Math. Ann. 256, 439–447 (1981)

Mathematische Annalen

The Last Possible Place of Unitarity for Certain Highest Weight Modules

Hans Plesner Jakobsen*

Mathematics Institute, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark

Introduction

The modules in question are those corresponding to holomorphically induced representations on Hermitian symmetric spaces of the non-compact type. Specifically let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition. By assumption $\mathfrak{k}$ has a non-trivial center $\eta=\mathbb{R}\cdot h_0$ and $\mathfrak{k}=\mathfrak{k}_1\oplus\mathbb{R}\cdot h_0$ where $\mathfrak{k}_1=[\mathfrak{k},\mathfrak{k}]$. The modules W_Λ considered are determined by a pair (Λ_0,λ) where Λ_0 is $\mathfrak{k}_1$-dominant and integral and $\lambda\in\mathbb{R}$. That is, $\Lambda=(\Lambda_0,\lambda)$ determines a finite-dimensional $\mathscr{U}(\mathfrak{k}^{\mathbb{C}})$-module V_Λ and W_Λ is the irreducible quotient of $\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \underset{\mathscr{U}(\mathfrak{k}^{\mathbb{C}}+\mathfrak{p}^+)}{\otimes} V_\Lambda$, where $\mathfrak{p}^+=\{z\in\mathfrak{p}^{\mathbb{C}}|[h_0,z]=iz\}$.

W_Λ may be represented as a space of V_Λ-valued polynomials on $\mathfrak{p}^+$ and the $\mathfrak{g}$-invariant Hermitian form on W_Λ, restricted to the space of first order polynomials, depends linearly on λ. For λ sufficiently negative W_Λ is unitarizable and thus, for Λ_0 fixed, the smallest λ such that W_Λ does not contain all first order polynomials determines the last possible place at which W_Λ can be unitarizable.

This philosophy was used in [5] to prove a conjecture of Kashiwara and Vergne, in the case of $\mathrm{SU}(p,q)$, by means of geometrical methods.

In this article our methods are algebraic. Specifically, the main tool is Bernstein-Gelfand-Gelfand [1, Th. 7.6.23] applied in a manner similar to, and motivated by, Shapovalov's in [8]. The technical side of this article, in fact, consists of adapting a theorem of Shapovalov to our situation.

As a corollary we prove the Kashiwara-Vergne conjecture for $\mathrm{Mp}(n,\mathbb{R})$.

Finally we mention that after we had realized that [8] could be applied, but before having completed the proof, a preprint of an article by Enright and Parthasarathy [2] was channelled to us. In their article a criterion for unitarizability is developed and applied to give a proof of the same conjecture. Though their criterion leads to the same result, the philosophy behind it, and in particular the proof of it, is quite different from ours.

* This research was partly supported by a grant from the NSF

0025-5831/81/0256/0439/$01.80

1. Notation

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$. We assume that $\mathfrak{k}$ has a nonempty center η; in this case $\eta=\mathbb{R}\cdot h_0$ for an $h_0\in\eta$ whose eigenvalues under the adjoint action on $\mathfrak{p}^{\mathbb{C}}$ are $\pm i$. Let

$$\mathfrak{p}^+=\{z\in\mathfrak{p}^{\mathbb{C}}|[h_0,z]=iz\},$$

and

$$\mathfrak{p}^-=\{z\in\mathfrak{p}^{\mathbb{C}}|[h_0,z]=-iz\}.$$

Let $\mathfrak{k}_1=[\mathfrak{k},\mathfrak{k}]$ and let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{k}=\mathfrak{k}_1\oplus\mathbb{R}\cdot h_0$, $\mathfrak{h}=(\mathfrak{h}\cap\mathfrak{k}_1)\oplus\mathbb{R}\cdot h_0$, $(\mathfrak{h}\cap\mathfrak{k}_1)^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{k}_1^{\mathbb{C}}$, and $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We let σ denote the conjugation in $\mathfrak{g}^{\mathbb{C}}$ relative to the real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. The sets of compact and non-compact roots of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$ are denoted Δ_c and Δ_n, respectively. $\Delta=\Delta_c\cup\Delta_n$. We choose an ordering of Δ such that

$$\mathfrak{p}^+=\sum_{\alpha\in\Delta_n^+}\mathfrak{g}^\alpha,$$

and set

$$\mathfrak{g}^+=\sum_{\alpha\in\Delta^+}\mathfrak{g}^\alpha,$$

$$\mathfrak{g}^-=\sum_{\alpha\in\Delta^-}\mathfrak{g}^\alpha,$$

and

$$\varrho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha.$$

Throughout β denotes the unique simple non-compact root. For $\gamma\in\Delta$ let H_γ be the unique element of $i\mathfrak{h}\cap[(\mathfrak{g}^{\mathbb{C}})^\gamma,(\mathfrak{g}^{\mathbb{C}})^{-\gamma}]$ for which $\gamma(H_\gamma)=2$. Then for all γ_1 in Δ

$$\langle\gamma_1,\gamma\rangle=\frac{2(\gamma_1,\gamma)}{(\gamma,\gamma)}=\gamma_1(H_\gamma),\tag{1.1}$$

where $(\cdot,\cdot)$ is the bilinear form on $(\mathfrak{h}^{\mathbb{C}})^*$ obtained from the Killing form of $\mathfrak{g}^{\mathbb{C}}$. For $\alpha\in\Delta_n^+$ choose $x_\alpha\in(\mathfrak{g}^{\mathbb{C}})^\alpha$ such that

$$[x_\alpha,x_\alpha^\sigma]=H_\alpha.\tag{1.2}$$

Following the notation of [7] we let γ_r denote the highest root. Then $\gamma_r\in\Delta_n^+$, and $H_{\gamma_r}\notin[\mathfrak{h}\cap\mathfrak{k}_1]^{\mathbb{C}}$.
Finally we let $u\to u^*$ be the antilinear antiautomorphism of $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$ that extends the map $x\to-x^\sigma$ of $\mathfrak{g}^{\mathbb{C}}$.

2. Modules

Corresponding to the decomposition

$$\mathscr{U}(\mathfrak{g}^{\mathbb{C}})=(\mathscr{U}(\mathfrak{g}^{\mathbb{C}})\mathfrak{g}^+\oplus\mathfrak{g}^-\mathscr{U}(\mathfrak{g}^{\mathbb{C}}))\oplus\mathscr{U}(\mathfrak{h}^{\mathbb{C}})$$

we let, for $u \in \mathscr{U}(\mathfrak{g}^{\mathbb{C}})$, $\gamma(u)$ denote the unique element of $\mathscr{U}(\mathfrak{h}^{\mathbb{C}})$ for which $u-\gamma(u)$ is in $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})\mathfrak{g}^+ \oplus \mathfrak{g}^- \mathscr{U}(\mathfrak{g}^{\mathbb{C}})$.

Let $\chi \in (\mathfrak{h}^{\mathbb{C}})^*$. The Verma module M_χ of highest weight $\chi - \varrho$ is defined to be $M_\chi = \mathscr{U}(\mathfrak{g}^{\mathbb{C}})/I_{\chi-\varrho}$, where $I_{\chi-\varrho}$ is the left ideal generated by the elements $(H-\chi(H)+\varrho(H))$, $H \in \mathfrak{h}^{\mathbb{C}}$, and $\mathfrak{g}^+$. We denote the image of 1 in M_χ by $1_{\chi-\varrho}$.

If Λ_0 is a dominant integral weight of $\mathfrak{k}_1$ and if $\lambda \in \mathbb{R}$ we denote by $\Lambda = (\Lambda_0, \lambda)$ the linear functional on $\mathfrak{h}^{\mathbb{C}}$ given by

$$\Lambda|_{(\mathfrak{h}\cap\mathfrak{k}_1)^{\mathbb{C}}} = \Lambda_0, \qquad \Lambda(H_{\gamma_r}) = \lambda .$$

Further we let V_Λ denote the irreducible finite-dimensional $\mathscr{U}(\mathfrak{k}^{\mathbb{C}})$-module of highest weight Λ. As $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-modules, clearly $V_\Lambda = V_{\Lambda_0}$.

The sesquilinear form B_Λ on $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$,

$$B_\Lambda(u,v) = \Lambda(\gamma(v^*u)) \tag{2.2}$$

is $\mathfrak{g}$-invariant. We let N_Λ denote the kernel of B_Λ;

$$N_\Lambda = \{u \in \mathscr{U}(\mathfrak{g}^{\mathbb{C}}) | \forall v \in \mathscr{U}(\mathfrak{g}^{\mathbb{C}}) : \Lambda(\gamma(v^*u)) = 0\}, \tag{2.3}$$

and set

$$N_\Lambda(\mathfrak{k}) = N_\Lambda \cap \mathscr{U}(\mathfrak{k}^{\mathbb{C}}). \tag{2.4}$$

Let $J_\Lambda = I_\Lambda + \mathscr{U}(\mathfrak{g}^{\mathbb{C}})N_\Lambda(\mathfrak{k})$. Since

$$\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) = \mathscr{U}(\mathfrak{p}^-)\mathscr{U}(\mathfrak{k}^{\mathbb{C}})\mathscr{U}(\mathfrak{p}^+),$$

and

$$M_{\Lambda+\varrho} = \mathscr{U}(\mathfrak{p}^-)\mathscr{U}(\mathfrak{k}^{\mathbb{C}})1_\Lambda$$

we have

Lemma 2.1. $\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \bigotimes_{\mathscr{U}(\mathfrak{k}^{\mathbb{C}}+\mathfrak{p}^+)} V_\Lambda = \mathscr{U}(\mathfrak{g}^{\mathbb{C}})/J_\Lambda$.

Definition 2.2. Let $u \in M_\chi$. Corresponding to the decomposition $M_\chi = \mathscr{U}(\mathfrak{p}^-)\mathscr{U}(\mathfrak{k}^{\mathbb{C}})1_{\chi-\varrho}$ we define $P^-(u)$ to be the projection of u onto $\mathscr{U}(\mathfrak{p}^-)1_{\chi-\varrho}$.

The unique irreducible quotient W_Λ of $\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \bigotimes_{\mathscr{U}(\mathfrak{k}^{\mathbb{C}}+\mathfrak{p}^+)} V_\Lambda$ is given as

$$W_\Lambda = \mathscr{U}(\mathfrak{g}^{\mathbb{C}})/N_\Lambda . \tag{2.5}$$

Any $\mathfrak{g}$-invariant Hermitian form on W_Λ is proportional to B_Λ.

For further background information we refer to [7; Sect. 1] and [4; Sect. 2].

The situations in which $J_\Lambda \neq N_\Lambda$ are quite interesting. In point of fact, by looking at a special case where these two ideals are different, we can describe a simple condition that must be fulfilled in order to have B_Λ define a positive-definite inner product on W_Λ. For Λ_0 fixed this will be a description of the biggest λ for which W_Λ possibly can be unitarizable:

Let Λ_0 be fixed. The module $\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \bigotimes_{\mathscr{U}(\mathfrak{k}^{\mathbb{C}}\oplus\mathfrak{p}^+)} V_\Lambda$ may be thought of as a space of V_Λ-valued polynomials on $\mathfrak{p}^+$. Consider in particular the space $\mathscr{P}_1$ of first order polynomials. Under the action of $\mathscr{U}(\mathfrak{k}^{\mathbb{C}})$, $\mathscr{P}_1$ breaks into a finite sum of irreducible subspaces; $\mathscr{P}_1=\bigoplus_{i=1}^{n} S_i$. For each i, B_Λ determines a Hermitian $\mathfrak{k}$-invariant form on S_i, and it is clear that this form is proportional to a fixed non-trivial such by a polynomial $(a_i\lambda+b_i)$ where $a_i\neq 0$. Let $\lambda_0=\min_{i=1,\dots,n}\left(-\frac{b_i}{a_i}\right)$. Then: W_Λ unitarizable $\Rightarrow \lambda\leqq\lambda_0$.

We shall describe λ_0 more precisely:

3. Bernstein, Gelfand and Gelfand

The celebrated theorem of Bernstein, Gelfand and Gelfand [1, Th. 7.6.23] describes the situations in which one Verma module can be imbedded into another. A special case is the following: If χ in $(\mathfrak{h}^{\mathbb{C}})^*$ satisfies $\chi(H_\alpha)=1$ for $\alpha\in\Delta^+$, then $M_{\chi-\alpha}\subset M_\chi$. As was shown, among other things, by Shapovalov in [8], this leads to the following.

Proposition 3.1 ([8]). i) *Let $\alpha\in\Delta^+$, $\chi\in(\mathfrak{h}^{\mathbb{C}})^*$, and assume $\chi(H_\alpha)=1$. Then there exists an element $\Theta_\alpha(\chi)\in\mathscr{U}(\mathfrak{g}^-)^{-\alpha}$ such that*

$$\forall\gamma\in\Delta^+ : [(\mathfrak{g}^{\mathbb{C}})^\gamma,\Theta_\alpha(\chi)]\in I_{\chi-\varrho}.$$

ii) *If $\alpha\in\Delta^+$ is simple, $\Theta_\alpha(\chi)=x_{-\alpha}$, where $x_{-\alpha}=x_\alpha^\sigma$.*

iii) *If $\alpha\in\Delta^+$ has $\varrho(H_\alpha)>1$ let $\varepsilon\in\Delta^+$ be a simple root such that $\alpha_1=s_\varepsilon(\alpha)\in\Delta^+$ has $\varrho(H_{\alpha_1})<\varrho(H_\alpha)$. If χ is integral and $\chi(H_\varepsilon)<0$, $\Theta_\alpha(\chi)$ is determined by the equation*

$$\Theta_\alpha(\chi)x_{-\varepsilon}^q=x_{-\varepsilon}^{q+p}\Theta_{\alpha_1}(\psi) \tag{3.1}$$

where $\psi=s_\varepsilon(\chi)$, $q=-\chi(H_\varepsilon)$, and $p=\alpha(H_\varepsilon)$.

iv) *In any basis of $\mathscr{U}(\mathfrak{g}^-)$ the coefficients of $\Theta_\alpha(\chi)$ depend polynomially on χ.*

It follows that (3.1) determines $\Theta_\alpha(\chi)$ for any $\chi\in(\mathfrak{h}^{\mathbb{C}})^*$ with $\chi(\alpha)=1$.

Proposition 3.1 can be applied to the present situation. However, to do so needs some preparation.

4. Concerning $\mathfrak{p}^-\otimes V_{\Lambda_0}$

Proposition 4.1. *Let $\mathbb{C}\cdot v_0$ be the subspace of highest weight in V_{Λ_0}. Any non-zero $\mathscr{U}(\mathfrak{k}_1{}^{\mathbb{C}})$-invariant subspace of $\mathfrak{p}^-\otimes V_{\Lambda_0}$ contains elements with non-zero coefficients in $\mathfrak{p}^-\otimes\mathbb{C}\cdot v_0$.*

Proof. Let M be an invariant subspace and assume M is perpendicular to $\mathfrak{p}^-\otimes\mathbb{C}\cdot v_0$. Then M is perpendicular to $\mathscr{U}(\mathfrak{k}_1{}^{\mathbb{C}})\cdot(\mathfrak{p}^-\otimes\mathbb{C}\cdot v_0)=\mathfrak{p}^-\otimes V_{\Lambda_0}$. □

Corollary 4.2. *A highest weight of $\mathfrak{p}^-\otimes V_{\Lambda_0}$ is of the form $\Lambda_0-\alpha$ for some $\alpha\in\Delta_n^+$.*

For $\alpha\in\Delta_n^+$ we write $z_{-\alpha}$ for the element x_α^σ defined by (1.2).

Corollary 4.3. *If a highest weight $\Lambda_0-\alpha$ occurs in $\mathfrak{p}^-\otimes V_{\Lambda_0}$, there are elements $q_{-\mu}$ of $\mathscr{U}(\mathfrak{k}_1{}^{\mathbb{C}})^{-\mu}$, $\mu\in\Delta_c^+$, such that the corresponding highest weight vector is given as*

$$z_{-\alpha}\otimes v_0+\sum_{\mu\in\Delta_c^+} z_{-\alpha+\mu}\otimes q_{-\mu}v_0.$$

Corollary 4.4. *There are no multiplicities.*

5. A Special Case in $\mathfrak{g}=\mathrm{so}(2n-1,2)$

As we shall see below there are two cases that demand special treatment. One of these is a particular non-compact positive root in the root system for $\mathfrak{g}=\mathrm{so}(2n-1,2)$:

Following the notation of [2] let $e_1, \ldots, e_n$ be the standard orthonormal basis of $\mathbb{R}^n$. The sets of positive roots are

$$\Delta_c^+ = \{e_i \pm e_j | 2 \leqq i < j \leqq n\} \cup \{e_i | 2 \leqq i \leqq n\},$$

and

$$\Delta_n^+ = \{e_1 \pm e_j | 2 \leqq j \leqq n\} \cup \{e_1\}.$$

If $\Lambda=(\lambda_1, \ldots, \lambda_n)$, Λ is $\mathfrak{k}_1$-integral if and only if $\{\lambda_2, \ldots, \lambda_n\} \subseteq \mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$. $\varrho=(2n-1/2, 2n-3/2, \ldots, \frac{1}{2})$ and Λ is Δ_c^+ dominant if and only if $\lambda_2 \geqq \ldots \geqq \lambda_n \geqq 0$. Observe that $\forall \mu \in \Delta_c^+ : s_\mu(e_1)=e_1$, i.e. e_1 is Δ_c^+ isolated from the rest of Δ_n^+.

We want to compute $P^-(\Theta_{e_1}(\chi))$ and observe that

$$e_1 = s_{e_1-e_2} s_{e_2-e_3} \cdots s_{e_{n-1}-e_n}(e_n).$$

By using one of the standard forms of $\mathrm{so}(2n-1,2)$ it is easy to see that there are non-zero elements k_{ij}, m_s, and z_t of $\mathfrak{g}^{\mathbb{C}}$ such that

$$\begin{aligned} k_{ij} &\in \mathfrak{g}^{e_j - e_i}; \quad 2 \leqq i < j \leqq n, \\ m_s &\in \mathfrak{g}^{-e_s}; \quad 2 \leqq s \leqq n, \\ z_t &\in \mathfrak{g}^{-e_1+e_t}; \quad 2 \leqq t \leqq n, \end{aligned}$$

and

$$z_1 \in \mathfrak{g}^{-e_1}$$

and that they can be normalized is a manner, insignificant for the result, such that

$$\begin{aligned} [k_{ij}, k_{jr}] &= k_{ir}; \quad [k_{ij}, m_n] = m_i \\ [z_t, k_{ij}] &= \delta_{i,t} z_j; \quad [z_t, m_s] = -\delta_{s,t} z_1. \end{aligned}$$

It is then straightforward to compute $P^-(\Theta_{e_1}(\chi))$ (or $\Theta_{e_1}(\chi)$ for that matter). In fact, one needs only pay attention to the terms m_s in the various expressions. We omit the details and give the result:

Proposition 5.1. *Let $\chi_1=\frac{1}{2}$. Up to a non-zero constant*

$$P^-(\Theta_{e_1}(\chi)) = \left(\prod_{i=2}^{n} (\chi_i - \chi_1)\right) z_1.$$

Proposition 5.2. *Let $\chi=\Lambda+\varrho$ and assume $\chi_1=\frac{1}{2}$. Then $P^-(\Theta_{e_1}(\chi)) \neq 0$ if and only if $\Lambda_0 - e_1$ is a highest weight for the $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-module $\mathfrak{p}^- \otimes V_{\Lambda_0}$.*

Proof. If $P^-(\Theta_{e_1}(\chi)) \neq 0$ it is clear that $\Lambda_0 - e_1$ is a highest weight. Suppose $P^-(\Theta_{e_1}(\chi))=0$. Since $\lambda_1 = 1-n$ this implies that $\lambda_n=0$. But $e_1=(e_1-e_n)+e_n$ and e_n is simple, so if $\lambda_n=0$, $\Lambda_0 - e_1$ cannot be a highest weight (cf. Corollary 4.3 and the proof of Proposition 7.3 below). □

6. Concerning the Root System

Since $\mathfrak{p}^-$ is a $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-module with highest weight $-\beta$, the following is obvious.

Lemma 6.1. *Let $\alpha \in \Delta$. The coefficient to β in α is* 1,0, *or* -1.

Proposition 6.2. *Root strings are of length at most* 3.

Proof. It is clear that strings through roots of equal length are of length at most 2. Equally obvious is the fact that a string $\beta + i\alpha, \beta + (i+1)\alpha, \ldots$, trough β will have either $i=1$ and $\alpha \in \Delta_c^+$, or $\alpha \in \Delta_n^+$ and $i \in \{-1, -2\}$. In the last case, by Lemma 6.1, the string has length at most 3. Let l and s abbreviate long and short. A hypothetical string of length 4 will then either be of the form slls or lssl. Since roots of equal length are conjugate under the Weyl group the first case, however, is ruled out because β is long. The only case to be examined, then, is the string β, $\beta+\alpha$, $\beta+2\alpha$, $\beta+3\alpha$ with $\alpha \in \Delta_c^+$. In this case $\langle \beta, \alpha \rangle = -3$ and thus $\langle \alpha, \beta \rangle = -1$. Hence $s_\beta(\beta+3\alpha) = 2\beta + 3\alpha$. Contradiction. □

Proposition 6.3. *Let $\varepsilon_1, \ldots, \varepsilon_t$ denote the set of positive simple roots in Δ_c^+, let $\alpha \in \Delta_n^+$, and assume $\varrho(H_\alpha) > 1$. If $\forall i : \alpha(H_{\varepsilon_i}) \leqq 0$ then either $\mathfrak{g} = \mathfrak{so}(2n-1, 2)$ and $\forall i : \varepsilon_i(H_\alpha) = 0$ or $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ and $\alpha = \beta + \varepsilon_i$ for a unique ε_i.*

Proof. The assumptions on α imply that $\langle \alpha, \beta \rangle > 0$. Then, by Lemma 6.1, $\langle \alpha, \beta \rangle > = 1$. Write $\alpha = \alpha_1 + \beta$ with $\alpha_1 \in \Delta_c^+$ and assume $-q = \langle \alpha, \alpha_1 \rangle < 0$. Then $s_{\alpha_1}(\alpha) = \alpha + q\alpha_1 \in \Delta_n^+$ and we have got a string β, $\beta + \alpha_1$, $\beta + (q+1)\alpha_1$. By Proposition 6.2 $q=1$ and thus $\beta + \alpha_1$ and $\beta + 2\alpha_1$ are of equal length. However, this implies that $\beta + 3\alpha_1$ should be a root and thus $\langle \alpha, \alpha_1 \rangle = 0$. This fact combined with $\langle \alpha, \beta \rangle = 1$ easily implies that $(\alpha, \alpha) = \frac{1}{2}(\beta, \beta)$. In other words, we are in a Hermitian symmetric space with two root lengths, that is, $\mathfrak{g} = \mathfrak{so}(2n-1, 2)$ or $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$. The rest is then standard. □

Finally we shall also need the following elementary fact about general root systems:

Lemma 6.4. *Let α_0 be a positive root with $\varrho(H_{\alpha_0}) > 1$. Let $\varepsilon_1, \varepsilon_2, \ldots$* be simple roots and define *$\alpha_j = s_{\varepsilon_j}(\alpha_{j-1})$ for $j = 1, 2, \ldots$. If for all $j : \alpha_j \in \Delta^+$ and $\varrho(H_{\alpha_j}) < \varrho(H_{\alpha_{j-1}})$, then $s_{\varepsilon_1} \ldots s_{\varepsilon_{j-1}}(\varepsilon_j) \in \Delta^+$ for $j = 2, \ldots$.*

Proof. By assumption $\langle \alpha_{j-1}, \varepsilon_j \rangle$ is positive. Thus $\langle s_{\varepsilon_{j-1}}(\alpha_{j-2}), \varepsilon_j \rangle > 0$. Since also $\langle \alpha_{j-2}, \varepsilon_{j-1} \rangle > 0$ it follows that $\varepsilon_{j-1} \neq \varepsilon_j$. Hence [cf. 3, p. 50] $s_{\varepsilon_{j-1}}(\varepsilon_j) \in \Delta^+$ and $\langle \alpha_{j-2}, s_{\varepsilon_{j-1}}(\varepsilon_j) \rangle > 0$. Now write $\alpha_{j-2} = s_{\varepsilon_{j-2}}(\alpha_{j-3})$ and proceed analogously. □

7. The General Case

Proposition 7.1. *Let α be a positive non-compact root with $\varrho(H_\alpha) > 1$ and let $\chi \in (\mathfrak{h}^{\mathbb{C}})^*$ be integral with $\chi(H_\alpha) = 1$. Assume the existence of a simple compact root ε such that $\varrho(H_{\alpha_1}) < \varrho(H_\alpha)$ for $\alpha_1 = s_\varepsilon(\alpha)$, and such that $\chi(H_\varepsilon) < 0$. Let $p = \alpha(H_\varepsilon)$, $q = -\chi(H_\varepsilon)$, and $\psi = s_\varepsilon(\chi)$ (cf. Sect. 3). If $d_\alpha(\chi)$ is defined by the equation $P^-(\Theta_\alpha(\chi)) = d_\alpha(\chi) z_{-\alpha}$, and $d_{\alpha_1}(\psi)$ analogously,*

$$d_\alpha(\chi) = c(\alpha, \alpha_1) \binom{q+p}{p} d_{\alpha_1}(\psi) \tag{7.1}$$

where $c(\alpha, \alpha_1)$ is a non-zero constant.

Proof. $\Theta_{\alpha_1}(\psi)$ is a sum of terms of the form $z_{-\alpha_i}u_{-\mu_i}$ where $u_{-\mu_i}\in\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})^{-\mu_i}$ and $\alpha_i+\mu_i=\alpha_1$. Obviously

$$x_{-\varepsilon}^{q+p}(z_{-\alpha_i}u_{-\mu_i})=\sum_{s=0}^{q+p}\binom{q+p}{s}z_{-\alpha_i-s\varepsilon}u_{-\mu_i-(q+p-s)\varepsilon}$$

for some elements $u_{-\mu_i-(q+p-s)\varepsilon}\in\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})^{-\mu_i-(q+p-s)\varepsilon}$. To obtain a term of the form $z_{-\alpha}u_{-q\cdot\varepsilon}$ we need $\mu_i=(s-p)\varepsilon$. Thus $s\geqq p$. Assume $s>p$. Since $\alpha=\alpha_i+s\varepsilon$ and $\alpha_1=\alpha-p\varepsilon=\alpha_i+(s-p)\varepsilon$ we have a string from α_i to $\alpha_i+s\varepsilon$. Hence, by Proposition 6.2, $s=2$, $p=1$. But α_1 and α are of equal length and we reach the contradiction that $\langle\alpha_i,\varepsilon\rangle=-3$. Hence $s=p$, and $\mu_i=0$. Finally, since $-\alpha_1-s\varepsilon$ is a root for $s=0,1,\ldots,p$, $[\mathfrak{g}^{-\alpha_1-s\varepsilon},\mathfrak{g}^{-\varepsilon}]=\mathfrak{g}^{-\alpha_1-(s+1)\varepsilon}$ for $s=0,\ldots,p-1$, so

$$x_{-\varepsilon}^{q+p}z_{-\alpha_1}=\sum_{s=0}^{p}c_s\binom{q+p}{s}z_{-\alpha_1-s\varepsilon}x_{-\varepsilon}^{q+p-s}$$

for some non-zero constants c_s, and (7.1) follows. □

Analogously it follows that

Proposition 7.2. *If $\alpha=\beta+\varepsilon$ with ε simple and compact, and if $s_\beta(\alpha)=\varepsilon$, then*

$$P^-(\Theta_\alpha(\chi))=c(\alpha)\chi(H_\beta)z_{-\alpha},$$

where $c(\alpha)$ is a non-zero constant.

Proposition 7.3. *Let Λ_0 be $\mathfrak{k}_1$-dominant and integral and put $\chi=\Lambda+\varrho$. Let $\alpha\in\Delta_n^+$ and assume $\chi(H_\alpha)=1$. Then $P^-(\Theta_\alpha(\chi))\neq 0$ if and only if $\Lambda_0-\alpha$ is a highest weight in the $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-module $\mathfrak{p}^-\otimes V_{\Lambda_0}$.*

Proof. If $P^-(\Theta_\alpha(\chi))\neq 0$, $\Theta_\alpha(\chi)$ does not belong to the ideal generated by $N_\Lambda(\mathfrak{k})$ and thus projects onto a non-zero element of $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})/J_\Lambda$. It then follows from Proposition 3.1 *i)* and Lemma 2.1 that $\Lambda_0-\alpha$ is a highest weight in $\mathfrak{p}^-\otimes V_{\Lambda_0}$.

Now consider the converse, and assume that $\varepsilon_1,\ldots,\varepsilon_j$ are simple compact roots such that if

$$\alpha_i=s_{\varepsilon_i}(\alpha_{i-1});\qquad \alpha_0=\alpha,\quad \text{then}\quad \varrho(H_{\alpha_i})<\varrho(H_{\alpha_{i-1}})$$

for $i=1,\ldots,j$. Analogously let $\chi_0=\chi$ and $\chi_i=s_{\varepsilon_i}(\chi_{i-1})$. By (7.1) there exists a non-zero constant $K_j(\alpha)$ such that

$$d_\alpha(\chi)=K_j(\alpha)d_{\alpha_j}(\chi_j)\prod_{i=0}^{j-1}\binom{q_i+p_i}{p_i}$$

where $q_i=-\chi_i(\varepsilon_{i+1})$ and $p_i=\alpha_i(H_{\varepsilon_{i+1}})$. We observe that $\chi_j(\alpha_j)=1$. (It can also be shown that $p_0=p_1=\ldots=p_{j-1}$.) From Lemma 6.4, the assumption that $\Lambda_0-\alpha$ is dominant, and (2.1) we conclude:

$$\begin{aligned}-q_i=\chi_i(H_{\varepsilon_{i+1}})&=(s_{\varepsilon_i}\ldots s_{\varepsilon_1}(\Lambda+\varrho))(H_{\varepsilon_{i+1}})\\ &\geqq 1+(s_{\varepsilon_i}\ldots s_{\varepsilon_1}(\Lambda_0))(H_{\varepsilon_{i+1}})\\ &\geqq 1+(s_{\varepsilon_i}\ldots s_{\varepsilon_1}(\Lambda_0-\alpha))(H_{\varepsilon_{i+1}})+\alpha_i(H_{\varepsilon_{i+1}})\\ &\geqq 1+p_i.\end{aligned}$$

Thus, for $r=1,\ldots,p_i$, $q_i+r\leqq(r-1)-p_i\leqq -1$. Thus $\binom{q_i+p_i}{p_i}\neq 0$ for $i=0,\ldots,j-1$.

Suppose that $\alpha_j=\beta+\varepsilon$ for some simple compact root ε and that $s_\beta(\alpha_j)=\varepsilon$. Then, by Proposition 7.2, $d_{\alpha_j}(\chi_j)=c\chi_j(H_\beta)$, with $c\neq 0$. For the first time we now use the assumption that $\Lambda_0-\alpha$ not only is dominant, but also is a highest weight on $\mathfrak{p}^-\otimes V_{\Lambda_0}$: Suppose $\chi_j(H_\beta)=0$. Since $\beta=\alpha_j-\varepsilon$ and α_j and ε are of equal length, $\chi_j(H_\varepsilon)=1$ $(\chi_j(H_{\alpha_j})=1)$. Let $\varepsilon_0=s_{\varepsilon_1}\dots s_{\varepsilon_j}(\varepsilon)$ and $\gamma=s_{\varepsilon_1}\dots s_{\varepsilon_j}(\beta)$. Then $\alpha=\gamma+\varepsilon_0$ and $(\Lambda+\varrho)(H_{\varepsilon_0})=1$. Since ε_0 is compact it follows that ε_0 is simple and $\Lambda_0(H_{\varepsilon_0})=0$. Let $\mathbb{C}\cdot v_0$ denote the subspace of highest weight in V_{Λ_0}. According to Corollary 4.3, if $\Lambda_0-\alpha$ is a highest weight the corresponding subspace is given as $\mathbb{C}(z_{-\alpha}\otimes v_0+\sum_{\mu\in\Delta_c^+} z_{-\alpha+\mu}\otimes q_{-\mu}v_0)$. But ε_0 is simple and $x_{-\varepsilon_0}v_0=0$. Thus, the coefficient in $\mathfrak{p}^-\otimes v_0$ of $x_{\varepsilon_0}(z_{-\alpha}\otimes v_0+\sum_{\mu\in\Delta_c^+} z_{-\alpha+\mu}\otimes q_{-\mu}v_0)$ is given as $[x_{\varepsilon_0},z_{-\alpha}]\otimes v_0=Kz_{-\gamma}\otimes v_0$ with $K\neq 0$. It follows from Proposition 6.3 and Proposition 5.2 that this contradiction completes the proof. □

8. A Criterion

If Λ_0 is $\mathfrak{k}_1$-dominant and integral and if $\alpha\in\Delta_n^+$ the equation $(\Lambda+\varrho)(H_\alpha)=1$, with Λ_0 fixed, has a unique solution in λ. It follows from Proposition 3.1 that for this $\lambda=\lambda_\alpha$, $\Theta_\alpha(\Lambda+\varrho)\in N_\Lambda$. Thus, by Proposition 7.3, we arrive at the following criterion:

Proposition 8.1. *Let $\Lambda_0-\alpha_1,\dots,\Lambda_0-\alpha_t$ be the set of highest weights in the $\mathcal{U}(\mathfrak{k}_1^{\mathbb{C}})$-module $\mathfrak{p}^-\otimes V_{\Lambda_0}$; $\alpha_1,\dots,\alpha_t\in\Delta_n^+$. Let, for $i=1,\dots,t$, λ_i be determined by the equation $((\Lambda_0,\lambda_i)+\varrho)(H_{\alpha_i})=1$ and let $\lambda_0=\min\{\lambda_1,\dots,\lambda_t\}$. If $\Lambda=(\Lambda_0,\lambda)$, $\lambda>\lambda_0$, then W_Λ is not unitarizable.*

Remark. With the exception of some easily handled cases in $\mathrm{so}(2n-1,2)$ and $\mathrm{sp}(n,\mathbb{R})$, $\Lambda_0-\alpha$ is a highest weight in $\mathfrak{p}^-\otimes V_{\Lambda_0}$ if and only if $\Lambda_0-\alpha$ is dominant.

9. Mp$(n,\mathbb{R})$

Let $e_1,\dots,e_n$ denote the standard orthonormal basis of $\mathbb{R}^n$. Then

$$\Delta_c^+=\{e_i-e_j \mid 1\leqq i<j\leqq n\},$$

and

$$\Delta_n^+=\{e_i+e_j \mid 1\leqq i<j\leqq n\}\cup\{2e_j \mid 1\leqq j\leqq n\}.$$

$\Lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ is $\mathfrak{k}_1$-integral and dominant if and only if $\lambda_1\geqq\lambda_2\geqq\dots\geqq\lambda_n$ and $\lambda_i-\lambda_j\in\mathbb{Z}$. Moreover, $\varrho=(n,n-1,\dots,1)$. Also observe that $\lambda=\lambda_1$.

Let a,b be non-negative integers with $a\geqq b$. Define $\Lambda_{a,b}=(0,\dots,0,1,\dots,1,2,\dots,2)$ where the string of non-zero integers has length a and the string of 2's has length b.

Proposition 9.1. *Let Λ be $\mathfrak{k}_1$-integral and dominant and let a, b be the largest possible integers such that $\Lambda+\Lambda_{a,b}$ is $\mathfrak{k}_1$-dominant. Then $\lambda=-\left(\frac{a+b}{2}\right)$ is the last possible place of unitarity.*

Proof. First observe that it follows from the proof of Proposition 7.3 that the only case in which an $\alpha \in \Delta_n^+$ can satisfy $\Lambda_0 - \alpha$ dominant but $\Lambda_0 - \alpha$ not a highest weight in $\mathfrak{p}^- \otimes V_{\Lambda_0}$ is when $\alpha = e_i + e_{i+1}$, $i = 1, \ldots, n-1$, and $\Lambda_0(H_{e_i - e_{i+1}}) = 0$. It is then straightforward to see that for $\alpha = e_{a+1} + e_{b+1}$, $\Lambda_0 - \alpha$ is a highest weight in $\mathfrak{p}^- \otimes V_{\Lambda_0}$ and that the λ determined by $(\Lambda + \varrho)(H_\alpha) = 1$ is the last possible place of unitarity. Finally, $\lambda_{a+1} = \lambda$ and $\lambda_{b+1} = \lambda - 1$. (This proof also covers the case $a = b$.) □

By direct comparison with [6; Th. 6.9 and Th. 6.13] we then conclude:

Proposition 9.2. *The Kashiwara-Vergne conjecture is true for* $Mp(n, \mathbb{R})$.

References

1. Dixmier, J.: Algebrès enveloppantes. Paris: Gauthier-Villars 1972
2. Enright, T.J., Parthasarathy, R.: A proof of a conjecture of Kashiwara and Vergne. Preprint, 1980
3. Humphreys, J.E.: Introduction to Lie algebras and representation theory. Berlin, Heidelberg, New York: Springer 1972
4. Jakobsen, H.P., Vergne, M.: Restrictions and expansions of holomorphic representations. J. Functional Analysis **34**, 29–53 (1979)
5. Jakobsen, H.P.: On singular holomorphic representations. Invent Math. **62**, 67–78 (1980)
6. Kashiwara, M., Vergne, M.: On the Segal-Shale-Weil repiesentation and harmonic polynomials. Invent. Math. **44**, 1–47 (1978)
7. Rossi, H., Vergne, M.: Analytic continuation of the holomorphic discrete series of a semi-simple Lie group. Acta Math. **136**, 1–59 (1976)
8. Shapovalov, N.N.: On a bilinar form on the universal enveloping algebra of a complex semi-simple Lie algebra. Functional Analysis Appl. **6**, 307–312 (1972)

Received December 17, 1980