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A New Class of Unitarizable Highest Weight Representations of Infinite-Dimensional Lie Algebras, II

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0. INTRODUCTION

In [7] we introduced a new class of unitarizable representations. More than that, we gave in fact enough details to solve completely the classification problem. However, we were rather sketchy on several points, and the list of unitary representations we gave had, in retrospect, an appalling shortcoming; it was not complete under tensor products. Another question, that of the integrability of the exceptional representations, was solved so late in the development that we were only able to mention the affirmative answer in a one line postscript. Finally, the approach in some of the chapters was so that more general algebras R than $\mathbb{C}[z, z^{-1}]$ were allowed. However, in the applications they had to be commutative. But the question remained: Does some of this make sense for non-commutative algebras, and if so, does it lead to unitarity?

In the present article we return to these issues. Since the writing of our first article, we have learned that F. A. Berezin [1] has treated the representations that we call exceptional, and in particular has obtained the integrated version. In return, our articles answer some questions asked there.

The present article is organized as follows: Section 1 contains the rigorous details for the classification problem. In Section 2 we reprove the unitarity of our exceptional representation. We do it in such a way that it becomes possible to deduce unitarity for some non-commutative *-algebras

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R. As examples we consider R to be the algebraic part of the irrational rotation algebra. The algebra R_F of finite rank operators is also considered. Section 3 is devoted to tensor products, and, finally, Section 4 treats integrability. Again, emphasis is placed on non-commutativity.

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1. The Classification Problem

We begin by reviewing the pertinent results and definitions from [7]. For basic definitions and facts from the theory of Kac–Moody algebras we refer to the book [8].

Let $\mathfrak{g} = \mathfrak{g}'(A)$ be a Kac–Moody algebra associated to the generalized Cartan matrix A. Let e_i, f_i $(i = 0, 1, ..., l)$ denote its Chevalley generators and let Δ be the set of roots of $\mathfrak{g}$.

Definition 1.1. A subset Δ_+ of Δ is called a *set of positive roots* if the following three properties hold:

(i) If $\alpha, \beta \in \Delta_+$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta_+$.

(ii) If $\alpha \in \Delta$ then either α or $-\alpha$ belongs to Δ_+.

(iii) If $\alpha \in \Delta_+$ then $-\alpha \notin \Delta_+$.

Given a set Δ_+ of positive roots one associates to it the *Borel subalgebra*

$$\mathscr{b} = \bigoplus_{\alpha \in \Delta_+ \cup \{0\}} \mathfrak{g}_\alpha. \tag{1.2}$$

Denote by $g \mapsto g_+$ the projection of $\mathfrak{g}$ on $\mathscr{b}$ along $\mathscr{n} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}$.

A subalgebra $\mathscr{p}$ of $\mathfrak{g}$ containing a Borel subalgebra is called a parabolic subalgebra.

Example. Let $\Pi^{st} = \{\alpha_0, \alpha_1, ..., \alpha_l\}$ be the set of roots (ordered as in [8]) corresponding to the generators $e_0, e_1, ..., e_l$ of $\mathfrak{g}$. We call Π^{st} the *standard set of simple roots*. Let $\Delta_+^{st} = \{\sum k_i \alpha_i \mid k_i = 0, 1, 2, ...$ and $\alpha_i \in \Pi^{st}\}$. This is the *standard set of positive roots* and the corresponding Borel subalgebra is denoted by $\mathscr{b}^{st}$.

Let W denote the Weyl group of the Kac–Moody algebra $\mathfrak{g}$. Put

$$\dot{\Pi} = \{\alpha_1, ..., \alpha_l\} \qquad \text{and} \qquad \dot{\Delta} = \left\{\alpha \in \Delta \mid \alpha = \sum_{i=1}^{l} k_i \alpha_i\right\}. \tag{1.3}$$

$$\dot{\Delta}_+ = \dot{\Delta} \cap \Delta_+^{st}. \tag{1.4}$$

Then $\dot{\Delta}$ is the root system of the underlying finite-dimensional simple Lie algebra $\dot{\mathfrak{g}}$. Let δ denote the unique indivisible imaginary root from Δ_+^{st} and let X be a subset of $\dot{\Pi}$. We associate to X a subset of positive roots Δ_+^X of Δ as follows. In the nontwisted case,

$$\Delta_+^X = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+ \backslash \mathbb{Z} \cdot X,\ n \in \mathbb{Z}\} \\ \cup \dot{\Delta}_+ \cup \{\alpha + n\delta \mid \alpha \in (\dot{\Delta} \cap \mathbb{Z}X) \cup \{0\},\ n > 0\}. \tag{1.5}$$

In the twisted case we put

$$\Delta_+^X = (\Delta_+ \cup \tfrac{1}{2}\Delta_+) \cap \Delta \tag{1.6}$$

with Δ_+ as in (1.5).

PROPOSITION 1.2 [7]. *If Δ is an affine root system then every set of positive roots is $W \times \{\pm 1\}$-conjugate to one of the sets Δ_+^X.*

An antilinear anti-involution ω of $\mathfrak{g}$ is called *consistent* if for all $\alpha \in \Delta$, $\omega \cdot \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$. It is clear that replacing e_i by some λe_i and f_i by $\lambda^{-1} f_i$ one can bring ω to the form

$$\omega \cdot e_i = \pm f_i \qquad (i = 0, ..., l). \tag{1.7}$$

An important example is the *compact antilinear anti-involution* ω_c,

$$\omega_c \cdot e_i = f_i \qquad (i = 0, ..., l). \tag{1.8}$$

We extend a given ω to the enveloping algebra $\mathcal{U}(\mathfrak{g})$ in the obvious way.

Let ω be a consistent antilinear anti-involution and let Π be a representation of $\mathfrak{g}$ on a vector space V. A hermitian form H on V is called *contravariant* (w.r.t. Π and ω) if

$$\forall u, v \in V,\ \forall x \in \mathfrak{g}\colon H(\pi(x)u, v) = H(u, \pi(\omega \cdot x)v). \tag{1.9}$$

In case H is positive (semi-)definite, π is said to be *unitarizable* (w.r.t. ω).

Let λ be a 1-dimensional representation of a Borel subalgebra $\mathscr{b}$ and assume that λ satisfies the reality condition

$$\lambda(g_+) = \overline{\lambda((\omega \cdot g)_+)} \qquad \text{for all} \quad g \in \mathfrak{g}. \tag{1.10}$$

Let

$$\mathscr{b}^\lambda = \{x \in \mathscr{b} \mid \lambda(x) = 0\} \tag{1.11}$$

and let ω be a consistent antilinear anti-involution. Then, on the module

$$M(\lambda) = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\,\mathscr{b}^\lambda \tag{1.12}$$

there is a unique contraviariant hermitian form H [7]. Let $I(\lambda)$ denote the kernel of H on $M(\lambda)$ and put

$$L(\lambda) = M(\lambda)/I(\lambda). \tag{1.13}$$

We shall occasionally denote $L(\lambda)$ by $L_{\mathscr{b},\omega}(\lambda)$. The corresponding highest weight representation will be denoted by Π_λ or by $\Pi_{\lambda;\mathscr{b},\omega}$.

Remark. One may more generally construct highest weight representations starting from 1-dimensional representations of parabolic subalgebras. Even though this does not lead to any more unitarizable representations than what is obtained starting from Borel subalgebres [7], we shall often find it more convenient to work with this situation. We denote the representation obtained in analogy with the above by $\Pi_{\lambda;\mathscr{p},\omega}$ and the corresponding space by $L_{\mathscr{p},\omega}(\lambda)$.

We now present all the unitarizable highest weight modules: Let $\lambda: \mathscr{b}^{st} \to \mathbb{C}$ be a 1-dimensional representation defined by

$$\lambda(e_i) = 0 \qquad \text{and} \qquad \lambda(\alpha_i^\vee) = m_i \in \mathbb{Z}_+ \ (i = 0, ..., l). \tag{1.14}$$

Then the representation $\Pi_{\lambda;\mathscr{b},\omega_c}$ is unitarizable [8, Chap. 11]. These representations are called *integrable* highest weight representations. In particular, if g is finite-dimensional, these are precisely the finite-dimensional representations.

It is well known that if g is a finite-dimensional simple Lie algebra, then an infinite-dimensional highest weight representation $\Pi_{\lambda,\omega}$ is unitarizable only if ω is a consistent antilinear anti-involution which corresponds to a hermitian symmetric space. Those λ's that lead to unitarity were determined by Jakobsen [5] (see also [2]).

There is the following "elementary" way to construct a unitarizable highest weight representation Π_λ of an affine Lie algebra g: First we put $\Pi_\lambda(c) = 0$ so that Π_λ can be viewed as a representation of the Lie algebra $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \dot{\mathfrak{g}}$ in the nontwisted case or of its subalgebra in the twisted case. Now fix N non-zero complex numbers $z_1, ..., z_N$ of modules one, and denote by $\varphi_i: \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \dot{\mathfrak{g}} \to \dot{\mathfrak{g}}$ the evaluation map at z_i, i.e.,

$$\varphi_i(z^k \otimes x) = z_i^k x. \tag{1.15}$$

Fix a Borel subalgebra $\dot{\mathscr{b}}$ of $\dot{\mathfrak{g}}$ and a consistent antilinear anti-involution $\dot{\omega}$ of $\dot{\mathfrak{g}}$. Let Π_{λ_i} be a unitarizable highest weight representation of $\dot{\mathfrak{g}}$ on $L(\lambda_i) = L_{\dot{\mathscr{b}},\dot{\omega}}(\lambda_i)$ $(i = 1, ..., N)$. Evidently then, $\Pi_{\lambda_i} \circ \varphi_i$ is a unitarizable highest weight representation of g. Let $\mathscr{p} = \mathbb{C}[z, z^{-1}] \otimes \dot{\mathscr{b}}$, define a representation $\mathscr{p} \to \mathbb{C}$ by

$$\lambda(z^k \otimes b) = \sum_i z_i^k \lambda_i(b) \tag{1.16}$$

and define an antilinear anti-involution ω of $\mathfrak{g}$ by

$$\omega \cdot (z^k \otimes x) = z^{-k} \otimes \dot{\omega} \cdot x. \tag{1.17}$$

Then,

$$\begin{aligned} L_{\not{p},\omega}(\lambda) &= L(\lambda_1) \otimes \cdots \otimes L(\lambda_N) \\ \Pi_{\lambda,\not{p},\omega} &= (\Pi_{\lambda_1} \circ \varphi_1) \otimes \cdots \otimes (\Pi_{\lambda_N} \circ \varphi_N). \end{aligned} \tag{1.18}$$

Thus the representation $\Pi_{\lambda,\not{p}}$ of $\mathfrak{g}$ on $L_{\not{p},\omega}(\lambda)$ is unitarizable. We call these representations *elementary.*

Finally, let $\mathfrak{g} = sl_{l+1}(\mathbb{C}[z, z^{-1}])$ (i.e., we assume again that the center $\mathbb{C} \cdot c$ acts trivially). Let

$$\not{p} = \{(a_{ij}(z)) \in \mathfrak{g} \mid a_{ij} = 0 \text{ if } i > j\}, \tag{1.19}$$

and let

$$\omega \cdot (a_{ij}(z)) = (\varepsilon_{ij} \overline{a_{ji}}(z^{-1})), \tag{1.20}$$

where $\varepsilon_{ij} = 1$ if $i \neq 1$ or $j \neq 1$ or $i = j = 1$, and $\varepsilon_{ij} = -1$ otherwise. (If $a(z) = \sum a_i z^i \in \mathbb{C}[z, z^{-1}]$ we write $\bar{a}(z) = \sum \overline{a_i} z^i$).

Let m be a finite positive measure on the unit circle $S^1 \subset \mathbb{C}$. Define a linear functional $\varphi_m : \mathbb{C}[z, z^{-1}] \to \mathbb{C}$ by

$$\varphi_m(a(z)) = \int_{S^1} a(e^{i\theta}) \, dm(\theta), \tag{1.21}$$

and define a representation $\lambda_m : \not{p} \to \mathbb{C}$ by

$$\lambda_m((a_{ij}(z))) = -\varphi_m(a_{11}(z)). \tag{1.22}$$

Then the representation Π_{λ_m} of $sl_{l+1}(\mathbb{C}[z, z^{-1}])$ is called *exceptional.* We will show, as a part of a more general result, that they are unitarizable.

We can now state our first main result.

THEOREM 1.3 (cf. [7]). *Let $\mathfrak{g}$ be an affine Lie algebra, let ω be a consistent antilinear anti-involution of $\mathfrak{g}$, and let $\mathscr{b}$ be a Borel subalgebra. Let $\lambda : \mathscr{b} \to \mathbb{C}$ be a 1-dimensional representation of $\mathscr{b}$. Then the representation $\Pi_{\lambda,\mathscr{b},\omega}$ of $\mathfrak{g}$ on the space $L_{\mathscr{b},\omega}(\lambda)$ is unitarizable if and only if it is equivalent to either an integrable representation, an elementary representation, an exceptional representation, or to the highest component of a $\otimes$-product of an elementary with an exceptional representation.*

We now begin to give the details of selected points of the proof of this theorem. We follow the general plan as outlined for the corresponding proof in [8]. Indeed, using the first elementary considerations of that proof, we are immediately reduced to studying the following situation:

$$\mathring{\mathfrak{g}} = \mathring{\mathfrak{p}}^- \oplus \mathring{\mathfrak{k}} \oplus \mathring{\mathfrak{p}}^+ = \mathring{\mathfrak{p}}^- \oplus \mathring{\mathfrak{k}}^- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{k}}^+ \oplus \mathring{\mathfrak{p}}^+ \tag{1.23}$$

is a simple finite-dimensional Lie algebra corresponding to a hermitian symmetric space, and $\mathfrak{g} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathring{\mathfrak{g}}$. Moreover,

$$\begin{aligned} \omega \cdot (z^n \otimes k_i^+) &= z^{-n} \otimes k_i^-, \qquad i > 1,\ n \in \mathbb{Z} \\ \omega \cdot (z^n \otimes p_1^+) &= -z^{-n} \otimes p_1^-, \end{aligned} \tag{1.24}$$

where $p_1^+, k_2^+, ..., k_l^+$ are those Chevalley generators that were previously denoted by $e_1, ..., e_l$. Here $k_2^+, ..., k_l^+$ belong to $\mathfrak{k}$, and p_1^+ belongs to the root space inside $\mathfrak{p}^+$ corresponding to the unique simple non-compact root. The elements $p_1^-, k_2^-, ...$ of course are defined analogously.

The representations in question are highest weight representations obtained from 1-dimensional representations of the parabolic subalgebra

$$\not{p} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} (\mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{k}}^+ \oplus \mathring{\mathfrak{p}}^+). \tag{1.25}$$

Let $h_1 = [p_1^+, p_1^-]$ and $h_i = [k_i^+, k_i^-]$; $i = 2, ..., l$. Then $\not{p}^\lambda \supset \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} (\mathring{\mathfrak{k}}^+ \oplus \mathring{\mathfrak{p}}^+)$ and

$$\lambda(z^n \otimes h_i) = \varepsilon_i \int_{S^1} e^{in\theta}\, d\mu_i(\theta) \tag{1.26}$$

for finite positive (possibly zero) measures $\mu_1, ..., \mu_l$ on S^1. To have unitarity the analysis in [7] gives that $\varepsilon_2 = \cdots = \varepsilon_l = 1$ and that the measures $\mu_2, ..., \mu_l$ all are finitely supported with integral mass in each point of the support. Moreover, just to have unitarity on $\mathring{\mathfrak{g}}$, we must have $\varepsilon_1 = -1$.

The following proposition takes care of a major part of the proof:

PROPOSITION 1.4. *Let $l = \dim_{\mathbb{C}} \mathring{\mathfrak{h}}$ and let $\mu_1, ..., \mu_l$ be as above. Assume that $l > 1$ and that μ_1 is not finitely supported. Then if $\mu_2 = \cdots = \mu_l = 0$, the corresponding highest weight representation is not unitarizable.*

Proof. Denote the contravariant form H by $\langle \cdot, \cdot \rangle$. We need to show that this form is not positive semi-definite. The crucial observation is the following, in which $\lambda_0 = -\int_{S^1} d\mu_1$, Δ_n^+ denote those positive roots whose root vectors lie in $\mathring{\mathfrak{p}}^+$, root vectors in $\mathring{\mathfrak{p}}^-$ are denoted by $e_{-\alpha}$; $\alpha \in \Delta_n^+$, and v_λ denotes the highest weight vector:

LEMMA 1.5. *If $l>1$ there is at least one* 2nd *order polynomial q in* $\mathscr{U}(\dot{\mathfrak{p}}^-)$,

$$q = \sum_{\alpha_i, \alpha_j \in \Delta_n^+} a_{\alpha_i, \alpha_j} e_{-\alpha_i} e_{-\alpha_j} \qquad (a_{\alpha_i, \alpha_j} = a_{\alpha_j, \alpha_i}) \tag{1.27}$$

such that $\langle q, q\rangle = \lambda_0(\lambda_0 - c_1)$ for some strictly negative real number c_1.

(For a proof of this lemma see, e.g., Wallach [12] or Rossi and Vergne [9]. (Cf. [3])).

In the above statement we have used the important fact that the hermitian form H, when restricted to $\mathscr{U}(\dot{\mathfrak{g}}) \cdot v_\lambda$ coincides with the hermitian form of the highest weight representation of $\dot{\mathfrak{g}}$ defined by $\lambda(h_1) = \lambda_0 = -\int_{S^1} d\mu_1$.

We now use the assumption on the support of μ_1 to guarantee the existence of an orthogonal family $\{g_N\}_{N=1}^\infty$ of *real* functions in $L^2(S^1, d\mu_1)$, where each g_N, $N = 1, \ldots$, is the restriction of a Laurent polynomial to S^1. Define

$$q_N = \sum_{i,j} a_{\alpha_i, \alpha_j} (g_N e_{-\alpha_i})(g_N e_{-\alpha_j}), \tag{1.28}$$

obtained from q as in (1.27) and where we from now on suppress the $\otimes$-sign. We also suppress v_λ.

Now compute

$$\begin{aligned} \langle q_N, q_M\rangle &= \sum_{i,j} \sum_{s,t} a_{\alpha_i,\alpha_j} \overline{a_{\alpha_s,\alpha_t}} \\ &\quad \times (-1) \langle (g_N e_{-\alpha_j}), (g_N e_{\alpha_i})(g_M e_{-\alpha_s})(g_M e_{-\alpha_t})\rangle \\ &= \sum_{i,j} \sum_{s,t} a_{\alpha_i,\alpha_j} \overline{a_{\alpha_s,\alpha_t}} \\ &\quad \times (-1) \langle (g_N^2 g_M^2 e_{-\alpha_j}), [e_{\alpha_i}, e_{-\alpha_s}]\, e_{-\alpha_t}\rangle \\ &\quad + \sum_{i,j} \sum_{s,t} a_{\alpha_i,\alpha_j} \overline{a_{\alpha_s,\alpha_t}} \\ &\quad \times (\langle g_N g_M e_{-\alpha_j}, e_{-\alpha_t}\rangle \langle g_N g_M e_{-\alpha_i}, e_{-\alpha_s}\rangle \\ &\quad + \langle g_N g_M e_{-\alpha_j}, e_{-\alpha_s}\rangle \langle g_N g_M e_{-\alpha_i}, e_{-\alpha_t}\rangle). \end{aligned} \tag{1.29}$$

If we compare this to the analogous computation for $\langle q, q\rangle$ and keep track of how the explicit dependence of this on λ_0 originates, it follows easily that

$$\langle q_N, q_M\rangle = \delta_{M,N}\, \lambda_0^2 + c_1 \int g_N^2 g_M^2 \, d\mu_1. \tag{1.30}$$

Let

$$f_i = g_1^2 + \cdots + g_i^2, \qquad i = 1, 2, \ldots . \tag{1.31}$$

Now consider

$$\left\langle \sum_{i=1}^{N} q_i, \sum_{i=1}^{N} q_i \right\rangle = \lambda_0^2 N + 2c_1 \int_{S^1} f_N^2 \, d\mu_1 . \tag{1.32}$$

Since $\int_{S^1} f_N \, d\mu_1 = N$ and $f_N \geqslant 0$, it follows easily from, e.g., Fatou's Lemma, that $\int_{S^1} f_N^2 \, d\mu_1$ grows more rapidly than any constant multiple of N, as $N \to \infty$. Since $c_1 < 0$ the expression (1.32) will thus eventually become negative. ∎

Remark. One might consider, in place of $\mathbb{C}[z, z^{-1}]$ some other *-algebra R (with or without unit) and, instead of μ_1, some self-adjoint trace φ on R. Highest weight representations can then clearly be defined for $R \otimes \dot{\mathfrak{g}}$. However, if we assume that R is commutative and that φ is faithful then unitarity, through a GNS construction based on φ leads us back to measure on some (locally) compact space. But, clearly, a similar non-unitarity result holds on any measure space as soon as the rank l of $\dot{\mathfrak{g}}$ is greater than one.

COROLLARY 1.6. *Let* $\dim \mathfrak{h} = l > 1$ *and assume that* $\mu_2, \ldots, \mu_l$ *are finitely supported measures on* S^1 *and that each point in the support of each measure has a mass equal to a positive integer. Then, if* μ_1 *is not finitely supported, the corresponding highest weight representation is not unitarizable.*

Proof. It suffices to consider the case in which all the measures are supported (at most) in one and the same point $a \in S^1$. The representation of $\mathbb{C}[z, z^{-1}] \otimes \mathfrak{k}$ then corresponds to evaluation inside a single unitary representation τ of $\mathfrak{k}$. Let τ' denote the contragradient representation. Thus, $\tau \otimes \tau'$ contains the identity representation.

Now consider an elementary representation of $\mathbb{C}[z, z^{-1}] \otimes \dot{\mathfrak{g}}$ corresponding to evaluation at a inside a unitary representation of $\dot{\mathfrak{g}}$ obtained through holomorphic induction from τ'. The tensor product of this representation with the one unitary representation we assume given is then again unitary. The decomposition of this tensor product into irreducibles (cf. Section 3) then yields unitary representations. In particular, the top term (which is obtained by "restriction to the diagonal" analogously to [6]) gives one piece which is trivial on $\mathbb{C}[z, z^{-1}] \otimes \mathfrak{k}$. But then Proposition 1.4 is contradicted, since this representation clearly is not finitely supported. ∎

What remains now is to prove the following

PROPOSITION 1.7. *Any unitary highest weight representation of* $\mathbb{C}[z, z^{-1}] \otimes su(n, 1)$ *based on the non-compact involution* (1.20) *is the highest component of the tensor product of an exceptional representation with an elementary representation.*

Remark. The unitarity of the exceptional representations is described in Section 2.

Proof. The essentials of the argument are already present for $n=2$, so we restrict ourselves to that case. Let $k = k_2^-$, $f = p_1^-$, and $e = p_1^+$ in the notation of (1.24). Further, let $k^m = (k \otimes \cdots \otimes k)$ (m copies). Let us now assume given a unitarizable highest weight representation. Then, suppressing as usual the highest weight vector, inside this representation, we have;

$$\begin{aligned} &\langle (z^t \otimes f)\, k^m, (z^s \otimes f)\, k^m \rangle \\ &\quad = m \cdot \langle k^{m-1}, k^{m-1} \rangle \\ &\quad\quad \times \left(\int_{S^1} e^{i(t-s)\theta}\, d\mu_1(\theta) - \int_{S^1} e^{i(t-s)\theta}\, d\mu_2(\theta) \right), \end{aligned} \tag{1.33}$$

where $m = \int_{S^1} d\mu_2(\theta) \in \mathbb{N}$, as required by $\mathfrak{k}$. Furthermore, it is easy to see that the support of $d\mu_2$ is finite (cf. [7]).

It follows that $d\mu_1 \geqslant d\mu_2$. In particular, $d\mu_1$ has positive mass in each of the points in the support of $d\mu_2$. Of course, we need to assert more than that, namely that the masses of $d\mu_1$ at the points of the support of $d\mu_2$ are so big that the finitely supported representation of $\mathbb{C}[z, z^{-1}] \otimes su(2, 1)$ defined by $d\mu_2$ and by the restriction of $d\mu_1$ to the support of $d\mu_2$, is unitarizable. (This will define the elementary representation.) However, this follows easily from the assumed unitarity of the originally given representation. We know, namely, that inner products are positive between elements of the form $(g \otimes x_1) \cdots (g \otimes x_k)$ (k arbitrary) with $x_1, \ldots, x_k \in su(2, 1)$ and $g \in \mathbb{C}[z, z^{-1}]$ an approximation in $L^1(S^1, d\mu_1)$-norm of a box-shaped function which is 1 near one of points in the support of $d\mu_2$ and which is zero outside an ε-neighbourhood of that point. In other words, we can "localize" our representation at the support of $d\mu_2$. Hence the assertion follows. What remains is the complement of the support of $d\mu_2$. We can also localize here, and the assumed unitarity of the given representation then yields an exceptional representation supported by that set. ∎

With this, we have completed the proof of Theorem 1.3.

2. Unitarizable Modules

The fact that the exceptional representations of $\mathbb{C}[z, z^{-1}] \otimes su(n, 1)$ are unitarizable was already in [7] seen as a consequence of a specific computation inside a more general framework. Since we wish to give some more applications, we shall now repeat this computation, correcting at the same time a few misprints.

Let R denote a (non-commutative) associative algebra over $\mathbb{C}$ and let φ be a *trace* on R, i.e., a linear map of R into $\mathbb{C}$ which satisfies

$$\varphi(ab) = \varphi(ba) \qquad \text{for all} \quad a, b \in R. \tag{2.1}$$

We define the Lie algebra

$$sl_2(R, \varphi) = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| \, a_i \in R, \; i = 1, 2, 3, 4 \text{ and } \varphi(a_1 + a_4) = 0 \right\}, \tag{2.2}$$

and we consider the Verma module $M(\varphi)$ defined by the property that there exists a non-zero vector v_φ such that

$$\begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix} v_\varphi = \varphi(a_1) \cdot v_\varphi. \tag{2.3}$$

Let $i \in \mathbb{N}$. We say that $\gamma = (\gamma_1, ..., \gamma_s) \in \mathbb{N}^s$ is an *s-partition of* i if

$$i = \gamma_1 + \gamma_2 + \cdots + \gamma_s, \qquad \text{and} \qquad \gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_s > 0. \tag{2.4}$$

We let $\mathrm{Par}_s(i)$ denote the set of all such s-tuples.

Let $\gamma \in \mathrm{Par}_s(N)$. Utilizing the fact that φ is a trace, we will say that $\Pi_1^1 \times \Pi_2^1 \in S_N \times S_N$ is equivalent to $\Pi_1 \times \Pi_2 \in S_N \times S_N$, where S_N denotes the group of permutations of N letters, if for all $z_1, ..., z_i, w_1, ..., w_i \in R$,

$$\varphi(z_{\pi_1^1(1)} w_{\pi_2^1(1)} \cdots z_{\pi_1^1(\gamma_1)} w_{\pi_2^1(\gamma_1)}) \cdots \varphi(z_{\pi_1^1(\gamma_1 + \cdots + \gamma_{s-1} + 1)} w_{\pi_2^1(\gamma_1 + \cdots + \gamma_{s-1} + 1)} \cdots) \tag{2.5}$$

can be obtained from the analogous expression for $\Pi_1 \times \Pi_2$ by a permutation of the s factors $\varphi(\cdots)$ and/or by cyclic permutation of the variables (e.q., $\varphi(z_3 w_3 z_1 w_1 z_2 w_2) = \varphi(z_2 w_2 z_3 w_3 z_1 w_1)$).

The set of equivalence classes is denoted by $(S_N \times S_N)(\gamma)$.

Lemma 2.1. *Let $z_1, ..., z_N, w_1, ..., w_N \in R$. Then in $M(\varphi)$,*

$$\begin{aligned}(-1)^N & \begin{pmatrix} 0 & z_1 \\ 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & z_N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ w_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 \\ w_N & 0 \end{pmatrix} \cdot v_\varphi \\ &= \sum_{s=1}^{N} \sum_{\gamma \in \mathrm{Par}_s(N)} \sum_{[\pi_1 \times \pi_2] \in (S_N \times S_N)(\gamma)} (-1) \\ &\quad \times \varphi(z_{\pi_1(1)} w_{\pi_2(1)} \cdots z_{\pi_1(\gamma_1)} w_{\pi_2 \gamma_1)}) \cdots (-1) \\ &\quad \times \varphi(\cdots z_{\pi_1(N)} w_{\pi_2(N)}) \cdot v_\varphi. \end{aligned} \tag{2.6}$$

Proof. We proceed by induction. $N=1$ is trivial, so assume (2.6) is true up to N. We have, by (2.3),

$$\begin{aligned}
&(-1)^{N+1}\begin{pmatrix}0 & z_1\\ 0 & 0\end{pmatrix}\cdots\begin{pmatrix}0 & z_{N+1}\\ 0 & 0\end{pmatrix}\\
&\quad\times\begin{pmatrix}0 & 0\\ w_1 & 0\end{pmatrix}\cdots\begin{pmatrix}0 & 0\\ w_{N+1} & 0\end{pmatrix}\cdot v_\varphi\\
&=\sum_{i=1}^{N+1}(-1)^N\begin{pmatrix}0 & z_1\\ 0 & 0\end{pmatrix}\cdots\begin{pmatrix}0 & z_N\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ w_1 & 0\end{pmatrix}\\
&\quad\cdots\widehat{\begin{pmatrix}0 & 0\\ w_i & 0\end{pmatrix}}\cdots\begin{pmatrix}0 & 0\\ w_{N+1} & 0\end{pmatrix}(-1)\varphi(z_{n+1}w_i)\cdot v_\varphi\\
&\quad+(-1)^{N+1}\sum_{j>i}\begin{pmatrix}0 & z_1\\ 0 & 0\end{pmatrix}\cdots\begin{pmatrix}0 & z_N\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ w_1 & 0\end{pmatrix}\\
&\quad\cdots\widehat{\begin{pmatrix}0 & 0\\ w_i & 0\end{pmatrix}}\cdots\begin{pmatrix}0 & 0\\ -w_iz_{N+1}w_jz_{N+1}w_i & 0\end{pmatrix}\cdots\begin{pmatrix}0 & 0\\ w_{N+1} & 0\end{pmatrix}\cdot v_\varphi.
\end{aligned}\tag{2.7}$$

This, then, can be evaluated using (2.6), and as a result one will get an expression analogous to this. To treat the constants rigorously it is, by symmetry, enough to examine terms of the form

$$(-1)\cdot\varphi(z_1w_1\cdots z_{\gamma_1}w_{\gamma_1})\cdots(-1)\cdot\varphi(z_{(\gamma_1+\cdots+\gamma_{s-1}+1)}\cdots z_{N+1}w_{N+1}).\tag{2.8}$$

Assuming $\gamma_s>1$ ($\gamma_s=1$ is trivial), the only way in which such a term can emerge is clearly by replacing w_N by $-w_Nz_{N+1}w_{N+1}$ in the analogous expression

$$(-1)\varphi(z_1w_1\cdots z_{\gamma_1}w_{\gamma_1})\cdots(-1)\varphi(z_{\gamma_1+\cdots+\gamma_{s-1}+1}\cdots z_Nw_N)$$

in (2.6).

COROLLARY 2.2 [7]. *The exceptional representations of* $\mathbb{C}[z,z^{-1}]\otimes$ *su*$(n,1)$ *are unitarizable.*

Proof. Let R above be the algebra of polynomial functions from S^1 into $gl(n+1,\mathbb{C})$. Let tr denote the usual trace on $gl(n+1,\mathbb{C})$ and define

$$\varphi_\mu(p)=-\int_{S^1}tr(p(e^{i\theta}))\,d\mu(\theta)\tag{2.9}$$

for some positive Radon measure $d\mu$ on S^1. Take the z's in (2.6) to be of the form of polynomial functions $S^1\to gl(n+1,\mathbb{C})$ whose only non-zero entries are in the first row, and let the w's be of the form z^* with z as before. Then (2.6) is easily seen to express the hermitian form on $M(\lambda) = M(\varphi_\lambda)$ as the sum of tensor products of positive definite hermitian forms, and thus as something positive. ▮

Let us put $M(R, \varphi) = M(\varphi)$ to stress the dependence of $M(\varphi)$ on R.

Assume that there is an antilinear anti-involution $a \to a^*$ of R such that

$$\varphi(a^*) = \overline{\varphi(a)}. \tag{2.10}$$

Define an antilinear anti-involution ω of $sl_2(R, \varphi)$ by

$$\omega\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix}. \tag{2.11}$$

COROLLARY 2.3. *Suppose that R is commutative. Then $M(\varphi, R)$ is unitarizable if and only if the form*

$$(a, b) = -\varphi(b^* a) \tag{2.12}$$

is positive definite.

Proof. In this case the expressions in (2.6) simplify considerably, and it is easy to see that the form is a sum of tensor products of positive definite forms when (2.12) defines a positive definite form. The converse is trivially satisfied. ▮

Remark. Usually, it is a part of the definition of a trace that φ should be positive on positive elements. Thus (2.12) says that $-\varphi$ must satisfy this requirement to ensure unitarity.

For our next example we take R to be the algebraic part of the *irrational rotation algebra.* Specifically, R is generated by two unitary operators U and V (and their adjoints) satisfying

$$V \cdot U = e^{2\pi i \rho} U \cdot V \tag{2.13}$$

for some irrational number ρ. We write

$$U^n = \begin{cases} U \cdot \cdots \cdot U \ (n \text{ copies}) & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ (U^*) \cdots (U^*) \ (n \text{ copies}) & \text{if } n < 0 \end{cases} \tag{2.14}$$

and similarly for V^n.

This algebra has got a distinguished one-parameter family of traces

$$(\varphi_\lambda)(U^m V^m) = \lambda \cdot \delta_{n,0} \cdot \delta_{m,0}, \tag{2.15}$$

and φ_λ clearly satisfies (2.10) if and only if λ is real. Let the antilinear anti-automorphism * on R be the map of an element into its adjoint and let a be as in (2.11).

COROLLARY 2.4. *For the algebraic part R of the irrational algebra, $M(R, \varphi_\lambda)$ is unitarizable if and only if $\lambda \leqslant 0$.*

Proof. Consider the expression (2.6). Let $z_i = U^{n_i}V^{n_i}$ and $w_j = V^{-s_j}U^{-t_j}$ for $i, j = 1, ..., N$. Each factor in each summand is then of the form $(-\varphi_\lambda(z_{i_1}w_{j_1} \cdots z_{i_r}w_{j_r}))$. By means of (2.13) and (2.15), each of these factors can be explicitly determined. To see the general pattern, it suffices to consider the following example:

$$\begin{aligned} -\varphi_\lambda(z_1 w_1 \cdots z_r w_r) &= -\varphi_\lambda(U^{n_1}V^{m_1}V^{-s_1}U^{-t_1}U^{n_2}V^{m_2}V^{-s_2}U^{-t_2}\cdots) \\ &= -e^{2\pi i p\cdot\alpha}\delta_{n_1+\cdots+n_r,\, t_1+\cdots+t_r}\cdot\delta_{m_1+\cdots+m_r,\, s_1+\cdots+s_r}, \end{aligned} \tag{2.16}$$

where $\alpha = [(n_2 - t_1)(m_1 - s_1) + (n_3 - t_2)(m_1 + m_2 - (s_1 + s_2)) + \cdots]$. Consider the linear transformation $T_{a,b}$ of R determined by

$$\begin{aligned} &T_{a,b}(U^{c_1}V^{d_1}U^{c_2}V^{d_2}\cdots U^{c_s}V^{d_s}) \\ &\qquad = U^{c_1+a}V^{d_1+b}U^{c_2+a}V^{d_2+b}\cdots U^{c_s+a}V^{d_s+b} \end{aligned} \tag{2.17}$$

$(a, b \in \mathbb{Z})$. Extend this operator to a linear operator $\tilde{T}_{a,b}$ on $M(R, \varphi_\lambda)$ by

$$\begin{aligned} &\tilde{T}_{a,b}\left[\begin{pmatrix} 0 & 0 \\ r_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 \\ r_s & 0 \end{pmatrix} v_{\varphi_\lambda}\right] \\ &\qquad = \begin{pmatrix} 0 & 0 \\ T_{a,b}(r_1) & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 \\ T_{a,b}(r_s) & 0 \end{pmatrix} v_{\varphi_\lambda} \end{aligned} \tag{2.18}$$

for $r_1, ..., r_s \in R$. It then follows from the computation in (2.16) that $\tilde{T}_{a,b}$ preserves the canonical inner product on $M(R, \varphi_\lambda)$.

Let us agree to say that elements of the form

$$\begin{pmatrix} 0 & 0 \\ r_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 \\ r_s & 0 \end{pmatrix} v_{\varphi_\lambda} \tag{2.19}$$

belong to level s in $M(R, \varphi_\lambda)$. We need to prove positivity at all levels. We begin by making two obvious observations: (i) Inner products between different levels are zero and (ii) the leading term in λ of the inner product of an s-level element q_s with itself is $(-\lambda)^s \cdot c$ for some constant $c > 0$. (Observe that each summand in q_s is a scalar multiple of an expression of the form (2.19) with each r_i being of the form $U^{n_i}V^{m_i}$.)

Let us for a moment assume that some q_s, for some fixed λ, has got a negative inner product with itself. Since $\tilde{T}_{a,b}$ preserves inner products we may then assume that each summand in q_s only involves elements $U^{n_i}V^{m_i}$ of R with $n_i \geqslant 0$ and $m_i \geqslant 0$. This means that if we already have proved positivity on the later kind of elements, then positivity follows globally. Specifically, let

$$L_s^+(N, M) = \left\{ \begin{pmatrix} 0 & 0 \\ U^{n_1}V^{m_1} & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 \\ U^{n_s}V^{m_s} & 0 \end{pmatrix} \times v_{\varphi_\lambda} \middle| \begin{array}{l} \text{all exponents are } \geqslant 0, \\ n_1 + \cdots + n_s = N, \text{ and } m_1 + \cdots + m_s = M \end{array} \right\}. \tag{2.20}$$

Then it is enough to prove positivity of the form restricted to each of these finite-dimensional subspaces.

By the previous remarks, the hermitian form restricted to $L_s^+(N, M)$ is positive definite for λ sufficiently negative. Let us again assume that the form is not positive definite for some (possibly all) $\lambda < 0$. Let s_0 be the lowest level at which there is non-unitarity. Clearly, $s_0 > 1$ (cf. Corollary 2.3). Let λ_0 be the first place (going from $-\infty$ towards 0) at which the form changes sign on some $L_{s_0}^+(N, M)$. Since the form varies smoothly with λ, at λ_0 the kernel must be non-trivial. Thus,

$$\exists \tilde{q} \in L_{s_0}^+(M, N), \qquad \forall q \in L_{s_0}^+(M, N): \quad \langle \tilde{q}, q \rangle_{\lambda_0} = 0. \tag{2.21}$$

Now let q_{s_0-1} be an arbitrary element of $L_{s_0-1}^+(N, M)$ and let $c \in \mathbb{C}$. Then,

$$\left\langle \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \tilde{q}, q_{s_0-1} \right\rangle_{\lambda_0} = 0, \tag{2.22}$$

since the form is contravariant. Hence, due to the minimality assumption on s_0,

$$\forall c \in \mathbb{C}: \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \tilde{q} = 0. \tag{2.23}$$

Let $\bar{n}$ and $\bar{m}$ denote the smallest coefficients of U and V, respectively, that occur in $\tilde{q}$. By, perhaps, replacing $\tilde{q}$ by $\tilde{T}_{-\bar{n}, -\bar{m}}\tilde{q}$, which is in the kernel of the form om $L_s^+(N - \bar{n}, M - \bar{m})$, we may assume that $\bar{n} = \bar{m} = 0$. Thus

$$\tilde{q} = \sum_{i=1}^{s} a_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^i x_i v_{\varphi_\lambda}, \tag{2.24}$$

where $x_i = \Pi_{j=1}^{s-i} \left(\begin{smallmatrix} 0 & 0 \\ v_{i,j} & 0 \end{smallmatrix}\right)$ and each $v_{i,j}$ is of the form $U^n V^m$ for some non-negative integers n and m (depending on i and j) satisfying $n+m>0$. Let i_0 be the smallest value for which $a_{i_0}=0$ in (2.24). Then $i_0 \geqslant 1$. We now compute (2.23) inside $M(R, \varphi_\lambda)$. We see that we get exactly one term

$$\beta \cdot a_{i_0+1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{i_0} x_{i_0+1} v_{\varphi_\lambda} \tag{2.25}$$

containing $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ to the power i_0. The constant β can easily be determined,

$$\beta = (i_0+1)(\lambda - 2s + i_0 + 2). \tag{2.26}$$

In particular, since $i_0 < s$, $s > 1$, and $\lambda < 0$ it follows that $\beta \neq 0$. Since we must have (2.23) satisfied, this is a contradiction with the minimality assumption on $s = s_0$. ∎

We conclude this section with an example which does not utilize (2.6) directly, but still leads to unitarizable modules.

Let R_F be the algebra of all finite-rank operators on a separable Hilbert space, and let $\varphi_\lambda = \lambda \cdot \mathrm{tr}$, where tr denotes the usual trace.

PROPOSITION 2.5. *$M(R_F, \varphi_\lambda)$ is unitarizable if and only if $\lambda = -1, -2, ..., -n$ (or $\lambda = 0$).*

Proof. Any computation using (2.6) can be chosen to take place inside some $su(N, N)$ for N sufficiently big. The result then follows from the known unitarity for these algebras [12, 9]. ∎

Remark. This seems to be the infinitesimal version of the model constructed in [13] (see also [11]). However, we stress that generically, namely for given λ, as soon as the ranks involved are greater than $-\lambda$, the representations involved are singular.

3. TENSOR PRODUCTS

We now return to the situation described in (1.23)–(1.26). Let $\lambda_1, ..., \lambda_N$ be highest weights of unitarizable highest weight modules of $\mathring{\mathfrak{g}}$, let $e^{i\theta_1}, ..., e^{i\theta_N}$ be points in S^1, and let μ be an infinitely supported positive finite measure on S^1. Let λ be the 1-dimensional representation of $\mathfrak{h}$ determined by

$$\lambda\left(z^n \otimes \left(\sum_{i=1}^{l} a_i h_i\right)\right)$$
$$= -a_1 \int_{S^1} e^{in\theta}\, d\mu(\theta) + \sum_{i=1}^{l} \sum_{j=1}^{N} a_i e^{in\theta_j} \lambda_j(h_i), \tag{3.1}$$

and let $\Pi(\mu, \lambda_1, \theta_1, ..., \lambda_N, \theta_N)$ denote the corresponding representation Π_λ. We extend this notation to also cover the cases in which either $\mu = 0$ or the λ_i's are zero, simply by leaving them out in the appropriate cases. Since, for $\dot{\mathfrak{g}} = su(l, 1)$ the above representation obviously occurs inside the tensor product of two (or, rather, $N+1$) unitarizable representations, we clearly have

PROPOSITION 3.1. *For* $\dot{\mathfrak{g}} = su(l, 1)$, $\Pi(\mu, \lambda_1, \theta_1, ..., \lambda_N, \theta_N)$ *is unitarizable.*

We proved in [7] that $\Pi(\mu)$ is irreducible. The representation $\Pi(\lambda_1, \theta_1, ..., \lambda_N)$ clearly is also irreducible since by localizing in $\mathbb{C}[z, z^{-1}]$ around the points $e^{i\theta_1}, ..., e^{i\theta_N}$, we can view this representation as a representation of $\dot{\mathfrak{g}} \times \cdots \times \dot{\mathfrak{g}}$ (N copies). Finally, again by localizing to open sets, we obtain the general result:

PROPOSITION 3.2. $\Pi(\mu, \lambda_1, \theta_1, ..., \lambda_N, \theta_N)$ *is irreducible.*

Let us now consider $\Pi(\mu^1) \otimes \Pi(\mu^2)$. To decompose this representation into irreducibles, we follow the ideas of [6]. Let v_1 and v_2 denote the highest weight vectors for $\Pi(\mu^1)$ and $\Pi(\mu^2)$, respectively. We shall write elements of $\dot{\mathfrak{p}}^-$ as f, and elements of $\dot{\mathfrak{p}}^+$ as e. Elements $z^n \otimes x$ in $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \dot{\mathfrak{g}}$ are written as $z^n \cdot x$. Now observe that

$$(z^n f \otimes 1) = \tfrac{1}{2}(z^n f \otimes 1 - 1 \otimes w^n f) + \tfrac{1}{2}(z^n f \otimes 1 + 1 \otimes w^n f) \tag{3.2}$$

and similarly for $1 \otimes w^n f$. It follows that any element in the tensor product can be written as a sum of terms of the form

$$\left(\prod_{j=1}^{T} (z^{n_j} f_j \otimes 1 + 1 \otimes w^{n_j} f_j) \right) \times \left(\prod_{i=1}^{N} (z^{n_i} f_i \otimes 1 - 1 \otimes w^{n_i} f_i) \right) (v_1 \otimes v_2). \tag{3.3}$$

Let us for simplicity denote an element of the form (3.3) by $S_T D_N$. Then,

$$(z^n e \otimes 1 + 1 \otimes w^n e)\, S_T D_N = S_{T-1} D_N + S_{T+1} D_{N-2}. \tag{3.4}$$

Let

$$I_M = \operatorname{span}\{ S_T D_N \mid N \leqslant M \}. \tag{3.5}$$

Then we have a filtration

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_M \subseteq \cdots \tag{3.6}$$

of the tensor product, which is invariant under $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \dot{\mathfrak{g}}$. Furthermore, since if f_i has got weight α_i,

$$(z^n h \otimes 1 + 1 \otimes w^n h)(z^{n_i} f_i \otimes 1 - 1 \otimes w^{n_i} f_i)(v_1 \otimes v_2)$$
$$= \alpha_i(h)(z^{n_i+n} f_i - 1 \otimes w^{n_i+n} f_i)(v_1 \otimes v_2)$$
$$+ \left(\int_{S^1} e^{in\theta}(d\mu^1(\theta) + d\mu^2(\theta)) \right) (z^{n_i} f_i \otimes 1 - 1 \otimes w^{n_i} f_i)(v_1 \otimes v_2), \qquad (3.7)$$

$z^n h \otimes 1 + 1 \otimes w^n h$ acts through a mixture of $\mu^1 + \mu^2$ and a shift operator. Depending on the sign of n, we get either a shift forward or a shift backwards. Hence, these shift operators generate a self-adjoint abelian algebra.

Finally, let k denote an element of $\dot{\mathfrak{k}}$. Then

$$(z^n k \otimes 1 + 1 \otimes w^n k)(z^{n_i} f_i \otimes 1 - 1 \otimes w^{n_i} f_i)(v_1 \otimes v_2)$$
$$= (z^{n_i+n}[k, f_i] \otimes 1 - 1 \otimes w^{n_i+n}[k, f_i])(v_1 \otimes v_2)$$
$$+ (z^{n_i} f_i \otimes 1 - 1 \otimes w^{n_i} f_i)(z^n k \otimes 1 + 1 \otimes w^n k)(v_1 \otimes v_2). \qquad (3.8)$$

Assume from now on, that $\dot{\mathfrak{g}} = su(l, 1)$. Let $\Pi(-\alpha_1)$ be the representation of $\dot{\mathfrak{g}}$ of highest weight $-\alpha_1$, where α_1, as before, denotes the unique simple non-compact root. Then $\Pi(-\alpha_1)$ is unitarizable [4], and the above analysis gives that

PROPOSITION 3.3. *For* $\dot{\mathfrak{g}} = su(l, 1)$,

$$\Pi(\mu^1) \otimes \Pi(\mu^2)$$
$$= \sum_{N=0}^{\infty} \underbrace{\int_{S^1} \cdots \int_{S^1}}_{N} \sum_{\sigma \in S_N} \frac{1}{N!}$$
$$\times \Pi(\mu^1 + \mu^2, -\alpha_1, \theta_{\sigma(1)}, ..., -\alpha_1, \theta_{\sigma(N)})\, d\theta_1 \cdots d\theta_N. \qquad (3.9)$$

Remark 1. We have, naturally, that all of the summands in the sum over the symmetric group S_N are equivalent. The formula is, of course, an abbreviation of a more complicated expression which takes into account what happens when some of the θ's coincide. Specifically, assume that out of $\theta_1, ..., \theta_N$; $\theta_1 = \cdots = \theta_i = \theta$, $i \leqslant N$. Then the results of [6] together with the explicit decomposition of $\mathscr{U}(\dot{\mathfrak{p}}^-)$ as determined in [10] (for $\dot{\mathfrak{g}} = su(l, 1)$, cf. [3]), gives that $\Pi(\mu^1 + \mu^2, -\alpha_1, \theta, ..., -\alpha_1, \theta, -\alpha_1, \theta_{i+1}, ..., -\alpha_1, \theta_N)$ should be replaced by $\Pi(\mu^1 + \mu^2, -i\alpha_1, \theta, -\alpha_1, \theta_{i+1}, ..., -\alpha_1, \theta_N)$.

Remark 2. In the above, we decompose the representation. We shall not attempt to decompose the Hilbert space explicitly.

Finally, let λ be the highest weight of a unitarizable highest weight representation $\Pi(\lambda)$ of $\dot{\mathfrak{g}} = su(l, 1)$, and consider $\Pi(\mu) \otimes \Pi(\lambda, \theta)$.

To describe the decomposition of this tensor product we need some notation.

Let $\Pi_{sc}(\varepsilon)$ denote the highest weight module of $su(l, 1)$ corresponding to the highest weight ("sc" stands for "scalar"),

$$\varepsilon(h_i) = \delta_{i,1} \cdot \varepsilon \tag{3.10}$$

for $\varepsilon < 0$. Then $\Pi_{sc}(\varepsilon)$ is unitarizable, and $\Pi_{sc}(\varepsilon) \otimes \Pi(\lambda)$ decomposes into a countable sum of irreducible unitary representations as described in [6],

$$\Pi_{sc}(\varepsilon) \otimes \Pi(\lambda) = \sum_{i=0}^{\infty} \Pi(i, \varepsilon, \lambda). \tag{3.11}$$

It is easy to see that each $\Pi(i, \varepsilon, \lambda)$ has got a limit as $\varepsilon \to 0$; we let

$$\Pi(\lambda_i) = \lim_{\varepsilon \to 0} \Pi(i, \varepsilon, \lambda) \qquad i = 0, 1, \ldots . \tag{3.12}$$

We now proceed as in (3.7) and (3.8), but we disregard all elements of the form

$$g(z) f \otimes 1 - 1 \otimes g(w) f, \tag{3.13}$$

in which $g(z) \in \mathbb{C}[z, z^{-1}]$ satisfies that $g(e^{i\theta}) = 0$. Thus

PROPOSITION 3.4. *For* $\dot{\mathfrak{g}} = su(l, 1)$,

$$\Pi(\mu) \otimes \Pi(\lambda, \theta) = \sum_{i=0}^{\infty} \Pi(\mu, \lambda_i, \theta).$$

4. INTEGRABILITY

Even through the results of [1] give the integrated versions of our exceptional representations, we will here present the details of how one can integrate our infinitesimal representations to the group. We do this because we feel that our computations have independent interest and because they reach out beyond the realm of commutativity.

Let us then consider a *-algebra R and let $\varphi: R \to \mathbb{C}$ be a linear map which satisfies (2.1), (2.10), and the condition

$$\varphi(aa^*) > 0 \qquad \text{for all} \quad a \in R. \tag{4.1}$$

We assume further that there is a Banach *-algebra norm $\|\cdot\|$ on R such that

$$\varphi(aa^*) \leqq \|aa^*\|. \tag{4.2}$$

This norm may, or may not, be induced by φ. Finally, we put

$$\varphi_\lambda = \lambda \cdot \varphi \qquad \text{for} \quad \lambda \in R. \tag{4.3}$$

Let us now assume that we have a unitarizable representation of $sl_2(R, \varphi_\lambda)$ on $M(R, \varphi_\lambda)$. We will assume that the hermitian form has got no kernel and that R has got a unit. Both assumptions exclude the case $R = R_F$ of Section 2. However, with rather obvious modifications, the following arguments can be made to cover also this case. Put

$$Gl(R) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in R, \text{ and } g \text{ invertible} \right\}. \tag{4.4}$$

Let

$$SU(1,1)^R = \{ g \in Gl(R) \mid \tilde{\omega}(g) = g^{-1} \}, \tag{4.5}$$

where $\tilde{\omega}$ denotes the anti-involution on $GL(R)$ which corresponds to the ω on $sl_2(R)$. This is then also given by (2.11).

Remark. At this point it seems an unnecessary complication to work with a λ-dependence of the groups. This, then, compels us to replace $sl_2(R, \varphi_\lambda)$ by $sl_2(R)$.

We identify elements $\left(\begin{smallmatrix} 0 & 0 \\ r & 0 \end{smallmatrix}\right)$ of $sl_2(R)$ with $r \in R$; more generally we let $\mathscr{P}(R)$ denote the space of polynomials in $\mathscr{U}(sl_2(R))$ generated by these. Let $\mathscr{H}_\lambda$ denote the Hilbert space completion of $M(R, \varphi_\lambda)$.

LEMMA 4.1. *Let* $\tilde{V}_\lambda = \{ pe^s \cdot v_{\varphi_\lambda} \mid p \in \mathscr{P}(R),\ s \in R, \text{ and } \|s\| < 1 \}$. *Then,* $\tilde{V}_\lambda$ *is contained in* $\mathscr{H}_\lambda$.

Proof. If we have proved that $e^s v_{\varphi_\lambda}$ belongs to $\mathscr{H}_\lambda$ then an easy power series argument applied to $e^{zr + ws} v_{\varphi_\lambda} (z, w \in \mathbb{C})$ proves the general claim. Thus we must examine

$$\sum_{N=0}^{\infty} \frac{\langle s^N, s^N \rangle_\lambda}{(N!)^2}, \tag{4.6}$$

where $\langle \cdot, \cdot \rangle_\lambda$ denotes the Hermitian form on $M(R, \varphi_\lambda)$, and where we from now on suppress v_{φ_λ}. To do this, we return to (2.6) and observe that each factor in each summand of $\langle s^N, s^N \rangle_\lambda$ is of the form

$$-\lambda \cdot \varphi((ss^*)^i) \tag{4.7}$$

for some i. By the assumptions (4.1) and (4.2) on φ, and since evidently $\lambda \leqslant 0$, it follows that

$$\langle s^N, s^N \rangle_\lambda \leqq \langle \|s\|^N, \|s\|^N \rangle_\lambda, \tag{4.8}$$

where in the right hand side, $\|s\|$ denotes the corresponding lower diagonal element in $sl_2(\mathbb{C})$, and where the inner product is computed in the highest weight representation of that algebra, of weight λ.

This is quite easily done, and the result is

$$\langle \|s\|^N, \|s\|^N \rangle_\lambda = \|s\|^{2N} N \cdot (N+1-\lambda) \cdot (N-\lambda) \cdot \cdots \cdot 1 \cdot (2-\lambda). \tag{4.9}$$

Since the ratio between successive terms in the series (4.6) is given by

$$\langle s^{N+1}, s^{N+1} \rangle_\lambda / (\langle s^N, s^N \rangle_\lambda \cdot (N+1)^2) = \|s\|^2 (N+1-\lambda)/(N+1), \tag{4.10}$$

the claim follows. ∎

Remark. The condition on the norm may seem unnatural and restrictive. However, if we compare with the situation in $SU(1,1)$ itself, we do have that if $g \in SU(1,1)$ is written as

$$g = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \tag{4.11}$$

(which is always possible), then $w = c_0/a_0$ and since, by (4.5) $a_0 a_0^* - c_0 c_0^* = 1$, it follows that $ww^* < 1$.

By utilizing the essential uniqueness of a highest weight representation of a given highest weight, the following proposition then leads directly to a projective unitary representation of $SU(1,1)^R$.

Let Π denote the action of $sl_2(R)$ on $\tilde{V}$. For $g \in GL(R)$ and $a \in sl_2(R)$ we put

$$\Pi_g(a) = \Pi(gag^{-1}). \tag{4.12}$$

PROPOSITION 4.2. *Let $g \in SU(1,1)^R$. There exists a vector $v_g \in \tilde{V}$, unique up to multiplication by scalars of modulus* 1, *such that*

$$\begin{aligned} &\text{(i)} && \langle v_g, v_g \rangle_\lambda = 1 \\ &\text{(ii)} && \Pi_g \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} \cdot v_g = \varphi_\lambda(r) \cdot v_g, \qquad \textit{and} \\ &\text{(iii)} && \Pi_g \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} \cdot v_g = 0, \end{aligned} \tag{4.13}$$

for all r and s in R.

Proof. It suffices to prove an analogous claim for each of the three factors in a decomposition of g as in (4.11), but here based on R, and with $\|w\| < 1$.

To begin with, the terms $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ are easily disposed of; as the vector v_g we can just use the original highest weight vector $v_{\varphi\lambda}$. Let us then consider

$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, \qquad w \in R \qquad \text{and} \qquad \|w\| < 1. \tag{4.14}$$

It is easy to see that the vector

$$v_{\tilde{g}} = e^{w} \cdot v_{\varphi\lambda} \tag{4.15}$$

satisfies (ii) and (iii). Suppose then, that an element

$$\bar{v}_{\tilde{g}} = p e^{s_2} \cdot v_{\varphi\lambda}, \tag{4.16}$$

where $p \in \mathscr{P}(R)$, also meets the requirements. Then, since

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} = \begin{pmatrix} s & 0 \\ ws + sw & -s \end{pmatrix} \tag{4.17}$$

it follows easily that p must be the constant polynomial since otherwise there can be no bound on its degree. Furthermore, since

$$\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ s_2 & 0 \end{pmatrix}^{i} \right] = \begin{pmatrix} 0 & \\ s_2 s + s s_2 & \end{pmatrix} \begin{pmatrix} 0 & 0 \\ s_2 & 0 \end{pmatrix}^{i-1} \tag{4.18}$$

inside $\mathscr{U}(sl_2(R))$, it follows that

$$(w - s_2) \cdot s = -s(w - s_2) \tag{4.19}$$

for all $s \in R$. Hence, $w = s_2$.

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