau \circ s_{i_{r+1}}$.

Case 1: $\omega_{\tau}(\alpha_{i_{r+1}}) = \gamma \in \Delta^+(\mathfrak{u})$. Then $\ell(w_1) = \ell(\omega_{\tau}) + 1$ and

$$\Delta^+(w_1) = \{\beta_{i_1}, \dots, \beta_{i_r}\} \cup \{\gamma\} \text{ and } \gamma = \beta_{r+1} = \omega_{\tau}(\alpha_{i_{r+1}}). \quad (5)$$

Thus, $w_1 \in W^p$.

Case 2: $\omega_{\tau}(\alpha_{i_{r+1}}) = -\gamma \in \Delta^-(\mathfrak{u})$. Then $\ell(w_1) = r - 1$ and

$$\Delta^+(w_1) = \{\beta_{i_1}, \dots, \beta_{i_r}\} \setminus \{\gamma\} \text{ and } \gamma = \beta_{i_s} \text{ for some } \beta_{i_s} \in \Delta^+(\omega_{\tau}). \quad (6)$$

We must always be in at least one of these cases since otherwise ω_{τ} would map all simple roots, hence $\Delta(\mathfrak{g})$, to $\Delta(\mathfrak{l})$. If ω_{τ} is maximal and α_i is a simple root such that $\omega_{\tau}(\alpha_i) \in \Delta^-(\mathfrak{u})$ then $\alpha_i \in \Delta^+(\mathfrak{u})$. It is easy to see that under the same assumptions, $\omega_{\tau}(\alpha_i) \in \Delta^-(\mathfrak{l})$ is not possible. Furthermore, if $\alpha_i \in \Delta^+(\mathfrak{l})$ then $\omega_{\tau}(\alpha_i) \in \Delta^+(\mathfrak{g})$. In conclusion, a maximal ω_{τ} maps $\Delta^+(\mathfrak{l})$ to $\Delta^+(\mathfrak{l})$ and $\Delta^+(\mathfrak{u})$ to $\Delta^-(\mathfrak{u})$. Thus, if ω_{τ} is maximal, $\omega_L \omega_{\tau} = \omega_{\tau} \omega_L = \omega_0$. Hence $\omega_{\tau} = \omega^p$.

It follows easily that we have the following conclusion: Let $\omega_{\tau} \in W^p$ with $\ell(\omega_{\tau}) = r$. Then we may write

$$\omega_{\tau} = s_{i_1} \circ \dots \circ s_{i_s} \circ \dots \circ s_{i_r} \text{ where for all } j = 1 \dots, r : w_s = s_{i_1} \circ \dots \circ s_{i_j} \in W^p. \quad (7)$$

Furthermore,

$$\Delta^+(\omega_{\tau}) = \{\beta_{i_1}, \dots, \beta_{i_s}, \dots, \beta_{i_r}\} \quad (8)$$

where for each i_s :

$$\beta_{i_s} = s_{i_1} \circ \dots \circ s_{i_{s-1}}(\alpha_{i_s}) \text{ for } s > 1, \text{ and } \beta_{i_1} = \alpha_{i_1}. \quad (9)$$

Moreover,

$$\omega_{\tau} = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_r} = \sigma_{\beta_r} \cdots \sigma_{\beta_2} \sigma_{\beta_1}. \quad (10)$$

From now on, q is a fixed element of \mathbb{C} which is not a root of unity, $w_{\tau} \in W^p$ is given with a fixed decomposition as in (10), and $\Delta^+(w_{\tau})$ is our universe.

Definition 4.1. Let \mathbf{b} denote the map $\Pi \rightarrow \{1, 2, \dots, R\}$ defined by $\mathbf{b}(\alpha_i) = i$. Let $\bar{\pi}_{\tau} : \{1, 2, \dots, r\} \rightarrow \Pi$ be given by

$$\bar{\pi}_{\tau}(j) = \alpha_{i_j}. \quad (11)$$

If $\bar{\pi}_{\tau}(j) = \alpha$ we say that α (or σ_{α}) occurs at position j in w_{τ} , and we say that $\bar{\pi}_{\tau}^{-1}(\alpha)$ are the positions at which α occurs in w . Set

$$\pi_{\tau} = \mathbf{b} \circ \bar{\pi}_{\tau}. \quad (12)$$

Let, for $1 \leq n \leq r$, $\omega_n = \sigma_{\alpha_{i_1}} \sigma_{\alpha_{i_2}} \cdots \sigma_{\alpha_{i_n}}$. Thus, we have a 1-dimensional presentation of the situation given by the (ordered) set $\{1, 2, \dots, r\}$.

The following 2-dimensional presentation is even more useful and informative:

Definition 4.2.

$$\mathbb{U}(\mathfrak{r}) = \{(s, t) \in \mathbb{N} \times \mathbb{N} \mid \exists n \text{ such that } s = \pi_{\mathfrak{r}}(n) \text{ and } \omega_n = \omega_1 \sigma_{i_n} \omega_2 \dots \omega_t \sigma_{i_n}\}. \quad (13)$$

In the above, it is understood that each $\omega_i \in W \setminus \{e\}$ is reduced and does not contain any σ_{i_n} .

We also identify $n \leftrightarrow (s, t)$ (and $\beta_n \leftrightarrow \beta_{s,t}$) if n, s, t are connected as above. We denote the identifications between $\{1, 2, \dots, r\}$ and $\mathbb{U}(\mathfrak{r})$ simply as $n \leftrightarrow (s, t)$. (And $\beta_n \leftrightarrow \beta_{s,t}$)

We define a map π_{ω_n} for such ω_n in analogy with that of $\pi_{\mathfrak{r}}$.

If $\omega_{\mathfrak{r}} = \omega_m \tilde{\omega}$ and $\omega_m = \omega_n \hat{\omega}$ with $\omega_n, \omega_m \in W^P$ and all Weyl group elements reduced, we say that $\omega_n < \omega_m$ if $\hat{\omega} \neq e$.

Definition 4.3. If $n \leftrightarrow (s, t)$ and $m \leftrightarrow (c, d)$ we define

$$(s, t) < (c, d) \Leftrightarrow \omega_{s,t} < \omega_{c,d}. \quad (14)$$

For a fixed $s \in \{1, 2, \dots, R\}$ we let s_{max} denote the maximal such t . This is the number of times σ_s occurs in $\omega_{\mathfrak{r}}$. We then have

$$\mathbb{U}(\mathfrak{r}) = \{(s, t) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq s \leq R \text{ and } 1 \leq t \leq s_{max}\}. \quad (15)$$

Finally, notice that if $(s, t) \in \mathbb{U}(\mathfrak{r})$ then we may construct a subset $\mathbb{U}(s, t)$ of \mathbb{U} by the above recipe, replacing $\omega_{\mathfrak{r}}$ by $\omega_{s,t}$. In this subset t is maximal.

5. THE QUASI-POLYNOMIAL ALGEBRA

5.1. The first definitions and computations.

Definition 5.1. $\overline{\mathcal{M}}_q(\mathfrak{r})$ denotes the \mathbb{C} -algebra generated by elements $\{z_j; j = 1, \dots, r\}$ indexed by the elements β_j defined as above and with relations

$$z_i z_j = q^{(\beta_i, \beta_j)} z_j z_i \text{ if } i < j. \quad (16)$$

We let $\overline{\mathcal{L}}_q(\mathfrak{r})$ denote the associated quasi-Laurent algebra, and we let $\overline{\mathcal{X}}_q(\mathfrak{r})$ denote the center of $\overline{\mathcal{L}}_q(\mathfrak{r})$. We will also label the generators by $z_{s,t}$ as discussed in Section 4.

Let $s \in \text{Im}(\pi_\tau)$. It is then straightforward to see that

$$-\omega(\Lambda_s) + \Lambda_s = \beta_{s,1} + \beta_{s,2} + \cdots + \beta_{s,s_{\max}}. \quad (17)$$

Definition 5.2. Let $s \in \text{Im}(\pi_\tau)$. We define the element $\overline{C}_s(\mathbf{r})$ in the quasi-polynomial algebra $\overline{\mathcal{M}}_q(\mathbf{r})$ by

$$\overline{C}_s(\mathbf{r}) = z_{s,1} z_{s,2} \cdots z_{s,s_{\max}}. \quad (18)$$

Proposition 5.3. The following holds for all $s \in \text{Im}(\pi_\tau)$ and all $(a, b) \in \mathbb{U}(\mathbf{r})$:

$$z_{a,b} \overline{C}_s(\mathbf{r}) = q^{-(\beta_{a,b}, (1+\omega_\tau)(\Lambda_s))} \overline{C}_s(\mathbf{r}) z_{a,b}, \quad (19)$$

Proof. We will be using repeatedly that $\beta_{s,t} = -\omega_{s,t}(\alpha_{i_s})$. Consider a decomposition $\omega_\tau = \omega_A \sigma_{\alpha_{i_\ell}} \omega_B$ with $\pi_\tau(\ell) = a \neq s$ and suppose that $\beta_{s,t} < \beta_{a,b} < \beta_{s,t+1}$ for some t . It is here understood that $\ell = (a, b)$. The elements ω_A, ω_B of course depend on ℓ , indeed, $\omega_A \sigma_{i_\ell} = \omega_\ell$. Then

$$\begin{aligned} z_\ell z_{s,t+1} \cdots z_{s,s_{\max}} &= q^{(\beta_{i_\ell}, (-\omega_\tau + \omega_A)(\Lambda_s))} z_{s,t+1} z_{s,t+2} \cdots z_{s,s_{\max}} z_\ell \\ &= q^{(\alpha_{i_\ell}, (-\sigma_{\alpha_{i_\ell}} \omega_B(\Lambda_s) + \Lambda_s))} z_{s,t+1} z_{s,t+2} \cdots z_{s,s_{\max}} z_\ell \end{aligned} \quad (20)$$

$$= q^{(\alpha_{i_\ell}, \omega_B(\Lambda_s))} z_{s,t+1} \cdots z_{s,s_{\max}} z_\ell. \quad (21)$$

Similarly,

$$z_\ell z_{s,1} \cdots z_{s,t} = q^{-(\alpha_{i_\ell}, \omega_A^{-1}(\Lambda_s))} z_{s,1} \cdots z_{s,t} z_\ell. \quad (22)$$

The statement then follows directly. To complete this part of the picture, we need to consider $z_\ell < z_{s,1}$ and $z_\ell > z_{s,s_{\max}}$

The case $z_\ell < z_{s,1}$ easily results in the exponent $(\beta_\ell, (1 - \omega_\tau)(\Lambda_s))$, but here $(\beta_\ell, \Lambda_s) = 0$. The case $z_\ell > z_{s,s_{\max}}$ gives an exponent $-(\beta_\ell, (1 - \omega_\tau)(\Lambda_s))$, and here $(\beta_\ell, \omega_\tau(\Lambda_s)) = 0$.

Next, we observe the following simple formulas, where ω_A and ω_B now are determined by (s, t) :

$$z_{s,t} z_{s,t+1} \cdots z_{s,s_{\max}} = q^{-1 + (\alpha_s, \omega_B(\Lambda_s))} z_{s,t+1} \cdots z_{s,s_{\max}} z_{s,t}, \quad (23)$$

and, similarly,

$$z_{s,t} z_{s,1} \cdots z_{s,t-1} = q^{1 - (\alpha_s, \omega_A^{-1}(\Lambda_s))} z_{s,1} \cdots z_{s,t-1} z_{s,t}. \quad (24)$$

These formulas also hold at the extreme positions of $z_{s,1}$ and $z_{s,s_{\max}}$, where either $\omega_A = 1$ or $\omega_B = 1$.

So, indeed for any simple root $\alpha = \pi_\omega(\ell)$ and decomposition $\omega_\tau = \omega_\ell \omega_B = \omega_A \sigma_\alpha \omega_B$, we get, for the corresponding z_ℓ ,

$$z_\ell \overline{C}_s(\mathbf{r}) = q^{(\alpha, (\omega_B - \omega_A^{-1})(\Lambda_s))} \overline{C}_s(\mathbf{r}) z_\ell, \quad (25)$$

which, by the previous definitions is equivalent to the statement in the proposition. \square

If $s \notin \text{Im}(\pi_\tau)$, we set $\overline{C}_s(\mathbf{r}) = 1$. To any linear combination

$$\sum_i n_i \Lambda_i$$

with integer coefficients we may consider the element in the quasi-Laurent algebra

$$(\overline{C}_{s_1}(\mathbf{r}))^{n_1} \dots (\overline{C}_{s_k}(\mathbf{r}))^{n_k}. \quad (26)$$

If $\overline{C}_s(\mathbf{r}) = 1$ we set $n_s = 0$.

Let

$$S_\tau = \text{Span}\{\alpha_s \mid s \in \text{Im}(\pi_\tau)\}.$$

It is obviously invariant under ω_τ . We view tacitly the elements Λ_s as restricted to this space.

Proposition 5.4. *Let n_1, \dots, n_k be integers and let $s_1, \dots, s_k \in \text{Im}(\pi_\tau)$.*

$$(\overline{C}_{s_1}(\mathbf{r}))^{n_1} \dots (\overline{C}_{s_k}(\mathbf{r}))^{n_k} \in \overline{\mathcal{Z}}_q(\mathbf{r}) \Leftrightarrow (1 + \omega_\tau) \left(\sum_{j=1}^k n_j \Lambda_{s_j} \right) = 0.$$

Proof. The commutation between this and any z_ℓ is given by

$$z_\ell (\overline{C}_{s_1}(\mathbf{r}))^{n_1} \dots (\overline{C}_{s_k}(\mathbf{r}))^{n_k} = q^{-(\beta_\ell, (1 + \omega_\tau)(\sum_i n_i \Lambda_i))} (\overline{C}_{s_1}(\mathbf{r}))^{n_1} \dots (\overline{C}_{s_k}(\mathbf{r}))^{n_k} z_\ell. \quad (27)$$

\square

This actually determines the center as will be proved below after some preparation.

Remark 5.5. *Since $1 + \omega_\tau$ is an integer matrix, there is an \mathbb{R} basis of the null space given by vectors with integer coordinates in the basis of fundamental weights.*

5.2. More definitions and computations. The center of the quasi-polynomial algebra.

We first make a very useful observation:

Lemma 5.6. *Let $\alpha_i \in \Phi$. Then*

$$(s_i + 1)(\Lambda_i) + \sum_{j \neq i} a_{ji}(\Lambda_j) = 0.$$

Proof. Let $\Lambda = (s_i + 1)(\Lambda_i) + \sum_{j \neq i} a_{ji}(\Lambda_j)$. We have that

$$\langle \Lambda_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}.$$

It follows easily that $\Lambda(\alpha_i) = 0$ and it only remains to consider $\Lambda(\alpha_k)$ for $k \neq i$ and $a_{ki} \neq 0$. (recall that $s_i(\Lambda_i) = \Lambda_i - \alpha_i$). Now

$$\Lambda(\alpha_k) = (-\alpha_i + a_{ki}\Lambda_k) \left(\frac{2\alpha_k}{(\alpha_k, \alpha_k)} \right) = 0$$

by the definition of a_{ki} . □

Definition 5.7. *Let $(s, t) \in \mathbb{U}(\mathfrak{r})$. Set*

$$\overline{M}_{s,t}^\downarrow = z_{s,1} \dots z_{s,t}. \quad (28)$$

This element has weight

$$p_{s,t} = -w_{s,t}(\Lambda_s) + \Lambda_s = \beta_{s,1} + \dots + \beta_{s,t}. \quad (29)$$

Proposition 5.8. *Let $(a, b), (s, t) \in \mathbb{U}(\mathfrak{r})$. Then*

$$z_{a,b} \overline{M}_{s,t}^\downarrow = q^{E_{a,b}} \overline{M}_{s,t}^\downarrow z_{a,b}, \quad (30)$$

where the exponent $E_{a,b}$ is given as follows: *CASE 1* : $(a, b) \leq (s, t)$:

$$E = -(\beta_{a,b}, (1 + w_{s,t})(\Lambda_s)). \quad (31)$$

CASE 2 : $(a, b) > (s, t)$:

$$E = -(\beta_{a,b}, (1 - w_{s,t})(\Lambda_s)). \quad (32)$$

Proof. CASE 1 is equivalent to Proposition 5.3 and CASE 2 follows by very similar arguments. □

As a special case we get (because we here are in Case 1 only)

Corollary 5.9.

$$\forall (s, t) \in \mathbb{U}(\mathfrak{r}), \forall j \in \text{Im}(\pi_{\mathfrak{r}}) : \overline{M}_{(s,t)}^{\downarrow} \overline{C}_j(\mathfrak{r}) = q^{-((1-\omega_{s,t})(\Lambda_s), (1+\omega_{\mathfrak{r}})(\Lambda_j))} \overline{C}_j \overline{M}_{s,t}^{\downarrow}. \quad (33)$$

The following formula, which is only seemingly more general, and in which $(a, b), (c, d) \in \mathbb{U}(\mathfrak{r})$ is also useful.

Corollary 5.10.

$$1 \leq (s, t) \leq (c, d) \Rightarrow \overline{M}_{s,t}^{\downarrow} \overline{M}_{c,d}^{\downarrow} = q^{-((1-\omega_{s,t})(\Lambda_s), (1+\omega_{c,d})(\Lambda_c))} \overline{M}_{c,d}^{\downarrow} \overline{M}_{s,t}^{\downarrow}. \quad (34)$$

Definition 5.11. We set $\omega_{s,0}(\Lambda_s) = \Lambda_s$ and $\overline{M}_{s,0}^{\downarrow} = 1$ for all $s \in \text{Im}(\pi_{\mathfrak{r}})$.

Notice that $w_{s,s_{max}}(\Lambda_s) = \omega_{\mathfrak{r}}(\Lambda_s)$.

For a fixed, but arbitrary, $(s, t) \in \mathbb{U}(\mathfrak{r})$ and $a \in \text{Im}(\pi_{\omega_{s,t}})$ set $\overline{p}_{\mathfrak{r}}(a, s, t) = \max(\pi_{\omega_{s,t}}^{-1}(a))$. If $a \notin \text{Im}(\pi_{\omega_{s,t}})$ we set $\overline{p}_{\mathfrak{r}}(a, s, t) = 0$.

Then we define

Definition 5.12.

$$\overline{F}(s, t) = \overline{M}_{s,t}^{\downarrow} \overline{M}_{s,t-1}^{\downarrow} \prod_{a_{js} < 0} \left(\overline{M}_{j, \overline{p}_{\mathfrak{r}}(j, s, t)}^{\downarrow} \right)^{a_{js}}. \quad (35)$$

The key result is:

Proposition 5.13. The following holds holds for all $(c, d), (s, t) \in \mathbb{U}(\mathfrak{r})$:

$$z_{c,d} \overline{F}(s, t) = q^{E_{c,d}} \overline{F}(s, t) z_{c,d}, \quad (36)$$

where

$$E_{c,d} = -(\beta_{c,d}, \alpha_s) + 2(\Lambda_s, \alpha_s) \delta_{(c,d), (s,t)}. \quad (37)$$

Furthermore,

$$\overline{M}_{c,d}^{\downarrow} \overline{F}(s, t) = q^{G_{c,d}} \overline{F}(s, t) \overline{M}_{c,d}^{\downarrow}, \quad (38)$$

where

$$G_{c,d} = -((\beta_{c,1} + \cdots + \beta_{c,d}), \alpha_s) + 2(\Lambda_s, \alpha_s) (\delta_{(c,1), (s,t)} + \cdots + \delta_{(c,d), (s,t)}). \quad (39)$$

Proof. Our strategy is to sum up the exponents of the the q -commutation for the minors of weights $-\omega_{s,t} \circ \sigma_s(\Lambda_s) + \Lambda_s$, $-\omega_{s,t}(\Lambda_s) + \Lambda_s$ and $a_{js}(-\omega_{s,t}(\Lambda_j) + \Lambda_j)$, $j \neq s$ and then use the formulas of Proposition 5.8 in combination with Lemma 5.6. This goes well and yields exponent $E = -(\beta_{c,d}, \alpha_s)$ in all cases where $(c, d) \neq (s, t)$ since there is no mixing of Case 1 and Case 2 positions for the various minors. When $(c, d) = (s, t)$ we do mix Case 1 and Case 2, but only in reference to the minor of weight $-\omega_{s,t} \circ \sigma_s(\Lambda_s) + \Lambda_s$. Here we notice that $2\Lambda_s + \sum_{j \neq s} a_{js}\Lambda_j = (1 - \sigma_s)\Lambda_s = \alpha_s$ by the same lemma. The results follow. \square

Now we combine two (adjacent) minors based on the same $\alpha_{\pi_\omega(n)}$ but of opposite signs. In the following we have to assume that $t < s_{max}$ since we work with both t and $t + 1$.

The following follows then easily:

Lemma 5.14.

$$\begin{aligned} & \forall (c, d) \in \mathbb{U}(\mathfrak{r}), \forall (s, t) \in \mathbb{U}(\mathfrak{r}) \text{ satisfying } t < s_{max}, : \\ & z_{c,d} \overline{F}(s, t) \overline{F}(s, t + 1)^{-1} = q^{\nabla_{c,d}} \overline{F}(s, t) \overline{F}(s, t + 1)^{-1} z_{c,d} \end{aligned} \quad (40)$$

where

$$\nabla_{c,d} = 2(\Lambda_s, \alpha_s) \delta_{(c,d),(s,t)} - 2(\Lambda_s, \alpha_s) \delta_{(c,d),(s,t+1)}. \quad (41)$$

We can thus construct some elements, that commute with everything except a single element:

Definition 5.15. $\forall (s, t) \in \mathbb{U}(\mathfrak{r})$ satisfying $t < s_{max}$ set

$$\overline{B}_{s,t} = \overline{F}(s, t) \overline{F}(s, t + 1)^{-1}. \quad (42)$$

Proposition 5.16. *Whenever $\overline{B}_{s,t}$ is defined, the following holds for all $(c, d) \in \mathbb{U}(\mathfrak{r})$:*

$$\overline{M}_{c,d}^\downarrow \overline{B}_{s,t} = q^{2(\Lambda_s, \alpha_s) \delta_{(c,d),(s,t)}} \overline{B}_{s,t} \overline{M}_{c,d}^\downarrow.$$

Definition 5.17. *We define a symplectic form $\overline{\mathbb{L}}_0$ by, for $\beta_i, \beta_j \in \Delta^+(\omega_{\mathfrak{r}})$*

$$\beta_i < \beta_j \Rightarrow \overline{\mathbb{L}}_{0,ij} = (\beta_i, \beta_j) \quad (43)$$

We let $\overline{\mathbb{L}}$ denote the symplectic form defined in terms of the elements $\overline{M}_{c,d}^\downarrow$:

$$\overline{M}_{c,d}^\downarrow \overline{M}_{s,t}^\downarrow = q^{\overline{\mathbb{L}}_{(c,d),(s,t)}} \overline{M}_{s,t}^\downarrow \overline{M}_{c,d}^\downarrow. \quad (44)$$

We will also find it convenient to introduce an auxiliary sesquilinear form in which the elements $\overline{M}_{c,d}^\downarrow$ form an orthonormal basis.

We define a matrix \mathbb{A} by

$$\mathbb{A}_{(c,d),(s,t)} = \begin{cases} 1 & \text{if } s = c, t = 1, 2, \dots, d \\ 0 & \text{else} \end{cases}. \quad (45)$$

It follows from (29) that

Lemma 5.18. $\bar{\mathbb{L}}$ is obtained, as a form, by a change of basis by the formula

$$\bar{\mathbb{L}} = \mathbb{A}\bar{\mathbb{L}}_0\mathbb{A}^t. \quad (46)$$

It now follows from Proposition 5.16 that we have

Corollary 5.19. For any $(s, t) \in \mathbb{U}(\mathfrak{r})$ for which $t < s_{max}$, the (basis) vector $2(\Lambda_s, \alpha_s)\bar{M}_{s,t}^\downarrow$ belongs to the image of $\bar{\mathbb{L}}$.

Since

$$\text{Range}(\bar{\mathbb{L}})^\perp = \text{Ker}(\bar{\mathbb{L}}) \quad (47)$$

it follows that the kernel is contained in the space spanned by the elements $\bar{C}_s(\mathfrak{r})$. Since q is not a root of unity, the center of $\bar{\mathcal{M}}_q(\mathfrak{r})$ is contained in the kernel of $\bar{\mathbb{L}}$. Combining Corollary 5.19 with Proposition 5.4, we have proved the following:

Theorem 5.20. The center of $\bar{\mathcal{L}}_q(\mathfrak{r})$ is given by the kernel of $(1 + \omega_{\mathfrak{r}})$ on $S_{\mathfrak{r}}$.

Proof. We need only consider the eigenspaces of $\omega_{\mathfrak{r}}$ (in the span of the covariant elements), and here it is only the ± 1 eigenspaces that merit attention: Since we are working with matrices with integer coefficients, we see that the mentioned eigenspaces are spanned by elements with integer coefficients. Observe also that $1 + \omega_{\mathfrak{r}} = 2 - (1 - \omega_{\mathfrak{r}})$. Consider

$\bar{C}_+(\mathfrak{r}) = \prod_{i \in \text{Im}(\pi_{\mathfrak{r}})} (\bar{C}_i(\mathfrak{r}))^{n_i} \in \bar{\mathcal{M}}_q(\mathfrak{r})$ for which

$$(1 - \omega_{\mathfrak{r}}) \left(\sum_{i \in \text{Im}(\pi_{\mathfrak{r}})} n_i \Lambda_i \right) = 0. \quad (48)$$

Then, by Proposition 5.3, for all (a, b) ,

$$z_{a,b} \bar{C}_+(\mathfrak{r}) = q^{2\beta_{a,b}, (\sum_{i \in \text{Im}(\pi_{\mathfrak{r}})} n_i \Lambda_i)} \bar{C}_+(\mathfrak{r}) z_{a,b}. \quad (49)$$

Recall that q is not a root of unity. It follows that $\bar{C}_+(\mathfrak{r})$ can only commute with all $z_{a,b}$ if $\sum_{i \in \text{Im}(\pi_{\mathfrak{r}})} n_i \Lambda_i = 0$. \square

Remark 5.21. *The kernel of $1 + \omega_{\mathfrak{r}}$ is of course unchanged if we enlarge $S_{\mathfrak{r}}$ to \mathbf{E}_{Π} by the elements Λ_i that are left invariant by $\omega_{\mathfrak{r}}$.*

5.3. Example: The full nilpotent - the longest element $w_0 \in W$. If $\omega = \omega_0$ is the longest word in the Weyl group, we know that it is either -1 or implemented by a diagram symmetry of order 2. Indeed, we get $\omega_0 = -1$ in all cases except

$$A_n, D_{2n+1}(n > 1), \text{ and } E_6. \quad (50)$$

(See eg. exercises 18, 32 in Chapter 2 in ([31]).)

Proposition 5.22. *If the simple Lie algebra is not in the list (50), the center of $\overline{U}_q(\mathfrak{n})$ is generated by the elements $\overline{C}_s(\mathfrak{r})$, with $s = 1 \dots, R$. In the following we use the numbering of simple roots from ([17]). In type \mathbf{A}_{ℓ} , with simple roots $\alpha_1, \alpha_2, \dots, \alpha_{\ell}$ we have that*

$$\omega_m(\Lambda_{\alpha_s}) = -\Lambda_{\alpha_{\ell-s+1}}, \quad (51)$$

and the center is generated by the elements

$$\begin{aligned} &\overline{C}_s(\mathfrak{r})\overline{C}_{\ell-s+1}(\mathfrak{r}) \text{ for } 2s \neq \ell + 1 \\ &\overline{C}_s(\mathfrak{r}) \text{ for } 2s = \ell + 1 \end{aligned}$$

In type $\mathbf{D}_{2\ell+1}$, the center is generated by the elements

$$\overline{C}_s(\mathfrak{r})(s = 1, \dots, 2\ell - 1), \text{ and } \overline{C}_{2\ell}(\mathfrak{r})\overline{C}_{2\ell+1}(\mathfrak{r}). \quad (52)$$

In type \mathbf{E}_6 , the center is generated by the elements

$$\overline{C}_1(\mathfrak{r})\overline{C}_6(\mathfrak{r}), \overline{C}_2(\mathfrak{r})\overline{C}_5(\mathfrak{r}), \overline{C}_2(\mathfrak{r}), \text{ and } \overline{C}_4(\mathfrak{r}). \quad (53)$$

6. DIOPHANTINE INTERLUDE

Consider a simple Lie algebra of type $A_{a+b+c-1}$ and let $N = a + b + c$. The simple roots are

$$\prod = \{e_i - e_{i+1} \mid i = 1, 2, \dots, N - 1\}. \quad (54)$$

We choose a Levi subalgebra defined by

$$\Sigma = \prod \setminus \{(e_a, e_{a+1}), (e_{a+b}, e_{a+b+1})\}. \quad (55)$$

We let I_n denote the $n \times n$ identity matrix. We are interested in

$$\omega_{\mathfrak{r}} = \begin{pmatrix} 0 & 0 & I_c \\ 0 & I_b & 0 \\ I_a & 0 & 0 \end{pmatrix}, \quad (56)$$

but introduce a more general family of $(a + b + c) \times (a + b + c)$ matrices

$$w([a, \varepsilon_a]; [b, \varepsilon_b]; [c, \varepsilon_c]) = I_{a+b+c} + \begin{pmatrix} 0 & 0 & \varepsilon_c I_c \\ 0 & \varepsilon_b I_b & 0 \\ \varepsilon_a I_a & 0 & 0 \end{pmatrix}. \quad (57)$$

Here ε_d denotes an integer, $d = a, b, c$, such that $\varepsilon_d^2 = 1$. We will always start with $\varepsilon_a = \varepsilon_b = \varepsilon_c = 1$ but we will later encounter more general signs when we perform Gaussian Elimination moves. In retrospect, it can be seen that all the matrices encountered here satisfy $\varepsilon_a \varepsilon_b \varepsilon_c = 1$. Our task is to try to determine $\text{corank}(w([a, \varepsilon_a]; [b, \varepsilon_b]; [c, \varepsilon_c]))$.

For arbitrary signs we have, by easy Gaussian moves, the following reductions:

- For $a \geq b + c$ ¹:

$$\text{corank}(w([a, \varepsilon_a]; [b, \varepsilon_b]; [c, \varepsilon_c])) = \text{corank}(w([a-b-c, \varepsilon_a]; [b, -\varepsilon_a \varepsilon_b]; [c, -\varepsilon_a \varepsilon_c]))$$

- For $c \geq a + b$:

$$\text{corank}(w([a, \varepsilon_a]; [b, \varepsilon_b]; [c, \varepsilon_c])) = \text{corank}(w([a, -\varepsilon_c \varepsilon_a]; [b, -\varepsilon_c \varepsilon_b]; [c-a-b, \varepsilon_c]))$$

- For $b + c \geq a > c$:

$$\text{corank}(w([a, \varepsilon_a]; [b, \varepsilon_b]; [c, \varepsilon_c])) = \text{corank}(w([a, -\varepsilon_b \varepsilon_a]; [b-(a-c), \varepsilon_b]; [c, -\varepsilon_b \varepsilon_c]))$$

- For $a + b \geq c > a$:

$$\text{corank}(w([a, \varepsilon_a]; [b, \varepsilon_b]; [c, \varepsilon_c])) = \text{corank}(w([a, -\varepsilon_b \varepsilon_a]; [b-(c-a), \varepsilon_b]; [c, -\varepsilon_b \varepsilon_c]))$$

There are many more moves which we shall not pursue here. Also, there is an evident tensorial nature to the set-up in the sense that a common factor of a, b , and c will also turn up as a factor in the resulting corank.

Now introduce variables $p = a + b$ and $q = b + c$. Then the four moves above can be reformulated, in the same order of appearance, and with the same stipulations, as follows:

- $(p, q, b) \rightarrow (p - q, q, b)$.
- $(p, q, p) \rightarrow (p, q - p, b)$.
- $(p, q, b) \rightarrow (q, 2q - p, b - (p - q))$.
- $(p, q, b) \rightarrow (q, 2q - p, b - (q - p))$.

Thus, the moves preserve the lattice generated by (p, q) in \mathbb{Z}^2 , and the equivalence class of b modulo the \mathbb{Z} lattice generated by p and q .

¹We will allow a matrix I_d to occur in the list even if $d = 0$. This just means that the row and column containing such a symbol is to be removed. In the first item, this means that if $a = b + c$ we get right hand side which is a $(b + c) \times (b + c)$ matrix. The sign ε_a together with $a - b - c$ is also removed in this case.

6.1. Special case: $b = 1$. We focus on the cases that can be computed without using the moves changing the b . It is straightforward to see that after a certain number of a or c moves (number 1 and 2 on the list) we get (induction), with the notation from above,

$$\begin{aligned} \text{corank}(w([a, 1]; [b, 1]; [c, 1])) = & \quad (58) \\ \text{corank}(w([X_L p - Y_L q - b, (-1)^{X_R + Y_R - 1}]; [b, (-1)^{X_L + X_R + Y_L + Y_R}]; & \\ [Y_R q - X_R p - b, (-1)^{X_L + Y_L - 1}])) . & \end{aligned}$$

Set $\Delta = g.c.d.(p, q)$. Assume $\Delta > 1$. Then we can only end in a configuration $(a', 1, a')$ with $\Delta = a' + 1$. To have a corank equal to $a' = \Delta - 1$ we need the signs $(-1)^{X_R + Y_R - 1}$ and $(-1)^{X_L + Y_L - 1}$ to equal. This happens exactly when $X_L + X_R + Y_L + Y_R$ is even. If it is odd, then we get a (-1) with the middle piece which cancels the $b = 1$, so in this case, the corank is 1. Write $p = x\Delta$ and $q = y\Delta$. Then the condition can be stated as follows:

$$\begin{aligned} x + y \text{ even} : \text{corank} &= \Delta - 1, & (59) \\ x + y \text{ odd} : \text{corank} &= 1. \end{aligned}$$

Now assume $\Delta = 1$. Then we end up with a configuration $X_L p - Y_L q = 1 = Y_R q - X_R p$, though the two equations, naturally, need not appear simultaneously. Thus $X_L + X_R + Y_L + Y_R$ has the same parity as $p + q$ and we get a non-trivial corank exactly if $p + q$ is odd. Here, the corank is 1.

Thus, we can state in all cases:

Proposition 6.1. *The corank $crk(1 + \omega_\tau)$ of the matrix*

$$I_{a+1+c} + \begin{pmatrix} 0 & 0 & I_c \\ 0 & I_1 & 0 \\ I_a & 0 & 0 \end{pmatrix} \quad (60)$$

is given as follows: Let $p = a + 1$, $q = c + 1$, let $\Delta = g.c.d.(p, q)$, and write $p = x\Delta$ and $q = y\Delta$. Then the condition can be stated as follows:

$$\begin{aligned} x + y \text{ even} : \text{crk}(1 + \omega_\tau) &= \Delta - 1, & (61) \\ x + y \text{ odd} : \text{crk}(1 + \omega_\tau) &= 1. \end{aligned}$$

Remark 6.2. *If we instead start with (for $a > c$) $(a, b + a - c, c)$ then the first b move yields $(a, (-1), 1, 1, c, (-1))$ and all the following a, c moves preserve these signs. This more general case is thus covered too.*

7. THE QUANTUM PARABOLICS

7.1. Quantized minors. Following a construction of classical minors by S. Fomin and A. Zelevinsky [14], the last mentioned and A. Berenstein have introduced a family of quantized minors $\Delta_{u \cdot \lambda, v \cdot \lambda}$ in [3]. These are elements of the quantized coordinate ring $\mathcal{O}_q(G)$.

The element $\Delta_{u \cdot \lambda, v \cdot \lambda}$ is determined by two non-zero vectors $\underline{w}_u, \underline{w}_v$ of weights $u \cdot \lambda$ and $v \cdot \lambda$ respectively, in the finite-dimensional module determined by the highest weight vector \underline{w}_λ . We will always assume that $u \leq v$. Then $\Delta_{u \cdot \lambda, v \cdot \lambda}(x) = (\underline{w}_u, x \underline{w}_v)_\lambda$.

The construction in ([3]) is even more sophisticated than what can be glimpsed here, since it is given by “divided powers” generators, say $X^{(k)}$. Since we are working with q generic, all such powers are set equal to (a constant times) X^k .

Lemma 7.1. [3] *The element $\Delta_{u \cdot \lambda, v \cdot \lambda}$ indeed depends only on the weights $u \cdot \lambda, v \cdot \lambda$, not on the choices of u, v and their reduced words.*

Theorem 7.2 (A version of Theorem 10.2 in [3]). *For any $\lambda, \mu \in P^+$, and $s, s', t, t' \in W$ such that*

$$\ell(s's) = \ell(s') + \ell(s), \ell(t't) = \ell(t') + \ell(t),$$

the following holds:

$$\Delta_{s's\lambda, t'\lambda} \Delta_{s'\mu, t't\mu} = q^{(s\lambda|\mu) - (\lambda|t\mu)} \Delta_{s'\mu, t't\mu} \Delta_{s's\lambda, t'\lambda}.$$

Definition 7.3. *A Levendorskii-Soibelman quadratic algebra is a quadratic algebra with generators W_β labeled by a subset $\Phi_0^+ \subseteq \Phi^+$ of the positive roots together with a total ordering $<$ of Φ_0^+ such that for all $\beta_1, \beta_2 \in \Phi_0^+$,*

$$\beta_1 < \beta_2 \Rightarrow W_{\beta_1} W_{\beta_2} = q^{\alpha_{ij}} W_{\beta_2} W_{\beta_1} + \text{L.O.T.s.} \quad (62)$$

The L.O.T.s stand for certain sums of products of elements W_β where $\beta_1 < \beta < \beta_2$, and thus are of lower order (hence the abbreviation) in the lexicographical order. The exponents α_{ij} are assumed to be integers.

Definition 7.4. *Let $\mathcal{A}_q(\mathfrak{n}_\omega) \subset \mathcal{U}^{\geq}(\mathfrak{n}_\omega) := \mathcal{U}(\mathfrak{n}_\omega) \cdot \mathcal{U}^0$ be a Levendorskii-Soibelman quadratic algebra with a set of generators W_β where β runs through the roots of $\mathfrak{u} = \mathfrak{n}_\omega$ and ordered according to ω . We will say that $\mathcal{A}_q(\mathfrak{n}_\omega)$ splits $\mathcal{U}^{\geq}(\mathfrak{n}_\omega)$, or that we have a splitting furnished by $\mathcal{A}_q(\mathfrak{n}_\omega)$ if*

$$\mathcal{U}^{\geq}(\mathfrak{n}_\omega) = \mathcal{A}_q(\mathfrak{n}_\omega) \times_s \mathcal{U}^0. \quad (63)$$

We now return to the setting of Section 4 and consider an element $\omega_{\mathbf{r}} \in W^p$.

Definition 7.5. $\mathcal{M}_q(\mathbf{r})$ denotes the algebra generated by the previously defined elements $Z_{\beta_j} \equiv Z_{s,t}$.

It is well known that this is a Levendorskii-Soibelman quadratic algebra ([34]). More detailed proofs have been given in [8] and [25]. This is equal to the algebra in [15] except $q \rightarrow q^{-1}$.

One considers in [15], and transformed to our terminology, more general elements

$$D_{\xi,\eta} = \Delta_{\xi,\eta} K^{-\eta}. \quad (64)$$

The family $D_{\xi,\eta}$ satisfies equations analogous to those in Theorem 7.2 subject to the same restrictions on the relations between the weights. The elements

$$W_{s,t} = Z_{s,t} K^{\omega_{s,t}(\Lambda_s)} \quad (65)$$

thus generate a quadratic algebra,

Definition 7.6. $\mathcal{W}_q(\mathbf{r})$ denotes the algebra generated by the elements $W_{s,t}$.

Its center will be discussed later.

Furthermore, a quasi-polynomial algebra $\overline{\mathcal{W}}_q(\mathbf{r})$ is associated to $\mathcal{W}_q(\mathbf{r})$ in the following obvious way: The generators are

$$w_{s,t} = z_{s,t} K^{\omega_{s,t}(\Lambda_s)}, \quad (66)$$

where the elements $z_{s,t}$ are the old ones of Section 5.

The relations are easily

$$(s,t) < (c,d) \Rightarrow w_{s,t} w_{c,d} = q^{R_{s,t}^{c,d}} w_{c,d} w_{s,t}, \quad (67)$$

where

$$R_{s,t}^{c,d} = (\beta_{s,t}, \beta_{c,d}) + (\Lambda_s, \beta_{c,d}) - (\Lambda_c, \beta_{s,t}) - \left(\sum_{i=1}^t \beta_{s,i}, \beta_{c,d} \right) + \left(\sum_{j=1}^d \beta_{c,j}, \beta_{s,t} \right). \quad (68)$$

We end this section with the following easy consequence of Theorem 7.2:

Proposition 7.7. Each element $\Delta_{\Lambda_s, \omega_{\mathbf{r}}(\Lambda_s)}$, $s \in \text{Im}(\pi_{\mathbf{r}})$, quasi-commutes with all elements of the form

$$\Delta_{\omega_{c,d}(\Lambda_c), \omega_{c,f}(\Lambda_c)}. \quad (69)$$

A similar result holds for the elements $D_{\Lambda_s, \omega_{\mathbf{r}}(\Lambda_s)}$.

It is proved in ([15]) that the elements $Z_{a,b}$ are of the form

$$D_{\omega_{c,d}(\Lambda_c), \omega_{c,f}(\Lambda_c)} \quad (70)$$

for some elements $(c, d), (c, f) \in \mathbb{U}(\mathfrak{r})$. Hence we conclude in particular, cf. (64), that

Proposition 7.8. *The elements $C_s(\mathfrak{r}) = D_{\Lambda_s, \omega_{\mathfrak{r}}\Lambda_s}$ q -commute with all elements in the algebra $\mathcal{M}_q(\mathfrak{n}_\omega)$.*

Definition 7.9. *An element $C \in \mathcal{M}_q(\mathfrak{n}_\omega)$ that q -commutes with all elements in the algebra $\mathcal{M}_q(\mathfrak{n}_\omega)$ is said to be covariant.*

8. THE CENTERS OF THE NILPOTENT PART OF QUANTIZED PARABOLICS

It follows easily from Theorem 7.2 that we have the following formula:

Proposition 8.1 ([15]). *Let $(s, t) \leq (c, d)$, then*

$$D_{\Lambda_s, \omega_{s,t}\Lambda_s} D_{\Lambda_c, \omega_{c,d}\Lambda_c} = q^{((1-\omega_{s,t})(\Lambda_s), (1+\omega_{c,d})(\Lambda_c))} D_{\Lambda_c, \omega_{c,d}\Lambda_c} D_{\Lambda_s, \omega_{s,t}\Lambda_s}. \quad (71)$$

There is a sign difference in the q exponent in relation to Corollary 5.9, otherwise they are identical.

Secondly, the critical step in the proof of Proposition 5.13 is where one considers $\omega_{s,t} = \omega' \circ \omega_s$ in the pairs

$$(\Lambda_s, \omega' \Lambda_s), ((\Lambda_s, \omega' \circ \sigma_s \Lambda_s), a_{sj}(\Lambda_j, \omega' \Lambda_j) \text{ (for all } a_{sj} < 0)). \quad (72)$$

Combined with equation (34) we obtained the important Proposition 5.16. Now we have equation (71) which is the same hence furnishes the same conclusions.

Thirdly, the elements $D_{\Lambda_s, \omega_{s,t}\Lambda_s}$ are ordered in the same way as the elements $\Delta_{\Lambda_s, \omega_{s,t}\Lambda_s}$, and as the elements $Z_{s,t}$ and $z_{s,t}$.

Fourthly, it follows easily from [15, Lemma 11.4 and Corollary 12.4] that the elements $C_s(\mathfrak{r})$ are regular and that all elements $M_{c,d}^\downarrow = D_{\Lambda_c, \omega_{c,d}\Lambda_c}$ are polynomials in the generators $Z_{s,t}$. We also know by proposition 7.8 that they generate a quasi-polynomial subalgebra. We can then invert them and consider now the algebra

$$\tilde{\mathcal{M}}_q(\mathfrak{n}_\omega) = \mathcal{M}_q(\mathfrak{n}_\omega)[C_1(\mathfrak{r})^{-1}, \dots, C_R(\mathfrak{r})^{-1}]. \quad (73)$$

We can then state, cf. Remark 5.21,

Theorem 8.2. *The center of the algebra $\tilde{\mathcal{M}}_q(\mathfrak{n}_\omega)[C_1(\mathfrak{r})^{-1}, \dots, C_R(\mathfrak{r})^{-1}]$ is given by the kernel of $(1 + \omega_\tau)$.*

8.1. Example: The full nilpotent - the longest element $w_0 \in W$.
We can now revisit Subsection 5.3, using the same numbering, and easily obtain:

Proposition 8.3. *If the simple Lie algebra is not in the list (50), the center of $\mathcal{U}_q(\mathfrak{n})$ is generated by the elements $C_s(\mathfrak{r})$, with $s = 1 \dots, R$. In type \mathbf{A}_ℓ , with simple roots $\alpha_1, \alpha_2, \dots, \alpha_\ell$ we have that*

$$\omega_m(\Lambda_{\alpha_s}) = -\Lambda_{\alpha_{\ell-s+1}}, \quad (74)$$

and the center is generated by the elements

$$\begin{aligned} &C_s(\mathfrak{r})C_{\ell-s+1}(\mathfrak{r}) \text{ for } 2s \neq \ell + 1 \\ &C_s(\mathfrak{r}) \text{ for } 2s = \ell + 1 \end{aligned}$$

In type $\mathbf{D}_{2\ell+1}$, the center is generated by the elements

$$C_s(\mathfrak{r})(s = 1, \dots, 2\ell - 1), \text{ and } C_{2\ell}(\mathfrak{r})C_{2\ell+1}(\mathfrak{r}). \quad (75)$$

In type \mathbf{E}_6 , the center is generated by the elements

$$C_1(\mathfrak{r})C_6(\mathfrak{r}), C_2(\mathfrak{r})C_5(\mathfrak{r}), C_2(\mathfrak{r}), \text{ and } C_4(\mathfrak{r}). \quad (76)$$

Example 8.4. *For the $a \times b$ quantized matrix algebra matrix we get, provided $a > 1$ and $b > 1$,*

$$\dim(\mathcal{C}_{a \times b}) = 2 + (g.c.d.(a - 1, b - 1) - 1). \quad (77)$$

In case $a \geq b = 1$, the algebra is commutative; $\dim(\mathcal{C}_{a \times 1}) = a$.

We now turn our attention to the Berenstein-Zelevinski algebra $\mathcal{W}_q(\mathfrak{r})$. We will for simplicity assume that

$$\omega_\tau = \omega_1 \omega_2 \dots \omega_\ell, \quad (78)$$

where

$$\forall i, j : Z_{i,j} = -\omega_1 \dots \omega_j(\alpha_i). \quad (79)$$

This is satisfied for all types except D_n, E_6, E_7, E_8 .

The precise meaning of this condition is that each ω_i can be written as a product of pairwise different reflexions $\sigma_{i,k}$. Furthermore, if $\sigma_{i,k}$ takes part in ω_i then it also takes part in ω_{i-1} (etc.). So, in particular,

$$Im(\pi_{\omega_i}) \subseteq Im(\pi_{\omega_{i-1}}).$$

This gives a natural ordering, for each j , of the elements $Z_{i,j}$ as well as the elements

$$\Delta_{s,t} = \Delta_{\Lambda_s, \omega_{s,t} \Lambda_s}, \quad (80)$$

where $\omega_{s,t} = \omega_1 \dots \omega_t$. We also remark that if ω_1 has σ_x to the right; $\omega_1 = \omega^{(1)} \sigma_x$ with $\omega^{(1)}$ reduced, then $\Delta_{\Lambda_x, \omega_1 \Lambda_x}$ commutes with all other elements $\Delta_{s,t}$. Any rewriting of ω_1 that results in another reflection σ_y at the right end will similarly give rise to a central element.

We see immediately that we have the following result:

Proposition 8.5. *Under the above assumption on $\omega_{\mathfrak{r}}$, there is a non-trivial center.*

For the algebras $\tilde{\mathcal{M}}_q(\mathfrak{n}_\omega)$ it may happen that there is a trivial center.

Let $\mathcal{L}_\Delta(\mathfrak{r})$ denote the Laurent quasi-polynomial algebra generated by the elements in the set $\{\Delta_{s,t} \mid (s,t) \in \mathbb{U}(\mathfrak{r})\}$, and let \mathbb{L}_Δ denote the symplectic form defined by these elements. We may also assume that there is an auxiliary inner product in which these elements form an orthonormal set. A very important difference between the current case and the previous is that the elements $\Delta_{\Lambda_s, \Lambda_s}$ are not equal to the constant function. This gives some modifications. However, using the assumption (78) it is straightforward to see that one may make a construction similar to the case of $\bar{\mathcal{L}}_q(\mathfrak{r})$, especially Definition 5.12, to obtain:

Lemma 8.6. *Let (i, j_0) be given with $j_0 \neq 1$. Then there exists an element ∇_{i,j_0} in $\mathcal{L}_\Delta(\mathfrak{r})$ such that*

$$\Delta_{s,t} \nabla_{i,j_0} = q^{\Gamma_{i,j_0,s,t}} \nabla_{i,j_0} \Delta_{s,t} \quad (81)$$

with

$$\Gamma_{i,j_0,s,t} = \begin{cases} 0 & \text{if } s \neq i \\ -(\Lambda_i, \alpha_i) & \text{if } s = i \text{ and } j_0 \leq t \leq i_{max} \\ +(\Lambda_i, \alpha_i) & \text{if } s = i \text{ and } t < j_0 \end{cases} . \quad (82)$$

By considering elements of the form $\nabla_{s,t} \nabla_{s,t-1}^{-1}$ we get (q generic)

Lemma 8.7.

$$\forall i, \forall 1 < j_0 < i_{max} : \Delta_{i,j_0} \in \text{Range}(\mathbb{L}). \quad (83)$$

Combining Lemmas 8.6 and 8.7 we also get

Lemma 8.8.

$$\forall i : \Delta_{i,1}^{-1} \Delta_{i,i_{max}} \in \text{Range}(\mathbb{L}). \quad (84)$$

This, similarly to the previous case, easily implies

Lemma 8.9. *The center is generated by elements of the form*

$$\prod_i (\Delta_{i,1} \Delta_{i,i_{max}})^{n_i}. \quad (85)$$

It is understood that in case $\Delta_{i,1} = \Delta_{i,i_{max}}$ we omit one of these factors. Naturally, a central element must commute with all elements $\Delta_{s,1}$. On the other hand, it follows that it commutes with all elements of the form ∇_{s,t_0} ($t_0 > 1$). Thus, it is also sufficient that it commutes with the elements $\Delta_{s,1}$. In other words, the following must hold:

Lemma 8.10. *The element $\prod_i (\Delta_{i,1} \Delta_{i,i_{max}})^{n_i}$ is central if and only if*

$$\forall s : (\Lambda_s, (1 - \omega_1^{-1} \omega) \sum_i n_i \Lambda_i) = 0. \quad (86)$$

Consider the algebra

$$\tilde{\mathcal{W}}_q(\mathbf{n}_\omega) = \mathcal{W}_q(\mathbf{n}_\omega)[(\Delta_{1,1} \Delta_{1,1_{max}})^{-1}, \dots, (\Delta_{R,1} \Delta_{R,R_{max}})^{-1}]. \quad (87)$$

We can then state

Theorem 8.11. *The center of the algebra $\tilde{\mathcal{W}}_q(\mathbf{r})$ is given by the kernel of $(1 - \omega_1^{-1} \omega_{\mathbf{r}})$ on $S_{\mathbf{r}}$.*

Let $\check{C}_i(\mathbf{r}), i = 1, \dots, R$ denote the elements in $\mathcal{U}_q(\mathbf{n}_\omega)$ that correspond to the functions $C_i(\mathbf{r}), i = 1, \dots, R$.

In view of our discussion in Section 2 we can also state for any $\omega \in W$:

Theorem 8.12. *The center of the algebra $\mathcal{U}_q(\mathbf{n}_\omega)[\check{C}_1(\mathbf{r})^{-1}, \dots, \check{C}_R(\mathbf{r})^{-1}]$ is given by the kernel of $(1 + \omega)$.*

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