INDECOMPOSABLE FINITE-DIMENSIONAL REPRESENTATIONS OF A CLASS OF LIE ALGEBRAS AND LIE SUPERALGEBRAS

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1. Introduction

The topic of indecomposable finite-dimensional representations of the Poincaré group was first studied in a systematic way by S. Paneitz ([5][6]). In these investigations only representations with one source were considered, though by duality, one representation with 2 sources was implicitly present.

The idea of nilpotency was mentioned indirectly in Paneitz's articles, but a more down-to-earth method was chosen there.

The results form a part of a major investigation by S. Paneitz and I. E. Segal into physics based on the conformal group. Induction from indecomposable representations plays an important part in this theory. See ([7]) and references cited therein.

The defining representation of the Poincaré group, when given as a subgroup of SU(2,2) (see below), is indecomposable. This representation was studied by the present author prior to the articles by Paneitz in connection with a study of special aspects of Dirac operators and positive energy representations of the conformal group ([4]).

Indecomposable representations in theoretical physics have also been used in a major way in a study by G. Cassinelli, G. Olivieri, P. Truini, and V. S. Varadarajan ([1]). The main object is the Poincaré group. In an appendix to the article, the indecomposable representations of the 2-dimensional Euclidean group are considered, and many results are obtained. This group can also be studied by our method, but we will not pursue this here. One small complication is that the circle group is abelian.

In the article at hand, we wish to sketch how, by utilizing nilpotency to its fullest extent while using methods from the theory of universal enveloping algebras, a complete description of the indecomposable representations may be reached. In practice, the combinatorics is still formidable, though.

It turns out that the method applies to both a class of ordinary Lie algebras and to a similar class of Lie superalgebras.

Besides some examples, due to the level of complexity we will only describe a few precise results. One of these is a complete classification of which ideals

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can occur in the enveloping algebra of the translation subgroup of the Poincaré group. Equivalently, this determines all indecomposable representations with a single, 1-dimensional source. Another result is the construction of an infinite-dimensional family of inequivalent representations already in dimension 12. This is much lower than the 24-dimensional representations which were thought to be the lowest possible. The complexity increases considerably, though yet in a manageable fashion, in the supersymmetric setting. Besides a few examples, only a subclass of ideals of the enveloping algebra of the super Poincaré algebra will be determined in the present article.

2. Finite-dimensional indecomposable representations of the Poincaré group

We are here only interested in what happens on the level of the Lie algebra. Equivalently, we consider a double covering of the Poincaré group given by

$$P = \left\{ \begin{pmatrix} a & 0 \\ k & a^{\star -1} \end{pmatrix} \mid a \in SL(2,\mathbb{R})\ ; k^\star = a^\star k a^{-1} \in gl(2,\mathbb{C}) \right\}.$$

This is a subgroup of $SU(2,2)$ when the latter is defined by the hermitian form

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

For our purposes, we may equivalently even consider the group

$$P = \left\{ \begin{pmatrix} u & 0 \\ z & v \end{pmatrix} \mid u, v \in SL(2,\mathbb{C})\ ; z \in gl(2,\mathbb{C}) \right\}.$$

Let G_0 denote the group $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Thus,

$$G_0 = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \mid u, v \in SL(2,\mathbb{C}) \right\}.$$

For what we shall be doing, it does not matter if we work with this group, its Lie algebra, or with $SU(2) \times SU(2)$.

It is important to consider the abelian Lie algebra

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \mid z \in gl(2,\mathbb{C}) \right\}.$$

along with its enveloping algebra

$$\mathcal{U}(\mathfrak{p}^-) = \mathcal{S}(\mathfrak{p}^-) = \mathbb{C}[z_1, z_2, z_3, z_4].$$

The last equality comes from writing the 2×2 matrix z above as

$$z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}.$$

In passing we make the important observation that the polynomial $\det z = z_1 z_4 - z_2 z_3$ is invariant in the sense that $\det uzv = \det z$ for all $u, v \in SL(2, \mathbb{C})$.

We make the basic assumption that all representations and equivalences are over $\mathbb{C}$. This has the powerful consequence that the abelian algebra $\mathfrak{p}^-$ acts nilpotently.

The general setting is the following: We consider a reductive Lie algebra $\mathfrak{g}_0$ in which the elements of the abelian ideals are given by semi-simple elements and a nilpotent Lie algebra $\mathfrak{n}$ together with a Lie algebra homomorphism α of $\mathfrak{g}_0$ into the derivations $Der(\mathfrak{n})$ of $\mathfrak{n}$. This gives rise, in the usual fashion, to the semi-direct product

$$\mathfrak{g} = \mathfrak{g}_0 \times_\alpha \mathfrak{n}.$$

In this situation a well-known result from algebra ([3],[2]) can easily be generalized to include the $\mathfrak{g}_0$ invariance.

Recall that a flag in a vector space V is a sequence of subspaces $0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_r = V$.

Theorem 2.1. *Suppose given a representation of $\mathfrak{g}$ in some finite-dimensional vector space V. Then there is a flag of subspaces such that $\mathfrak{n}$ maps W_i into W_{i-1} for $i = 1, \dots, r$ and such that each W_i is invariant and completely decomposable under $\mathfrak{g}_0$.*

We associate a graph to the indecomposable representation V as follows: The nodes are the $\mathfrak{g}_0$ irreducible representations that occur. Two nodes, labeled by irreducibles V_1 and V_2, are connected by an arrow pointing from V_1 to V_2 if $V_2 \subset \mathfrak{n}^- \cdot V_1$ inside V. If there are multiplicities in the isotypic components the situation becomes more complicated. If the multiplicity at the node i is n_i one can simply place n_i black dots at the node. They can be placed in a stack or in a circle. In case $n_i > 1$, $n_j > 1$ there may also be a number $a_{i,j} > 1$ of arrows from i to j, and this needs also to be indicated. The simplest way is just to attach the n_i to each node and to attach the $a_{i,j}$ to the arrow from i to j with the further stipulation that only numbers strictly greater that 1 need to be given. We shall not pursue such details here; see, however, the third of the simple examples below.

The theorem above has the immediate consequence that there are no closed paths in this graph.

Remark 2.2. *The assumption of finite dimension is essential here. Already on the level of the polynomial algebra $\mathbb{C}[z_1, z_2, z_3, z_4]$, if one quotients out by the ideal generated by an inhomogeneous polynomial in* $\det z$ *there will be closed loops, but the resulting module is infinite-dimensional. If one insists on finite-dimensionality, one must have all homogeneous polynomials in the quotient after a certain degree. Thus* $\det z^n$ *for some n must be in the ideal. This precludes an* *inhomogeneous polynomial in* $\det z$ *since in* $\mathbb{C}[T]$ *(where* $T = \det z$*), any inhomogeneous polynomial* $p(T)$ *is relatively prime to* T^n *for any* $n = 0, 1, 2, \ldots$.

Given any such directed graph, any node with arrows only pointing out is as usual called a source. The opposite is called a sink. There is a simple way whereby one may reverse all arrows, thereby turning sources into sinks, and vice versa: Replace V by its dual module V'.

Simple situations:

$$\text{One generator}\quad \begin{array}{ccc} & & V_2 \\ & \nearrow & \\ V_1 & & \\ & \searrow & \\ & & V_3 \end{array} \qquad\qquad \text{(or its dual...)}\quad \begin{array}{ccc} & & V_2' \\ & \swarrow & \\ V_1' & & \\ & \nwarrow & \\ & & V_3' \end{array}$$

This leads to decomposable representations if the targets (sinks) are equal (respectively if the origins (sources) are equivalent). Otherwise they are indecomposable.

$$\text{One source:}\quad \begin{array}{ccc} & & \bullet_{(sink)} \\ & \nearrow & \\ \bullet_{(source)} & & \\ & \searrow & \\ & & \bullet_{(sink)} \end{array} \qquad\qquad \text{(or its dual...)}\quad \begin{array}{ccc} & & \bullet_{(source)} \\ & \swarrow & \\ \bullet_{(sink)} & & \\ & \nwarrow & \\ & & \bullet_{(source)} \end{array}$$

$$\text{Two sources, three sinks}\qquad \begin{array}{ccccc} & & \bullet^2 & & \\ & \swarrow & \downarrow & \searrow & \\ \bullet & & \bullet & & \bullet \end{array}$$

$$\text{Two generators - two sinks}\quad \begin{array}{ccccc} & & \bullet & & \\ & \nearrow & & \nwarrow & \\ \bullet & & & & \bullet \\ & \searrow & & \swarrow & \\ & & \bullet & & \end{array}$$

2.1. **One generator.** We consider only the Poincaré algebra. Let V_0 denote an irreducible finite-dimensional representation of $\mathfrak{g}_0 = sl(2,\mathbb{C}) \times sl(2,\mathbb{C})$, given by non-negative integers (n,m) so that the spins are $(\frac{n}{2},\frac{m}{2})$ and the dimension is $(n+1)(m+1)$.

Let Π denote an indecomposable finite-dimensional representation of $\mathcal{S}(\mathfrak{p}^-) \times_s \mathfrak{g}_0$ in a space V_Π, generated by a $\mathfrak{g}_0$ invariant source V_0. Let $\mathcal{P}(V_0)$ denote the space of polynomials in the variables z_1, z_2, z_3, z_4 with values in V_0. This is generated by polynomials of the form $p_0(z_1,z_2,z_3,z_4)\cdot v$ for $v \in V_0$ and p_0 a complex polynomial. We consider this a left $\mathcal{S}(\mathfrak{p}^-) \times_s \mathfrak{g}_0$ module in the obvious way. The map

$$p_0(z_1,z_2,z_3,z_4)\cdot v \mapsto \pi(p_0)v \tag{1}$$

is clearly $\mathcal{S}(\mathfrak{p}^-) \times_s \mathfrak{g}_0$ equivariant.

The decomposition of the $\mathfrak{g}_0$ module $\mathcal{S}(\mathfrak{p}^-)$ is well-known and is given by the representations spin$(\frac{n}{2},\frac{n}{2})$ for $n = 0,1,2,\dots$. Each occurs with infinite multiplicity due to the invariance of $\det z$ under $\mathfrak{g}_0$. We will describe these representations in detail below.

The decomposition of $\mathcal{P}(V_0)$ into irreducible $\mathfrak{g}_0$ representations follows easily from this using the well-known decomposition of the tensor product of two irreducible representations of $su(2)$. The decomposition of $\mathcal{P}(V_0)$ is in general more degenerate than what results from the invariance of $\det z$.

It is clear that there exists a sub-module $\mathcal{I} \subseteq \mathcal{P}(V_0)$ such that

$$\mathcal{P}(V_0)/\mathcal{I} \equiv V_\Pi.$$

The finite-dimensionality assumption on V_Π then implies that $\mathcal{I}$ contains all homogeneous polynomials of a degree greater than or equal to some fixed degree, say d_0. Since there are only a finite number of linearly independent homogeneous polynomials in $\mathcal{P}(V_0)$ of degree d_0, it follows that there exists a finite number of elements $p_1, p_2, \dots, p_j$ in $\mathcal{P}(V_0)$ (these may be chosen for instance as highest weight vectors) such that if $\mathcal{I}\langle p_1,p_2,\dots,p_j\rangle$ denotes the $\mathcal{S}(\mathfrak{p}^-) \times_s \mathfrak{g}_0$ submodule generated by these elements, then

$$V_\Pi \equiv \mathcal{P}(V_0)/\mathcal{I}\langle p_1,p_2,\dots,p_j\rangle.$$

We assume that the number j of polynomials is minimal.

Once the elements $p_1,p_2,\dots,p_j$ are known, it is possible to construct the whole graph as above. In particular, **the sinks in V_Π can be directly determined from this**. See Proposition 2.4 below for a simple example that indicates how. In case $\dim V_0$ is large the task, of course, will be more cumbersome.

Example 2.3. *As is well known, we have that*

$$\mathfrak{p}^- \otimes spin(\frac{1}{2},0) = spin(1,\frac{1}{2}) \oplus spin(0,\frac{1}{2}).$$

If we mod out by all second order polynomials, and possibly one of the first order polynomial representations, we get the following 3 indecomposable representations:

- *spin*$(\frac{1}{2},0) \to$ *spin*$(0,\frac{1}{2})$. *This 4-dimensional representation comes from the the defining representation.*
- *spin*$(\frac{1}{2},0) \to$ *spin*$(1,\frac{1}{2})$. *This is an 8-dimensional representation.*
- *spin*$(\frac{1}{2},0) \to$ *spin*$(0,\frac{1}{2})$, *spin*$(1,\frac{1}{2})$. *This is a 10-dimensional representation which includes the two former.*

Proceeding analogously,

$$\mathfrak{p}^- \otimes spin(1,0) = spin(\frac{3}{2},\frac{1}{2}) \oplus spin(\frac{1}{2},\frac{1}{2})$$

leads to inequivalent representations in dimensions 7,11,15.

Similarly,

$$\mathfrak{p}^- \otimes spin(\frac{1}{2},\frac{1}{2}) = spin(0,0) \oplus spin(1,0) \oplus spin(0,1) \oplus spin(1,1)$$

leads to indecomposable representations in dimensions 5, 7, 8, 10, 11, 13, 16, 17, 19, 20. Several dimensions here carry a number of inequivalent representations.

Together with duals of these or versions obtained as mirror images by interchanging the spins, these exhaust all the known representations in dimensions less than or equal to 8 with the exception of a 6-dimensional representation which we describe in Example 2.5.

2.2. **Special case: Ideals in $\mathcal{U}(\mathfrak{p}^-)$.** It is well-known that there is a decomposition

$$\mathcal{U}(\mathfrak{p}^-) = \oplus W_{r,s}$$

into $\mathfrak{g}_0$ representations, where the subspace $W_{r,s}$ may be defined through its highest weight vector, say $z_1^r \det z^s$. This is possible since each representation occurs with multiplicity one. Denote this representation simply by $[r,s]$. It is elementary to see that the action of $\mathfrak{p}^-$ on the left on $\mathcal{U}(\mathfrak{p}^-)$, when expressed in terms of representations, is given as follows. All arrows represent non-trivial maps.

$$
\begin{array}{ccccccc}
\downarrow & & & & & & \\
[1,0] & & & & & & \\
\downarrow & \searrow & & & & & \\
[2,0] & & [0,1] & & & & \\
\downarrow & \searrow & \downarrow & & & & \\
[3,0] & & [1,1] & & & & \\
\downarrow & \searrow & \downarrow & \searrow & & & \\
[4,0] & & [2,1] & & [0,2] & & \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow & & \\
[5,0] & & [3,1] & & [1,2] & & \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
[6,0] & & [4,1] & & [2,2] & & [0,3] \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array} \tag{2}
$$

Any ideal $\mathcal{I} \subseteq \mathcal{U}(\mathfrak{p}^-)$ that has finite codimension must clearly contain some $z_1^{r_1}$ for some minimal $r_1 \in \mathbb{N}$ (we omit the trivial case of codimension 0). Since we are assuming that the ideals are $\mathfrak{g}_0$ invariant, if some other element $z_1^{r_2} p(\det z)$ is in the ideal then first of all we can assume $r_2 < r_1$ and secondly we can assume that the polynomial p is homogeneous; $p(\det z) = \det z^{s_2}$ for some $s_2 > 0$. The latter inequality follows by the minimality of r_1.

Thus the following is clear:

Proposition 2.4. *Any $\mathfrak{g}_0$ invariant ideal $\mathcal{I} \subseteq \mathcal{U}(\mathfrak{p}^-)$ of finite codimension is of the form*

$$\mathcal{I} = \mathfrak{g}_0 \cdot \langle z_1^{r_1} \det z^{s_1}, z_1^{r_2} \det z^{s_2}, \cdots, z_1^{r_t} \det z^{s_t} \rangle \tag{3}$$

for some positive integers $r_1 > r_2 > \cdots > r_t$ and integers $0 = s_1 < s_2 < \cdots < s_t$. If the set is minimal, then furthermore

$$\forall j = 2, 3, \ldots, t : s_1 + s_2 + \cdots + s_j \leq r_1 - r_j.$$

Any set of such integers determine an invariant ideal of finite codimension. The sinks in the quotient module are

$$z_1^{r_1 - s_2} \det z^{s_2 - 1}, z_1^{r_1 - s_2 - s_3} \det z^{s_3 - 1} \ldots z_1^{r_1 - s_2 - \cdots - s_t} \det z^{s_t - 1}, \text{ and } \det z^{s_t + r_t - 1}.$$

Example 2.5. *If we mod out by all second order polynomials, that is by z_1^2 and $\det z$, we get the 5-dimensional indecomposable representation*

$$spin(0,0) \to spin(\frac{1}{2}, \frac{1}{2}).$$

If we instead mod out by z_1^2 and $z_1 \det z$ we get the 6-dimensional indecomposable representation

$$spin(0,0) \to spin(\frac{1}{2},\frac{1}{2}) \to spin(0,0).$$

Example 2.6. *The representations determined by ideals are easily written down, though some finer details from the representation theory of $su(2)$ will have to be invoked to get the precise form. In simple examples like the last in the previous example, everything follows immediately since there is no need to be precise about the relative scales in the 3 spaces:*

$$\mathfrak{p}^- \ni \underline{p} = (p_1,p_2,p_3,p_4) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ p_1 & 0 & 0 & 0 & 0 & 0 \\ p_2 & 0 & 0 & 0 & 0 & 0 \\ p_3 & 0 & 0 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_4 & -p_3 & -p_2 & p_1 & 0 \end{pmatrix}. \tag{4}$$

An element (u,v) in the diagonal subgroup G_0 (see Section 2) acts as $0 \oplus u \otimes (v^t)^{-1} \oplus 0$.

Notice that the matrix with just p_1 corresponds to a map which sends the constant 1 to the polynomial $p_1 z_1$ and sends the polynomial z_4 to $p_1 \det z$ and all other first order polynomials z_1, z_2, z_3 to 0.

2.3. **Two sources and 2 sinks.** We here consider the Poincaré algebra.

Consider the situation previously depicted under 'Simple situations' where there is one source and two sinks. The resulting representations may be written as

$$\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ F(w) & 0 & 0 \\ G(w) & 0 & 0 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \mapsto \begin{pmatrix} \tau_1(u,v) & 0 & 0 \\ 0 & \tau_2(u,v) & 0 \\ 0 & 0 & \tau_3(u,v) \end{pmatrix}. \tag{5}$$

With this one can easily write down a representation with 2 sources and 2 sinks, indeed a 4-parameter family given by elements $(\alpha,\beta,\gamma,\delta) \in \mathbb{C}^4$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha \cdot F(w) & \beta \cdot F(w) & 0 & 0 \\ \gamma \cdot G(w) & \delta \cdot G(w) & 0 & 0 \end{pmatrix} \text{ resp. } \begin{pmatrix} \tau_1(u,v) & 0 & 0 & 0 \\ 0 & \tau_1(u,v) & 0 & 0 \\ 0 & 0 & \tau_2(u,v) & 0 \\ 0 & 0 & 0 & \tau_3(u,v) \end{pmatrix}. \tag{6}$$

In this case, there is a continuum of inequivalent representations and they are generically indecomposable. This lead to a continuum already in dimension 12 where the two sources are equal and 2-dimensional - say spin($\frac{1}{2}$,0), and the two sinks are spin(1,$\frac{1}{2}$) and spin(0,$\frac{1}{2}$) or, also in dimension 12, the 2 sources are the 4-dimensional spin($\frac{1}{2}$,$\frac{1}{2}$), and the sinks are spin(1,0) and spin(0,0), or in dimension 16 where one is the 2-dimensional spin ($\frac{1}{2}$,0) and the other is the 6-dimensional spin($\frac{1}{2}$,1) and the targets are spin(0,$\frac{1}{2}$) and spin(1,$\frac{1}{2}$). The moduli space in these cases is $\mathbb{CP}^1$. Specifically, the indecomposable is determined by a point $(p,q) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ giving relative scales on the arrows. Here, $p \equiv (\alpha,\beta)$ and $q \equiv (\gamma,\delta)$ in the above representation. Two such points (p_1,q_1) and (p_2,q_2), are equivalent if there is an element $g \in GL(2,\mathbb{C})$ such that $(p_2,q_2) = (gp_1, gp_2)$.

3. Supersymmetry

The previous considerations are now extended to the supersymmetric setting as follows: Let H_1, H_2, and H_3 be reductive matrix Lie groups with Lie algebras $\mathfrak{h}_1, \mathfrak{h}_2$, and $\mathfrak{h}_3$, respectively. We assume that possible abelian ideals are represented by semi-simple elements. We consider an irreducible representation of each of these Lie algebras; V_1, V_2, and V_3. We identify the representation with the space in which it acts. We denote furthermore the dual representation of a representation V by V' (this is the $\mathbb{C}$ linear dual). Let

$$W_1 = \hom(V_1,V_2) \equiv V_1' \otimes V_2 \quad ; \quad W_2 = \hom(V_2,V_3) \equiv V_2' \otimes V_3 \tag{7}$$

$$\text{and} \qquad Z = \hom(V_1,V_3) \equiv V_1' \otimes V_3. \tag{8}$$

The Lie superalgebra $\mathfrak{g}_{super} = \mathfrak{g}_{super}(\mathfrak{h}_1,\mathfrak{h}_2,\mathfrak{h}_3,V_1,V_2,V_3)$ is defined as

$$\mathfrak{g}_{super} = \tag{9}$$
$$\left\{ \begin{pmatrix} a & 0 & 0 \\ w_1 & g & 0 \\ z & w_2 & b \end{pmatrix} \mid a \in \mathfrak{h}_1 \ , g \in \mathfrak{h}_2 \ , b \in \mathfrak{h}_3 \ , w_1 \in W_1 \ , w_2 \in W_2 \ , \text{ and } z \in Z \right\}.$$

The odd part is given as

$$\mathfrak{g}^1_{super} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ w_1 & 0 & 0 \\ 0 & w_2 & 0 \end{pmatrix} \mid w_1 \in W_1 \ , \text{ and } w_2 \in W_2 \right\}.$$

Let

$$\mathfrak{n}_{super} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ w_1 & 0 & 0 \\ z & w_2 & 0 \end{pmatrix} \mid w_1 \in W_1 \ , w_2 \in W_2 \ , \text{ and } z \in Z \right\}.$$

and

$$\mathfrak{g}^r_{super} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & b \end{pmatrix} \mid a \in \mathfrak{h}_1 \ , g \in \mathfrak{h}_2 \ , \text{ and } b \in \mathfrak{h}_3 \right\}.$$

Obviously, $\mathfrak{n}_{super}$ is a maximal nilpotent ideal and $\mathfrak{g}^r_{super}$ is the reductive part. We let

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix} \mid z \in Z \right\}.$$

Then $\mathfrak{g}^0_{super} = \mathfrak{g}^r_{super} \oplus \mathfrak{p}^-$.

We then have the following super algebraic generalization of Theorem 2.1:

Theorem 3.1. *Consider a finite-dimensional representation V_{super} of $\mathfrak{g}_{super}$. Then there is a flag of subspaces $\{0\} = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{r-1} \subsetneq W_r = V_{super}$ such that each W_i is invariant and completely reducible under $\mathfrak{g}^0_{super}$ and such that $\mathfrak{n}_{super}$ maps W_i into W_{i-1} for $i = 1, \dots, r$.*

Thus, the previous treatment with directed graphs and ideals carry over. Naturally, the picture gets more complicated with the odd generators.

The most simple thing to consider would be the $\mathfrak{g}_{super}$ module $\mathcal{U}(\mathfrak{n}_{super})$, but even here the situation is complex, though in principle tractable.

Consider as an example the case of the simplest super Poincaré algebra,

$$\mathfrak{g}^P_{super} = \tag{10}$$
$$\left\{ \begin{pmatrix} a & 0 & 0 \\ w_1 & 0 & 0 \\ z & w_2 & b \end{pmatrix} \mid a, b \in sl(2,\mathbb{C}), w_1 \in M(1,2), w_2 \in M(2,1) \ , \text{ and } z \in M(2,2) \right\}.$$

Let $W_1 = M(1,2)$ and $W_2 = M(2,1)$. We number the spaces

$$\begin{matrix} 1 & W_2 & W_2 \wedge W_2 \\ W_1 & W_1 \wedge W_2 & W_1 \wedge W_2 \wedge W_2 \\ W_1 \wedge W_1 & W_1 \wedge W_1 \wedge W_2 & W_1 \wedge W_1 \wedge W_2 \wedge W_2 \end{matrix} = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}.$$

We then have that

$$\mathcal{U}(\mathfrak{n}_{super}) = \sum_{i=1}^{9} \mathcal{U}(\mathfrak{p}^-) \otimes \mathcal{U}(\mathfrak{q}^1_{super})_i. \tag{11}$$

Each of the 9 summands is invariant under $\mathfrak{g}^r_{super}$. The representations corresponding to this are given right below . Here, n, d are independent non-negative integers (in a few obvious cases, n must furthermore be non-zero). The labels $\uparrow$ and $\downarrow$ may be taken just as part of a short hand notation that are defined by the stated equations. They are listed here for convenience even though they are not used directly. One can ascertain useful information from them about how the various pieces occur in the tensor products of $su(2)$ representations.

$$\begin{array}{rcl} 1\,[n,n,d] & \oplus & \\ 2\uparrow[n,d]=2\,[n-1,n,d] & \oplus & 2\downarrow[n,d]=2\,[n+1,n,d] \\ 3\,[n,n,d] & \oplus & \\ 4\uparrow[n,d]=4\,[n,n-1,d] & \oplus & 4\downarrow[n,d]=4\,[n,n+1,d] \\ 5\uparrow\uparrow[n,d]=5\,[n-1,n-1,d] & \oplus & 5\uparrow\downarrow[n,d]=5\,[n-1,n+1,d] \\ 5\downarrow\uparrow[n,d]=5\,[n+1,n-1,d] & \oplus & 5\downarrow\downarrow[n,d]=5\,[n+1,n+1,d] \\ 6\uparrow[n,d]=6\,[n,n-1,d] & \oplus & 6\downarrow[n,d]=6\,[n+1,n,d] \\ 7\,[n,n,d] & \oplus & \\ 8\uparrow[n,d]=6\,[n-1,n,d] & \oplus & 8\downarrow[n,d]=8\,[n+1,n,d] \\ & \oplus & 9\,[n,n,d] \end{array}$$

A further complication is that there are representations in different spaces that are equivalent under $\mathfrak{g}^r_{super}$:

$$\begin{array}{rcl} 8\uparrow[n,d] & \leftrightarrow & 4\uparrow[n-1,d+1] \\ 8\downarrow[n,d] & \leftrightarrow & 4\,[n+1,d] \\ 5\downarrow\downarrow[n,d] & \leftrightarrow & 1\,[n+1,d] \\ 5\uparrow\uparrow[n,d] & \leftrightarrow & 1\,[n-1,d+1] \\ 5\downarrow\downarrow[n-1,d+1]\,,5\uparrow\uparrow[n+1,d] & \leftrightarrow & 9\,[n,d] \\ 5\downarrow\downarrow[n-1,d+1]\,,5\downarrow\downarrow[n+1,d] & \leftrightarrow & 9\,[n,d] \\ 6\uparrow[n,d] & \leftrightarrow & 2\downarrow[n-1,d+1] \\ 6\uparrow[n,d] & \leftrightarrow & 2\downarrow[n-1,d+1] \\ 6\downarrow[n,d] & \leftrightarrow & 2\uparrow[n+1,d] \\ 6\downarrow[n,d] & \leftrightarrow & 2\uparrow[n+1,d] \end{array}$$

To each finite-dimensional representation V_r of $\mathfrak{g}^r_{super}$ (may be reducible), the general object of interest is the left module

$$\mathcal{U}(\mathfrak{n}_{super})\cdot V_r=\sum_{i=1}^{n}\mathcal{U}(\mathfrak{p}^-)\otimes\mathcal{U}(\mathfrak{q}^1_{super})_i\cdot V_r. \qquad (12)$$

To further analyze this we have to choose a PBW-type basis. We will do this in the indicated fashion with $\mathcal{U}(\mathfrak{p}^-)$ to the left and with furthermore W_1 always to the left of W_2.

Example 3.2. *Assume that $\mathcal{U}(\mathfrak{p}^-)$ acts trivially on the space V_r. The resulting module is then*

$$\bigwedge(\mathfrak{g}^1_{super}) \cdot V_r, \tag{13}$$

or some of the subrepresentations thereof. The exterior algebra $\bigwedge(\mathfrak{g}^1_{super})$ occurs because the W_1, W_2 anticommute in the considered quotient.

Observe that in the sum (12) the summand

$$\mathcal{U}_{2,3,5,6,8,9}(\mathfrak{n}_{super}) \cdot V_r = \sum_{i=2,3,5,6,8,9} \mathcal{U}(\mathfrak{p}^-) \otimes \mathcal{U}(\mathfrak{q}^1_{super})_i \cdot V_r \tag{14}$$

is invariant. We may then pass to a general subclass of indecomposable modules by first taking the quotient by this. The vector space that results is

$$\mathcal{U}_{rest}(\mathfrak{n}_{super}) \cdot V_r = \sum_{i=1,4,7} \mathcal{U}(\mathfrak{p}^-) \otimes \mathcal{U}(\mathfrak{q}^1_{super})_i \cdot V_r. \tag{15}$$

If we let $\mathcal{U}(\mathfrak{p}^-)_{\geq s}$ be the ideal generated by all homogeneous elements of degree s, it is easy to see that for each $s = 0, 1, 2, \dots,$ the space

$$\mathcal{U}^s_{rest}(\mathfrak{n}_{super}) \cdot V_r = \mathcal{U}(\mathfrak{p}^-)_{s+2} \cdot V_r \oplus \mathcal{U}(\mathfrak{p}^-)_{s+1} \cdot W_1 \cdot V_r \oplus \mathcal{U}(\mathfrak{p}^-)_s \cdot (W_1 \wedge W_1) \cdot V_r \tag{16}$$

is invariant.

Example 3.3. *Let V^1_r be the irreducible 2-dimensional representation which is only non-trivial on the $\mathfrak{h}_1$ piece of the reductive part. The defining representation of $\mathfrak{g}^P_{super}$ is a subrepresentation of the quotient*

$$\mathcal{U}_{rest}(\mathfrak{n}_{super}) \cdot V^1_r / \mathcal{U}^0_{rest}(\mathfrak{n}_{super}) \cdot V^1_r.$$

Indeed, we just have to limit ourselves further by removing two appropriate $\mathfrak{g}^r_{super}$ representations.

Returning to the more general situation, let us assume from now on that V_r is the trivial 1-dimensional module. We are thus left with the space

$$\mathcal{U}_{rest}(\mathfrak{n}_{super}) = \mathcal{U}(\mathfrak{p}^-) \oplus \mathcal{U}(\mathfrak{p}^-) \cdot W_1 \oplus \mathcal{U}(\mathfrak{p}^-) \cdot (W_1 \wedge W_1). \tag{17}$$

The general form of the representation is (we give only the W_1, W_2 operators)

$$\begin{pmatrix} 0 & (w_2^1 p_{11} + w_2^2 p_{21})1_{\mathcal{P}} & (w_2^1 p_{12} + w_2^2 p_{22})1_{\mathcal{P}} & 0 \\ w_1^1 1_{\mathcal{P}} & 0 & 0 & -(w_2^1 p_{12} + w_2^2 p_{22})1_{\mathcal{P}} \\ w_1^2 1_{\mathcal{P}} & 0 & 0 & (w_2^1 p_{11} + w_2^2 p_{21})1_{\mathcal{P}} \\ 0 & -w_1^2 1_{\mathcal{P}} & w_1^1 1_{\mathcal{P}} & 0 \end{pmatrix}. \tag{18}$$

Here, each block corresponds to a space of polynomials. The operators $p_{i,j}$ are multiplication operators in $\mathcal{U}(\mathfrak{p}^-)$ - and hence also such operators in the given space. Notice that they increase the degree of the target by 1. The symbol $1_{\mathcal{P}}$ denotes the identity operator.

The finer details are given as follows, where the arrows point upwards come from W_2 and those pointing downwards come from W_1.

$$
\begin{array}{ccccc}
 & & 1\,[n, d+1] & & \\
 & \nearrow & & \nwarrow & \\
4\downarrow [n-1, d+1] & & & & 4\uparrow [n+1, d] \\
 & \nwarrow & & \nearrow & \\
 & & 7\,[n, d] & &
\end{array}
\tag{19}
$$

$$
\begin{array}{ccccc}
 & & 1\,[n, d] & & \\
 & \swarrow & & \searrow & \\
4\downarrow [n, d] & & & & 4\uparrow [n, d] \\
 & \nwarrow & & \nearrow & \\
 & & 7\,[n, d] & &
\end{array}
\tag{20}
$$

Any invariant ideal $\mathcal{I}_{super}$ in $\mathcal{U}_{rest}(\mathfrak{n}_{super})$ contains a sum of the form

$$\mathcal{I}_1(\mathfrak{p}^-) \oplus \mathcal{I}_4(\mathfrak{p}^-) \cdot W_1 \oplus \mathcal{I}_7(\mathfrak{p}^-) \cdot (W_1 \wedge W_1), \tag{21}$$

where $\mathcal{I}_1(\mathfrak{p}^-) \subseteq \mathcal{I}_4(\mathfrak{p}^-) \subseteq \mathcal{I}_7(\mathfrak{p}^-) \subseteq \mathcal{U}(\mathfrak{p}^-)$ are $\mathfrak{p}^-$ ideals. These are precisely the ideals determined in Section 1.

Proposition 3.4. *If we have a representation $7\,[n, d] \in \mathcal{I}_7(\mathfrak{p}^-)$ then $4\,[n, d+1] \in \mathcal{I}_4(\mathfrak{p}^-)$ and $1\,[n, d+1] \in \mathcal{I}_1(\mathfrak{p}^-)$. Furthermore, $4\uparrow [n, d]\,, 4\downarrow [n, d] \in \mathcal{I}_{super}$. In particular,*

$$\mathfrak{p}^- \cdot \mathfrak{p}^- \cdot \mathcal{I}_7(\mathfrak{p}^-) \subseteq \mathcal{I}_1.$$

This result in principle solves the problem but there are still extremely many cases - even if we start with an ideal $\mathcal{I}_7$ and ask for how many configurations of the ideals $\mathcal{I}_1, \mathcal{I}_4$ that are possible. We refrain from pursuing this further and just give a low-dimensional example.

Example 3.5. *Let $\mathcal{I}_7 = \mathcal{I}\langle z_1 \rangle$. The following list is exhaustive and each case occurs.*

- $\mathcal{I}_1 = \mathcal{I}\langle z_1 \rangle$ *then* $\mathcal{I}_4 = \mathcal{I}\langle z_1 \rangle$.
- $\mathcal{I}_1 = \mathcal{I}\langle z_1^2 \rangle$ *then either* $\mathcal{I}_4 = \mathcal{I}\langle z_1^2 \rangle$, $\mathcal{I}_4 = \mathcal{I}\langle z_1^2, \det z \rangle$, *or* $\mathcal{I}_4 = \mathcal{I}_7$.
- $\mathcal{I}_1 = \mathcal{I}\langle z_1^2, \det z \rangle$ *then* $\mathcal{I}_4 = \mathcal{I}\langle z_1^2, \det z \rangle$.
- $\mathcal{I}_1 = \mathcal{I}\langle z_1^3, \det z \rangle$ *then* $\mathcal{I}_4 = \mathcal{I}\langle z_1^2, \det z \rangle$.
- $\mathcal{I}_1 = \mathcal{I}\langle z_1^3, z_1 \det z \rangle$ *then* $\mathcal{I}_4 = \mathcal{I}\langle z_1^2, \det z \rangle$ *or* $\mathcal{I}_4 = \mathcal{I}_1^2$.

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