THE EXPONENTIAL NATURE AND POSITIVITY

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ABSTRACT. In the present article, a basis of the coordinate algebra of the multi-parameter quantized matrix is constructed by using an elementary method due to Lusztig. The construction depends heavily on an anti-automorphism, the bar action. The exponential nature of the bar action is derived which provides an inductive way to compute the basis elements. By embedding the basis into the dual basis of Lusztig's canonical basis of $U_q(n^-)$, the positivity properties of the basis as well as the positivity properties of the canonical basis of the modified quantum enveloping algebra of type A, which has been conjectured by Lusztig, are proved.

1. INTRODUCTION

The coordinate algebra of the multi-parameter quantized matrix has been introduced by Artin, Schelter and Tate [1]. In [8], Jakobsen and Zhang introduced a class of quadratic algebras which are not bi-algebras in general, but are quite similar to the coordinate algebra of the quantum matrix with one-parameter in the representation aspect. In the present paper, we construct a class of quadratic algebras $\mathcal{O}_{q,P,Q}(M(n))$ which includes the above two classes of algebras as special cases, by using bi-character deformation. The algebra $\mathcal{O}_{q,P,Q}(M(n))$ is not a bi-algebra in general. However, since we only consider the basis of the coordinate algebra, we need not to deal with the multiplication of the matrices. We will show that our algebra is a bi-character deformation of the quantum matrix algebra algebra considered in [19]. Hence, the quantum determinant and quantum minors can be defined in the exactly the same way as in [19] and all of the properties of the quantum minors of the so-called official quantum matrix algebras transfer to the minors of the present algebra after slight modifications. By using a method by Lusztig, we construct a nice basis of the algebra – the dual canonical basis – which is "invariant" under the multiplication of certain quantum minors. The main ingredient of the construction is a nice anti-automorphism, the bar action. We prove that the bar action has an exponential nature by introducing some simple operators $T_{ij}^{st}, \overline{T_{ij}^{st}}$. Hence, our construction provides an inductive algorithm for computing the basis elements. We then compute the bases for certain special cases, and propose a conjecture for the general case. Embedding the algebra $O_q(M(n))$ into the negative part $U_q(A_{2n-1})^-$ of the quantum enveloping algebra $U_q(A_{2n-1})$, we show that our dual canonical basis is a subset of the dual canonical basis (after a slight modification) of Lusztig's canonical basis; this provides an interpretation of the coefficients of the expansion of our dual canonical basis elements in terms of the modified monomials $Z(A)$ and enables us to prove the positivity properties

of our basis and, by duality, the positivity property of the canonical basis of the modified quantum enveloping algebra of type A is proved which was conjectured by Lusztig [16].

2. LUSZTIG'S CONSTRUCTION OF THE BASIS

Our method to construct the basis is a modification of Lusztig's construction in [15], see also [3]. We refer to this construction as Lusztig's elementary method.

Let Γ be an abelian group with a total ordering which is compatible with the group structure on Γ. Let Γ_+ be the set of elements of Γ which are strictly positive for this ordering and let $\Gamma_- = (\Gamma_+)^{-1}$. Let $a \mapsto \bar{a}$ be the involution of the group ring $\mathbb{Z}[\Gamma]$ which takes γ to γ^{-1}. Let V be a free $\mathbb{Z}[\Gamma]$ module with a basis $\{t_i | i \in I\}$, where the index set I has an ordering $\leq$. Assume that there is a map

$$- : V \longrightarrow V,$$

satisfying $\overline{a.v} = \bar{a}\bar{v}$ for all $a \in \mathbb{Z}[\Gamma]$ and $v \in V$. Furthermore, assume that

$$\bar{t}_i = \sum a_{ij} t_j$$

with $a_{ii} = 1$ and $a_{ij} \neq 0$ only if $j \leq i$. In [15], Lusztig proved that

Proposition 2.1. *Given $i \in I$, there is a unique element $b_i \in V$ such that*

$$\overline{b_i} = b_i,$$

and

$$b_i = \sum_{j \leq i} h_{ij} b_j,$$

where $h_{ii} = 1$ and for any $j < i$, $h_{ij} \in \mathbb{Z}[\Gamma^+]$. The elements b_i form a basis of V.

The coefficients h_{ij} satisfy a system of equations

(2.1)
$$
\begin{aligned}
h_{ii} &= 1, \\
\overline{h_{ij}} - h_{ij} &= \sum_{i<k<j} a_{ik} h_{kj}.
\end{aligned}
$$

Hence, the coefficients h_{ij} can be computed inductively, provided the a_{ij} are known.

3. THE BI-CHARACTER DEFORMATION AND THE MULTI-PARAMETER QUANTUM MATRIX SPACE

Let $P = (p_{ij})$ and $Q = (q_{ij})$ be matrices whose entries satisfy

$$p_{ii} = q_{ii} = p_{ij}p_{ji} = q_{ij}q_{ji} = 1,$$

where p_{ij}, q_{ij} $(1 \leq i < j \leq n)$, and, later, q, are independent variables. The base field in the rest of the paper is $K = \mathbb{Q}(q, p_{ij}, q_{ij} | i < j)$.

Let G be an abelian semi-group. A semi-group homomorphism from $G \times G$ to the multiplicative group K^* is called a bi-character of G.

Let A be an associative algebra with a $G \times G$ gradation:

$$A = \oplus_{g,h \in G} A_{g,h},$$

satisfying

$$A_{g_1,h_1} A_{g_2,h_2} \subset A_{g_1+g_2,h_1+h_2}$$

for all $g_1, g_2, h_1, h_2 \in G$.

Let ϕ and ψ be two bi-characters of G. For $a \in A_{g_1,h_1}$ and $b \in A_{g_2,h_2}$, one may define a new multiplication

$$a * b = \phi(g_1, g_2)\psi(h_1, h_2)ab.$$

It is easy to check that the new multiplication $*$ is associative and this defines a new associative algebra structure on A which is called a bi-character deformation of the algebra A.

Define the coordinate algebra $\mathcal{O}_{q,P,Q}(M(n))$ to be the associative algebra generated by n^2 generators Z_{ij} subject to the defining relations:

$$
\begin{aligned}
(3.1) \qquad Z_{st}Z_{ij} &= p_{si}^2 q_{tj}^2 Z_{ij}Z_{st} + (q^2 - 1)p_{si}^2 Z_{it}Z_{sj}, \text{ if } s > i, t > j, \\
Z_{st}Z_{ij} &= q^2 p_{si}^2 q_{jt}^{-2} Z_{ij}Z_{st}, \text{if } s > i, t \leq j, \\
Z_{it}Z_{ij} &= q_{tj}^2 Z_{ij}Z_{it},
\end{aligned}
$$

Remark 3.1. *We get the official 2×2 matrix algebra with the following choices:* $p_{21} = q^{-1/2}$; $q_{21} = q^{1/2}$ *(with "q^{-1} relations").*

Proposition 3.2. *The algebra $\mathcal{O}_{q,P,Q}(M(n))$ is a bi-character deformation of the coordinate algebra of the quantum matrix space of Dipper-Donkin ([2]).*

Proof: The coordinate algebra of the quantum matrix space of Dipper-Donkin is an associative algebra $D_q(n)$ generated by n^2 generators Z_{ij} subject to the defining relations:

$$
\begin{aligned}
(3.2) \qquad Z_{st}Z_{ij} &= Z_{ij}Z_{st} + (q^2 - 1)Z_{it}Z_{sj}, \text{ if } s > i, t > j, \\
Z_{st}Z_{ij} &= q^2 Z_{ij}Z_{st}, \text{if } s > i, t \leq j, \\
Z_{it}Z_{ij} &= Z_{ij}Z_{it},
\end{aligned}
$$

Let G be the semi-group $\mathbb{Z}_+^n$ with standard basis $e_1, e_2, \cdots, e_n$. The algebra $D_q(n)$ is $G \times G$-graded with

$$deg Z_{ij} = (e_i, e_j).$$

Let ϕ and ψ be semi-group homomorphisms defined by

$$
\begin{aligned}
(3.3) \qquad \phi : G \times G &\longrightarrow K^*, \\
(e_i, e_j) &\mapsto p_{ij}, \text{ for all } i, j.
\end{aligned}
$$

and

$$(3.4) \qquad\qquad \psi : G \times G \ \longrightarrow \ K^*,$$
$$(e_i, e_j) \ \mapsto \ q_{ij}, \text{ for all } i, j.$$

From the defining relations, one can see clearly that the algebra $\mathcal{O}_{q,P,Q}(M(n))$ is a bi-character deformation of the algebra $D_q(n)$. $\qquad\qquad\qquad\square$

Remark 3.3. *The algebra* $\mathcal{O}_{q,P,Q}(M(n))$ *is an iterated Ore extension and hence a noetherian domain. If we put* $p_{ij} = q_{ji}$ *we get the algebras constructed by Artin-Schelter-Tate. The algebras considered in* [8] *are also special cases of the present algebras.*

Our construction of the basis depends heavily on a bar action. The following can be obtained easily from the defining relations of the algebra $\mathcal{O}_{q,P,Q}(M(n))$.

Lemma 3.4. *The assignment*

$$(3.5) \qquad\qquad ^- : \mathcal{O}_{q,P,Q}(M(n)) \longrightarrow \mathcal{O}_{q,P,Q}(M(n)),$$
$$Z_{ij} \mapsto Z_{ij}, q \mapsto q^{-1}, p_{ij} \mapsto p_{ji}, q_{ij} \mapsto q_{ji}$$

extends to an algebra anti-automorphism over $\mathbb{Q}$.

Two elements x, y in the algebra $\mathcal{O}_{q,P,Q}(M(n))$ are called equivalent if there is a monomial m of $q, q^{-1}, p_{ij}, q_{ij}$ such that $x = my$. In this case, we write $x \sim y$.

For any matrix $A = (a_{ij}) \in M_n(\mathbb{Z}_+)$, we define the monomial $Z^A = \Pi Z_{ij}^{a_{ij}}$, where the factors are arranged according to the lexicographic ordering.

Using Bergman's diamond lemma, we see that the algebra $\mathcal{O}_{q,P,Q}(M(n))$ has the nice basis

$$\{Z^A | A \in M_n(\mathbb{Z}_+)\}.$$

However, the above basis is only almost the right choice for our construction; another normalization will be needed:

Let

$$D(A) = q^{\sum_{s>i,j\leq t} a_{st}a_{ij}} \Pi_{s>i} p_{si}^{a_{st}a_{ij}} \Pi_{t>j,s\leq i} q_{tj}^{a_{st}a_{ij}} \Pi_{s>i,j>t} q_{jt}^{-a_{st}a_{ij}},$$

we define the normalized monomial $Z(A) = D(A)Z^A$. For matrices A and B in $M_n(\mathbb{Z}_+)$, we define $B \leq A$ if B can be obtained from A by a sequence of 2×2 sub-matrix moves of the form

$$(3.6) \qquad\qquad \begin{pmatrix} a_{ij} & a_{it} \\ a_{sj} & a_{st} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{ij} - 1 & a_{it} + 1 \\ a_{sj} + 1 & a_{st} - 1, \end{pmatrix}$$

where $a_{ij}, a_{st} \geq 1$. Denote by c_i the sum of the elements in the ith column and r_i the sum of the elements in the ith row. Notice that if $B \leq A$, then A, B have the same row sums and column sums.

From the defining relations of the algebra we have

$$\overline{Z(A)} = Z(A) + \sum_B c_{AB} Z(B)$$

where $c_{AB} \in \mathbb{Z}[q, q^{-1}, p_{ij}, q_{ij}]$ and $c_{AB} \neq 0$ only if $B \leq A$.

Let Γ be the subgroup of K^* generated by q, p_{ij}, q_{ij} for all $i, j = 1, 2, \cdots, n$. Define an ordering on the monomials of parameters by $q < p_{ij} < q_{st} < 0$ for $i < j; s < t$, and extend to a lexicographic ordering on Γ which is compatible with the group structure on Γ. Denote by Γ_+ the set of strictly positive elements and Γ_- the set of strictly negative elements. Clearly, $\Gamma_- = (\Gamma_+)^{-1}$.

Using Lusztig's method in section 2, we get

Theorem 3.5. *For each $A \in M_n(\mathbb{Z}_+)$, there is a unique element $b(A)$ characterized by the following properties:*

(1) $b(A) = Z(A) + \sum_B h_{AB} Z(B)$, *where $h_{AB} \in \mathbb{Z}[\Gamma_+]$ and $h_{AB} \neq 0$ only if $B \leq A$.*

(2) $\overline{b(A)} = b(A)$.

The set $B^* = \{b(A) | A \in M_n(\mathbb{Z}_+)\}$ is a $\mathbb{Q}(q, p_{ij}, q_{ij} | i, j = 1, 2 \cdots, n)$ basis of the algebra $\mathcal{O}_{q,P,Q}(M(n))$.

Remark 3.6. *Later on, after proving the exponential nature we shall see that the coefficients h_{AB} are in fact polynomials of q^{-1}. This fact justifies the word canonical.*

4. THE CASE OF 2×2

The bar action on the monomials $Z(A)$ is simply a reordering of the generators. The purpose of introducing the normalized monomial is that it allows us to ignore the quasi-polynomial moves. Clearly, only the first relation in the defining relations produces new terms in the process of reordering the generators. However, this relation only involves four generators. In other words, only a 2×2 sub-matrix of the matrix A is involved. Hence, the bar action can somehow be computed locally which means we should first consider the 2×2 case. The coordinate algebra of 2×2 quantum matrix is an algebra with four generators and relations:

$$
\begin{aligned}
(4.1) \qquad Z_{22}Z_{11} &= p_{21}^2 q_{21}^2 Z_{11}Z_{22} + (q^2 - 1)p_{21}^2 Z_{12}Z_{21}, \\
Z_{22}Z_{21} &= q_{21}^2 Z_{21}Z_{22}, \\
Z_{12}Z_{11} &= q_{21}^2 Z_{11}Z_{12} \\
Z_{22}Z_{12} &= q^2 p_{21}^2 Z_{12}Z_{22}, \\
Z_{21}Z_{12} &= q^2 p_{21}^2 q_{21}^{-2} Z_{12}Z_{21}, \\
Z_{21}Z_{11} &= q^2 p_{21}^2 Z_{11}Z_{21}.
\end{aligned}
$$

For $A = (a_{ij})_{2 \times 2} \in M_2(\mathbb{Z}_+)$, set

$$Z^A = Z_{11}^{a_{11}} Z_{12}^{a_{12}} Z_{21}^{a_{21}} Z_{22}^{a_{22}}.$$

The set $\{Z^A | A = (a_{ij})_{2 \times 2} \in M_2(\mathbb{Z}_+)$ is a basis of the algebra. Denote by $\mathrm{tr}(A) = a_{11} + a_{22}$ the trace of A and $\mathrm{str}\, A = a_{11} - a_{22}$ the super trace of A.

Define

$$D(A) = q^{(a_{11}a_{21}+a_{12}a_{21}+a_{12}a_{22})} p_{21}^{(a_{11}a_{21}+a_{11}a_{22}+a_{12}a_{21}+a_{12}a_{22})} q_{21}^{(a_{11}a_{12}+a_{11}a_{22}-a_{12}a_{21}+a_{21}a_{22})}$$

and

$$Z(A) = D(A) Z^A$$

The 2×2 quantum determinant is

$$\det_q = Z_{11} Z_{22} - q_{21}^{-2} Z_{12} Z_{21}$$

which satisfies

$$(4.2) \quad \det_q \cdot Z_{11} = p_{21}^2 q_{21}^2 \cdot Z_{11} \cdot \det_q \quad , \quad Z_{22} \cdot \det_q = p_{21}^2 q_{21}^2 \cdot \det_q \cdot Z_{22}$$
$$\det_q \cdot Z_{12} = q^2 p_{21}^2 q_{21}^{-2} \cdot Z_{12} \cdot \det_q \quad , \quad Z_{21} \cdot \det_q = q^2 p_{21}^2 q_{21}^{-2} \cdot \det_q \cdot Z_{21}.$$

The normalized determinant

$$(4.3) \quad \Delta = p_{21} q_{21} \cdot \det_q$$

$$(4.4) \quad = p_{21} q_{21} \cdot Z_{11} Z_{22} - p_{21} q_{21}^{-1} \cdot Z_{12} Z_{21} = Z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - q^{-1} Z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is more appropriate for our computations.

Examples:

$$b \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = Z \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - q^{-2} Z \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$b \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = Z \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - q^{-1} Z \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},$$

$$b \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = Z \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - q^{-1} Z \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

In the sequel we let $E = E_{12} + E_{21}$. Notice that the quantities $r_2 - r_1 = a_{21} + a_{22} - a_{11} - a_{12}$, $c_2 - c_1 = a_{12} + a_{22} - a_{11} - a_{21}$, and $a_{21} - a_{12}$ are the same for A, $A \pm I$, and $A \pm E$. Set $f_A = q^{a_{21}-a_{12}} p_{21}^{r_2-r_1} q_{21}^{c_2-c_1}$.

Lemma 4.1. $Z(A) \Delta = f_A \left(Z(A + I) - q^{-\mathrm{tr}(A)-1} Z(A + E) \right).$

Proof: This follows by an elementary computation using (4.1) and (4.2). □

The next result is proved by similar arguments:

Lemma 4.2. $\overline{\Delta} = \Delta$. *Moreover,*

$$Z^A \Delta = f_A^2 \Delta Z^A.$$

Some of the following is well-known ([4], [5], [12]), but the connection to canonical bases seems to be new.

Definition 4.3. *We set*
$$(n)_q = 1 + q^2 + \cdots + q^{2n-2} \text{ if } n \geq 2; (1)_q = 1, \text{ and } (0)_q = 1.$$

Definition 4.4. *We set*
$$\exp_q(X) = \sum_{n=0}^{\infty} \frac{X^n}{(n)_q!}.$$

Lemma 4.5. *If* $XY = q^2 YX$, *then*
$$(X + Y)^n = \sum_{i=0}^{n} \binom{n}{i}_q X^i Y^{n-i},$$
where the coefficients satisfy
$$\binom{n+1}{i}_q = \binom{n}{i}_q + q^{2n-2i+2} \binom{n}{i-1}_q,$$
and are given by
$$\binom{n}{i}_q = \frac{(n)_q!}{(i)_q(n-i)_q}.$$

Lemma 4.6. *These quantized binomial coefficients satisfy the following identity as follows easily by induction:*
$$\sum_{m=0}^{s} (-1)^m q^{m(m-1)} \binom{s}{m}_q = 0.$$

Proposition 4.7. *If* $YX = q^2 XY$, *then*
$$\exp_q(X + Y) = \exp_q(X) \exp_q(Y).$$
This follows easily from Lemma 4.5 above.

Proposition 4.8.
$$\exp_q(X) \exp_{q^{-1}}(-X) = 1.$$
Proof: This follows easily from Lemma 4.6 above. □

If we introduce change-of-basis matrices between the canonical basis and the PBW basis,

$$
\begin{aligned}
Z(A) &= \sum_C T_{CA} b(C) \\
\overline{Z(A)} &= \sum_D h_{DA} Z(D) \\
&= \sum_C \overline{T_{CA}} b(C), \quad \text{thus,} \\
TH &= \overline{T}, \text{ i.e.} \\
H &= T^{-1}\overline{T}.
\end{aligned}
$$

Define operators t and $\bar{t}$ as follows: For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{Z}_+)$ with $a_{11}a_{22} > 0$, define

$$
A' = \begin{pmatrix} a_{11} - 1 & a_{12} + 1 \\ a_{21} + 1 & a_{22} - 1 \end{pmatrix}
$$

The linear operator t is given by

$$
t(Z(A)) = \left(\sum_{s=|\operatorname{str} A|+1}^{\operatorname{tr}(A)-1} q^{-s} \right) Z(A')
$$

if $a_{11}a_{22} > 0$, and zero otherwise. The linear operator $\bar{t}$ is given by

$$
\bar{t}(Z(A)) = \left(\sum_{s=|\operatorname{str} A|+1}^{\operatorname{tr}(A)-1} q^{s} \right) Z(A')
$$

if $a_{11}a_{22} > 0$, and zero otherwise.

The first result is straightforward and we omit the proof:

Lemma 4.9. $\bar{t}t = q^{-2}t\bar{t}$.

Let us define $\tau_C^{(i)}$ and $\mu_C^{(i)}$ by

$$
\begin{aligned}
t^i(Z(C)) &= \tau_C^{(i)} \cdot Z(C - iI + iE) \text{ and} \\
(-t + \bar{t})^i(Z(C)) &= \mu_C^{(i)} \cdot Z(C - iI + iE),
\end{aligned}
$$

respectively. The first fundamental observation is

Lemma 4.10 (Key).

$$(4.5) \quad \frac{\tau_{A+I}^{(r)}}{(r_{q^{-1}})!} = \left(\frac{\tau_A^{(r)}}{(r_{q^{-1}})!} + q^{-\operatorname{tr}(A-(r-1)I)-1} \frac{\tau_A^{(r-1)}}{((r-1)_{q^{-1}})!} \right),$$

$$(4.6) \quad \frac{\mu_{A+I}^{(s)}}{(s_{q^{-1}})!} =$$

$$\frac{\mu_A^{(s)}}{(s_{q^{-1}})!} - q^{-\operatorname{tr}(A-(s-1)I)-1} \frac{\mu_A^{(s-1)}}{((s-1)_{q^{-1}})!} + q^{\operatorname{tr}(A)+1} \frac{\mu_A^{(s-1)}}{((s-1)_{q^{-1}})!} .$$

Proof: Let us set $A_{11} = a$ and $a_{22} = b$. We assume that $a \geq b$. Then,

$$tZ(A) = q^{-a+b-1}(b)_{q^{-1}} Z(A - I + E).$$

Equation (4.5) follows from this and the simple identity

$$(b+1)_{q^{-1}} = (b-r+1)_{q^{-1}} + (r)_{q^{-1}} q^{-2b+2r-2}.$$

The second identity follows by a simple computation from the observation that

$$(\bar{t} - t)Z(A) = \frac{(q^a - q^{-a})(q^b - q^{-b})}{q - q^{-1}} Z(A - I + E).$$

$\square$

Proposition 4.11. *In the case at hand,*

$$H = \exp_{q^{-1}}(-t + \bar{t}).$$

Moreover, $T = \exp_q(t)$, $\overline{T} = \exp_{q^{-1}}(\bar{t})$, *and* $T^{-1} = \exp_{q^{-1}}(-t)$.

Proof: The claim is that

$$(4.7) \quad \overline{Z(A+I)} = \sum_s \frac{\mu_{A+I}^{(s)}}{(s_{q^{-1}})!} Z(A + I - sI + sE).$$

It follows easily from Lemma 4.1 and Lemma 4.2 that

$$(4.8) \quad \overline{Z(A+I)} = f_A^{-1} \overline{Z(A)} \Delta + q^{\operatorname{tr}(A)+1} \overline{Z(A+E)}.$$

Applying (4.6), and Lemma 4.1 once again, the claim follows by induction on $\operatorname{tr}(A)$. $\square$

We have:

Lemma 4.12. *For any matrix* $D \in M_n(\mathbb{Z}_+)$,

$$b(D) = \sum_C (T^{-1})_{C,D} Z(C).$$

Proof: It is clear that $\sum_C (T^{-1})_{C,D} Z(C)$ is bar invariant with the right leading term $Z(D)$. $\qquad\square$

Proposition 4.13.

(4.9) $$b(A) \cdot (f_A^{-1} \Delta) = b(A + I).$$

Proof: The claim is that

$$\sum_r \frac{(-1)^r \tau_{A+I}^{(r)}}{(r_{q^{-1}})!} Z(A + I - rI + rE) = \sum_s \frac{(-1)^s \tau_A^{(s)}}{(s_{q^{-1}})!} Z(A - sI + sE) \cdot (f_A^{-1} \Delta).$$

Using Lemma 4.1, this follows from (4.5) by induction or $\mathrm{tr}(A)$. $\qquad\square$

5. The exponential nature

In this section, we compute the matrix of the bar action with respect to the basis $\mathbb{B}_0$ consisting of normalized monomials for general n. For $1 \le i, j, s, t \le n$ with $i < s, j < t$ we define a linear operator T_{ij}^{st} as follows:

(5.1) $$T_{ij}^{st}(Z(A)) = \left(\sum_{s=|a_{ij}-a_{st}|+1}^{a_{ij}+a_{st}-1} q^{-s} \right) Z(A'), \text{ if } a_{ij} a_{st} \ge 1,$$

$$T_{ij}^{st}(Z(A)) = 0, \text{ if } a_{ij} a_{st} = 0.$$

Similarly, an operator $\overline{T_{ij}^{st}}$ is defined as:

(5.2) $$\overline{T_{ij}^{st}}(Z(A)) = \left(\sum_{s=|a_{ij}-a_{st}|+1}^{a_{ij}+a_{st}-1} q^{s} \right) Z(A') \text{ if } a_{ij} a_{st} \ge 1,$$

$$\overline{T_{ij}^{st}}(Z(A)) = 0 \text{ if } a_{ij} a_{st} = 0.$$

The bar action maps $\mathbb{B}_0$ into the "ultimately opposite" basis $\mathbb{B}_u$ given by the elements $\overline{Z(A)}$.

We know that for $n = 2$, the matrix of the bar action is given by $\mathbb{H}_2 = \exp_q(-T + \overline{T})$ as described in the last section.

To facilitate the study of the bar action, we introduce a series of intermediate PBW bases $\mathbb{B}_{k+1} = S_{i_k,j_k}^{s_k,t_k} \mathbb{B}_k$, where $S_{i_k,j_k}^{s_k,t_k}$ is a linear map which is applied to each of the vectors in the given basis. It is the map which sends any **normalized** monomial of the form $\ldots Z_{i_k j_k}^{a_{i_k j_k}} \ldots Z_{i_k,t_k}^{a_{i_k t_k}} \ldots Z_{s_k j_k}^{a_{s_k j_k}} \ldots Z_{s_k t_k}^{a_{s_k t_k}}$ into the corresponding normalized expression $\ldots Z_{s_k t_k}^{a_{s_k t_k}} \ldots Z_{s_k,j_k}^{a_{s_k j_k}} \ldots Z_{i_k t_k}^{a_{i_k t_k}} \ldots Z_{i_k j_k}^{a_{i_k j_k}}$. If we assume, and this condition will always be satisfied in the applications, that the interior ellipses represent terms which quasi-commute with the elements $Z_{i_k j_k}, Z_{i_k,t_k}, Z_{s_k j_k}$, and

$Z_{s_k t_k}$, then the matrix $\mathbb{S}_{i_k,j_k}^{s_k,t_k}$ of the map $S_{i_k,j_k}^{s_k,t_k}$ with respect to the basis $\mathbb{B}_k$ is precisely $\mathbb{H}_2$ tensored appropriately with an identity operator representing all the variables which stay fixed. $\mathbb{S}_{i_k,j_k}^{s_k,t_k}$ is also the change of basis matrix from $\mathbb{B}_{k+1}$ to $\mathbb{B}_k$.

To give the matrix $\mathbb{H}$ of the bar action in the basis $\mathbb{B}_0$ is the same as giving the change of basis matrix from $\mathbb{B}_u$ to $\mathbb{B}_0$.

Theorem 5.1. *For any matrix $A \in M_n(\mathbb{Z}_+)$,*

$$(5.3) \qquad \mathbb{H} = \Pi_{(i,j),(s,t),i<s,j<t} \mathbb{S}_{ij}^{st}$$

where the factors are arranged according to the double lexicographic ordering which means that we first use lexicographic ordering on the indices (i,j) and then lexicographic ordering of indices (s,t). The matrices $\mathbb{S}_{ij}^{st}$ are viewed as above.

Proof: One only needs to verify that at each step, the ellipses mentioned in the discussion above indeed do represent quasi-commuting element. This is elementary. $\square$

Remark 5.2. *There are at least three other such decomposition, namely where one uses the opposite ordering in one or both places.*

Theorem 5.1 shows that the bar action on the normalized monomials only depends on q, so the canonical basis only depends on the parameter q. This also means that for the multi-parameter case, the expression of the dual canonical basis is exactly the same as that expression in the one-parameter case. In [21], it is proved that the dual canonical basis is invariant under the multiplications of certain covariant quantum minors which is also true by the above theorem.

Letting $p_{ij} = q_{ij}$, we get the algebra constructed in [1] which is a bi-algebra with the usual coproduct. Inverting the quantum determinant we get the quantum function algebra which is dual to the quantum enveloping algebra $U_{q,P}(gl_n)$ (see [1] for more detail), as well as a basis $\bar{B}$ of this quantum function algebra. The dual basis of $\bar{B}$ is a basis of $U_{q,P}(gl_n)$ and this basis also only depends on the parameter q.

The upper triangular case enable us to construct a basis of $U_q(n^+)$ which should be the canonical basis constructed by Lusztig. To this end we need to show that the basis consisting of the images of $Z(A)$ is dual to the PBW basis consisting of divided powers.

6. AN INDUCTIVE PROGRAM

We only need to consider the official one. For any matrices $A, B \in M_n(\mathbb{Z}_+)$, there exist $d_{AB}, d_{BA} \in \mathbb{Z}$ such that

$$Z(A)Z(B) = q^{d_{AB}} Z(A+B) + \text{ lower order terms}$$

and
$$Z(B)Z(A) = q^{d_{BA}} Z(A + B) + \text{ lower order terms.}$$

A direct computation shows that $d_{AB} = -d_{BA}$.

Assume that both $b(A)$ and $b(B)$ are known, then $\frac{qb(A)b(B) - q^{-1}b(B)b(A)}{q - q^{-1}}$ is invariant under the bar action and so
$\frac{q^{1-d_{A,B}}b(A)b(B) - q^{d_{A,B}-1}b(B)b(A)}{q - q^{-1}} = b(A + B) + \sum_{D < A+B} c^D_{A,B} b(D)$ with the coefficients $c^D_{A,B} \in \mathbb{Z}[q + q^{-1}]$. Hence the element $b(A + B)$ can be determined uniquely by the above equation.

After showing the exponential nature, we see that the expression of the dual canonical basis elements do not depend on the choice of the parameters. Hence, the dual canonical basis is stable (up to the equivalence relation) under multiplication by covariant minors.

Now, let us compute the basis for 2×3 matrices. After the removal of the covariant minors, the only case we need to compute directly is for the sub-matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \end{pmatrix}$$

(1) If $a \leq b$, the basis element is
$$q^{-bc}(Z_{11}Z_{22} - q^2 Z_{12}Z_{21})^a Z_{22}^{b-a} Z_{23}^c$$

(2) If $a > b$ and $a \geq b + c$, then the basis element is
$$q^{-bc} Z_{11}^{a-b-c}(Z_{11}Z_{22} - q^2 Z_{12}Z_{21})^b(Z_{11}Z_{23} - q^2 Z_{13}Z_{21})^c$$

(3) if $a > b$ but $a < b + c$, then the basis element is
$$q^{-bc}(Z_{11}Z_{22} - q^2 Z_{12}Z_{21})^b(Z_{11}Z_{23} - q^2 Z_{13}Z_{21})^{a-b} Z_{23}^{c-a+b}$$

7. A CONJECTURE

Given a matrix $A \in M_n(\mathbb{Z}_+)$ we can draw a graph $H(A)$. The nodes of the graph are the matrices B which can be obtained from A by 2×2 matrix transformations of the form (3.6). We place B_1 on a level above B_2 if B_2 can be obtained from B_1. We draw a line between two nodes if we get the lower node from the upper one by one transformation; we attach to the line the upper indices (i, j) and lower indices (s, t) with the obvious meaning. Notice that, like at the top, there is just one graph at the bottom of the graph, say T_A; the tail of A.

A path in $H(A)$ is called principal path from A to B if it is a longest path from A to B and is maximal (according to the lexicographic order) among all of the longest paths in $H(A)$ from A to B.

We use $p_{ij}^{st}(B)$ to denote the number of lines in the principal path from A and B with indices $(i,j),(s,t)$. Denote by l_{AB} the number of lines (the length) from A to B in the principal path.

Conjecture 7.1.

$$b(A) = \sum_{B \leq A} (-q^{-1})^{l_{AB}} \Pi_{(i,j),(s,t)i<s,j<t} q^{-|a_{ij}-a_{st}|} \begin{pmatrix} min\{a_{ij}, a_{st}\} \\ p_{ij}^{st}(B) \end{pmatrix}_{q^{-2}} Z(B).$$

We only need to prove that the elements are bar invariant. A direct computation shows that the conjecture hold for the cases 2×2 and 2×3.

8. THE POSITIVITY

The negative part $U_q(n^-)$ is the subalgebra of quantum enveloping algebra $U_q(A_{2n-1})$ generated by $F_1, F_2, \cdots, F_{2n-1}$ subject to the quantum Serre relations:

$$(8.1) \qquad\qquad\qquad F_i F_j = F_j F_i, \text{ if } |i - j| > 1,$$
$$F_i^2 F_j - (q^2 + q^{-2})F_i F_j F_i + F_j F_i^2 = 0, \text{ if } |i - j| = 1.$$

For a homogeneous element x, denote by $wt(x)$ the weight of x.

Let $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}\}$ be the set of simple roots of the Lie algebra of type A_{2n-1}. For a positive root $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$, the corresponding root vector is defined as

$$F_{\alpha_i} = F_i, F_{ij} = F_{\alpha_{ij}} := [F_j, [F_{j-1}, [\cdots, [F_{i+1}, F_i]_q \cdots]_q]_q]_q.$$

The quantum commutator is given by

$$[x, y]_q = xy - q^{-2(\alpha,\beta)}yx,$$

for homogeneous elements x and y with weights α and β respectively.

We introduce an ordering on the set of positive roots Δ_+ by

$$\alpha_{ij} \leq \alpha_{kl}, \text{ if } j < l \text{ or } j = l \text{ and } i < k.$$

Notice that this ordering coincides with the ordering given by the reduced expression of the longest element $w_0 = r_1 r_2 \cdots r_{2n-1} r_2 \cdots r_{2n-1} \cdots r_{2n-2} r_{2n-1} r_1$ in the Weyl group, where r_i is the simple reflection determined by the simple root α_i. The PBW basis of $U_q(n^-)$ is indexed by the set $\mathbb{Z}_+^{\Delta_+}$.

For an $\mathbf{m} = (m_{ij}) \in \mathbb{Z}_+^{\Delta_+}$, denote by $\deg \mathbf{m} = \sum m_{ij}(j - i + 1)$. The PBW basis element indexed by $\mathbf{m}$ is

$$F(\mathbf{m}) := \Pi \frac{F_{ij}^{m_{ij}}}{[m_{ij}]_{q^2}!},$$

where the factors are arranged according to the above ordering on the set Δ_+. Denote by $|\mathbf{m}|$ the weight of $F(\mathbf{m})$.

The tensor product $U_q(n^-) \otimes U_q(n^-)$ can be regarded as a $\mathbb{Q}(q)$-algebra with multiplication

$$(x_1 \otimes x_2)(x_1' \otimes x_2') = q^{2(wt(x_2), wt(x_1'))} x_1 x_1' \otimes x_2 x_2',$$

for homogeneous elements $x_1, x_2, x_1', x_2' \in U_q(n^-)$.

In [17], it was proved that

Lemma 8.1. *The following assignment*

$$(8.2) \qquad\qquad r : U_q(n^-) \longrightarrow U_q(n^-) \otimes U_q(n^-)$$
$$F_i \mapsto F_i \otimes 1 + 1 \otimes F_i, \text{ for all } i$$

extends to an algebra homomorphism.

Remark 8.2. *The algebra homomorphism r is co-associative.*

There is a scalar product on $U_q(n^-)$ (see [17]) satisfying

$$(F_i, F_j) = \delta_{ij}, (x, y_1 y_2) = (\Delta(x), y_1 \otimes y_2), (x_1 x_2, y) = (x_1 \otimes x_2, \Delta(y)),$$

where the scalar product on $U_q(n^-) \otimes U_q(n^-)$ is given by

$$(x_1 \otimes x_2, y_1 \otimes y_2) = (x_1, y_1)(x_2, y_2).$$

On the PBW basis, the scalar product is given by

$$(F(\mathbf{m}), F(\mathbf{n})) = \frac{(1 - q^4)^{\deg \mathbf{m}}}{\Pi_{i \leq j} \phi_{m_{ij}}(q^4)} \delta_{\mathbf{m}, \mathbf{n}},$$

where $\phi_k(z) = (1 - z)(1 - z^2) \cdots (1 - z^k)$.

Let

$$\mathcal{L} := \oplus_{m \in \mathbb{Z}_+^{\Delta_+}} \mathbb{Z}[q] F(\mathbf{m}).$$

Denote by $-$ the ring automorphism:

$$(8.3) \qquad\qquad - : U_q(n^-) \longrightarrow U_q(n^-),$$
$$\overline{F_i} = F_i, \overline{q} = q^{-1} \text{ for all } i.$$

The canonical basis $B = \{G(\mathbf{m})\}$ (the lower global crystal basis in Kashiwara's terminology) of $U_q(n^-)$ is a $\mathbb{Z}[q]$ basis of $\mathcal{L}$ such that

$$\overline{G(\mathbf{m})} = G(\mathbf{m}), G(\mathbf{m}) = F(\mathbf{m}) \qquad \mod q\mathcal{L}.$$

Lusztig proved that the canonical basis enjoys some remarkable properties:

Theorem 8.3. *(Positivity, [17])The following hold.*

(1) *For any $b, b' \in B$, we have*

$$bb' = \sum_{b'' \in B, n \in \mathbb{Z}} c_{b,, b', b'', n} q^n b'',$$

where $c_{b,, b', b'', n} \in \mathbb{Z}_+$ are zero except for finitely many b'', n.

(2) *For any $b \in B$, we have*

$$r(b) = \sum_{b', b'' \in B, n \in \mathbb{Z}} d_{b, b', b'', n} q^n b' \otimes b'',$$

where $d_{b, b', b'', n} \in \mathbb{Z}_+$ are zero except for finitely many b', b'', n.

(3) *For any $b, b' \in B$ we have*

$$(b, b') = \sum_{n \in \mathbb{Z}_+} f_{b, b', n} q^n$$

where $f_{b, b', n} \in \mathbb{Z}_+$.

The canonical basis B is almost orthogonal with respect to the above scalar product. By almost orthogonal one means that

$$(G(\mathbf{m}), G(\mathbf{n})) = \delta_{\mathbf{mn}} \quad \mod q\mathcal{A},$$

where $\mathcal{A}$ is the subring of $\mathbb{Q}(q)$ consisting of the rational functions regular at $q = 0$.

Denote by $\{F^*(\mathbf{m})\}$ and $\{G^*(\mathbf{m})\}$ the dual bases of $\{F(\mathbf{m})\}$ and $\{G(\mathbf{m})\}$ respectively with respect to the above scalar product. Since $\{F(\mathbf{m})\}$ is orthogonal, the basis $\{F^*(\mathbf{m})\}$ is simply a rescaling of $\{F(\mathbf{m})\}$, namely

$$F^*(\mathbf{m}) = \Pi q^{\binom{m_{ij}}{2}} F_{ij}^{* \, m_{ij}},$$

where

$$F_{ij}^* = (1 - q^4)^{i - j} F_{ij}.$$

The dual canonical basis $\{G^*(\mathbf{m})\}$ can be characterized by conditions similar to those defining the canonical basis. Let

$$\mathcal{L}^* := \oplus_{\mathbf{m}} \mathbb{Z}[q] F^*(\mathbf{m}).$$

Let Φ be the anti-automorphism of $U_q(n^-)$ such that $\Phi(F_i) = F_i$, and $\Phi(q) = q^{-1}$. In [14], it was proved that

Proposition 8.4. *Let $\mathbf{m} \in \mathbb{Z}_+^{\Delta_+}$ and write $|\mathbf{m}|^2 := (wt(\mathbf{m}), wt(\mathbf{m}))$. Then $G^*(\mathbf{m})$ is the unique homogeneous element of degree $wt(\mathbf{m})$ of $U_q(n^-)$ satisfying*

$$\Phi(G^*(\mathbf{m})) = q^{2 \deg \mathbf{m} - |\mathbf{m}|^2} G^*(\mathbf{m}), G^*(\mathbf{m}) = F^*(\mathbf{m}) \quad \mod q\mathcal{L}^*.$$

The dual canonical basis of $U_q(n^-)$ can be constructed using Lusztig's elementary method by modifying the above construction. Let $F_N^*(\mathbf{m}) = q^{\frac{1}{2}|\mathbf{m}|^2 - \deg \mathbf{m}} F^*(\mathbf{m})$ and let $G_N^*(\mathbf{m}) = q^{\frac{1}{2}|\mathbf{m}|^2 - \deg \mathbf{m}} G^*(\mathbf{m})$. Let

$$\mathcal{L}_N^* = \oplus_m \mathbb{Z}[q] F_N^*(\mathbf{m}).$$

Then the above proposition can be rewritten as

Proposition 8.5. *Let* $\mathbf{m} \in \mathbb{Z}_+^{\Delta_+}$. *Then* $G_N^*(\mathbf{m})$ *is the unique homogeneous element of degree* $wt(\mathbf{m})$ *of* $U_q(n^-)$ *satisfying*

$$\Phi(G_N^*(\mathbf{m})) = G_N^*(\mathbf{m}), G_N^*(\mathbf{m}) = F_N^*(\mathbf{m}) \qquad \mod q\mathcal{L}_N^*.$$

The basis $\{G_N^*(\mathbf{m})|\mathbf{m} \in \mathbb{Z}_+^{\Delta_+}\}$ is called the normalized dual canonical basis of $U_q(n^-)$.

Theorem 8.6. *The multiplication of the basis* B^* *of the algebra* $O_q(M(n))$ *satisfies a positivity property analogous to 1 in Theorem 8.3.*

Proof: By duality, the positivity properties hold for the dual canonical basis of the canonical basis of $U_q(A_{2n-1})^-$. Hence, The multiplication of the normalized dual canonical basis $\{G_N^*(\mathbf{m})|\mathbf{m} \in \mathbb{Z}_+^{\Delta_+}\}$ has the positivity property.

Define inductively

$$(8.4) \qquad\qquad F_{i,i}^* = F_i \ , \qquad \text{and}$$

$$(8.5) \qquad\qquad F_{i,j}^* = \frac{qF_j F_{i,j-1}^* - q^{-1} F_{i,j-1}^* F_j}{q^2 - q^{-2}}.$$

A simple induction on $j - i$ gives that the F_{ij}^*'s are fixed by Φ. It is easy to see that the F_{ij}^* satisfy the defining equations of the quantum matrix indeed, that one may embed the algebra $O_q(M(n))$ into $U_q(n^-)$ as a subalgebra by:

$$Z_{ij} \mapsto F_{i,j+n}^* \text{ for all } i, j = 1, 2, \cdots, n.$$

Under this embedding, the PBW basis $\{Z(A)|A \in M_n(\mathbb{Z}_+)\}$ is a subset of the basis $\{F_N^*(\mathbf{m})\}$ By the descriptions of the basis B^* obtained from Lusztig's procedure, one sees that B^* is a subset of the normalized dual canonical basis of $U_q(n^-)$. $\square$

Definition 8.7. *An element* $b \in B^*$ *is called decomposable if there exist* $m \in \mathbb{Z}$ *and* $b_1, b_2 \in B^*$ *such that* $b = q^m b_1 b_2$. *The basis element is called indecomposable otherwise.*

By definition, any basis element $b \in B^*$ can be written as

$$(8.6) \qquad\qquad b = q^m b_1 b_2 \cdots b_s$$

for some $m \in \mathbb{Z}$ and indecomposables $b_1, b_2, \cdots, b_s \in B^*$. Notice that it follows from the positivity that any b written as in (8.6) as a product of more than two basis elements is, indeed, decomposable.

Conjecture 8.8. *The decomposition is unique up to a permutation.*

Proposition 8.9. *Let $b \in B^*$. Assume that $b = q^m b_1 b_2 \cdots b_s$. Then for any $i, j \in [1, n]$ there exists an integer h_{ij} such that $b_i b_j = q^{h_{ij}} b_j b_i$. Furthermore, for any $\{i_1 < i_2 < \cdots < i_r\} \subset [1, n]$ there exists an element $e \in B^*$ such that*

$$e = q^s b_{i_1} b_{i_2} \cdots b_{i_r}.$$

Proof: We use induction on s. The case of $s = 1$ is trivial. Assume that our hypothesis holds for $\leq s - 1$. We first prove that b_i q-commutes with b_{i+1}. By the positivity of the multiplication $b_i b_{i+1}$ must be of the form $a_{ij} b'$ for some $a_{ij} \in \mathbb{Z}_+[q, q^{-1}]$ and a_{ij} must be a power of q since multiplication with basis elements can not decrease the number of summands. Applying the bar action, we see that

$$b_j b_i = \bar{a}_{ij} b' = \bar{a}_{ij} a_{ij}^{-1} b_i b_j.$$

Hence, all of the factors b_i and b_j q-commute with each other. Because multiplication satisfies positivity, the statement follows. $\square$.

Hence, to understand the basis B^* completely, one must determine the indecomposables.

Proposition 8.10. *All quantum minors are indecomposable.*

Proof: Let D be a quantum minor. Specifically, D is the quantum determinant of a subalgebra A of $O_q(M(n))$ isomorphic to $O_q(M(m))$ for some $m \leq n$. If D is decomposable and is written as a product of two basis elements b_1, b_2, then b_1, b_2 are members of the dual canonical basis of A. Hence, we may only treat the case that D is the quantum determinant. With respect to the lexicographic order, the leading term of det_q is $Z_{11} Z_{22} \cdots Z_{nn}$. Hence, the leading terms of b_1 and b_2 produce the term $Z_{11} Z_{22} \cdots Z_{nn}$. But then the leading term of b_1 will be of the form $Z_{i_1, i_1} \ldots Z_{i_r, i_r}; i_1 < \cdots < i_r$, and the leading term of b_2 will be what remains. But then b_1 is a minor centered around the diagonal, and so is b_2. It is clear that the product of two such "disjoint" minors cannot give the full quantum determinant. $\square$

In [22], it was proved that the dual canonical basis B^* of the algebra $O_q(M(n))$ is invariant under multiplication by the quantum determinant. Setting the quantum determinant to 1, we get a basis K^* of the algebra $O_q(SL(n))$. Clearly, we have

Theorem 8.11. *The multiplication of the basis K^* of the algebra $O_q(SL(n))$ has the positivity property.*

In [22], it was proved that the basis K^* is dual to the canonical basis of the modified quantum enveloping algebra $\widetilde{U_q(A_{2n-1})}$ (one can refer to [16] for more details of the construction of the canonical basis of the modified quantum enveloping algebra). By duality again, we have

Theorem 8.12. *The co-product of the canonical basis of $\widetilde{U_q(A_{2n-1})}$ has the positivity property.*

This result was originally conjectured by Lusztig [16]. We believe that this result holds in all simply-laced cases. We will deal with this in a forthcoming paper.

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