QUANTIZED SYMMETRIZATIONS AND $\star$-PRODUCTS

HANS PLESNER JAKOBSEN
Mathematics Institute
Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark
E-mail: jakobsen@math.ku.dk

The interrelation between certain quadratic algebras occurring in quantized enveloping algebras on the one hand side, and Poisson structures and deformation quantization on the other, is discussed. It is shown that there are two different methods of constructing $\star$-products available. Further implications in the direction of quantized wave- and Dirac-operators are investigated.

1 Content

The purpose of this article is to make precise some of the results we have reported in talks in Bedlevo (Sep. 2000) and Krakow (July 2001). We have pointed out there that for any algebra having a PBW-type basis, it is possible to construct a $\star$-product. At the conference this has also been made pertinent by J. Wess in his talk and R. Twarock also uses Bergman's Diamond Lemma [1]. On the other hand, in our article [7] we have shown, in a work that falls naturally in the line of investigations of Procesi and de Concini [2], and also of the Diamond Lemma, that it is possible to construct (apparently) another $\star$-product for a class of quadratic algebras. In this connection we wish to stress, in view of Kontsevich's result [10], the importance of determining the gauge group. For all of the known cases, these algebras are generated by certain elements of the quantized enveloping algebra corresponding to a hermitian symmetric space, and actually, a close inspection - and fine tuning - of the construction reveals that the results extend to both the quantized Borel subalgebra as well as the full quantized enveloping algebra. Here, the result of Levendorskii [12,13] is also needed. Even an arbitrary (classical) Lie algebra falls under this and we rediscover here the Kontsevich $\star$-product [10]. The $\star$-products have direct implications for the quantized Dirac-operator - an operator we want to act on "classical" functions on a "classical" (indeed, linearly flat) space.

2 The first $\star$-products; Algebras with PBW-type bases

Let A be an iterated twisted Ore extension, or, more generally, an algebra over a field $\mathbb{F}$ with a PBW-type basis $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. Let

$$x^\alpha \cdot x^\beta = \sum_\gamma a^\gamma_{\alpha,\beta} x^\gamma$$

be the expansion in the basis x^α of the product of x^α and x^β in A. Then

$$z^\alpha \star_1 z^\beta = \sum_\gamma a^\gamma_{\alpha,\beta} z^\gamma$$

defines an associative product which we may interpret as a product in $\mathbb{F}[z_1, z_2, \dots, z_n]$.

3 $\star$-products arising from a certain class of algebras

Consider an algebra $\mathcal{A}$ generated by (linearly independent) elements $X_1, \dots, X_N$. For each $i = 1, \dots, N$ let $\mathcal{A}_i$ denote the algebra generated by $X_1, \dots, X_i$. We assume that the defining relations are of the form:

$$\text{(Rel)} \quad \text{If } i > j \text{ then } X_i X_j = b_{ij} X_j X_i + p_{ij} \quad \text{with } p_{ij} \text{ of “lower order”}.$$

Let V denote the N-dimensional complex vector space spanned by the elements $X_1, \dots, X_N$, let $T = T(V)$ denote the tensor algebra over V, and let I_R denote the ideal in T generated by elements $X_i X_j - (b_{ij} X_j X_i + p_{ij})$. Then

$$\mathcal{A} := T/I_R. \tag{1}$$

Remark 3.1 *The last condition could e.g. be of the form $p_{ij} \in \mathcal{A}_{i-1}$ The algebra is quadratic if each p_{ij} is quadratic. Originally, this was the most interesting case and it should be noted that there are extremely many examples of this situation in contemporary mathematical physics. To each quantized hermitian symmetric space there is such an algebra. The quantized enveloping algebra is, by Levendorskii's result [12],[13], another example. We think of the relation as $X_i X_j = b_{ij} X_j X_i$ with “an error term p_{ij}” and the decisive feature of p_{ij} is that it should be of lower order in some appropriate sense. With this in mind it may be seen that even a classical Lie algebra with $XY = YX + [X, Y]$ fits into the framework and is covered by the results to come.*

We may at first think of (Rel) as a reduction system with the reductions

$$(X_i X_j, b_{ij} X_j X_i + p_{ij}) \quad (\text{ for all } i > j). \tag{2}$$

The essential assumption (EA) below is needed to avoid situations where, due to some special cancellations, a sum of elements in I_R might add up to an element which strictly precedes all the summands in the order. Specifically, assume for any element $u \in I_R$

$$\text{(EA):} \qquad u \in \mathrm{Span}\{a \cdot (X_iX_j - b_{ij}X_jX_i - p_{ij}) \cdot b \mid a, b \in T \text{ and } a \cdot (X_iX_j) \cdot b \leq_l u\}.$$

It can be seen that the requirements for the Diamond Lemma [1] to be applicable to our situation indeed are satisfied. Thus,

Proposition 3.2 *All elements of T are reduction unique. Moreover, the set $\{X_1^{i_1} \cdots X_N^{i_N} \mid 1_1, \dots, i_n \in \mathbb{N}_0\}$ is a basis for $\mathcal{A}$ and $\mathcal{A}$ is a domain and is in fact an iterated twisted polynomial (Ore) algebra. In particular, the assumptions of Procesi and de Concini [2] are satisfied. Conversely, one can recover our situation from theirs.*

Remark 3.3 *It would be interesting to classify all quadratic algebras that satisfy this reduction assumption (EA). It is clearly a quite strong assumption, on the order of complication of e.g. the Jacobi Identity in the enveloping algebra.*

4 Projections and quantized symmetries.

The $\star$-product of Section 2 only assumed the existence of a PBW-type basis. In this section we indicate briefly how an algebra $\mathcal{A}$ as in Section 3 gives rise to yet another kind of $\star$-product:

Definition 4.1 *We define a linear map $\Sigma : V \otimes V \longrightarrow V \otimes V$ by*

$$\Sigma(X_i \otimes X_j) = b_{ij}X_jX_i + p_{ij} \text{ if } i > j, \tag{3}$$

$$\Sigma(X_j \otimes X_i) = (b_{ij})^{-1}(X_iX_j - p_{ij}) \text{ if } i > j, \text{ and} \tag{4}$$

$$\Sigma(X_i \otimes X_i) = X_i \otimes X_i \text{ for all } i = 1, \dots, N. \tag{5}$$

Observe that p_{ij} may be a higher – or lower – order tensor. From now on, we assume that $\forall i, j : b_{ij} = q^{\alpha_{ij}}$ where q is a non-zero complex number which is not a root of unity. Recall that the associated quasi-polynomial algebra is the quadratic algebra $\overline{\mathcal{A}}$, generated by elements $\overline{x}_1, \dots, \overline{x}_N$ with relations $\overline{x}_i\overline{x}_j = q^{\alpha_{ij}}\overline{x}_j\overline{x}_i$.

Definition 4.2 *For $i \in \mathbb{N}, \sigma_i$ denotes the linear map $T \longrightarrow T$ given by*

$$\sigma_i(v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n) = \tag{6}$$

$$v_1 \otimes \ldots \otimes v_{i-1} \otimes \Sigma(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_n \text{ and} \tag{7}$$

$$\sigma_i(v_1 \otimes \ldots \otimes v_r) = v_1 \otimes \ldots \otimes v_r \text{ if } r \leq i. \tag{8}$$

Finally, we define operators $\overline{\sigma}_i$, $T \longrightarrow T$, analogously, but based on maps $\overline{\Sigma}$ in which all p_{ij} are zero.

Lemma 4.3 *For each $i \in \mathbb{N}$, $\overline{\sigma}_i\ \overline{\sigma}_{i+1}\ \overline{\sigma}_i = \overline{\sigma}_{i+1}\ \overline{\sigma}_i\ \overline{\sigma}_{i+1}$, and hence $\overline{\sigma}_1, \ldots, \overline{\sigma}_{n-1}$ define a representation, called* **quasi-permutation**, *of the symmetric group S_n on T^n.*

Lemma 4.4 *For each $i \in \mathbb{N}$, $\sigma_i = \overline{\sigma}_i$ modulo lower order.*

Lemma 4.5 *The following hold: 1) For each $i \in \mathbb{N}$, σ_i preserves I_R. 2) For each $i \in \mathbb{N}$, if for $u \in T : \overline{\sigma}_i(u) = u$, then $\sigma_i(u) = u$. 3) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ modulo I_R or modulo lower order terms.*

We wish to introduce an analogue of the usual symmetrization map on T. Let us first consider the representation of S_n described in Lemma 4.3. For any $\sigma \in S_n$ we denote the resulting operator on T^n as $\overline{\sigma}$ and we set

$$P_{\text{quasi-sym}} = \frac{1}{n!} \sum_{\sigma \in S_n} \overline{\sigma} \qquad (\textbf{ quasi-symmetrization}). \tag{9}$$

Lemma 4.6

$$\forall i : \qquad \overline{\sigma}_i \cdot P_{\textit{quasi-sym}} = P_{\textit{quasi-sym}} \cdot \overline{\sigma}_i = P_{\textit{quasi-sym}}.$$

We set $S_{\text{quasi-sym}} = P_{\text{quasi-sym}}(T)$. As a vector space, this is clearly equivalent to $\overline{\mathcal{A}}$.

We next want to define a similar operator on T^n with respect to the σ_i's. The problem is, of course, that we do not have a bona fide representation. In spite of this we proceed by defining for each $\sigma \in S_n$ an operator $\widehat{\sigma} = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}$ if $\sigma = s_{i_1}s_{i_2}\cdots s_{i_r}$, where $s_j, j = 1, \ldots, n-1$, denotes the elementary transpositions in S_n and we set, for each such set of decompositions of elements,

$$P(n) = \frac{1}{n!} \sum_{\sigma \in S_n} \widehat{\sigma}. \tag{10}$$

Definition/Proposition 4.7 *Set*

$$\mathcal{P}^{q\text{-}sym} = \lim_{N \longrightarrow \infty} (\lim_{n \longrightarrow \infty} P(n))^N. \tag{11}$$

Then $\mathcal{P}^{q\text{-}sym}$ is a well-defined projection satisfying $\mathcal{P}^{q\text{-}sym}(I_R) = 0$. The image of $\mathcal{P}^{q\text{-}sym}$ is equal to the image of $\mathcal{P}_{quasi\text{-}sym}$, i.e. $\overline{\mathcal{A}}$. Moreover, $\forall i : \mathcal{P}^{q\text{-}sym} \circ \sigma_i = \mathcal{P}^{q\text{-}sym}$.

Proposition 4.8

$$(\star) \quad \forall r, s : (I_r \otimes \mathcal{P}_k^{q\text{-}sym} \otimes I_s)\mathcal{P}^{q\text{-}sym} = \mathcal{P}^{q\text{-}sym},$$
$$(\star\star) \quad \forall r, s : \mathcal{P}^{q\text{-}sym}(I_r \otimes \mathcal{P}_k^{q\text{-}sym} \otimes I_s) = \mathcal{P}^{q\text{-}sym}.$$

If $u, v \in S_{\text{quasi-sym}}$ (construed as a subset of T) we define

$$u \star_2 v = \mathcal{P}^{q\text{-sym}}(u \otimes v). \tag{12}$$

Remark 4.9 *For a classical Lie algebra it follows, using* [11] *and* [15], *that* $\star_2$ *is the star product one gets from Kontsevich's general construction* [10].

Remark 4.10 *For the purely quadratic algebras, the Poisson structure is also quadratic. It will be interesting to relate the general* $\star_2$*-product to that of Kontsevich. Furthermore, Shklyarov* [14] *has related the algebraic structures more directly to the co-product* $\triangle$ *and it remains to be seen if there is a direct way of relating* $\star_2$ *to that.*

Let $\mathcal{B}_q^+$ denote the Borel part of $\mathcal{U}_q$ and let $\mathcal{B}_q^-$ be defined analogously. Let χ_λ be a 1-dimensional representation of $\mathcal{B}_q^+$ corresponding to the weight λ. Let $H^0(\chi_\lambda) = \{f : \mathcal{U}_q \mapsto \mathbb{C} \mid f(ub^+) = \chi_\lambda^{-1}(b^+)f(u) \text{ and } f \in \text{FIN}(\mathcal{U}_q(\mathfrak{k})\}$ with the action $(x \cdot f)(u) = f(S(x)u)$, S being the antipode. The condition FIN denotes the usual finiteness condition. This is the quantized analogue of a holomorphically induced representation. Covariant (quantized) differential operators [3,4,5,6] arise in this context. Indeed, any element $u_0 \in \mathcal{U}_q$ may in some sense be viewed as an intertwiner through the map $f \mapsto f_{u_0}$ $f_{u_0}(u) = f(u \cdot u_0)$, but to be an intertwiner, there are severe restrictions on u_0 (and χ).

4.1 Translation into ordinary functions; quadratic algebras

We now specialize to the quantized hermitian symmetric spaces - in particular to the one corresponding to the conformal algebra.

Here, $\mathcal{B}_q^- = \mathcal{N}_q^- \otimes \mathbf{C}[K_1^{\pm 1}, \ldots, K_r^{\pm 1}]$ and $\mathcal{N}_q^- = \mathcal{A}_q^- \mathcal{U}_q(\mathfrak{k}_q^-)$. It is a very interesting question to decide between which of these spaces (which all fit into the general framework) should be used, but for now we choose $\mathcal{A}_q^-$: For $su_q(2,2)$ it is generated by w_1, w_2, w_3, w_4 with the usual relations (see also [4]):

$$\begin{aligned} w_1w_2 = qw_2w_1, w_1w_3 = qw_3w_1, w_3w_4 = qw_4w_3, \\ w_2w_4 = qw_4w_2, w_2w_3 = w_3w_3, \\ w_1w_4 - w_4w_1 = (q - q^{-1})w_2w_3. \end{aligned}$$

The elements of $H^0(\chi_\lambda)$ may be viewed as (vector valued) functions on $\mathcal{A}_q^-$, so let us just work with polynomials on the classical space $\mathfrak{p}_0^-$ (identifiable with $\mathfrak{p}_0^+$. For the conformal algebra this is simply polynomials on $\mathbb{C}^4$. Let $z^\gamma = z^{\gamma_1}z^{\gamma_2}z^{\gamma_3}z^{\gamma_4}$, let w^γ in $\mathcal{A}_q^- \subseteq \mathcal{U}_q^-$ be defined analogously, and let B denote the map

$$z^\gamma \mapsto \frac{z^\gamma}{[\gamma]!_q} w^\gamma.$$

We shall not argue here for inserting the factor $[\gamma]!_q$ "for good measure" but proceed to define a linear map S as the composite $S = P_{\text{quasi-sym}} \circ S_{classical}$ where $S_{classical}$ is the classical symmetrization map.

We now basically have two maps B, S from polynomials on $\mathbb{C}^4$ into $\mathcal{U}_q^-$. If $\circ$ denotes the product in $\mathcal{U}_q^-$, then the two $\star$-products are given as follows:

$$x_1 \star_1 x_2 = (x_1, x_2)_B := B^{-1}\left((Bx_1) \circ (Bx_2)\right)$$
$$x_1 \star_2 x_2 = (x_1, x_2)_S := S^{-1}\left((Sx_1) \circ (Sx_2)\right)$$

If $R := B^{-1}S$ then $(x_1, x_2)_S = R^{-1}\left((Rx_1, Rx_2)_B\right)$ - but this map is presumably not "geometrical" (a gauge-equivalence). Both maps B, S take first order expressions into first order expressions. We may then translate left multiplication in $\mathcal{U}_q^-$ into first order differential operators:

$$\left(\partial_x^B p\right)(y) := p(y \star_1 x). \quad \text{Likewise,}$$
$$\left(\partial_x^S p\right)(y) := p(y \star_2 x)$$
$$= \left(\partial_x^B (p(R^{-1}(\cdot))\right)(Ry)$$

Of course, any $\star$-product with these properties might be used. Even a rescaling of the basis will in some sense (c.f. below) yield another. For now, we end the article by showing how the map B may be used to define the differential operators occurring in the quantized Dirac operator and we discuss possible interpretations of these, thereby continuing the investigation in [9].

The natural first order differential operators.

Consider operators of the form $((w_0^R)^\dagger \cdot f)(w^\gamma) = f(w^\gamma w_0)$. Presently, w_0 will be either w_1, w_2, w_3, w_4, or $w_1 w_4 - q w_2 w_3$. Transformed into operators on functions on $\mathbb{C}^4$ via (4.1) these become q-differential operators expressible in terms of, among other things, the usual q-differential operators on $\mathbb{C}^1$ as well as scaling operators, that is

$$\left(\left[\frac{\partial}{\partial x}\right]_q \psi\right)(x) = \frac{f(xq) - f(xq^{-1})}{x(q - q^{-1})}, \text{ and}$$
$$K_i(z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4}) = q^{-\alpha_i} z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} \quad i = 1, 2, 3, 4.$$

In terms of un-quantized operators, if $q = e^\hbar$ then $K_i = e^{-\hbar S_i} = \sum_{n=0}^{\infty} \frac{(-\hbar S_i)^n}{n!}$, where $S_i = z_i \frac{\partial}{\partial z_i}$. Also observe that $K_i = \left[\frac{\partial}{\partial z_i}\right]_q z_i -$

$qz_i \left[\frac{\partial}{\partial z_i}\right]_q$. We obtain:

$$(w_4^R)^\dagger = \left[\frac{\partial}{\partial z_4}\right]_q, (w_2^R)^\dagger = K_4\left[\frac{\partial}{\partial z_2}\right]_q, (w_3^R)^\dagger = K_4\left[\frac{\partial}{\partial z_3}\right]_q, \text{ and}$$

$$(w_1^R)^\dagger = K_2K_3K_4^2\left[\frac{\partial}{\partial z_1}\right]_q + z_4(1-q^{-2})K_4\Box_q, \text{ where}$$

$$\Box_q \stackrel{Def.}{=} \left((w_1w_4 - qw_2w_3)^R\right)^\dagger = K_2K_3\left[\frac{\partial}{\partial z_1}\right]_q\left[\frac{\partial}{\partial z_4}\right]_q - \left[\frac{\partial}{\partial z_2}\right]_q\left[\frac{\partial}{\partial z_3}\right]_q.$$

The last defined operator is of course nothing else but the quantized wave operator (see also [3]).

If, instead, we use the classical $\gamma!$ we get quite similar formulas but with ordinary differentiations - except for the $\Box_q$. In this connection observe that the map $\Gamma : z^\gamma \mapsto \frac{[\gamma]_q!}{\gamma!}$ intertwines $\frac{\partial}{\partial z}$ and $\left[\frac{\partial}{\partial z}\right]_q$!

We may rewrite the equation $w_1w_4 - w_4w_1 = (q-q^{-1})w_2w_3$ as $qw_4w_1 - q^{-1}w_1w_4 = (q-q^{-1})\det_q$ where $\det_q = w_1w_4 - qw_2w_3$ generates the center of $\mathcal{A}_q^-$. Here, this becomes

$$q(w_4^R)^\dagger(w_1^R)^\dagger - q^{-1}(w_1^R)^\dagger(w_4^R)^\dagger = (q-q^{-1})\Box_q. \tag{13}$$

Since $\Box_q = c\cdot I$, with $c \in \mathbb{C}$ is any irreducible representation, this has the flavor of a quantized Heisenberg algebra.

Along a similar vein,

$$(w_1^R)^\dagger \mapsto K_2K_3K_4^2\left[\frac{\partial}{\partial z_1}\right]_q + c\cdot z_4(1-q^{-2})K_4.$$

which points towards a covariant derivative; an interpretation which may be further supported since the K_i's may be removed by a change of generators.

Another way of seeing the same phenomenon is by letting

$$G(f) = \left(f, \Box_q f, \Box_q^2 f, \Box_q^3 f, \dots\right)$$

and B the matrix whose $i,(i+1)$ entry is $z_4(1-q^{-2})K_4$ and zeros everywhere else.

Then we have

$$(K_2K_3K_4^2\left[\frac{\partial}{\partial z_1}\right]_q + B)G(f) = G\left(\left(K_2K_3K_4^2\left[\frac{\partial}{\partial z_1}\right]_q + z_4(1-q^{-2})K_4\Box_q\right)f\right).$$

This version is also well behaved with respect to the other generators.

Observe: $\left[\frac{\partial}{\partial z}\right]_q(zK) - q^{-2}(zK)\left[\frac{\partial}{\partial z}\right]_q = I$, i.e. $\left[\frac{\partial}{\partial z}\right]_q$ and zK are "quantized conjugate operators".

References

1. G.M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29**, 178 (1978).
2. C. de Concini and C. Procesi, *Quantum groups.* Lecture notes in mathematics **1565**, 31 (Springer 1993).
3. V.K. Dobrev: J. Phys. A **27**, 4841, 6633 (1994), & **28**, 7135 (1995).
4. V.K. Dobrev: Phys. Lett. **341B**, 133 (1994) & **346B**, 427 (1995).
5. M. Harris and H.P. Jakobsen *Singular holomorphic representations and singular modular forms*, Math. Ann. **259**, 227 (1983).
6. H.P. Jakobsen, *Basic covariant differential operators on hermitian symmetric spaces*, Ann. scient. Èc. Norm. Sup. **18**, 421 (1985).
7. H.P. Jakobsen, *Q-differential operators*, preprint math.QA/9907009.
8. H.P. Jakobsen and Jøndrup, *Quantized Rank R Matrices*, J. Alg. to appear.
9. H.P. Jakobsen, *Quantized Dirac operators*, Czech. J. Phys. **50**, 1265 (2000).
10. M. Kontsevich, *Deformation quantization of Poisson manifolds I*, preprint math.QA/9709040.
11. V. Kathotia, *Kontsevich universal formula for the quantization and the Campbell-Baker-Hausdorff formula*, preprint math.QA/9811174.
12. S.Z. Levendorskii, *A short proof of the PBW-theorem for quantized enveloping algebras*, preprint 1991.
13. S.Z. Levendorskii and Y.S. Soibelman, *Algebras of functions on compact quantum groups, Schubert cells and quantum tori*, Comm. Math. Phys. **139**, 141-170 (1991).
14. D.L. Shklyarov, in *Lectures on q-Analogues of Cartan Domains and Harish-Chandra Modules*, Ed. L. Vaksman, preprint, Kharkov (2001).
15. B. Shoikhet, *Vanishing of Konsetvich integrals of the wheels*, preprint math.QA/0007080.