ON INTERSECTING GEODESICS.

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Definition. [1]. Let $\gamma(t)$, $-\infty < t < \infty$, be a geodesic in a Riemannian manifold M. The geodesic is called closed if there exists a number $L > 0$ such that $\gamma(t+L) = \gamma(t)$ for all t. The geodesic is said to be simply closed if in addition $\gamma(t_1) \neq \gamma(t_2)$ for $0 < t_1 < t_2 \leq L$.

Lemma. Let G be a Lie group with the identity element e. Let a(t) and b(t) be one-parameter subgroups of G. If $a(1) = b(1)$, then the closed curve c(t):

$$c(t) = a(t)b(-t) \qquad (0 \leq t \leq 1)$$
$$c(0) = c(1) = e$$

is smooth at e.

Proof. Let $\mathfrak{G}$ denote the Lie algebra of G, identified with the tangent space at e. Let X and Y be elements of $\mathfrak{G}$ with $a(t) = \exp Xt$ and $b(t) = \exp Yt$. Let $\dot{c}(t)$ denote the tangent vector to the curve c(t) at t. Then we have obviously

$$\dot{c}(t) = dL_{c(t)}(\exp(\operatorname{ad} tY)X - Y)$$

where L_c denotes the left translation $x \longmapsto cx$ for $x, c \in G$, and $(\operatorname{ad} Y)Z = [Y, Z]$ for $Y, Z \in \mathfrak{G}$. Since $c(1) = e$ and $\exp X = \exp Y$

implies that $\exp(\mathrm{ad}\,X) = \exp(\mathrm{ad}\,Y)$ in virtue of the commutativity $\exp \circ \mathrm{ad} = \mathrm{Ad} \circ \exp$, we have that

$$\dot{c}(1) = \exp(\mathrm{ad}\,X) \cdot X - Y = X - Y = \dot{c}(0).$$

Q.E.D.

Next, let us apply the lemma to a semisimple Lie group G with a canonical Riemannian metric due to H. C. Wang in [2]. We choose a Cartan decomposition $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ of the semisimple Lie algebra $\mathfrak{G}$. (Cf. Helgason [1]). Corresponding to the decomposition we have a unique left invariant Riemannian metric such that every geodesic through e is given by

$$\exp(P - K)t \cdot \exp 2Kt \qquad (K \in \mathfrak{K},\ P \in \mathfrak{P}).$$

The following theorem follows directly from the lemma.

Theorem. Let G be a semisimple Lie group with a canonical Riemannian metric. Then every geodesic which intersects itself is a simply closed geodesic.

REFERENCES

1. S. Helgason, Differential geometry and symmetric spaces, Academic Press 1962.
2. H. C. Wang, Discrete nilpotent subgroups of Lie groups, J. Differential Geometry, vol. 3 no. 4 (1969), pp. 481 - 492.

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