Unitary Irreducible Representations of a Lie Algebra for Open Matrix Chains

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Abstract. We construct highest weight unitary irreducible representations of a Lie algebra for open quantum matrix chains akin to quotients of Verma modules for simple finite-dimensional Lie algebras. Those representations resembling typical unitary irreducible representations of $gl(n)$ turn out to be tensor products of the defining representation. They can be physically identified as multiple meson states or multiple open string states. Other representations are intimately related to the Cuntz algebra. They may be related to novel bound states.

INTRODUCTION

Strong interaction and quantum gravity are two fundamental branches of physics. Quantum chromodynamics (QCD) is an experimentally well-established theory for the former, whereas M(atrix)-theory is the latest candidate for an ultimate theory of everything incorporating the latter. The main tool for doing calculations in both theories is perturbative analysis, which does lead to a wealth of decent knowledge explaining, say, scattering phenomena both of high-energy partons induced by strong interaction, and of M-theory objects induced by classical supergravity. (Ref. [1], for instance, gives a detailed account of perturbative QCD. Listed in Ref. [2] are two very recent reviews on M-theory and its perturbative analysis. The reader can find lists of further literature from them.) Perturbation theory, however, cannot be used as an all-purpose tool to account for everything. In particular, it is invalid when we want to study low-energy phenomena of strong interaction like the hadron spectrum or color confinement, or when we want to understand large quantum effects in supergravity. (Again the reader can find relevant discussions of this point in Refs. [1] and [2], and the citations therein.) Other techniques are needed to study these important and interesting phenomena.

The study of symmetry is one such non-perturbative approach. This basically involves identifying a symmetry of a physical system and a set of generators gener-

1) funded by a grant from the Danish National Research Foundation.

ating this symmetry transformation, constructing an algebra from these generators and a representation theory for the algebra, and finally applying the representation theory to compute interesting physical quantities like the mass spectrum or correlation functions. Sometimes, the symmetry of a system puts such a powerful constraint on its behavior that it completely determines its physics. As an elementary but non-trivial example, the $so(4)$ symmetry of the hydrogenic atom dictates completely its energy spectrum [3].

Other examples abound. The Virasoro algebra is a famous one. This is the Lie algebra describing conformal symmetry [4]. Through its representation theory, we have learnt a lot about conformal field theory. For instance, consider a conformal field theory on a torus. Let τ be the ratio of the complex periods along two independent orientations of the torus. As the Hamiltonian and momentum can be written in terms of the Virasoro generators L_0 and $\tilde{L}_0$ of the holomorphic and anti-holomorphic parts of a conformal field theory, its partition function is

$$\mathrm{Tr}\; q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24},$$

where $q = \exp(2\pi i \tau)$, $\bar{q}$ is the complex conjugate of q, c is the conformal charge of the holomorphic part, and $\tilde{c}$ is the conformal charge of the anti-holomorphic part of the theory. The task of calculating this partition function then boils down to computing the Virasoro characters from a knowledge of its representation theory. Indeed, the result is

$$\frac{q^{h+(1-c)/24} \bar{q}^{\tilde{h}+(1-\tilde{c})/24}}{\eta(\tau)\eta(\bar{\tau})}.$$

In this formula, h and $\tilde{h}$ are positive real numbers called the highest weights, the values of which depend on the model we are studying, and

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind function. As another example, reducible representations of the Virasoro algebra provide us with a set of null vectors, which, in the language of conformal field theory, can be translated into a set of differential constraints on the correlation functions, which can then be explicitly computed. Therefore, studying the representation theory of the Virasoro algebra goes a long way towards understanding conformal field theory. Can we adopt a similar approach to QCD and M-theory?

A remarkable common feature between QCD and M-theory is that both are matrix models. In the case of QCD, this originates from the fact that the gluon fields are in the adjoint representation of the gauge group $SU(N)$, which is effectively the same as $U(N)$ in the large–N limit [5,6]; in the case of M-theory, this stems from a conjecture that in a light-front coordinate system, M-theory can be described by a supersymmetric matrix quantum mechanics in the large–N limit [7]. Abstractly speaking, we can paraphrase a corollary to M-theory called matrix string theory

[8] and large–N QCD in terms of quantum matrix oscillators [9,10], which, as an additional bonus, can be used to formulate quantum spin chains, too [11]. Physical states built out of these oscillators can be classified into two broad families — open matrix chains and closed matrix chains. The former can be interpreted as mesons, discretized open strings or open spin chains, and the latter as glueballs, discretized closed strings or closed spin chains.

Thus, what we need to do is to identify symmetry algebras for open and closed matrix chains, and develop representation theories for them.

Rajeev and the second author of this article have reported on the existence of two Lie (super)-algebras for open and closed matrix chains. (See Ref. [10] and the citations therein.) They are called the open string (super)-algebra and the closed string (super)-algebra, respectively. Since the former looks simpler, our study of the representation theories starts with the open string algebra first. Owing to the lack of space, we will only briefly recapitulate the operators that span the open string algebra. The reader is referred to Refs. [12], [9] and [13] for more complete discussions. Also, the reader can find an account of some basic notions of the representation theory of Lie algebras to be used below in Ref. [14].

BASIC FORMALISM

An open matrix chain is a matrix product of an N-dimensional row vector, a (possibly empty) series of $N \times N$ square matrices and an N-dimensional column vector. It can be abstractly written as

$$\bar{\phi}^{\rho_1} \otimes s^{\dot{K}} \otimes \phi^{\rho_2},$$

where ρ_1 is a positive integer, $\dot{K}$ a finite integer sequence and ρ_2 another positive integer (Fig.1(a)). They label the quantum states (other than the color quantum number) of the row vector, the series of square matrices and the column vector, respectively.

In the large–N limit, there are four families of operators acting on open matrix chains. They can be abstractly written as finite linear combinations of

1. $\bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes f^{\dot{I}}_{\dot{J}} \otimes \Xi^{\lambda_3}_{\lambda_4}$,
2. $\bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes l^{\dot{I}}_{\dot{J}} \otimes 1$,
3. $1 \otimes r^{\dot{I}}_{\dot{J}} \otimes \Xi^{\lambda_1}_{\lambda_2}$ and
4. $1 \otimes \sigma^{I}_{J} \otimes 1$.

(See Figs.1(b) to (e) for illustrations.) Roughly speaking, the first family of operators replaces a whole open matrix chain with another one (Fig.1(f)); the second family replaces the row vector and perhaps a few square matrices adjacent to the row vector with another row vector and perhaps other square matrices (Fig.1(g));

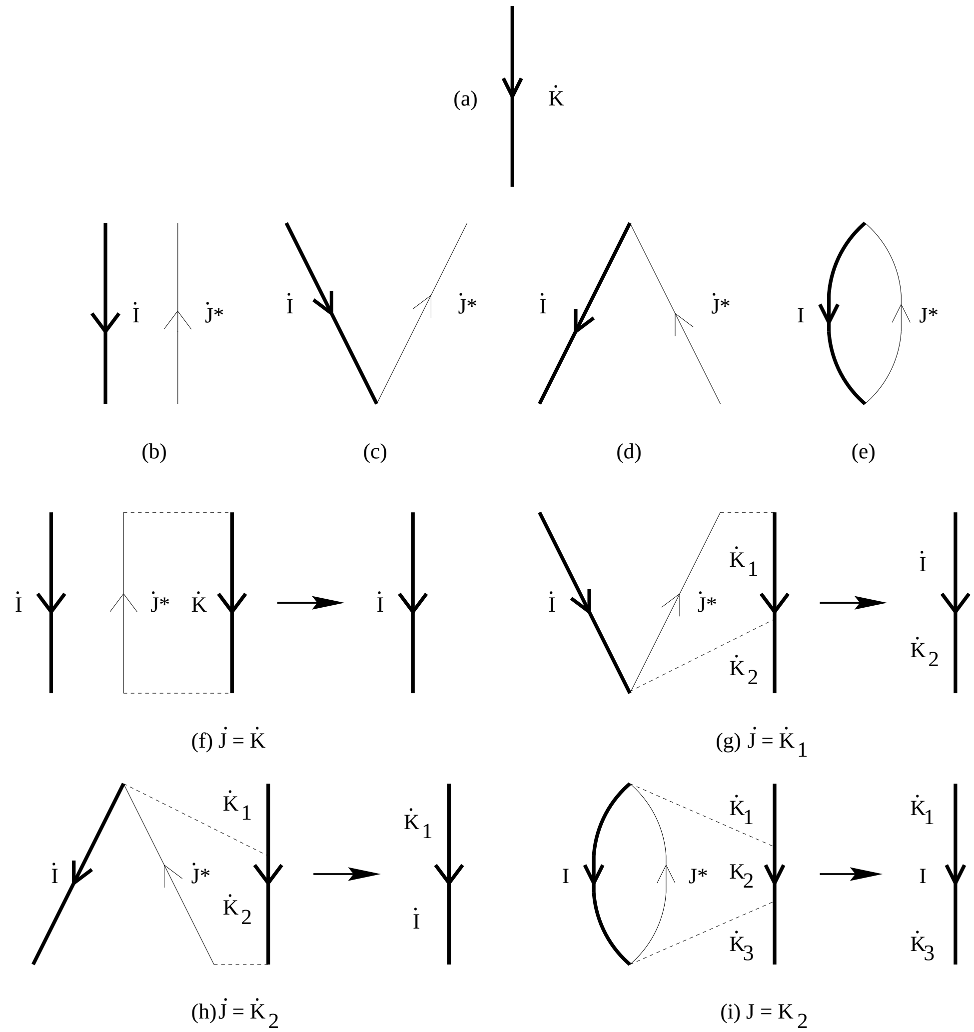

FIGURE 1. Figure (a) shows the part $s^{\dot{K}}$ of an open matrix chain. We ignore $\bar{\phi}^{\rho_1}$ and ϕ^{ρ_2} in this diagram. Figures (b), (c), (d) and (e) show the parts $f^{\dot{I}}_{\dot{J}}$, $l^{\dot{I}}_{\dot{J}}$, $r^{\dot{I}}_{\dot{J}}$ and σ^{I}_{J} of operators of the first, second, third and fourth kind, respectively. Figures (f), (g), (h) and (i) show the actions of these operators on an open matrix chain. (If a capital letter carries an asterisk, then the corresponding integer sequence is put in reverse.)

the third family works pretty much like the second family except that it acts on the column-vector end (Fig.1(h)); the last family replaces a product of square matrices in the middle of the open matrix chain with another product (Fig.1(i)). Ref. [9] contains diagrams illustrating these action. A typical physical observable like the momentum or the Hamiltonian can be written as a linear combination of these operators.

Note that these operators are *not* linearly independent; some finite linear combinations act identically on any open matrix chain. This results in a number of relations among the operators. Important examples are

$$\begin{aligned}
\sum_{\lambda=1}^{\Lambda_F} \bar{\Xi}^{\lambda}_{\lambda} \otimes l^I_J &= \sigma^I_J - \sum_{i=1}^{\Lambda} \sigma^{iI}_{iJ}, \\
\sum_{\lambda=1}^{\Lambda_F} r^I_J \otimes \Xi^{\lambda}_{\lambda} &= \sigma^I_J - \sum_{j=1}^{\Lambda} \sigma^{Ij}_{Jj}, \\
\sum_{\lambda_1,\lambda_2=1}^{\Lambda_F} \bar{\Xi}^{\lambda_1}_{\lambda_1} \otimes f^I_J \otimes \Xi^{\lambda_2}_{\lambda_2} &= \sigma^I_J - \sum_{i=1}^{\Lambda} \sigma^{iI}_{iJ} - \sum_{j=1}^{\Lambda} \sigma^{Ij}_{Jj} + \sum_{i,j=1}^{\Lambda} \sigma^{iIj}_{iJj}, \\
\sum_{\lambda_3=1}^{\Lambda_F} \bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes f^{\dot{I}}_{\dot{J}} \otimes \Xi^{\lambda_3}_{\lambda_3} &= \bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes l^{\dot{I}}_{\dot{J}} - \sum_{j=1}^{\Lambda} \bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes l^{\dot{I}j}_{\dot{J}j} \text{ and} \\
\sum_{\lambda_1=1}^{\Lambda_F} \bar{\Xi}^{\lambda_1}_{\lambda_1} \otimes f^{\dot{I}}_{\dot{J}} \otimes \Xi^{\lambda_2}_{\lambda_3} &= r^{\dot{I}}_{\dot{J}} \otimes \Xi^{\lambda_2}_{\lambda_3} - \sum_{i=1}^{\Lambda} r^{i\dot{I}}_{i\dot{J}} \otimes \Xi^{\lambda_2}_{\lambda_3}.
\end{aligned} \tag{1}$$

A notable feature from the symmetry viewpoint is that these operators form a Lie algebra, the open string algebra, with the space spanned by open matrix chains as the *defining representation*. The reader can find all the Lie brackets in Ref. [13].

VERMA-LIKE MODULES

In the case of the classical Lie algebras and the Virasoro algebra, the Weyl decomposition is a valuable tool for constructing highest weight unitary irreducible representations as quotients of Verma modules. We would like to adopt the same approach here. However, because the Cartan subalgebra together with all the root vectors do not span the open string algebra [13], we cannot construct a Verma module in the traditional sense. Nevertheless, there is still a useful decomposition out of which we can construct Verma-like modules. Let us describe them now.

Let G^{00} be the subalgebra of the open string algebra $\hat{G}_{\Lambda,\Lambda_F}$ spanned by all operators of the form

1. $\bar{\Xi}^{\lambda_1}_{\lambda_1} \otimes f^{\dot{I}}_{\dot{I}} \otimes \Xi^{\lambda_2}_{\lambda_2}$,

2. $\bar{\Xi}^{\lambda}_{\lambda} \otimes l^{\dot{I}}_{\dot{I}} \otimes 1$,

3. $1 \otimes r^{\dot{I}}_{\dot{I}} \otimes \Xi^{\lambda}_{\lambda}$ and

4. $1 \otimes \sigma^I_I \otimes 1$.

G^{00} turns out to be a Cartan subalgebra [13]. Let G^+ be the subalgebra of the open string algebra spanned by

1. $\bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes f^{\dot{I}}_{\dot{J}} \otimes \Xi^{\lambda_3}_{\lambda_4}$ such that $\dot{I}\lambda_1\lambda_3 > \dot{J}\lambda_2\lambda_4$ (see Ref. [9] or [13] for the definition of a lexicographical ordering of integer sequences);
2. $\bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes l^{\dot{I}}_{\dot{J}} \otimes 1$ such that $\dot{I}\lambda_1 > \dot{J}\lambda_2$;
3. $1 \otimes r^{\dot{I}}_{\dot{J}} \otimes \Xi^{\lambda_1}_{\lambda_2}$ such that $\dot{I}\lambda_1 > \dot{J}\lambda_2$; and
4. $1 \otimes \sigma^I_J \otimes 1$ such that $I > J$.

Finally, let G^- be the subalgebra defined similar to G^+ except that each $>$ in the above definition is changed to $<$. Then

$$\hat{G}_{\Lambda,\Lambda_F} = G^- \oplus G^{00} \oplus G^+.$$

Moreover,

$$\begin{aligned} [G^{00}, G^{00}] &= 0 \text{ and} \\ [G^{00}, G^{\pm}] &\subset G^{\pm}. \end{aligned}$$

Thus here $G^{00} \oplus G^+$ plays a role analogous to what the Borel subalgebra does for a simple Lie algebra.

Let v be a basis vector of a one-dimensional representation $\mathbb{C}_h$ of $G^{00} \oplus G^+$ satisfying

$$\begin{aligned} G^+(v) &= 0, \\ \bar{\Xi}^{\lambda_1}_{\lambda_1} \otimes f^{\dot{I}}_{\dot{I}} \otimes \Xi^{\lambda_2}_{\lambda_2}(v) &= h_I(\lambda_1; \dot{I}; \lambda_2)v, \\ \bar{\Xi}^{\lambda}_{\lambda} \otimes l^{\dot{I}}_{\dot{I}} \otimes 1(v) &= h_{II}(\lambda; \dot{I})v, \\ 1 \otimes r^{\dot{I}}_{\dot{I}} \otimes \Xi^{\lambda}_{\lambda}(v) &= h_{III}(\dot{I}; \lambda)v \text{ and} \\ 1 \otimes \sigma^I_I \otimes 1(v) &= h_{IV}(I)v, \end{aligned}$$

where h_I, h_{II}, h_{III} and h_{IV} are functionals on integer sequences. We will call them *weight functions*. Since the four kinds of operators are not linearly independent, *the weight functions are* not *independent either*. The preceding equations show that $\mathbb{C}_h$ is a left $G^{00} \oplus G^+$ module. Let $\mathcal{U}(\hat{G}_{\Lambda,\Lambda_F})$ and $\mathcal{U}(G^{00} \oplus G^+)$ be the universal enveloping algebras of the open string algebra and $G^{00} \oplus G^+$, respectively. Note that $\mathcal{U}(\hat{G}_{\Lambda,\Lambda_F})$ is a right $G^{00} \oplus G^+$ module. We call the highest weight representation defined by the quotient of

$$\mathcal{U}(\hat{G}_{\Lambda,\Lambda_F}) \otimes \mathbb{C}_h$$

by the subspace of this direct product generated by all

$$m_1 b \otimes m_2 - m_1 \otimes b m_2,$$

where $m_1 \in \mathcal{U}(\hat{G}_{\Lambda,\Lambda_F})$, $m_2 \in \mathbb{C}_h$ and $b \in \mathcal{U}(G^{00} \oplus G^+)$, a *Verma-like module*.

In general, a Verma-like module is neither irreducible nor unitary. We need to quotient out the maximal proper subrepresentation, and choose the weight functions judiciously to obtain a unitary irreducible highest weight representation.

The defining representation can be obtained from a Verma-like module as stated in

Theorem 1 *The quotient of the Verma-like module with*

$$\begin{aligned} h_I(\lambda_1; \dot{I}; \lambda_2) &= \delta_1^{\lambda_1} \delta_\emptyset^{\dot{I}} \delta_1^{\lambda_2}, \\ h_{II}(\lambda; \dot{I}) &= \delta_1^{\lambda} \delta_\emptyset^{\dot{I}}, \\ h_{III}(\dot{I}; \lambda) &= \delta_\emptyset^{\dot{I}} \delta_1^{\lambda} \text{ and} \\ h_{IV}(I) &= 0 \end{aligned}$$

by the kernel of the Hermitian form of the module is the defining representation of the open string algebra.

A proof can be found in Ref. [13].

We can get many representations from the defining representation by taking its tensor products. The following theorem describes interesting properties of these representations. To state this theorem, we need the notion of an *approximately finite* Verma-like module. This is a Verma-like module in which

1. $h_I(\lambda_1; \dot{I}; \lambda_2) - h_I(\lambda_3; \dot{J}; \lambda_4)$ should be a non-negative integer whenever $\dot{J}\lambda_3\lambda_4 > \dot{I}\lambda_1\lambda_2$;
2. $h_{II}(\lambda; \dot{I}) = \sum_{\dot{I}_1, \lambda_1} h_I(\lambda; \dot{I}\dot{I}_1; \lambda_1)$;
3. $h_{III}(\dot{I}; \lambda) = \sum_{\lambda_1, \dot{I}_1} h_I(\lambda_1; \dot{I}_1\dot{I}; \lambda)$; and
4. $h_{IV}(I) = \sum_{\lambda_1, \dot{I}_1, \dot{I}_2, \lambda_2} h_I(\lambda_1; \dot{I}_1 I \dot{I}_2; \lambda_2)$.

(Only a finite number of summands should be non-zero in the last three equations.)

Theorem 2 *The following statements pertaining to a unitary irreducible representation are equivalent:*

1. *The representation is a tensor product of the defining representation.*
2. *The representation is the quotient of an approximately finite Verma-like module by its maximal subrepresentation.*
3. *The representation is the quotient of a Verma-like module in which h_I, h_{II}, h_{III} and h_{IV} are all non-zero only on a finite number of arguments by its maximal subrepresentation.*

4. The representation is the quotient of a Verma-like module in which h_{IV} is non-zero only on a finite number of arguments by its maximal subrepresentation.

That *1.* implies *2.*, *2.* implies *3.* and *3.* implies *4.* should be obvious. Let us briefly explain how *4.* leads to *1.*. As we have underscored previously, the weight functions are not linearly independent. Indeed, it follows from Eqs.(1) that

$$
\begin{aligned}
\sum_{\lambda=1}^{\Lambda_F} h_{II}(\lambda; I) &= h_{IV}(I) - \sum_{i=1}^{\Lambda} h_{IV}(iI), \\
\sum_{\lambda=1}^{\Lambda_F} h_{III}(I; \lambda) &= h_{IV}(I) - \sum_{j=1}^{\Lambda} h_{IV}(Ij), \\
\sum_{\lambda_1,\lambda_2=1}^{\Lambda_F} h_I(\lambda_1; I; \lambda_2) &= h_{IV}(I) - \sum_{i=1}^{\Lambda} h_{IV}(iI) - \sum_{j=1}^{\Lambda} h_{IV}(Ij) + \sum_{i,j=1}^{\Lambda} h_{IV}(iIj), \\
\sum_{\lambda_2=1}^{\Lambda_F} h_I(\lambda_1; \emptyset; \lambda_2) &= h_{II}(\lambda_1; \emptyset) - \sum_{i=1}^{\Lambda} h_{II}(\lambda_1; i) \text{ and} \\
\sum_{\lambda_1=1}^{\Lambda_F} h_I(\lambda_1; \emptyset; \lambda_2) &= h_{III}(\emptyset; \lambda_2) - \sum_{i=1}^{\Lambda} h_{III}(i; \lambda_2).
\end{aligned} \tag{2}
$$

Then *4.* and Eqs.(2) together imply *3.*. *3.* and Eqs.(2) together, in turn, imply that all of h_{II}, h_{III} and h_{IV} can be completely determined from h_I by bootstrapping. h_I is the weight functions of the Cartan subalgebra of a proper ideal isomorphic to $gl(\infty)$. The claim now follows from the fact that all unitary irreducible highest weight representations of $gl(n)$ are tensor products of its defining representation. See Ref. [13] for a more detailed rigorous proof.

Theorem 3 *Any unitary irreducible highest weight representation of the open string algebra is a tensor product of a tensor product of the defining representation, and a representation of the quotient of the open string algebra by $gl(\infty)$.*

This basically results from the decompositions

$$
\begin{aligned}
\bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes l^{\dot{I}}_{\dot{J}} &\equiv \bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes \tilde{l}^{\dot{I}}_{\dot{J}} + \sum_{\lambda_3=1}^{\Lambda_F} \sum_{\dot{K}} \bar{\Xi}^{\lambda_1}_{\lambda_2} \otimes f^{\dot{I}\dot{K}}_{\dot{J}\dot{K}} \otimes \Xi^{\lambda_3}_{\lambda_3}, \\
r^{\dot{I}}_{\dot{J}} \otimes \Xi^{\lambda_1}_{\lambda_2} &\equiv \tilde{r}^{\dot{I}}_{\dot{J}} \otimes \Xi^{\lambda_1}_{\lambda_2} + \sum_{\lambda_3=1}^{\Lambda_F} \sum_{\dot{K}} \bar{\Xi}^{\lambda_3}_{\lambda_3} \otimes f^{\dot{K}\dot{I}}_{\dot{K}\dot{J}} \otimes \Xi^{\lambda_1}_{\lambda_2} \text{ and} \\
\sigma^I_J &\equiv \tilde{\sigma}^I_J + \sum_{\lambda_1,\lambda_2=1}^{\Lambda_F} \sum_{\dot{K},\dot{L}} \bar{\Xi}^{\lambda_1}_{\lambda_1} \otimes f^{\dot{K}I\dot{L}}_{\dot{K}J\dot{L}} \otimes \Xi^{\lambda_2}_{\lambda_2}.
\end{aligned} \tag{3}
$$

These decompositions are well-defined because the infinite sums on the right hand sides of Eqs.(3) are well-defined operators when acting on a highest weight module.

$\Xi^{\lambda_1}_{\lambda_2} \otimes \tilde{l}^{I}_{j}$, $\tilde{r}^{\dot{I}}_{j} \otimes \Xi^{\lambda_1}_{\lambda_2}$ and $\tilde{\sigma}^{I}_{J}$ all commute with any $f^{\dot{I}}_{j}$. They span the quotient of the open string algebra by $gl(\infty)$. The infinite sums in the above three equations together with all $f^{\dot{I}}_{j}$ are represented by a tensor product of the defining representation. Again see Ref. [13] for a fuller proof.

PHYSICAL INTERPRETATIONS AND OUTLOOK

What are the physical interpretations of these results? It is obvious that the tensor products of the defining representation in Theorem 2 describe multiple discretized open string states, or multiple meson states. As irreducible representations, they reflect once again the long-established fact that in the large–N limit, we cannot break an open string into many, or join several open strings into one [15]. The quotient representations of Theorem 3, if they exist, yield novel bound states. Since we know that the quotient is closely related to the Cuntz algebra [16,12], perhaps the representation theory of the Cuntz algebra will lead to novel physics.

ACKNOWLEDGEMENTS

We thank K. Bering, B. Durhuus, V. John and S. G. Rajeev for discussions and helpful suggestions at various stages of this work.

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