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A Classification of the Unitarizable Highest Weight Modules for Affine Lie Superalgebras

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We consider loop algebras over the basic classical Lie superalgebras, and central extensions thereof, and determine the full set of unitarizable highest weight modules. This is essentially equivalent to what is promised in the title. We allow the Borel subalgebra and the involution to be completely general except for a natural compatibility condition.

1. INTRODUCTION

The class we look at is the class of affine Lie superalgebras, i.e., the contragredient Lie superalgebras of finite growth (restricted by the assumption that the Cartan matrix is symmetrizable). These have been classified in [7]. However, we will take our starting point by looking at loop algebras based on basic classical Lie superalgebras and their (one-dimensional) central extensions. In fact, we start out by classifying these extensions. Our result, which seems to agree with the ones stated in [1], is that besides the usual one, based on the bilinear form, only $\mathbb{C}[t^1, t^{-1}] \otimes A(n, n)$ has a non-trivial one. The existence of the latter is not so surprising, since already $sl(n, n)$ is a non-trivial extension of $A(n, n)$.

It turns out almost immediately that the center, at least the one corresponding to the usual central extension, must act trivially, and this is in fact the reason we can make the statement about the contragredient Lie superalgebras of finite growth.

In many ways, the present investigation should be seen as analogous, and a continuation, of the study [3]. There, affine Kac–Moody algebras were investigated, and it was established that, besides the integrable representations, any unitarizable highest weight module must act trivially on the center. For any affine Lie algebra, there is always such a class, namely the so-called elementary representations, obtained by point evaluation in a unitary representation of the underlying simple finite-dimensional Lie algebra. Besides this, there is still another class for $A_n^{(1)}$ based on the

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real form $su(n, 1)$, and with this, the set of unitarizable highest weight modules for affine Kac–Moody algebras is exhausted.

Because of these observations, it follows that there can only be unitary representations (other than those corresponding to point evaluation) of the loop algebra based on a basic classical Lie superalgebra $\mathcal{G}_f$ if it has an A_n or a $\mathbb{C}$ summand in its even part. For this reason, following two sections giving some general definitions and background results, we start our investigation by looking at $sl^{(1)}(n, m)$. In many ways, the treatment here is analogous to the one in [3] though, of course, everything is further complicated by the fact that there may be two measures involved and because of the exterior algebra aspect. A key observation is that it is, for most purposes, enough to treat a special class of representations, the so-called "scalar" representations. These correspond, roughly speaking, to measures having no isolated atoms. The fact that they do not have these atoms make them a good deal easier to work with. One can then later on add a finite number of atoms by a tensor product procedure.

Having finished this, we finally turn to the exceptional cases in Section 5. It turns out that there are quite a lot of representations here for all the cases $D(2, 1, \alpha)$, $F(4)$, $G(3)$, $B(0, n)$, $B(1, n)$, $B(m, 1)$, $C(n)$, and $D(m, 1)$. Furthermore, there are special subcases for the algebras $C(2)$, $B(1, 1)$, and $B(0, 1)$, and also $D(2, n)$ and $D(3, n)$ have to be considered. We have decided to give the most detailed proofs for the algebras $D(2, 1, \alpha)$. The other cases follow by a mixture of the results for $sl^{(1)}(n, m)$ and the methods used for $D(2, 1, \alpha)$.

2. Superalgebras

2.1. *Basics.* We recall briefly the basic definitions. For details, see [5, 8]. A *Lie superalgebra* $\mathcal{G}$ is a complex vector space which is a direct sum of two subspaces $\mathcal{G}_0$ and $\mathcal{G}_1$, called the *even* and the *odd* part, respectively, endowed with a super Lie bracket $[\cdot, \cdot] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ satisfying

$$\begin{aligned} [x, y] &= -(-1)^{(\deg x)(\deg y)} [y, x], \qquad \text{and} \\ [x, [y, z]] &= [[x, y], z] + (-1)^{(\deg x)(\deg x)} [y, [x, z]]. \end{aligned} \tag{1}$$

Here, $\deg x = 0$ (1) if and only if x is even (odd).

A bilinear form B on $\mathcal{G}$ is called *supersymmetric* if

$$B(x, y) = (-1)^{(\deg x)(\deg y)} B(y, x), \tag{2}$$

and *skew-supersymmetric* if there is an extra minus on the right-hand side of (2). It is called *consistent* if

$$B(x, y) = 0 \qquad \text{for} \quad x \in \mathcal{G}_0 \quad \text{and} \quad y \in \mathcal{G}_1, \tag{3}$$

and *invariant* if

$$B([x,y], z) = B(x, [y, z]). \tag{4}$$

The Lie superalgebra is then called *basic classical* if

- $\mathscr{G}$ is simple.
- $\mathscr{G}_0$ is reductive.
- There exists a non-degenerate supersymmetric consistent invariant bilinear form B on $\mathscr{G}$.

The word *classical* is used when the action of $\mathscr{G}_0$ on $\mathscr{G}_1$ is completely reducible. Clearly, the existence of a form B on $\mathscr{G}$ with the above properties implies that $\mathscr{G}$ is classical. The list of basic classical Lie superalgebras is the following as determined by V. Kac [5]:

- Simple Lie algebras.
- $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$, $D(2, 1, \alpha)$, $F(4)$, and $G(3)$.

The *loop algebra* $L(\mathscr{G})$ over $\mathscr{G}$ is by definition $L(\mathscr{G}) = \mathbb{C}[t^1, t^{-1}] \otimes \mathscr{G}$. The commutator is given by

$$[t^r \otimes r, t^s \otimes y] = t^{r+s}[x, y]. \tag{5}$$

It follows that the part of the enveloping algebra $\mathscr{U}(L(\mathscr{G}))$ that corresponds to the odd generators can be taken to be $\bigwedge (\mathbb{C}[t^1, t^{-1}] \otimes (\mathscr{G})_1)$. Naturally, the part corresponding to a single odd generator can be identified with the span of the spaces of antisymmetric Laurent polynomials on $\underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{r}$ $(r = 0, 1, ...)$.

We now describe briefly the so-called *contragredient Lie superalgebras*. These arise from generalized Cartan matrices and are directly given in terms of generators and relations. The subclass of *affine* Lie superalgebras given by the requirement of finite growth and that the Cartan matrix be symmetrizable was determined in [7]. The above list, due to Kac, is exactly the family of finite-dimensional contragredient Lie superalgebras (with symmetrizable Cartan matrices). Moreover, the full class of affine algebras turns out to consist entirely of central extensions of (twisted) loop algebras based on Kac's list above. This is one reason for taking our point of departure at the loop algebras over the basic classical Lie superalgebras and for beginning with a study of central extensions. Another reason will be apparent when we see what restrictions unitarity impose.

Since it will be a major ingredient in the following, we finish this subsection by recalling the definition of $sl(n, m)$ and $A(n, m)$ [5],

$$sl(m,n)=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in M(m+n+2,\mathbb{C}) \mid \alpha \text{ is an } (m+1)\times(m+1),\right.$$
$$\beta \text{ an } (m+1)\times(n+1),\ \gamma \text{ an } (n+1)\times(m+1),$$
$$\left.\delta \text{ an } (n+1)+(n+1) \text{ matrix, and } \operatorname{tr}\alpha=\operatorname{tr}\delta\right\}. \tag{6}$$

When $n\neq m$, $A(n,m)\stackrel{\text{def}}{=}sl(n,m)$, but $sl(n,n)$ has a one-dimensional center, and $A(n,n)$ is defined to be the quotient by this center.

2.2. *Central Extensions.* Let $\mathscr{G}$ be an arbitrary Lie superalgebra. Suppose we have a central extension $\tilde{\mathscr{G}}=\mathscr{G}\oplus\mathbb{C}\cdot c$ defined by a Lie bracket $[\cdot,\cdot]'$,

$$[x\oplus\lambda_1\cdot c,\ y\oplus\lambda_2\cdot c]'=[x,y]\oplus F(x,y)\cdot c \tag{7}$$

for some bilinear map $F:\mathscr{G}\times\mathscr{G}\to\mathbb{C}$, where $[\cdot,\cdot]$ is the given Lie bracket on $\mathscr{G}$. It follows easily from (1) that

LEMMA 2.1. *F is skew-supersymmetric and satisfies*

$$F(x,[y,z])=F([x,y],z)+(-1)^{\deg x\deg y}F(y,[x,z]) \qquad \forall x,y,z\in\mathscr{G}. \tag{8}$$

Conversely, any such F defines a central extension by formula (7).

PROPOSITION 2.1. *Suppose $\tilde{\mathscr{G}}$ is a central extension of a loop algebra over a basic classical Lie superalgebra $\mathscr{G}_f$, the extension being given by a bilinear map $F:\mathscr{G}\times\mathscr{G}\to\mathbb{C}$ as in* (7). *Then either*

$$F(t^i x, t^k y)=\mathfrak{m}(t^{i+k})\,l([x,y]), \tag{9}$$

where $\mathfrak{m}$ is a linear functional on the Laurent polynomials and l is a linear functional on $\mathscr{G}_f$, or $\mathscr{G}_f=A(n,n)$ and

$$F(t^i x_1, t^j x_2)=\mathfrak{m}(t^{i+j})\operatorname{tr}(\beta_1\gamma_2+\beta_2\gamma_1) \tag{10}$$

for $x_i\equiv\left(\begin{smallmatrix}\alpha_i & \beta_i\\ \gamma_i & \delta_i\end{smallmatrix}\right)(\operatorname{mod}\mathbb{C}\cdot I_{2n+2})$, $i=1,2$, and $\mathfrak{m}$ a linear functional on the space of Laurent polynomials, or

$$F(t^i x_1, t^j x_2)=\delta_{i+j,0}\cdot i\cdot B(x_1,x_2), \tag{11}$$

where B denotes the relevant bilinear form on $\mathscr{G}_f$ (up to a non-zero constant).

Proof. Suppose there exists an i such that

$$F(x,t^i y)\neq 0. \tag{12}$$

Define

$$F_f(x,y)=F(x,t^i y). \tag{13}$$

Then it follows easily, by looking at $F_f(x,[y,z])=F(x,[y,t^i z])$, that F_f satisfies (8), i.e., that it defines a central extension of the underlying finite-dimensional superalgebra.

Furthermore it follows directly that

$$F(t^i x, t^k y) = \mathfrak{m}(t^{i+k})\, F_f(x, y), \tag{14}$$

where $\mathfrak{m}$ is a linear functional on the Laurent polynomials.

Let us look closer at F_f. It consists of two pieces, an even and an odd, which independently satisfy (8). The even piece lives on even × even and on odd × odd, whereas the odd pieces lives on even × odd.

Regarding central extensions of finite-dimensional superalgebras, it is well known that since the second cohomology group $H^2(\mathscr{G}_f, \mathbb{C})$ vanishes for a *simple* finite dimensional Lie algebra, on even × even we can at most have $F_f(x, y) = l([x, y])$ for some linear functional l on $\mathscr{G}_f$. This formula can clearly be extended to odd × odd to define an even (and trivial) central extension. The $\tilde{F}_f$ given by the formula $\tilde{F}_f(x, y) = l([x, y])$ can then be subtracted from F_f resulting in a new central extension which vanishes on even × even. It follows that this new form, on odd × odd, has got to be invariant under the action of the even part. A quick glance down the list of classical Lie superalgebras together with well-known results from invariant theory immediately gives that only for $sl(n, m)$ (and $C(n)$) can this occur, and exactly with the indicated form. Indeed, in all the cases where the odd part is irreducible under the even part, the even part consists of at least two pieces, and it never happens that these pieces at the same time leave invariant forms of the same kind of symmetry. Turning to the remaining cases, it is easy to see that when $n \neq m$ the expression $\mathrm{tr}(\beta_1 \gamma_2 + \beta_2 \gamma)$ in (10) again corresponds to a linear functional evaluated on the commutator. When $n = m$, the expression is exactly the one which gives $sl(n, n)$ as a (non-trivial) central extension of $A(n, n)$. In the case of $C(n)$, the restriction of the expression for $sl(1, 2n-3)$ gives the invariant symmetric form on the odd space (based on the symplectic form), which again leads to a trivial extension.

Let us finally assume that the F_f is odd. A small computation shows that on even × odd, we must similarly have

$$B(x, z) = l([x, z]), \tag{15}$$

for some linear functional l on the odd space. But this is a trivial extension.

Now assume that $\forall i : B_i(x, y) = F(x, t^i y) = 0$. It follows by a small computation (cf. below) that there must be an i such that

$$B(x, y) = F(t^{-i} x, t^i y) \neq 0. \tag{16}$$

Then,

$$\begin{aligned} B(x, [y, z]) &= F(t^i x, [y, t^{-i} z]) \\ &= F(t^i [x, y], t^{-i}) + 0 \\ &= B([x, y], z). \end{aligned} \tag{17}$$

In other words, B is *invariant*. To see that B is non-zero, and to further analyze the dependence upon i, it suffices to consider

$$c_{i,j} = F(t^i x_\alpha, t^j x_{-\alpha}) \tag{18}$$

for arbitrary root vectors x_α and $x_{-\alpha}$ since we can construct B from these. It follows easily that

$$c_{i+s,j+s} = c_{i,j} + c_{s,i+j-s} \qquad \forall s \in \mathbb{Z}. \tag{19}$$

It follows that $c_{a,k-a} = a \cdot c_{1,k-1}$; hence, in particular, $k \cdot c_{1,k-1} = 0$. All in all we conclude that

$$c_{i,j} = \delta_{i+j,0} \cdot i \cdot c_{1,-1}, \tag{20}$$

which gives the usual central extension corresponding to affine super algebras. ∎

Remark 2.1. Clearly, in the case of (9), the extension of the loop algebra, being expressible as a linear functional on the commutator, is trivial. The central extension of $\mathbb{C}[t^1, t^{-1}] \otimes A(n, n)$ defined in (10), on the other hand, is not trivial. We denote this extension by $A'''(n, n)$ and shall return to it at the end of Section 4. As usual, $\mathscr{G}_f^{(1)}$ denotes the central extension of $L(\mathscr{G}_f)$ defined by (11), and $sl^{(1)}(n, n)$ is defined analogously.

Remark 2.2. The set of twisted affined Lie superalgebras does not contribute anything non-trivial.

2.3. *Borel Subalgebras and Highest Weight Modules.* In the setup of loop algebras, or central extensions thereof, that we are going to consider, there will be the notion of a Cartan subalgebra $\mathfrak{H}$ and of the set of roots Δ, and $\mathscr{G}$ will be the sum of its root spaces together with $\mathfrak{H}$. As in the case of affine Kac–Moody algebras [3] we then make the following definitions

DEFINITION 2.1. A subset Δ^+ of Δ is called a *set of positive roots* if the following three properties hold:

(1) If $\alpha, \beta \in \Delta^+$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta^+$.

(2) If $\alpha \in \Delta$ then either α or $-\alpha$ belongs to Δ^+.

(3) If $\alpha \in \Delta^+$ then $-\alpha \notin \Delta^+$.

DEFINITION 2.2. Given a set Δ^+ of positive roots one associates to it the *Borel subalgebra*

$$\mathscr{B}^+ = \bigoplus_{\alpha \in \Delta^+ \cup \{0\}} \mathscr{G}_\alpha. \tag{21}$$

We shall not here classify all sets of positive roots, since it turns out not to be necessary to do so.

Naturally, we also have the notion of the opposite Borel subalgebra,

$$\mathscr{B}^- = \bigoplus_{\alpha \in \Delta^+ \cup \{0\}} \mathscr{G}_{-\alpha}. \tag{22}$$

Define $\mathscr{N}^+ = \mathscr{B}^+ \backslash \mathfrak{H}$ and $\mathscr{N}^- = \mathscr{B}^- \backslash \mathfrak{H}$.

DEFINITION 2.3. Let Λ be a linear functional on $\mathfrak{H}$. By a highest weight module of highest weight Λ, we mean a module V_Λ over the enveloping algebra $\mathscr{U}(\mathscr{G})$ generated by a non-zero vector v_Λ which further satisfies

$$\mathscr{N}^+ \cdot v_\Lambda = 0, \tag{23}$$
$$\forall h \in \mathfrak{H} : h \cdot v_\Lambda = \Lambda(h) \cdot v_\Lambda.$$

Just as in the case of affine Kac–Moody algebras, it becomes natural to consider generalized highest weight modules, which are defined in the obvious manner. We will return to these when looking at $sl(n, m)$.

2.4. *The Canonical Hermitian Form.* A map $\omega : \mathscr{G} \to \mathscr{G}$ is called an anti-involution if

$$\forall x, y \in \mathscr{G} : \omega([x, y]) = [\omega(y), \omega(x)]. \tag{24}$$

An anti-linear anti-involution ω of $\mathscr{G}$ is called *consistent* if for all $\alpha \in \Delta$:

$$\omega(\mathscr{G}_\alpha) = \mathscr{G}_{-\alpha}. \tag{25}$$

Let ω be a consistent anti-linear anti-involution of $\mathscr{G}$ and let π be a representation of $\mathscr{G}$ on a vector space V. Extend ω to $\mathscr{U}(\mathscr{G})$ in the obvious way.

DEFINITION 2.4. A hermitian form $\mathscr{H}$ on V is called *contravariant* (w.r.t. the given data) if

$$\forall u, v \in V, \forall x \in \mathscr{G} : \mathscr{H}(\pi(x)u, v) = \mathscr{H}(u, \pi(\omega(x))v). \tag{26}$$

If $\mathscr{H}$ is positive (semi-)definite, π is called *unitary* or *unitarizable*.

We then have

PROPOSITION 2.2. *Let V_Λ be a highest weight module and let ω be consistent. Assume that*

$$\forall h \in \mathfrak{H} : \Lambda(h) = \overline{\Lambda(\omega(h))}. \tag{27}$$

Then there is a unique contravariant form on V_Λ.

Proof. This follows just as in [3], including the actual construction of the hermitian form. ∎

Remark 2.3. We will always be in a situation where (27) applies. Furthermore, the results extends immediately to the generalized highest weight modules we consider later on.

3. UNITARITY

3.1. *c Has To Be Zero.* We will almost entirely be concerned with central extensions $\mathscr{G}_f^{(1)}$ of loop algebras over basic classical Lie superalgebras $\mathscr{G}_f$ where the extension is given as in (11). Let us first look at $A^{(1)}(n, m)$. The most general element in the *odd* part of $sl(n, m)$ has the form

$$\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & e_{1,1} & \cdots & e_{m+1,1} \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & e_{1,i} & \cdots & e_{m+1,i} \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & e_{1,n+1} & \cdots & e_{m+1,n+1} \\ f_{1,1} & \cdots & f_{1,i} & \cdots & f_{1,n+1} & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ f_{m+1,1} & \cdots & f_{m+1,i} & \cdots & f_{m+1,n+1} & 0 & \cdots & 0 \end{bmatrix}. \tag{28}$$

Throughout, we identify the variables $e_{i,j}$ and $f_{i,j}$ above with the corresponding matrices, that is, matrices as above but where the only non-zero entry is $e_{i,j}$ or $f_{i,j}$, respectively. Let us look at such a matching pair, $e_x = e_{i,j}$ and $f_x = f_{i,j}$. Moreover, let

$$h_x = [e_x, f_x] = e_x f_x + f_x e_x. \tag{29}$$

The Borel subalgebras we are looking at either have the form

$$B_{st}^+ = \text{Span}\{e_x, t^n e_x, t^n f_x \mid n > 0\},$$
$$B_{nst}^+ = \text{Span}\{t^n e_x \mid n \in \mathbb{Z}\}, \qquad \text{or}$$
$$B_{nst}^- = \text{Span}\{t^n f_x \mid n \in \mathbb{Z}\},$$

as far as the part of the odd space corresponding to the label x is concerned.

We now consider (generalized) highest weight modules of $A^{(1)}(n, m)$. These are by definition modules generated by a vector v_0 which satisfies:

- In the appropriate sense, v_0 is a highest weight vector for both $A_n^{(1)}$ and $A_m^{(1)}$.

- For the elements h_x defined above, v_0 is an eigenvector. Moreover,
  — In the case of B_{st}^+, $t^r h_x v_0 = 0$ for all $r > 0$.
  — In the case of B_{nst}^+ or B_{nst}^-, $t^r h_x v_0 = \mu_x(t^r)$ for some linear functional μ_x defined on the set of all Laurent polynomials.

LEMMA 3.1. *For $A^{(1)}(n, m)$, in case of unitarity, $c = 0$.*

Proof. Let us look at a highest weight module based on the first situation, and let v_x denote the highest weight vector. Furthermore, let the anti-linear anti-involution ω be determined by

$$\omega(t^r e_x) = \varepsilon_x \cdot t^{-r} f_x, \qquad \text{and} \qquad \omega(t^r f_x) = \varepsilon_x \cdot t^{-r} e_x, \tag{30}$$

where $\varepsilon_x^2 = 1$.

The inner product on the highest weight module is denoted by $(\cdot, \cdot)$. We then have

$$\forall n \geqslant 0 : ((t^n e_x) \cdot v_x, (t^n e_x) \cdot v_x) = \varepsilon_x \lambda_x + c \cdot n \cdot \varepsilon_x \cdot B(f_x, e_x),$$

and (31)

$$\forall n > 0 : ((t^n f_x) \cdot v_x, (t^n f_x) \cdot v_x) = \varepsilon_x \lambda_x + c \cdot n \cdot \varepsilon_x \cdot B(e_x, f_x),$$

where λ_x is a constant which is independent of n. Since the bilinear form B is skew on the odd piece, it follows that there can be no positivity, and hence no unitarity here, if $c \neq 0$.

In the case of the other Borel, we get

$$((t^n f_x) \cdot v_x, (t^n f_x) \cdot v_x) = \varepsilon_x \lambda_x + n \cdot c \cdot \varepsilon_x \cdot B(e_x, f_x) \tag{32}$$

for all $n \in \mathbb{Z}$, and hence, again, c has to be zero. ∎

Remark 3.1. It might have been argued that already by looking at $A_n^{(1)}$ and $A_m^{(1)}$ does it follow that $c = 0$ since the bilinear forms on A_n and A_m have opposite signs. However, this argument is not valid since this disease can be cured by using the opposite Borel on one of the even parts.

Remark 3.2. Notice that the proof of Lemma 3.1 is completely general in the sense that it does not use anything about the anti-involution on the even part.

PROPOSITION 3.1. *In a unitarizable module for a Lie algebra of the form $\mathscr{G}_f^{(1)}$ with $\mathscr{G}_f$ a basic classical Lie superalgebra, the center c acts trivially.*

Proof. This follows just as in the case of $A^{(1)}(n, m)$ since we can always imbed an $sl^{(1)}(0, 0)$ into a such. ∎

Remark 3.3. Of course, this generalizes immediately to the twisted Lie superalgebras.

3.2. *Elementary Representations.* The generalization of the notion of an elementary representation to the present framework is straightforward.

DEFINITION 3.1. Let π be a unitary representation of $\mathscr{G}_f$ in a Hilbert space $\mathscr{H}_\pi$, based on a given choice of a Borel subalgebra and a choice of a consistent anti-linear anti-involution ω. Let $\theta_0 \in S^1$. The representation $\tilde{\pi}_{\theta_0}$ of $\mathscr{G}_f^{(1)}$ defined by

$$\forall r \in \mathbb{Z}, \forall x \in \mathscr{G}_f : \tilde{\pi}_{\theta_0}(t^r x) = e^{i \cdot r \cdot \theta_0} \pi(x), \qquad \text{and} \qquad \tilde{\pi}_{\theta_0}(c) = 0, \tag{33}$$

is called the representation obtained from π by point-evaluation at θ_0. A finite tensor product of such representations is called an *elementary* representation.

It then follows immediately from [3] that

PROPOSITION 3.2. *Suppose $(\mathscr{G}_f)_0$ does not contain an A_n or a $\mathbb{C}$ as a summand. Then the only unitarizable highest weight modules are the elementary representations.*

Remark 3.4. The set of all unitary representations of $\mathscr{G}_f$ was determined in [2].

Remark 3.5. Throughout this article, we use the terminology *exceptional* to denote unitary highest weight representations which are not elementary.

LEMMA 3.2. *The set of irreducible unitary representations of the loop algebra $\mathbb{C}[t^1, t^{-1}]$ $(=\mathbb{C}[t^1, t^{-1}] \otimes \mathbb{C})$ is given by the set of (Radon) measures on S^1, where, for a given measure μ, the corresponding representation π_μ is given by*

$$\pi_\mu(t^r) = \int_{S^1} e^{i \cdot r \cdot \theta} \, d\mu(\theta). \tag{34}$$

Proof. This follows from the theory of maximal abelian $*$-algebras; see, eg., [9]. ∎

4. $sl(n, m)$

4.1. *Compact involutions.* We consider here $sl^{(1)}(n, m)$ and leave open the posibility that n and m may be equal, i.e., we do not pass to any quotient when $n = m$. The reason is that several results for the case $n \neq m$ carry over directly to the case $n = m$.

We known already (cf. [2]) that there are exactly two consistent anti-linear anti-involutions, ω_+ and ω_-, on $sl(n, m)$ that restrict to the compact anti-involution on the even part, namely

$$\omega_\pm \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha^* & \pm\gamma^* \\ \pm\beta^* & \delta^* \end{pmatrix}. \tag{35}$$

PROPOSITION 4.1. *Consider B_{st}^+. Unless $n = m = 0$, there are no unitarizable modules for $A^{(1)}(n, m)$ with respect to the involutions $\omega_\pm$ (35) except the trivial.*

Proof. It is obvious that B_{st}^+ induces the standard Borels on $A_n^{(1)}$ and $A_m^{(1)}$. But then we already know (cf. [3]) that since $c = 0$, the representations corresponding to $A_n^{(1)}$ and $A_m^{(1)}$ must be the trivial one-dimensional representation. Now suppose that we have a unitary highest weight representation and consider the elements $t^{-1}e_{i,1}$ for $i = 1, ..., n+1$ (cf. (28)). A small computation shows that these vectors are highest weight vectors for $A_n^{(1)}$ and hence they must be in the kernel of hermitian form. A similar argument shows that the elements $t^{-1}e_{m+1,j}$ are highest weight vectors for $A_m^{(1)}$. Combined with the triviality on A_n and A_m, this immediately shows that only the trivial representation survives in all cases except $sl(0, 0)$ (which we turn to in the next proposition). ∎

PROPOSITION 4.2. *Let $\mathscr{H}_\lambda$ be a highest weight module for $sl^{(1)}(0, 0)$ with respect to B_{st}^+. Let $v_\lambda \neq 0$ be the highest weight vector and assume that $h_x v_\lambda = \lambda v_\lambda$, where h_x is defined as in (29). Then*

$$\mathscr{H}_\lambda \text{ is unitary with respect to } \omega_+ \Leftrightarrow \lambda \geqslant 0,$$

and (36)

$$\mathscr{H}_\lambda \text{ is unitary with respect to } \omega_- \Leftrightarrow \lambda \leqslant 0.$$

Proof. The two cases are similar, so we only consider the first. Let $f = f_x = f_{1,1}$ and $e = e_x = e_{1,1}$. The highest weight module defined by v_λ (of which $\mathscr{H}_\lambda$ may be a quotient) is spanned by elements of the form

$$\begin{aligned} &t^{-n_r}f \cdot t^{-n_{r-1}}f \cdot \cdots \cdot t^{-n_1}f \cdot t^{-m_s}e \cdot t^{-m_{s-1}}e \cdot \cdots \\ &\cdot t^{-m_1}e \cdot t^{-p_t}h_x \cdot t^{-p_{t-1}}h_x \cdot \cdots \cdot t^{-p_1}h_x \cdot v_\lambda, \end{aligned} \tag{37}$$

where we may assume that $n_r > n_{r-1} > \cdots n_1 > 0$, $m_s > m_{s-1} > \cdots m_1 \geqslant 0$, and $p_t > p_{t-1} > \cdots > p_1 \geqslant 0$. Of course, we also allow for the possibility that there are, say, no terms corresponding to f in (37), etc. Indeed, a small computation shows immediately that the vectors $t^{-p_t}h_x \cdot v_\lambda$ for $p_t > 0$ are in the radical of the hermitian form (recall that h_x commutes with everything), and hence we can immediately suppress all terms involving h_x from

(37). What remains is then clearly an orthogonal set, and the norm of the element in (37) with the h_x-terms removed is easily computed as being λ^{r+s}. ∎

We now turn to the case of the non-standard Borel subalgebras. Together with B^+_{nst} we will also consider B^-_{nst}. To be precise,

$$B^+_{nst} = \mathrm{Span}\{t^n e_{i,j} \mid n \in \mathbb{Z},\ 1 \leqslant i \leqslant m+1 \text{ and } 1 \leqslant j \leqslant n+1\},$$

and (38)

$$B^-_{nst} = \mathrm{Span}\{t^n f_{i,j} \mid n \in \mathbb{Z},\ 1 \leqslant i \leqslant m+1 \text{ and } 1 \leqslant j \leqslant n+1\}.$$

Indeed, both choices are compatible with a choice of Borel subalgebras on the even pieces. We shall always assume that the even Borels are non-standard since the condition $c = 0$ only allows the trivial representation if the standard Borels are used and the trivial representation (for the even piece) is equally well treated with the non-standard Borel subalgebras.

Since we still are using the compact involutions on the even part, we known that the heighest vector v_0 of any unitarizable highest weight module must define elementary reprentations of $A^{(1)}_n$ and $A^{(1)}_m$ (cf. Definition 3.1). Let us now put

$$h_0 = e_{1,n+1} f_{1,n+1} + f_{1,n+1} e_{1,n+1}. \tag{39}$$

Any highest weight module of $A^{(1)}(n, m)$ that may lead to unitarity in the present setup is then completely determined by two elementary representations of $A^{(1)}_n$ and $A^{(1)}_m$, respectively, together with a linear functional, written tentatively as a (real-valued) measure $d\mu_\alpha$, such that

$$\forall r \in \mathbb{Z} : t^r h_0 \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta}\, d\mu_\alpha(\theta) \cdot v_0. \tag{40}$$

Remark 4.1. It follows actually, by looking at the terms $\sum_{r=-N}^{N} a_r t^r f_{1,1} \cdot v_0$ (or $\sum_{r=-N}^{N} a_r t^r e_{1,1} \cdot v_0$ for the opposite Borel) that $d\mu_\alpha$ either is a positive or a negative (depending on $\omega_\pm$) linear functional; hence it can be extended to ± 1 times a Radon measure.

PROPOSITION 4.3. *For $A^{(1)}(n, m)$, the elementary representations are unitary representations with respect to the Borel subalgebras and anti-involutions inherited from $sl(n, m)$. In this case, the measure $d\mu_\alpha$ is supported by a finite number of points ("finitely supported"). Conversely, if the representation defined by a finitely supported measure $d\mu_\alpha$ together with two elementary representations of the even part is unitary, then this is an elementary representation.*

Proof. This follows easily by unwinding the definitions. ∎

Remark 4.2. For an arbitrary $\mathscr{G}_f^{(1)}$, an elementary representation may likewise be viewed as being defined by $r=\dim(\mathfrak{H})$ finitely supported measures fulfilling the constraints imposed by unitarity.

For tactical reasons it is important to settle the unitarity for *scalar* representations first.

DEFINITION 4.1. A highest weight representation of $sl^{(1)}(n, m)$ in which the highest weight vector determines the trivial representations of $A_n^{(1)}$ and $A_m^{(1)}$ is said to be *scalar*. Equivalently, v_0 gives the same eigenvalue for all the operators h_x (29).

PROPOSITION 4.4. *Suppose a scalar highest weight module $\mathscr{H}_{\mu_\alpha}$ defined by a measure μ_α as in* (40) *and by a Borel subalgebra whose odd component is equal to either B_{nst}^+ or B_{nst}^- is given. Assume that μ_α has no atoms. Then the possible unitarity of $\mathscr{H}_{\mu_\alpha}$ defined either in terms of ω_+ or of ω_-, in relation to μ_α, is as follows*:

> (B_{nst}^+, ω_+): *There is no unitarity unless $m=0$. If so, there is unitarity exactly when μ_α is a positive measure.*
> (B_{nst}^+, ω_-): *There is no unitarity unless $n=0$. If so, there is unitarity exactly when μ_α is a negative measure.*
> (B_{nst}^-, ω_+): *There is no unitarity unless $n=0$. If so, there is unitarity exactly when μ_α is a positive measure.*
> (B_{nst}^+, ω_-): *There is no unitarity unless $m=0$. If so, there is unitarity exactly when μ_α is a negative measure.*

Proof. The four cases are entirely similar so we shall only prove the first. Suppose then to begin with that $m \geqslant 1$. Let $g=\sum_{r=-N}^{N} a_r e^{i\cdot r\cdot\theta}$ be a Laurent polynomial restricted to S^1. By looking just at the expressions

$$\langle gf_{1,1}, gf_{1,1}\rangle, \tag{41}$$

it follows that μ_α must be a positive measure (cf. Remark 4.1). Then an easy computation shows that

$$\langle gf_{2,1}\, gf_{1,1}, gf_{2,1}\, gf_{1,1}\rangle = \left(\int_{S^1} |g|^2\, d\mu_\alpha\right)^2 - \int_{S^1} |g|^4\, d\mu_\alpha. \tag{42}$$

Now consider the characteristic function $\xi_{\theta_0,\delta}$ of a (small) subset $]\theta_0, \theta_0+\delta[$ of S^1 (parametrized by $[0, 2\pi[$) and let $\bar{\delta}=\mu_\alpha(]\theta_0, \theta_0+\delta[)=\int_{S^1} \xi\, d\mu_\alpha$. Clearly, by Stone–Weierstrass, the Laurent polynomials approximate ξ well enough that we may replace the function g in (42) by $r\cdot\xi$ for any $r\in\mathbb{N}$. But then (42) together with unitarity implies that

$$(\bar{\delta})^2 \geqslant \bar{\delta}. \tag{43}$$

So, since we are ruling out point measures, $\bar{\delta}$ must be zero for δ sufficiently small; hence, finally, μ_α must be identically zero.

Now let us turn to the case $m=0$ and prove that if μ_α is a positive Radon measure, then we do get unitarity:

Consider the inner product

$$\langle((t^{a_{1,1}}f_1)\cdots(t^{a_{1,i_1}}f_1))\cdot\cdots\cdot((t^{a_{n+1,1}}f_{n+1})\cdots(t^{a_{n+1,i_{n+1}}}f_{n+1}))\cdot v_0,$$
$$((t^{b_{1,1}}f_1)\cdots(t^{b_{1,j_1}}f_1))\cdot\cdots\cdot((t^{b_{n+1,1}}f_{n+1})\cdots(t^{b_{n+1,j_{n+1}}}f_{n+1}))\cdot v_0\rangle. \tag{44}$$

It follows by a small computation that if $j>i$, then $[e_i, f_j]$ commutes with any f_k for $k\geqslant j$. Hence, $i_1=j_1$, $i_2=j_2$, ..., and $i_{n+1}=j_{n+1}$ if the inner product is to be non-zero. Now, evidently, for a fixed i, the inner product will be anti-symmetric in the second index in the $a_{i,j}$'s and likewise for the $b_{i,j}$'s. Let us then compute the inner product by first moving $(t^{a_{1,1}}f_1)$ from the left side of the inner product to the right-hand side as $(t^{-a_{1,1}}e_1)$. Since the latter expression annihilates v_0, the new right-hand side becomes

$$\sum_{k=1}^{n+1}(t^{b_{1,1}}f_1)\cdots(\widehat{t^{b_{1,k}}f_1})\cdots(t^{b_{1,j_1}}f_1)\cdots(t^{b_{n+1,j_{n+1}}}f_{n+1})$$
$$\cdot(-1)^{k-1}\langle t^{a_{1,1}}, t^{b_{1,k}}\rangle_{\mu_\alpha}\cdot v_0$$
$$+\sum_{k,l,s}(t^{b_{1,1}}f_1)\cdots(\widehat{t^{b_{1,k}}f_1})\cdots(t^{b_{1,j_1}}f_1)\cdots(-1)^{k-1}$$
$$\times(t^{b_{l,s}+b_{1,k}-a_{1,1}}f_l)\cdots(t^{b_{n+1,j_{n+1}}}f_{n+1})\cdot v_0, \tag{45}$$

where $\langle\cdot,\cdot\rangle_{\mu_\alpha}$ denotes the inner product in $L^2(d\mu_\alpha)$. Let us now for a moment contemplate what is going to happen to the the right-hand side when we continue to compute the inner product by moving terms of the form $(t^{a_{1,j}}f_1)$ from the left to the right, until at last there remain no terms containing f_1 to the left: There will be expressions of three different kinds. *The first kind* are like those in the first sum in (45). In these, the terms involving $(t^{b_{1,j}}f_i)$ for $i>1$ are unchanged. *The second kind* are those where one, or several, terms of the form $(t^{b_{i,j}}f_i)$ for $i>1$ have been replaced by expressions of the form $(t^{b_{i,j}+b_{1,s}-a_{1,k}}f_i)$, and where, at the same time the factor $t^{b_{1,s}}f_1$ has disappeared. Finally, *the third kind* are those that are not of the first two kinds. Here, at least one expression has the form $(t^{b_{1,s_1}+\cdots+b_{1,s_r}+b_{i,j}-a_{1,l_1}\cdots-a_{1,l_r}}f_i)$ with $r>1$. The crucial observation is that we can forget altogether the expressions of the third kind since they, in the end, due to the anti-symmetry, will cancel out among each other.

Bearing this in mind, we now return to the computation where we were lowering the degree of the left-hand side by just one. We wish to demonstrate that the positivity of the inner product at a given level (=degree) follows from the positivity at lower levels. For this purpose we define a linear map C_{1,i_1} (a contraction) from the space spanned by expressions of the form

$$((t^{a_{1,1}}f_1)\cdots(t^{a_{1,i_1}}f_1))\cdot\cdots\cdot((t^{a_{n+1,1}}f_{n+1})\cdots(t^{a_{n+1,i_{n+1}}}f_{n+1}))\cdot v_0 \tag{46}$$

with i_1 fixed, to a similar space, but where we only have i_1-1 terms involving f_1. We do this by first defining a map $\tilde{C}_{1,i_1}$ on an analogous space, namely the space spanned by similar expressions, but where we now have no relations of anti-commutativity among the $(t^a f_i)$'s (i.e., the tensor algebra):

$$\begin{aligned}\tilde{C}_{1,i_1}:&((t^{a_{1,1}}f_1)\cdots(t^{a_{1,i_1}}f_1))\cdot\cdots\cdot((t^{a_{n+1,1}}f_{n+1})\cdots(t^{a_{n+1,i_{n+1}}}f_{n+1}))\cdot v_0\\ &\to\sum_{k=2,l=1}^{n+1,i_k}((t^{a_{1,2}}f_1)\cdots(t^{a_{1,i_1}}f_1))\cdots(t^{a_{1,1}+a_{k,l}}f_k)\cdots\\ &\quad\times((t^{a_{n+1,1}}f_{n+1})\cdots(t^{a_{n+1,i_{n+1}}}f_{n+1}))\cdot v_0.\end{aligned} \tag{47}$$

The map C_{1,i_1} is then defined to be what we get when we let $\tilde{C}_{1,i_1}$ pass to the quotient where the various terms *do* anti-commute according to $sl(n,m)$.

It then follows from the preceding discussion that the inner product equals

$$\begin{aligned}&\langle C_{1,i_1}(((t^{a_{1,1}}f_1)\cdots(t^{a_{n+1,i_{n+1}}}f_{n+1}))\cdot v_0),\ C_{1,i_1}(((t^{b_{1,1}}f_1)\cdots(t^{b_{n+1,j_{n+1}}}f_{n+1}))\cdot v_0)\rangle\\ &\quad+\langle t^{a_{1,1}}\wedge\cdots\wedge t^{a_{1,i_1}},\ t^{b_{1,1}}\wedge\cdots\wedge t^{b_{1,i_1}}\rangle_{\mu_\alpha}\\ &\quad\cdot\langle((t^{a_{2,1}}f_1)\cdots(t^{a_{n+1,i_{n+1}}}f_{n+1}))\cdot v_0,\ ((t^{b_{2,1}}f_1)\cdots(t^{b_{n+1,j_{n+1}}}f_{n+1}))\cdot v_0\rangle,\end{aligned} \tag{48}$$

since the sign $(-1)^{k-1}$ in the second line of (45) is exactly right. Here, $\langle t^{a_{1,1}}\wedge\cdots\wedge t^{a_{1,i}},\ t^{b_{1,1}}\wedge\cdots\wedge t^{b_{1,i_1}}\rangle_{\mu_\alpha}$ denotes the standard inner product on the anti-symmetrized tensor product of i_1 copies of $L^2(d\mu_\alpha)$.

The positivity of the inner product then follows immediately from this. ∎

THEOREM 4.1. *The most general unitary highest weight representation of $sl^{(1)}(n,m)$ corresponding to an involution of the form $\omega_\pm$ occurs as the leading term in the scalar product between a scalar representation and an elementary representation.*

Proof. This follows by the same kind of argument as in [3]. See also the proof of Proposition 4.7 where the details of an analogous case are given. ∎

4.2. *Non-compact Involutions.* It was proved in [3] that besides the *integrable* representations (see, e.g., [6]) and the *elementary* representations of the affine Lie algebras, there is an *exceptional* series of unitarizable modules for $A_n^{(1)}$ $(n \geqslant 1)$ based on an involution ω_n which defines the real form $su(n, 1)$ of A_n. It was also proved that with this last series, the set of unitarizable highest weight modules for affine Lie algebras is exhausted.

We shall see that these representations give rise, in connection with the previous results, to some exceptional series for $sl^{(1)}(n, m)$. In analogy with (28) we wish to set the notation for some special elements of $sl(n, m)$:

$$\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 & z_1 & e_{1,1} & \cdots & e_{m+1,1} \\ \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & z_i & e_{1,i} & \cdots & e_{m+1,i} \\ \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & z_n & e_{1,n} & \cdots & e_{m+1,n} \\ w_1 & \cdots & w_i & \cdots & w_n & 0 & e_{1,n+1} & \cdots & e_{m+1,n+1} \\ f_{1,1} & \cdots & f_{1,i} & \cdots & f_{1,n} & f_{1,n+1} & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ f_{m+1,1} & \cdots & f_{m+1,i} & \cdots & f_{m+1,n} & f_{m+1,n+1} & 0 & \cdots & 0 \end{pmatrix}. \tag{49}$$

(The places with the zeros are then to be filled with elements from A_{n-1} and A_m in the obvious manner.) Let us now introduce the following spaces:

$$\begin{aligned} Z^+ &= \operatorname{Span}\{z_i \mid i = 1, ..., n\}, \\ P_1 &= \operatorname{Span}\{e_{i,j} \mid 1 \leqslant i \leqslant m+1,\ 1 \leqslant j \leqslant n\}, \\ P_2 &= \operatorname{Span}\{e_{i,n+1} \mid 1 \leqslant i \leqslant m+1\}, \\ W^- &= \operatorname{Span}\{w_i \mid i = 1, ..., n\}, \\ Q_1 &= \operatorname{Span}\{f_{i,j} \mid 1 \leqslant i \leqslant m+1,\ 1 \leqslant j \leqslant n\}, \\ Q_2 &= \operatorname{Span}\{f_{i,n+1} \mid 1 \leqslant i \leqslant m+1\}. \end{aligned}$$

One complication now is that there are quite a number of Borel subalgebras that must be considered. In fact, it follows easily from the condition of unitarity, together with Subsection 4.1, [3], and [2], that the following *odd* Borel subalgebras should be investigated:

$$
\begin{aligned}
B^+_{st,nst} &= \{t^r p_1 + t^s p_2 \mid p_1 \in P_1, p_2 \in P_2 \text{ and } r, s \in \mathbb{Z}\},\\
B^-_{st,nst} &= \{t^r q_1 + t^s q_2 \mid q_1 \in Q_1, q_2 \in Q_2 \text{ and } r, s \in \mathbb{Z}\},\\
B^+_{nst,nst} &= \{t^r p_1 + t^s q_2 \mid p_1 \in P_1, q_2 \in Q_2 \text{ and } r, s \in \mathbb{Z}\},\\
B^+_{1,mix} &= \{t^r q_2 + t^s x \mid q_2 \in Q_2, x \in P_1 \oplus Q_1, r, s \in \mathbb{Z},\\
&\qquad s > 0 \text{ if } x \in Q_1, \text{ and } s \geqslant 0 \text{ if } x \in P_1\} \text{ (sample)},\\
B^+_{2,mix} &= \{t^r p_1 + t^s x \mid p_1 \in P_1, x \in P_2 \oplus Q_2, r, s \in \mathbb{Z},\\
&\qquad s > 0 \text{ if } x \in Q_1, \text{ and } s \geqslant 0 \text{ if } x \in P_1\} \text{ (sample)}.
\end{aligned} \tag{50}
$$

In passing, we mention that in all cases, the contribution to the total Borel having to do with the spaces Z^+ and W^- is

$$\{t^l z \mid z \in Z^+, l \in \mathbb{Z}\}. \tag{51}$$

The reason for the word "sample" in connection with the Borels having a subscript "mix" is that there are in fact several more of these, correspondig to either an exchange of the P_i and the Q_i in "$P_i \oplus Q_i$," to a change of t to t^{-1}, or both. However, we allow ourselves to be a little sloppy at this point since we have the following result:

PROPOSITION 4.5. *That Borel subalgebras containing components with a subscript "mix" do not in any essential way contribute to the set of unitarizable highest weight modules.*

Proof. The proofs for the various cases are quite similar, so we shall only consider $B^+_{2,mix}$. For this, if the highest weight vector is denoted by v_0, it is easy to see that

$$Z^+(t^{-r_1}x_1 \cdots t^{-r_s}x_s \cdot v_0) = 0 \tag{52}$$

for all $r_1, ..., r_s \in \mathbb{N}$ and all $x_1, ..., x_s \in P_2 \oplus Q_2$. But, if the representation corresponding to P_2, Q_2 (with the restrictions imposed from Subsection 4.1) is non-trivial, then there are unitary representations of arbitrary big as well as arbitrary small highest weight of $su(n, 1)$, and this is impossible. ∎

Remark 4.3. The above does not lead to different modules either.

As in the case of the compact involutions, there are also here two anti-involutions, $\omega(\pm, \mp)$, defined as

$$
\begin{aligned}
\omega(\pm, \mp)(t^r z_i) &= -t^{-r} w_i,\\
\omega(\pm, \mp)(t^r p_1) &= \pm t^{-r} q_1,\\
\omega(\pm, \mp)(t^r p_2) &= \mp t^{-r} q_2,
\end{aligned} \tag{53}
$$

where the p_i's and the q_i's are matching elements of P_i and Q_i, $i=1, 2$. As in the case of the compact involutions, it is important to settle the issue of unitarity for scalar modules. In the present context, these are defined as follows: Let B_{odd} denote one of the remaining Borels in (50) and let, as usual, $E_{i,j}$ denote the matrix with a 1 for the ijth entry and zero elsewhere.

DEFINITION 4.2. The highest weight module $M(\mu_\alpha, \mu_\lambda)$ defined by two measures μ_α and μ_λ and a non-zero vector v_0 by

$$A_{n-1}^{(1)} \cdot v_0 = 0,$$

$$A_m^{(1)} \cdot v_0 = 0,$$

$$\forall r \in \mathbb{Z}\ \forall j = 1, ..., n:$$

$$(t^r z_j) \cdot v_0 = 0,$$

$$\forall r \in \mathbb{Z}, \forall j \in \{1, ..., n\}, \forall k = 1, ..., m+1:$$

$$(t^r(E_{j,j} + E_{n+1+k,n+1+k})) \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta}(d\mu_\alpha(\theta) + \tfrac{1}{2}\, d\mu_\lambda(\theta)) \cdot v_0,$$

$$\forall r \in \mathbb{Z}, \forall k = 1, ..., m+1:$$

$$(t^r(E_{n+1,n+1} + E_{n+1+k,n+1+k})) \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta}(d\mu_\alpha(\theta) - \tfrac{1}{2}\, d\mu_\lambda(\theta)) \cdot v_0,$$

$$\forall r \in \mathbb{Z}, \forall j = 1, ..., n-1:$$

$$(t^r(E_{j,j} - E_{n+1,n+1})) \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta}\, d\mu_\lambda(\theta) \cdot v_0,$$

$$B_{\mathrm{odd}} \cdot v_0 = 0, \tag{54}$$

is called the scalar module determined by μ_α and μ_λ.

PROPOSITION 4.6. *Let $M(\mu_\alpha, \mu_\lambda)$ be defined as above. Then, unless both measures are zero, there is no unitarizable quotient defined by the anti-involution $\omega(+,-)$, irrespective of which odd Borel is chosen.*

Proof. By considering the vectors $t^{r_1} x_1 \cdot v_0$ with x_1 appropriately either in Q_1 or P_1, it follows that $\mu_\alpha + \frac{1}{2}\mu_\lambda$ must be a positive measure. Likewise, by considering vectors $t^{r_1} y_1 \cdot v_0$ with y_1 appropriately either in Q_2 or P_2, it follows that $\mu_\alpha - \frac{1}{2}\mu_\lambda$ must be a negative measure. But it is known from [3] that μ_λ must be a negative measure; hence $\mu_\alpha + \frac{1}{2}\mu_\lambda$ is a sum of two negative measures. But then these two measures must be zero, and hence, $\mu_\alpha = \mu_\lambda = 0$. ∎

The following is a key observation (cf. [3]):

PROPOSITION 4.7. *Suppose $M(\mu_\alpha, \mu_\lambda)$ is unitarizable and that μ_α, μ_λ, or both have isolated atoms, say $a_i \in S^1$, $i = 1, ..., N$. Let $\alpha_i = \mu_\alpha(\{a_i\})$ and $\lambda_i = \mu_\lambda(\{a_i\})$ for $i = 1, ..., N$. Then the highest weight representations π_i of $sl(n, m)$ defined by (α_i, λ_i) (see [2, Sect. 4]) with the corresponding Borel and anti-linear involution, are unitarizable. Let, for $i = 1, ..., \pi(a_i)$ denote the corresponding elementary representation of $A^{(1)}(n, m)$, let $\tilde{\mu}_\alpha, \tilde{\mu}_\lambda$ be the measures obtained by removing the atoms from μ_α and μ_λ, respectively, and let $M(\tilde{\mu}_\alpha, \tilde{\mu}_\lambda)$ denote the corresponding highest weight representation. Then this is unitarizable and $M(\mu_\alpha, \mu_\lambda)$ is obtained as the leading term in the tensor product $M(\tilde{\mu}_\alpha, \tilde{\mu}_\lambda) \otimes \pi(a_1) \otimes \cdots \otimes \pi(a_N)$.*

Proof. Suppose that a_1 is an atom for μ_λ. Let ξ_δ be the characteristic function for a "δ-interval" around a_1. This can be approximated to any degree in $L^1(\mu_\lambda)$ as well as in $L^1(\mu_\alpha)$ by restrictions of Laurent polynomials to S^1. Moreover, by letting δ tend to zero, we can approximate the characteristic function 1_{a_1} arbitrarily well in the same L^1 spaces. All in all, we can say that except for an arbitrarily small error term, expressions of the form

$$(1_{a_1} b_1^-) \cdot \cdots \cdot (1_{a_1} b_s^-) \cdot v_0 \tag{55}$$

belong to $M(\mu_\alpha, \mu_\lambda)$. Here, v_0 denotes the highest weight vector, and $b_1^-, ..., b_s^-$ are elements in the opposite (total) Borel subalgebra. The first claim then follows easily from this. Observe, in passing, that if a_1 is not an atom for μ_α then $\alpha_1 = 0$. The remaining part now follows easily by the way tensor products behave under decomposition (cf. [3, 4]). ∎

Analogously, we clearly get

THEOREM 4.2. *The most general unitary highest weight representation occurs as the highest component in the tensor product of an elementary representation and a scalar representation where neither μ_α nor μ_λ have isolated atoms.*

PROPOSITION 4.8. *Consider $B^+_{st,nst}$ and the involution $\omega(-, +)$. Assume that neither μ_α nor μ_λ have isolated atoms. Then $M(\mu_\alpha, \mu_\lambda)$ is unitarizable only if $n = 1$. In this case, there is unitary exactly if μ_λ is a negative (Radon) measure and $\mu_\alpha = \frac{1}{2}\mu_\lambda$.*

Proof. By Poincare–Birkhoff–Witt, it follows that

$$M(\mu_\alpha, \mu_\lambda) = \mathscr{U}(L(Q_1)) \cdot \mathscr{U}(L(W^-)) \cdot \mathscr{U}(L(Q_2)) \cdot v_0. \tag{56}$$

Since we moreover have the commutators,

$$[Z^+, Q_2] = 0, \qquad [P_1, Q_2] \subseteq Z^+, \qquad [P_1, W^-] \subseteq P_2, \tag{57}$$

we can begin by considering $\mathscr{U}(L(Q_2)) \cdot v_0$. For this to be unitarizable, we must clearly have that

$$\mu_\alpha - \tfrac{1}{2}\mu_\lambda \text{ is a positive measure.} \tag{58}$$

Analogously,

$$\mu_\alpha + \tfrac{1}{2}\mu_\lambda \leqslant 0, \qquad \text{and} \qquad (\text{cf. } [3])\, \mu_\lambda \leqslant 0. \tag{59}$$

It follows from (57) that $\mathscr{U}(L(Q_2)) \cdot v_0$ consists entirely of highest weight vectors for Z^+. Moreover, under the adjoint action of the special element $h_0 = E_{n,n} - E_{n+1,n+1}$, Q_2 consists of eigenvectors corresponding to positive eigenvectors so it follows by looking at $A_n^{(1)}$ that a subspace of $\mathscr{U}(L(Q_2)) \cdot v_0$ of finite codimension must be contained in the radical of the hermitian form. But this clearly implies that $\mu_\alpha - \frac{1}{2}\mu_\lambda$ must be equal to a finitely supported positive measure,

$$\mu_\alpha - \tfrac{1}{2}\mu_\lambda = \mu_f. \tag{60}$$

To avoid that also $\mu_\alpha + \frac{1}{2}\mu_\lambda$ is finitely supported, we must, according to Proposition 4.4, assume that $n = 1$. The condition $\mu_\alpha + \frac{1}{2}\mu_\lambda \leqslant 0$ now translates into $\mu_\lambda \leqslant -\mu_f$. But then, by assumption, μ_f must be trivial since otherwise, μ_λ inherits the atoms of μ_f. It follows that $\mu_\alpha + \frac{1}{2}\mu_\lambda = \mu_\lambda \leqslant 0$ which is all right.

Now observe that $\mu_f = 0$ implies that all of the space $\mathscr{U}(L(Q_2)) \cdot v_0$ is in the radical of the hermitian form. Since, moreover, modulo this radical, P_1 and W^- commute (57), it follows easily that, in the remaining cases, the hermitian form is positive definite on

$$\mathscr{U}(L(Q_1)) \cdot \mathscr{U}(L(W^-)) \cdot v_0. \quad \blacksquare \tag{61}$$

PROPOSITION 4.9. *Consider $B^-_{st,nst}$ and the involution $\omega(-,+)$. Assume that neither μ_α nor μ_λ have isolated atoms. Then $M(\mu_\alpha, \mu_\lambda)$ is unitarizable exactly if μ_λ is a negative (Radon) measure and $\mu_\alpha = -\frac{1}{2}\mu_\lambda$.*

Proof. To begin with, this proceeds in analogy with the proof of Proposition 4.8. Indeed, by looking at P_1 it follows that $\mu_\alpha + \frac{1}{2}\mu_\lambda = -\mu_f$ where μ_f is a finitely supported positive measure, and from P_2 it follows that $\mu_\alpha - \frac{1}{2}\mu_\lambda = \mu^+$ for some positive measure μ^+. It follows from this that $\mu_\alpha = -\mu^+ - \mu_f$ and hence μ_λ inherits the atoms of μ_f. Thus, $\mu_f = 0$. The equation corresponding to (56) becomes, in the present context,

$$M(\mu_\alpha, \mu_\lambda) = \mathscr{U}(L(P_2)) \cdot \mathscr{U}(L(W^-)) \cdot \mathscr{U}(L(P_1)) \cdot v_0, \tag{62}$$

and by the above considerations, we can restrict our attention to

$$M(\mu_\alpha, \mu_\lambda) = \mathscr{U}(L(P_2)) \cdot \mathscr{U}(L(W^-)) \cdot v_0. \tag{63}$$

Since we moreover have commutators,

$$[Q_2, W^-] \subseteq Q_1 \qquad \text{and} \qquad [Q_1, W^-] = 0, \tag{64}$$

the unitarity of what remains, follows easily. ∎

Finally, we have

PROPOSITION 4.10. *Consider $B^+_{nst,nst}$ and the involution $\omega(-,+)$. Assume that neither μ_α nor μ_λ have isolated atoms. If $n>1$, $M(\mu_\alpha, \mu_\lambda)$ is unitarizable exactly if $\mu_\alpha = -\frac{1}{2}\mu_\lambda$ where μ_λ is a negative (Radon) measure. If $n=1$, there is unitarity exactly if μ_λ is a negative (Radon) measure and $|\mu_\alpha| \leqslant |\frac{1}{2}\mu_\lambda|$.*

Proof. In case $n>1$, this comes about exactly as in the proof of Proposition 4.9 by using Proposition 4.4. Turning to $n=1$, the appropriate replacement for (56) is here

$$M(\mu_\alpha, \mu_\lambda) = \mathscr{U}(L(W^-)) \cdot \mathscr{U}(L(P_2)) \cdot \mathscr{U}(L(Q_1)) \cdot v_0, \tag{65}$$

and we have the commutators

$$[Z^+, Q_1] \subseteq Q_2, \qquad [Z^+, P_2] \subseteq P_1, \qquad [Q_1, Q_2] = 0. \tag{66}$$

From this, the restrictions on the measures follow immediately, and it is also easy to see that there is no finiteness condition imposed on

$$M(\mu_\alpha, \mu_\lambda) = \mathscr{U}(L(P_2)) \cdot \mathscr{U}(L(Q_1)) \cdot v_0 \tag{67}$$

by $A_n^{(1)}$. Moreover, the hermitian form restricted to the space (67) is clearly positive definite when the measures satisfy the above restrictions.

To prove unitarity on the full space (65) we argue inductively on the degree of the elements coming from $\mathscr{U}(L(P_2))$. Let us first consider

$$\mathscr{U}(L(W^-)) \cdot \mathscr{U}(L(Q_1)) \cdot v_0. \tag{68}$$

Since it follows from (66) that $\mathscr{U}(L(Q_1)) \cdot v_0$ consists of vectors that are annihilated by Z^+, and are eigenvectors for $h_0 = E_{n,n} - E_{n+1,n+1}$ of non-positive eigenvalues, the unitarity here is clear. Consider now an element in (67) of the form

$$(p_2) \cdot (q_{1,1}) \cdot \cdots \cdot (q_{1,N}) \cdot v_0. \tag{69}$$

We wish to show that we can find an element $\tilde{q} \in \mathscr{U}(L(W^-)) \cdot \mathscr{U}(L(Q_1)) \cdot v_0$ such that

$$Z^+((p_2) \cdot (q_{1,1}) \cdot \cdots \cdot (q_{1,N}) \cdot v_0 - \tilde{q}) = 0. \tag{70}$$

It suffices to consider $p_2 = t^r e_x$ and $q_{1,1} = t^s f_y$, where e_x and f_y are $(n+m+2) \times (n+m+2)$ matrices, appropriately corresponding to P_2 and Q_1, respectively. Let $\hat{w} = e_x \cdot f_y$ (matrix product). Then $t^{r+s}\hat{w} \in W^-$ and the element

$$\tilde{q} = (t^{r+s}\hat{w})\, q_{1,2} \cdot \cdots \cdot (q_{1,N}) \cdot v_0 \tag{71}$$

clearly has the needed properties. We leave the rest of the induction to the reader, as it follows analogously, using

$$[W^-, P_2] = [P_1, P_2] = 0. \quad \blacksquare \tag{72}$$

PROPOSITION 4.11. *There is no unitarity corresponding to both A_n and A_m having non-compact real forms.*

Proof. This follows easily from the analogous result for $sl(n, m)$. In fact, all meausures must be trivial, since one otherwise, by approximating appropriate characteristic functions (with non-zero L^1-norms), would be able to get non-trivial unitary representations of $sl(n, m)$ of the indicated form, and such do not exist (see [2]). ∎

4.3. *The Other Central Extension.* The other central extension $A^{\mathfrak{m}}(n, n)$ defined in Remark 2.1 clearly satisfies

$$A^{\mathfrak{m}}(n, n) \equiv \mathbb{C}[t^1, t^{-1}] \otimes sl(n, n)/\mathrm{Span}\{(t^n - \mathfrak{m}(t^n)) \cdot I_{2n+2} \mid n \in \mathbb{Z}\}. \tag{73}$$

From this it follows that we can translate the results obtained for $\mathbb{C}[t^1, t^{-1}] \otimes sl(n, n)$ into corresponding ones for $A^{\mathfrak{m}}(n, n)$ exactly if we have

$$\forall n \in \mathbb{Z} : \int_{S^1} (t^n - \mathfrak{m}(t^n))\, d\mu_\alpha = 0, \tag{74}$$

i.e., if

$$\mathfrak{m} = \mu_\alpha / \mu_\alpha(S^1). \tag{75}$$

5. EXCEPTIONAL SUPERALGEBRAS

We now investigate the exceptional cases in which there is an A_1 or $\mathbb{C}$ component in the even part, i.e., $D(2, 1, \alpha)$, $F(4)$, $G(3)$, $B(0, n)$, $B(1, n)$, $B(m, 1)$, $C(n)$, and $D(m, 1)$.

To begin with, there seems to be a profusion of Borel subalgebras. However, it will be sufficient to investigate what corresponds to the scalar representations for $A^{(1)}(n, m)$ (Definition 4.1) and this will lead to an important reduction in the number of cases.

DEFINITION 5.1. Let ω be a given anti-linear anti-involution. A highest weight representation of $\mathbb{C}[t^1, t^{-1}] \otimes \mathscr{G}_f$ with $\mathscr{G}_f \neq A(n, m)$ is said to be *scalar* if it is trivial on each component of $\mathbb{C}[t^1, t^{-1}] \otimes (\mathscr{G}_f)_0$ which does not have exceptional representations with respect to the restriction of ω to the component.

LEMMA 5.1. *Let N^+_{odd} be the odd part of a Borel involved in the definition of a non-trivial scalar representation. Let $\mathbb{C}[t^1, t^{-1}] \otimes (\mathscr{G}_f)_{0,0}$ be the biggest subspace of the loop algebra over the even part which acts trivially. Suppose $\mathscr{G}_f \neq B(1, n)$. Then N^+_{odd} is invariant under this algebra.*

Proof. (Sketched) If N^+_{odd} is not invariant, then a subspace of N^-_{odd} also acts trivially, then commutators between this space and N^+_{odd} act trivially, and so on. In the end, everything acts trivially. The fact that this argument breaks miserably down for $B(1, n)$ indicates that one should check the list of superalgebras and make sure that this is the only place where it happens. This is indeed the case. ▮

Remark 5.1. We return to $B(1, n)$ in Subsection 5.5.

Below we describe in greater detail for the various algebras, and depending on the given involution, the scalar series of representations corresponding to those for $sl^{(1)}(n, m)$. To finish the classification we appeal here, once and for all, to the following structure theorem, which is easily proved in analogy with the case of $sl^{(1)}(n, m)$.

THEOREM 5.1 (Structure Theorem). *The most general unitary highest weight representation occurs as the highest component in the tensor product between a scalar and an elementary representation.*

5.1. $D(2, 1, \alpha)$. The Cartan matrix is

$$D_\alpha = \begin{bmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}. \tag{76}$$

The roots are, expressed in terms of linear functions $\varepsilon_1, \varepsilon_2$, and ε_3,

$$\Delta_0 = \{\pm 2\varepsilon_i, i = 1, 2, 3\}, \qquad \Delta_1 = \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}. \tag{77}$$

Up to W-equivalence there are four systems of simple roots. However, three of these are equivalent by permutation of the indices, and for this reason we will only consider two systems, one of the three together with the fourth.

We choose non-zero root vectors $e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4$, such that

$$e_1 \equiv -\varepsilon_1+\varepsilon_2-\varepsilon_3, \qquad h_1=\frac{1}{2}h_{2\varepsilon_1}+\frac{\alpha+1}{2}h_{2\varepsilon_2}+\frac{\alpha}{2}h_{2\varepsilon_3},$$

$$e_2 \equiv \varepsilon_1+\varepsilon_3-\varepsilon_3, \qquad h_2=-\frac{1}{2}h_{2\varepsilon_1}+\frac{\alpha+1}{2}h_{2\varepsilon_2}+\frac{\alpha}{2}h_{2\varepsilon_3},$$

$$e_3 \equiv -\varepsilon_1+\varepsilon_2+\varepsilon_3, \qquad h_3=\frac{1}{2}h_{2\varepsilon_1}+\frac{\alpha+1}{2}h_{2\varepsilon_2}-\frac{\alpha}{2}h_{2\varepsilon_3},$$

$$e_4 \equiv \varepsilon_1+\varepsilon_2+\varepsilon_3, \qquad h_4=-\frac{1}{2}h_{2\varepsilon_1}+\frac{\alpha+1}{2}h_{2\varepsilon_2}-\frac{\alpha}{2}h_{2\varepsilon_3}.$$

We choose f_i to have weight equal to minus the weight of e_i, and we assume that for $i=1, 2, 3, 4$,

$$[e_i, f_i]=h_i. \tag{78}$$

(This is different from the convention in [5].)

We let $k_{\pm 2\varepsilon_i}$ denote the usual root vectors for the three sl_2's defined by the roots $\pm 2\varepsilon_i$, $i=1, 2, 3$. We consider now an anti-linear anti-involution ω and we put

$$\omega(k_{2\varepsilon_i})=\bar{\varepsilon}_i k_{-2\varepsilon_i} \quad \text{for} \quad i=1, 2, 3, \qquad \text{and} \qquad \omega(e_1)=\bar{\varepsilon}_0 f_1. \tag{79}$$

An easy computation [2] gives:

LEMMA 5.2.

$$\begin{aligned} \omega(e_2)&=\bar{\varepsilon}_0\varepsilon_1 f_2,\\ \omega(\varepsilon_3)&=\bar{\varepsilon}_0\bar{\varepsilon}_3 f_3,\\ \omega(e_4)&=\bar{\varepsilon}_0\,\bar{\varepsilon}_1\varepsilon_3 f_4=-\bar{\varepsilon}_0\bar{\varepsilon}_2 f_4. \end{aligned} \tag{80}$$

In particular, $\bar{\varepsilon}_1\cdot\bar{\varepsilon}_2\cdot\bar{\varepsilon}_3=-1$.

We now return to the possible sets of positive root vectors in $\mathscr{G}_1$. As mentioned previously, there are essentially two. We choose these such that the orderings on the three sl_2's of $D(2, 1, \alpha)_0$ are the usual,

$$N_A^+=\text{Span}\{e_1, e_2, e_3, e_4\}, \qquad N_B^+=\text{Span}\{f_1, e_2, e_3, e_4\}. \tag{81}$$

For each of these there are four possibilities of the signs $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$, and $\bar{\varepsilon}_3$ as indicated by (a), (b), (c), and (d) below:

	$\bar{\varepsilon}_1$	$\bar{\varepsilon}_2$	$\bar{\varepsilon}_3$
(a)	-1	-1	-1
(b)	$+1$	-1	$+1$
(c)	$+1$	$+1$	-1
(d)	-1	$+1$	$+1$.

It is easy to see that in both case N_A^+ and N_B^+, the cases (c) and (d) are equivalent in the sense that N_A^+ and N_B^-, respectively, are symmetrical under a change of indices $1 \leftrightarrow 3$. Thus, we will omit case (d) from further consideration.

LEMMA 5.3. *There are no non-trivial unitarizable modules corresponding to* $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \bar{\varepsilon}_3 = -1$.

Proof. This follows easily from Lemma 5.2, since ω in this case is positive on some elements of N_{odd}^- and negative on others. ∎

PROPOSITION 5.1. *There are unitarizable scalar representations only corresponding to* $\mathbb{C}[t^1, t^{-1}] \otimes N_A^+$ *and to the signs in* (b). *In this case, there is unitarity exactly when the following are satisfied*:

- $\bar{\varepsilon}_0 = -1$.
- $\alpha > 0$.
- *The highest weight vector* v_0 *satisfies*: $\forall r \in \mathbb{Z} : t^r h_{2\varepsilon_1} \cdot v_0 = t^r h_{2\varepsilon_3} \cdot v_0 = 0$ *and* $t^r h_{2\varepsilon_2} \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta} \, d\mu(\theta)$ *where* μ *is a negative* (*Radon*) *measure on* S^1.

Proof. The claim involving N_A^+ and the signs follows immediately from Lemma 5.1. Now consider expressions of the form

$$(t^{n_1} f_1 \wedge \cdots \wedge t^{n_R} f_1) \cdot v_0. \tag{82}$$

These are highest weight vectors for the loop algebras based on the $sl(0, 0)$ defined by each of the pairs e_2, f_2, e_3, f_3, and e_4, f_4, and it follows easily that we must have, for all $R \in \mathbb{N}$: $2\bar{\varepsilon}_0 R < 0$, $2\bar{\varepsilon}_0 \alpha R < 0$, and $2\bar{\varepsilon}_0(\alpha + 1) R < 0$ to ensure unitarity. From this the claims involving α and $\bar{\varepsilon}_0$ follow immediately.

We now proceed to investigate the unitariy as far as $\mathbb{C}[t^1, t^{-1}] \otimes N_A^-$ is concerned; i.e., we consider the inner product on the span of vectors of the form

$$(t^{p_1} f_4 \wedge \cdots \wedge t^{p_P} f_4) \wedge (t^{q_1} f_3 \wedge \cdots \wedge t^{q_Q} f_3) \\ \wedge (t^{m_1} f_2 \wedge \cdots \wedge t^{m_M} f_2) \wedge (t^{n_1} f_1 \wedge \cdots \wedge t^{n_N} f_1) \cdot v_0. \tag{83}$$

The positivity of the inner product on the special elements in (82) follows from Proposition 4.2. Now observe that

$$t^n e_2(t^{n_1} f_1 \wedge \cdots \wedge t^{n_R} f_1) \cdot v_0 = 0$$

and

$$\begin{aligned} &t^n h_2(t^{n_1} f_1 \wedge \cdots \wedge t^{n_R} f_1) \cdot v_0 \\ &\quad = -4(t^{n+n_1} f_1 \wedge \cdots \wedge t^{n_R} f_1) \cdot v_0 - \cdots - 4(t^{n_1} f_1 \wedge \cdots \wedge t^{n+n_R} f_1) \cdot v_0 \\ &\qquad + \left(\int_{S^1} e^{i \cdot n \cdot \theta} \, d\mu(\theta) \right) \cdot (t^{n_1} f_1 \wedge \cdots \wedge t^{n_R} f_1) \cdot v_0. \end{aligned} \tag{84}$$

The representation of the abelian $*$-algebra $\mathfrak{a}_2 = \mathrm{Span}\{ t^n h_2 \mid n \in \mathbb{Z} \}$ in the space spanned by the elements (82) is then clearly a direct integral, over the set

$$\{ (p_1, ..., p_R) \in (S^1)^R \mid \forall i \neq j : p_i \neq p_j \}, \tag{85}$$

of one-dimensional representations

$$t^n h_2 v_{p_1, ..., p_R} = \left(-4(p_1)^n - \cdots - 4(p_R)^n + \int_{S^1} e^{i \cdot n \cdot \theta} \, d\mu(\theta) \right) \cdot v_{p_1, ..., p_R}, \tag{86}$$

where $(p_1, ..., p_R)$ are (ordered) points in S^1 satisfying: $\forall i \neq j : p_i \neq p_j$. All in all, we see that the span of the elements (82) is a direct integral of highest weight vectors for the loop algebra $\mathbb{C}[t^1, t^{-1}] \otimes sl(0, 0)$ based on the pair e_2, f_2 and these representations, again by Proposition 4.2, are unitary. It follows that the inner product is positive on the terms

$$(t^{m_1} f_2 \wedge \cdots \wedge t^{m_M} f_2) \wedge (t^{n_1} f_1 \wedge \cdots \wedge t^{n_N} f_1) \cdot v_0. \tag{87}$$

Furthermore, we can continue along these lines and establish the positivity on the span of elements of the form

$$(t^{q_1} f_3 \wedge \cdots \wedge t^{q_Q} f_3) \wedge (t^{m_1} f_2 \wedge \cdots \wedge t^{m_M} f_2) \wedge (t^{n_1} f_1 \wedge \cdots \wedge t^{n_N} f_1) \cdot v_0 \tag{88}$$

without problems.

The situation involving the pair e_4, f_4, however, deserves some comments. To begin with, it is clear that the action of the algebra $\mathfrak{a}_4 = \mathrm{Span}\{ t^n h_4 \mid n \in \mathbb{Z} \}$ again is the direct integral of one-dimensional actions, and it is sufficient to replace expressions from $\mathbb{C}[t^1, t^{-1}] \otimes sl(0, 0)$, based on e_4, f_4, by the corresponding expressions evaluated at a finite set of points in S^1. The problem now is that the $t^n e_4$'s do not annihilate the space

(83). Let us denote by x_p the expression which, for $x = e_1, ..., f_4$, formally corresponds to $\xi_p \otimes x$, where ξ_p denotes the characteristic function for the point $p \in S^1$. In its simplest form, the problem is that

$$(e_4)_p ((f_3)_{q_1} \wedge (f_2)_{q_2}) = k \cdot \delta_{q_1, q_2} \cdot \delta_{p, q_1} (f_1)_p. \tag{89}$$

More generally, let $F_{R,S,T}$ denote the space spanned by the wedge product of R expressions of the form $(f_3)_{p_i}$, S os the form $(f_2)_{q_j}$, and T of the form $(f_1)_{r_l}$. Then, $(e_4)_p F(R, S, T) \subseteq F(R-1, S-1, T+1)$.

Returning to the simple example (89) it is easy to see that

$$(e_4)_p ((f_3)_p \wedge (f_2)_p - a \cdot ((f_4)_p \wedge (f_1)_p)) = 0, \tag{90}$$

for a non-zero constant a, and here, $(f_1)_p$ is annihilated by all the elements $(e_4)_q$. In general, the crucial fact that h_4 has strictly negative eigenvalues on all the spaces $F(R, S, T)$ gives that we can write an element $\tilde{f} \in F(R, S, T)$ (where, say, $R \leqslant S$) as

$$\tilde{f}_R = \tilde{v}_R + (f_4)_{p_1} \wedge \tilde{v}_{R-1} + \cdots + ((f_4)_{q_1} \wedge \cdots \wedge (f_4)_{q_R}) \wedge \tilde{v}_0, \tag{91}$$

where each $\tilde{v}_i$ is annihilated by all the elements $(e_4)_q$. Notice that the $\tilde{v}_i$'s do not (necessarily) belong to one of the spaces $F(A, B, C)$, but $\tilde{v}_i$ has degree i as far as expressions involving the $(f_3)_q$'s are concerned. Noting that the different summands are homogeneous to different degrees in the $(f_4)_q$'s, the positivity on the whole span of the elements (83) follows from (91).

Finally, we need to consider the $\mathbb{C}[t^1, t^{-1}] \otimes sl_2$ that carries the exceptional representation defined by μ. If we let w denote the variable corresponding to the negative non-compact root, we must consider polynomials $t^{n_1}w \cdot \cdots \cdot t^{n_r}w$ multiplied from the left onto the elements from (83). The situation is here quite analogous to that for f_4. Again we get direct integrals of representations, and again we can set up an argument based on a filtration as above. Since all weight spaces behave appropriately, the whole unitarity follows easily from this. ▮

5.2. $F(4)$. We refer (again) to [5, 2] for the details concerning the results and notation used here.

Let $F(4)_0$ and $F(4)_1$ denote the even and odd part, respectively. Then

$$F(4)_0 = B_3 \oplus A_1, \quad \text{and} \quad F(4)_0|_{F(4)_1} = \text{spin}_7 \otimes sl_2. \tag{92}$$

Let e_1, e_2, e_3, and $\delta = e_4$ denote the usual basis of $\mathbb{C}_4$. Let Δ_0 and Δ_1 denote the even and the odd roots, respectively. Then

$$\Delta_0 = \{\pm e_i \pm e_j, \pm e_i, \pm \delta\}, \quad \text{and} \quad \Delta_1 = \{\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm \delta)\}. \tag{93}$$

Up to W equivalence there are five sets of simple roots, Σ_1, Σ_2, Σ_3, Σ_4, and Σ_5. However, because of Lemma 5.1, we do not really need to describe all these.

Choose non-zero elements $x_1, ..., x_8, y_1, ..., y_8$ in $F(4)_1$ such that

x_1 has weight $\frac{1}{2}(-e_1-e_2-e_3-\delta)$, y_1 has weight $\frac{1}{2}(-e_1-e_2-e_3+\delta)$,
x_2 has weight $\frac{1}{2}(-e_1-e_2+e_3-\delta)$, y_2 has weight $\frac{1}{2}(-e_1-e_2+e_3+\delta)$,
x_3 has weight $\frac{1}{2}(-e_1+e_3-e_3-\delta)$, y_3 has weight $\frac{1}{2}(-e_1+e_2-e_3+\delta)$,
x_4 has weight $\frac{1}{2}(-e_1+e_2+e_3-\delta)$, y_4 has weight $\frac{1}{2}(-e_1+e_2+e_3+\delta)$,
x_5 has weight $\frac{1}{2}(+e_1-e_2-e_3-\delta)$, y_5 has weight $\frac{1}{2}(+e_1-e_2-e_3+\delta)$,
x_6 has weight $\frac{1}{2}(+e_1-e_2+e_3-\delta)$, y_6 has weight $\frac{1}{2}(+e_1-e_2+e_3+\delta)$,
x_7 has weight $\frac{1}{2}(+e_1+e_2-e_3-\delta)$, y_7 has weight $\frac{1}{2}(+e_1+e_2-e_3+\delta)$,
x_8 has weight $\frac{1}{2}(+e_1+e_2+e_3-\delta)$, y_8 has weight $\frac{1}{2}(+e_1+e_2+e_3+\delta)$.

We choose the standard ordering on B_3 and A_1 and choose the bilinear form to be positive definite on B_3 and negative definite on A_1.

Let

$$N^+_{nst} = \mathbb{C}[t^1, t^{-1}] \otimes (\text{Span}\{y_1, ..., y_8\}). \tag{94}$$

LEMMA 5.4. *If there is an exceptional series of representations, the odd part of the Borel subalgebra must be* N^+_{nst}.

Proof. This follows immediately from Lemma 5.1. ∎

Choose vectors h_{e_1}, h_{e_2}, h_{e_3}, and $h_\delta \in \mathbb{C}^4$ (identified with a Cartan subalgebra of $F(4)$) such that $e_i(h_{e_j}) = 2\delta_{i,j}$, $\delta(h_{e_j}) = 0$, $e_i(h_\delta) = 0$, and $\delta(h_\delta) = 2$. We choose the $x_1, ..., y_8$ such that

$$\begin{aligned}
[y_1, x_8] &= h_{-e_1} + h_{-e_2} + h_{-e_3} - 3h_\delta,\\
&= -h_{e_1} - h_{e_2} - h_{e_3} - 3h_\delta,\\
[y_2, x_7] &= h_{-e_1} + h_{-e_2} + h_{e_3} - 3h_\delta,\\
&\vdots\\
[y_8, x_1] &= h_{e_1} + h_{e_2} + h_{e_3} - 3h_\delta.
\end{aligned} \tag{95}$$

We now quote the conditions that must be fulfilled by an anti-linear anti-involution ω. Let

$$\omega(y_1) = \varepsilon_0 x_8 \tag{96}$$

for an $\varepsilon_0 = \pm 1$, and assume that the sign of ω on the root spaces $e_2 - e_3$, $e_1 - e_2$, e_3, and δ is $\varepsilon_{2,3}$, $\varepsilon_{1,2}$, ε_3, and ε_δ, respectively.

PROPOSITION 5.2 [2].

$$\begin{aligned}
\omega(y_2) &= \varepsilon_0\varepsilon_3 x_7,\\
\omega(y_3) &= \varepsilon_0\varepsilon_{2,3}\varepsilon_3 x_6,\\
\omega(y_4) &= \varepsilon_0\varepsilon_{2,3} x_5,\\
\omega(y_5) &= \varepsilon_0\varepsilon_{1,2}\varepsilon_{2,3}\varepsilon_3 x_4,\\
\omega(y_6) &= \varepsilon_0\varepsilon_{1,2}\varepsilon_{2,3} x_3,\\
\omega(y_7) &= \varepsilon_0\varepsilon_{1,2} x_2,\\
\omega(y_8) &= \varepsilon_0\varepsilon_{1,2}\varepsilon_3 x_1.
\end{aligned} \tag{97}$$

Moreover,

$$\varepsilon_\delta = -\varepsilon_3\varepsilon_{1,2}.$$

PROPOSITION 5.3. *Consider N^+_{nst}. In this case, there is an exceptional series of scalar unitary representations exactly when the following are satisfied:*

- $\varepsilon_0 = 1$.
- *The highest weight vector v_0 satisfies: $\forall r \in \mathbb{Z}$, $\forall x \in B_3 : t^r x \cdot v_0 = 0$ and $t^r h_\delta \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta}\, d\mu(\theta)$ where μ is a negative (Radon) measure on S^1.*

Proof. This follows along the exact same lines as for $D(2, 1, \alpha)$. ∎

5.3. $G(3)$. We have

$$\mathscr{G}_0 = G_2 \oplus A_1 \qquad \text{and} \qquad \mathscr{G}_0|_{\mathscr{G}_1} = G_2 \otimes sl_2, \tag{98}$$

where the G_2 in the tensor product denotes the irreducible seven-dimensional representation of G_2.

Let ε_1, ε_2, ε_3, and δ denote vectors such that

$$\begin{aligned}
&(\varepsilon_1, \varepsilon_1) = (\varepsilon_2, \varepsilon_2) = (\varepsilon_3, \varepsilon_3) = 2 = (\delta, \delta),\\
&(\varepsilon_i, \varepsilon_j) = -1 \qquad \text{if} \quad i \neq j,\\
&(\varepsilon_i, \delta) = 0 \qquad \text{for all } i,
\end{aligned} \tag{99}$$

and such that, moreover, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$. The sets of even and odd roots are

$$\Delta_0 = \{\varepsilon_i - \varepsilon_j, \pm\varepsilon_i, \pm 2\delta\}, \qquad \Delta_1 = \{\pm\varepsilon_i \pm \delta, \pm\delta\}. \tag{100}$$

We represent G_1 as

$$\begin{array}{ccccccccccccc} \delta+\varepsilon_1 & \longrightarrow & \delta-\varepsilon_3 & \longrightarrow & \delta-\varepsilon_2 & \longrightarrow & \delta & \longrightarrow & \delta+\varepsilon_2 & \longrightarrow & \delta+\varepsilon_3 & \longrightarrow & \delta-\varepsilon_1 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ -\delta+\varepsilon_1 & \longrightarrow & -\delta-\varepsilon_3 & \longrightarrow & -\delta-\varepsilon_2 & \longrightarrow & -\delta & \longrightarrow & -\delta+\varepsilon_2 & \longrightarrow & -\delta+\varepsilon_3 & \longrightarrow & -\delta-\varepsilon_1. \end{array} \tag{101}$$

Up to W equivalence there are four sets of simple roots, Σ_1, Σ_2, Σ_3, and Σ_4. More importantly, let N^+ be the subspace of G_1 spanned by the root vectors corresponding to the set of roots $\{\delta+\varepsilon_1, \delta-\varepsilon_3, \delta-\varepsilon_2, \delta, \delta+\varepsilon_2, \delta+\varepsilon_3, \delta-\varepsilon_1\}$ (the top row in (101)), and let

$$N^+_{nst} = \mathbb{C}[t^1, t^{-1}] \otimes N^+. \tag{102}$$

Choose the standard ordering on G_2 and A_1. As for $F(4)$, we immediately get

Lemma 5.5. *If there is an exceptional series of representations, the odd part of the Borel subalgebra must be* N^+_{nst}.

Let $z_{2\delta}$ be a non-zero root vector belonging to the root 2δ in A_1 and let $z_{-2\delta}$ be defined analogously and such that $[z_{2\delta}, z_{-2\delta}] = h_{2\delta}$. Choose non-zero root vectors y_1, x_7 of weights $\delta+\varepsilon_1$ and $-(\delta+\varepsilon_1)$, respectively, and such that

$$[y_1, x_7] = h_{2\delta} - \tfrac{1}{2} h_{\varepsilon_1}. \tag{103}$$

Let

$$\omega(y_1) = \varepsilon_0 x_7, \qquad \text{and} \qquad \omega(z_{2\delta}) = \varepsilon_\delta z_{-2\delta}, \tag{104}$$

so that $\varepsilon_0^2 = \varepsilon_\delta^2 = 1$.

In order to have the exceptional unitarity, we must clearly have $\varepsilon_\delta = -1$. (It is furthermore clear that the sign ε_0 stays constant on the odd part.)

Proposition 5.4. *Consider* N^+_{nst}. *In this case, there is an exceptional series of scalar unitary representations exactly when the following are satisfied*:

- $\varepsilon_0 = -1$.
- *The highest weight vector* v_0 *satisfies*: $\forall r \in \mathbb{Z}$, $\forall x \in G_2 : t^r x \cdot v_0 = 0$ *and* $t^r h_{2\delta} \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta}\, d\mu(\theta)$ *where* μ *is a negative* (*Radon*) *measure on* S^1.

Proof. This follows as for $F(4)$. The sign on ε_0 is a result of the way we have normalized the vectors y_1, x_7. ∎

5.4. $C(n)$. It is convenient to write, for $n \geq 1$,

$$C(n+1) = \left\{ \begin{pmatrix} \beta & \gamma & \xi_1 & \xi_2 \\ \delta & -\beta^r & \eta_1 & \eta_2 \\ -\eta_2^r & \xi_2^r & a & 0 \\ -\eta_1^r & \xi_1^r & 0 & -a \end{pmatrix} \right\}, \tag{105}$$

where the ξ's are η's are (column) vectors in $\mathbb{C}^n$, $a \in \mathbb{C}$, and the r denotes reflection in the "opposite" diagonal. The submatrix defined by β, γ, δ, and $-\beta^r$, of course, is an element of C_n.

Let us first look at the case of a compact involution, i.e., the case where we use the compact real form of the C_n. Let

$$N_A^+ = \mathrm{Span}\{\xi_1, \eta_1\} \tag{106}$$

denote the subspace of $C(n+1)$ spanned by the elements in (105) corresponding to the symbols ξ_1, η_1. Let

$$\omega_\pm(\xi_1) = \mp \eta_2^r, \qquad \text{and} \qquad \omega_\pm(\eta_1) = \pm \xi_2^r. \tag{107}$$

Let us define a *compact scalar* highest weight representation to be a representation generated by a non-zero vector v_0 for which

- $\mathbb{C}[t^1, t^{-1}] \otimes x \cdot v_0 = 0$ for all $x \in C_n \oplus N_A^+$.
- $t^r \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta} \, d\mu_a(\theta)$, where μ_α is a (Radon) measure on S^1.

PROPOSITION 5.5. *There is unitarity for a compact scalar highest weight representation of $C(n+1)$ exactly when the involution is given by ω_+ and μ_a is a positive (Radon) measure.*

Proof. This follows by first considering the space S_2 spanned by wedges of elements $t_i^r \xi_2^{(i)}$ applied to v_0. The unitarity on this space follows directly from Proposition 4.4. We then consider the space spanned by wedges of element $t_i^r \eta_2^{(i)}$ applied to S_2. Again we can use the results of Subsection 4.1 since we get tensor products of elementary and exceptional representations. The restriction on the possible involutions comes about by noticing that the ξ_2's have non-zero weights both with respect to the usual Cartan in C_n and the elements $\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)$. ∎

Finally there is the possibility of having a non-compact real form exactly for $C(2)$: Let us rewrite $C(2)$ as

$$C(2) = \left\{ \begin{pmatrix} h & \gamma & p_1 & p_2 \\ \delta & -h & -q_2 & -q_1 \\ q_1 & p_2 & a & 0 \\ q_2 & p_1 & 0 & -a \end{pmatrix} \right\}. \tag{108}$$

Let

$$\omega^n_{\pm}\, p_i = \pm q_i. \tag{109}$$

Let us define a *non-compact scalar* highest weight representation of $C(2)$ to be a representation generated by a non-zero vector v_0 for which

- $\mathbb{C}[t^1, t^{-1}] \otimes x \cdot v_0 = 0$ for all $x = p_1, p_2, \gamma$.
- $t^r\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta}\, d\mu_a(\theta)$ where μ_a is a (Radon) measure on S^1.
- $t^r\left(\begin{smallmatrix} h & 0 \\ 0 & -h \end{smallmatrix}\right) \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta}\, d\mu_\lambda(\theta)$ where μ_λ is a negative (Radon) measure on S^1.

PROPOSITION 5.6. *The above choice of an odd Borel subalgebra is the only one that may lead to exceptional unitarity. There is unitarity for a compact scalar highest weight representation for $C(2)$ exactly when the involution is given by ω^n_- and $\mu_\lambda \leqslant -|\mu_a|$.*

Proof. The statement about the odd Borels follows by considering the weights of the element $\left(\begin{smallmatrix} h & 0 \\ 0 & -h \end{smallmatrix}\right)$ in (108). The remaining part follows easily by direct computation or from Proposition 4.10. ∎

5.5. *$B(0, n)$, $B(1, n)$, $B(m, 1)$, and $D(m, 1)$.*

LEMMA 5.6. *For $B(0, n)$ there can be no exceptional unitarity, compact or non-compact, unless $n = 1$.*

Proof. This follows easily from Lemma 5.1 together with the fact that only C_1 can give rise to exceptional representations. ∎

Let us write

$$B(0, 1) = \left\{ \begin{pmatrix} h & \gamma & p \\ \delta & -h & -q \\ q & p & 0 \end{pmatrix} \right\}. \tag{110}$$

Let

$$\omega^n_{\pm}\, p = \pm q. \tag{111}$$

Let us define a *non-compact scalar* highest weight representation of $B(0, 1)$ to be a representation generated by a non-zero vector v_0 for which

- $\mathbb{C}[t^1, t^{-1}] \otimes x \cdot v_0 = 0$ for $x = p, \gamma$.
- $t^r h \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta}\, d\mu_\lambda(\theta)$ where μ_λ is a negative (Radon) measure on S^1.

PROPOSITION 5.7. *The above choice of odd Borel is the only one that may lead to unitarity. The non-compact scalar highest weight representation of* $B(0, 1)$ *is unitarizable exactly for* ω^n_- *and for* μ_λ *a negative (Radon) measure on* S^1.

Proof. This follows analogously to (and actually from) the similar case for $C(2)$. ▮

Let us now turn to $B(1, n)$. We write this algebra as

$$B(1,n)=\left\{\begin{pmatrix}\beta & \gamma & 0 & x\\ \delta & 0 & -\gamma & y\\ 0 & -\delta & -\beta & z\\ \tilde{z} & \tilde{y} & \tilde{x} & C_n\end{pmatrix}\right\}, \tag{112}$$

where the x, y, z denote (row) vectors in $\mathbb{C}^{2n}$, the column vector $\tilde{x}$ is x transposed followed by a sign change on the first n coordinates, and similarly for $\tilde{y}$, $\tilde{z}$.

We can now see why $B(1, n)$ is an exceptional case in Lemma 5.1. Suppose, namely, that we consider a highest weight representation which is trivial on $\mathbb{C}[t^1, t^{-1}]\otimes C_n$. Clearly, there are no Borel subalgebras which are invariant under this.

The upper right-hand 3×3 matrix in (112) evidently corresponds to an A_1 which we order in the usual way. Then, clearly, any Borel must contain (some) elements corresponding to the x's and some corresponding to the y's in (112). Since the annihilator of the highest weight vector must be invariant under $\mathbb{C}[t^1, t^{-1}]\otimes C_n$, we are clearly led to the following observation:

LEMMA 5.7. *In order to have a unitarizable scalar module of* $B(1, n)$ *based on a compact real form of* C_n, *the highest weight vector* v_0 *must satisfy*:

- $(\mathbb{C}[t^1, t^{-1}]\otimes g)\cdot v_0 = 0$ *for all* g *in* $\mathrm{Span}\{x, y\}\oplus C_n$.
- $t^r\begin{pmatrix}\beta & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -\beta\end{pmatrix}\cdot v_0=\int_{S^1} e^{i\cdot r\cdot\theta}\, d\mu_\lambda(\theta)$ *where* μ_λ *is a negative (Radon) measure on* S^1.

Above, we denote by x *and* y *the elements of* (112) *defined by* x, $\tilde{x}$ *and* y, $\tilde{y}$, *respectively.*

Proof. By the above remarks, we need only observe that we must have a non-compact real form on the top right A_1 to avoid everything being trivial. ▮

Let us write the vectors x, z in (112) as $x=(p_1, p_2)$ and $z=(q_2, -q_1)$, where the vectors p_1, p_2, q_1, q_2 belong to $\mathbb{C}^n$. (In this way, the (p_1, p_2) corresponds to (q_1, q_2) under ordinary complex conjugation applied to the elements in (112).) Let

DEFINITION 5.2. $\omega_{\pm}(p_i)=\pm q_i$, $i=1, 2$.

PROPOSITION 5.8. *There is unitarity for the representations in Lemma 5.7 precisely for the involution ω_-. For this involution, all the representations are unitary.*

Proof. To begin with we can establish the positivity on the span S of wedges of elements of the form $t^n z$ applied to v_0 just as in the proof of Proposition 5.5. Indeed, with appropriate sign changes, we can take over the result directly. We then need to examine the loop algebra $\mathbb{C}[t^1, t^{-1}]\otimes A_1$ for the A_1 corresponding to β, γ, δ in (112). In particular, we need to examine the action of the (non-standard) Borel subalgebra. This can be done along the lines in the proof of Proposition 5.1; i.e., using the positivity on all of S we establish the positivity on the whole module inductively by using an appropriate filtration by degree of S. The reason this procedure is successful is that S consists of vectors of negative weight with respect to the Cartan subalgebra in A_1 corresponding to the β. ∎

The cases $B(m, 1)$ and $D(m, 1)$ are, with the proper identifications, identical to $B(1, n)$. For this reason we omit all details and merely state

PROPOSITION 5.9. *There is a unique exceptional series of representations for each of the algebras $B(m, 1)$ and $D(m, 1)$. It is similar to the one for $B(1, n)$ described in Lemma 5.7.*

Consider now the special case of $B(1, 1)$. Let us write

$$B(1,1)=\left\{\begin{pmatrix} \beta & \gamma & 0 & q_1 & -p_3 \\ \delta & 0 & -\gamma & q_2 & -p_2 \\ 0 & -\delta & -\beta & q_3 & -p_1 \\ p_1 & p_2 & p_3 & h & z \\ q_3 & q_2 & q_1 & w & -h \end{pmatrix}\right\}, \tag{113}$$

where each entry is a complex number.

In this case, $\mathscr{G}_0=A_1\times A_1$; hence there is the possibility of having an exceptional representation on the "bottom right"-A_1, A_1^{br}, together with either a non-compact form or a compact form of the "top left"-A_1, A_1^{tl}.

PROPOSITION 5.10. *There are no non-trivial unitarizable highest weight modules for which both $\mathbb{C}[t^1, t^{-1}]\otimes A_1$'s are represented by non-trivial exceptional representations.*

Proof. The odd part of the Borel that should define such a representation must consist of weight vectors whose weights all are non-negative or non-positive with respect to the two Cartan subalgebras. There are no such Borel subalgebras. ∎

Referring to (113), let

DEFINITION 5.3. $\omega_\pm(p_i) = \pm q_i$, $i = 1, 2, 3$.

Define a highest weight representation of $B(1, 1)$ generated by a non-zero vector v_0 for which

- $\mathbb{C}[t^1, t^{-1}] \otimes x \cdot v_0 = 0$ for all x in $\operatorname{Span}\{p_1, p_2, p_3\} \oplus A_1''$, and
- $t^r\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) \cdot v_0 = \int_{S^1} e^{i \cdot r \cdot \theta}\, d\mu_\lambda(\theta)$ where μ_λ is a negative (Radon) measure on S^1, to be a *non-compact scalar* representation.

PROPOSITION 5.11. *There is unitarity of the non-compact scalar representations of* $B(1, 1)$ *exactly for the involution* ω_-. *In this case, there is unitarity for all representations.*

Proof. Naturally, the involution must be constant on the orbits of A_1''. The claim now follows along the same lines as the proof of Proposition 5.8. ∎

Finally there are the exceptional cases $D(2, n)$ and $D(3, n)$ where there is a possibility for exceptional series of representations based on the $A_1 \times A_1$, respectively A_3, summand in the even part. However, by arguments similar to those in the proof of Proposition 5.10 it is easy to see that these cases do not, after all, contribute. On the other hand, there are, naturally, the series from $D(m, 1)$ with $m = 2$ or $m = 3$ and also the series from $D(2, 1, \alpha)$ with $\alpha = 1$. (The latter algebra is isomorphic to $D(2, 1)$.)

PROPOSITION 5.12. *There are no exceptional series of representations for* $D(2, n)$ *and* $D(3, n)$ *other than those previously found.*

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