MATRIX CHAIN MODELS AND KAC-MOODY ALGEBRAS

HANS PLESNER JAKOBSEN[1] AND C.-W. HERBERT LEE

ABSTRACT. The Lie Algebras for the open, respectively closed, Matrix Chain Models offer an interesting laboratory for the study of various non-trivial mixtures of Heisenberg, Kac-Moody, and Virasoro algebras as well as $sl(\infty)$.

The algebras are not only infinite dimensional but are rather unwieldy. Various restrictions may be imposed to cut down on the complexity, e.g. finite momentum P and small alphabets. In fact, this is what reveals the non-split extensions in their simplest form. We will in the talk describe the set of all unitary highest weight modules in the case of an alphabet of just one letter ("$\Lambda = 0$") and preliminary results for the case of two letters and momentum less than or equal to 1. In both cases it is the open model that is considered.

1. INTRODUCTION

Investigations of infinite dimensional Lie algebras seem so far to have focused on special series such as Kac-Moody algebras, natural limits of finite dimensional series, or other special classes. One interesting such class are the so-called locally finite Lie algebras; see Neeb ([22]) and references cited therein. See also Palev([23]) and its list of references. One common feature of many of these is that a Heisenberg algebra is an important building block.

Besides infinite dimensional Heisenberg algebras, there does not seem to have been much interest in infinite dimensional nilpotent Lie algebras for their own sake.

For the infinite dimensional Lie algebras, the first complication is the absence of the Killing form and all the theorems that rely on it. Even worse, the Radical Splitting Theorem fails. Classification thus becomes even more dubious than might näively have been expected.

We here sketch several simple cases that show how Kac-Moody algebras, Virasoro and Heisenberg algebras may get mixed up in infinite dimensions. These examples are even "natural" in the sense that these mathematical structures originate from a model rooted in theoretical physics.

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Much work has been done on the cohomology of conformal algebras ([1], [3], [4], [5], [7], [8], [9]), see in particular the remarks on p. 78–79 in "Vertex Algebras for Beginners" by V. Kac ([16]) where non-split extensions of the Virasoro Algebra by conformal modules are classified. Our work must evidently be related to this although our structures a priori are simpler. The details remain to be worked out.

We shall not offer any explanation of why and how this is related to physics except that it goes back to investigations of Rajeev and Lee ([20, 21]), related in some sense to work of Thorn from the 1970's ([24]). The former studied meson states by which they mean states consisting of one quark, one anti-quark, and an arbitrary number of gluons. The only operators that they allow are those that propagate single meson states into single meson states. in the leading order of the "planar large N limit". They found four classes of relevant operators. There also seem to be analogies to symbolic dynamical systems, Cuntz algebras, and shift algebras.

2. Definition of the algebra

Consider the set $\{0, 1, \ldots, (\Lambda - 1)\}$.

In the sequel, the symbols $\dot{I}, \dot{J}, \dot{K}, \dot{L}$ denote finite sequences of symbols from this set and where the empty sequence $\emptyset$ is also allowed. If the empty set is not allowed, the dot is omitted. We consider the infinite dimensional vector space of functions on such symbols and we let $s^{\dot{K}}$ be the function which is 1 on the symbol $\dot{K}$ and 0 on all different symbols.

If $\dot{K} = k_1, \ldots, k_r$ then in a natural way $s^{\dot{K}} \equiv e_{k_1} \otimes \cdots \otimes e_{k_r}$. We define four classes of operators:

$$f^{\dot{I}}_{\dot{J}}\left(s^{\dot{K}}\right) = \delta^{\dot{K}}_{\dot{J}} \otimes s^{\dot{I}} \quad , \quad l^{\dot{I}}_{\dot{J}} s^{\dot{K}} = \sum_{\dot{K}_1 \dot{K}_2 = \dot{K}} \delta^{\dot{K}_1}_{\dot{J}} s^{\dot{I}\dot{K}_2} \quad ,$$

$$r^{\dot{I}}_{\dot{J}} s^{\dot{K}} = \sum_{\dot{K}_1 \dot{K}_2 = \dot{K}} \delta^{\dot{K}_2}_{\dot{J}} s^{\dot{K}_1 \dot{I}} \quad , \quad \text{and} \quad \sigma^{I}_{J} s^{\dot{K}} = \left(\sum_{\dot{K}_1 K_2 \dot{K}_3 = \dot{K}} \delta^{K_2}_{J} s^{\dot{K}_1 I \dot{K}_3} \right) .$$

Notice the un-dotted indices on σ^I_J.

Definition 2.1. *The* **open string algebra** Ξ_o *is the Lie algebra generated by these operators under the usual operator bracket. The above representation is* **the defining representation**, *denoted by* $\mathcal{T}_o$.

The following identities hold:

$$r^I_J = \sigma^I_J - \sum_{j=0}^{\Lambda-1} \sigma^{Ij}_{Jj} \quad , \quad l^I_J = \sigma^I_J - \sum_{i=0}^{\Lambda-1} \sigma^{iI}_{iJ}, \tag{2.1}$$

$$f^{\dot{I}}_{\dot{J}} = l^{\dot{I}}_{\dot{J}} - \sum_{j=0}^{\Lambda-1} l^{\dot{I}j}_{\dot{J}j} \quad , \quad f^{\dot{I}}_{\dot{J}} = r^{\dot{I}}_{\dot{J}} - \sum_{i=0}^{\Lambda-1} r^{i\dot{I}}_{i\dot{J}} \tag{2.2}$$

$$l^{\emptyset}_{\emptyset} \quad = \quad r^{\emptyset}_{\emptyset} \qquad (\text{ and is central}) \tag{2.3}$$

Remark 2.2. *The four classes of operators form an overdetermined system and the equations (2.1) - (2.3) give a defining set of linear relations between them, though with a little redundancy; e.g. (2.3) is redundant. Due to the absence of dotted indices, the* f, r, *and* ℓ *operators are not completely determined by the* σ*'s. More importantly, the set* $\{f^{\dot{I}}_{\dot{J}}\}$ *spans an ideal, the sets* $\{f^{\dot{I}}_{\dot{J}}, \ell^{\dot{A}}_{\dot{B}}\}$ *and* $\{f^{\dot{I}}_{\dot{J}}, r^{\dot{C}}_{\dot{D}}\}$ *span ideals, and these ideals together again span a proper ideal.*

2.1. **Background.** Recall the following fundamental results in the theory of finite dimensional Lie algebras:

Theorem 2.3 (The Radical Splitting Lemma)**.** *Any finite dimensional Lie algebra* $\mathfrak{g}$ *can be written as the semi-direct sum* $\mathfrak{g} = \mathfrak{s} \oplus_\sigma \mathfrak{b}$ *of a semi-simple Lie algebra* $\mathfrak{s}$ *and a solvable Lie algebra* $\mathfrak{b}$

Theorem 2.4. *For a finite dimensional Lie algebra, there exists a fundamental bilinear form, the Killing form, which is invariant under* ad *and which is non-degenerate if the Lie algebra is semi-simple.*

Remark 2.5. *Both fail miserably in infinite dimensions. This makes life both more complicated and more intriguing.*

2.2. **Commutators.** The commutators in Ξ_o are notoriously given by complicated expressions. We give here one such - and here there are no "dotted indices", at least not in the first formula.

$$
\begin{aligned}
[\sigma^I_J, \sigma^K_L] &= \delta^K_J \sigma^I_L + \sum_{J_1 J_2 = J} \delta^K_{J_2} \sigma^I_{J_1 L} + \sum_{K_1 K_2 = K} \delta^{K_1}_J \sigma^{IK_2}_L \\
&+ \sum_{\substack{J_1 J_2 = J \\ K_1 K_2 = K}} \delta^{K_1}_{J_2} \sigma^{IK_2}_{J_1 L} + \sum_{J_1 J_2 = J} \delta^K_{J_1} \sigma^I_{L J_2} + \sum_{K_1 K_2 = K} \delta^{K_2}_J \sigma^{K_1 I}_L \\
&+ \sum_{\substack{J_1 J_2 = J \\ K_1 K_2 = K}} \delta^{K_2}_{J_1} \sigma^{K_1 I}_{L J_2} + \sum_{J_1 J_2 J_3 = J} \delta^K_{J_2} \sigma^I_{J_1 L J_3} + \sum_{K_1 K_2 K_3 = K} \delta^{K_2}_J \sigma^{K_1 I K_3}_L \\
&- (I \leftrightarrow K, J \leftrightarrow L),
\end{aligned}
$$

This may be written in a more compact form as

$$
\left[\sigma^I_J, \sigma^K_L\right] = \sum_{\substack{J = J_1 J_2 J_3 \\ K = K_1 K_2 K_3 \\ (\#J_1)\cdot(\#K_1) = 0 \\ (\#J_3)\cdot(\#K_3) = 0}} \delta_{J_2, K_2} \sigma^{K_1 I K_3}_{J_1 L J_3} - \sum_{\substack{I = I_1 I_2 I_3 \\ L = L_1 L_2 L_3 \\ (\#I_1)\cdot(\#L_1) = 0 \\ (\#I_3)\cdot(\#L_3) = 0}} \delta_{I_2, L_2} \sigma^{I_1 K I_3}_{L_1 J L_3}.
$$

In the last equation above we have omitted dotting the sequences J_1, K_1, J_3, K_3 since it is clear from the context that they may be empty, indeed that at least one of them must be so. We shall occasionally in the sequel, for instance in the beginning of the next section, omit the dotting if it is clear form the context if and when the empty set is allowed.

3. Orderings, diagonal subalgebra, Weyl decomposition, involutions, Verma-like modules, unitarity

We say about two sequences A, B that $A < B$ if either $\#(A) < \#(B)$ or, when $\#(A) = \#(B)$, if, at the first index from the left where they are different, A has the biggest index.

This induces an ordering on the operators X^A_B, where $X \equiv f, \ell, r, \sigma$, according to which X^A_B is **positive** if and only if $A < B$. The notion of negative of course is similar. Observe that we may introduce an ordering in the defining representation according to which

$$s^A > s^B \Leftrightarrow A < B \quad (!)$$

Then X^A_B is positive iff $\forall K : X^A_B(s^K) > s^K$. Similar sentiments may of course be expressed about the negative operators.

We now define

$$\begin{aligned} \Xi^+ &= \text{Span}\{X^A_B \mid A < B,\ X = f, \ell, r, \sigma\}, \\ \Xi^0 &= \text{Span}\{X^A_B \mid A = B,\ X = f, \ell, r, \sigma\}, \\ \Xi^- &= \text{Span}\{X^A_B \mid A > B,\ X = f, \ell, r, \sigma\}. \end{aligned}$$

The equations (2.1) - (2.3) clearly descends to these spaces. Each space is a subalgebra and $[\Xi^0, \Xi^\pm] \subseteq \Xi^\pm$. Moreover, $[\Xi^0, \Xi^0] = 0$ since Ξ^0 acts diagonally in the defining representation. On the other hand, e.g. Ξ_0 is not diagonalizable in the adjoint representation. We do have, though, a Weyl decomposition:

$$\Xi_o = \Xi^- \oplus \Xi^0 \oplus \Xi^+.$$

The **standard anti-involution** ω_s is defined as follows:

$$\omega_s(X^A_B) = X^B_A. \tag{3.4}$$

We shall discuss briefly later the possibility of having more elaborate, albeit natural, involutions.

Let λ be a linear functional on Ξ^0. Let $\mathbb{C}_\lambda$ be the $\Xi^0 \oplus \Xi^+$ module defined by extending λ trivially to Ξ^+. We may now define a **Verma-like module** $\mathcal{M}_\lambda$ of $\mathcal{U}(\Xi_o)$ of highest weight λ as the left $\mathcal{U}(\Xi_o)$-module

$$\mathcal{M}_\lambda = \mathcal{U}(\Xi_o) \otimes_{\mathcal{U}(\Xi^0 \oplus \Xi^+)} \mathbb{C}_\lambda.$$

We may furthermore define a highest weight representation of highest weight λ as any quotient of this representation. And we may define unitarity in the standard way.

4. Approximately finite

Let

$$\lambda_f(A) = \lambda(f^A_A) \quad , \quad \lambda_\ell(A) = \lambda(\ell^A_A) \; , \tag{4.5}$$

$$\lambda_r(A) = \lambda(r^A_A) \quad , \quad \lambda_\sigma(A) = \lambda(\sigma^A_A) \; . \tag{4.6}$$

We have the following crucial equations in the defining representation, where the right hand sides are (locally) finite:

$$\begin{aligned} \ell^{\dot I}_{\dot J} &= \sum_{\dot B} f^{\dot I \dot B_2}_{\dot J \dot B} \; , \\ r^{\dot I}_{\dot J} &= \sum_{\dot D} f^{\dot D \dot I}_{\dot D_1 \dot J} \; , \\ \sigma^I_J &= \sum_{\dot E_1, \dot E_3} f^{\dot E_1 I \dot E_3}_{\dot E_1 J \dot E_3} \; . \end{aligned}$$

For the moment we are only interested in unitarity defined with respect to the standard involution ω_s.

Definition 4.1. *A highest weight representation is called* **approximately finite** *if the following hold:*

$$\begin{aligned} \textit{If } A > B \textit{ then } \lambda_f(B) - \lambda_f(A) \qquad & \textit{is a non-negative integer} \; , \\ \forall A : \lambda_\ell(A) \quad &= \quad \textstyle\sum_{\dot B} \lambda_f(A \dot B) \; , \\ \forall A : \lambda_r(A) \quad &= \quad \textstyle\sum_{\dot D} \lambda_f(\dot D A) \; , \\ \forall A : \lambda_\sigma(A) \quad &= \quad \textstyle\sum_{\dot E_1 \dot E_3} \lambda_f(\dot E_1 A \dot E_3) \; . \end{aligned}$$

Theorem 4.2. *An approximately finite representation is unitary (w.r.t. ω_s) if and only if it is a sub-representation of the rth order tensor product of the defining representation $\mathcal{T}_o$ for some $r \in \mathbb{N}$.*

Definition 4.3. *$sl_\infty(f)$ denotes the the subalgebra spanned by all $[f^A_B, f^C_D]$.*

Theorem 4.4. *Any unitary irreducible highest weight representation of the open string algebra is a tensor product of a unitary irreducible approximately finite representation and a unitary irreducible lowest weight representation in which any element of $sl_\infty(f)$ acts trivially.*

Theorem 4.5 (Go and No-Go)**.** *If $\Lambda = 1$ there* **are** *unitary highest weight representations of Ξ_o other than the approximately finite ones (see below). If $\Lambda \geq 2$, there are* **none**.

Proof (For $\Lambda > 1$): We assume that we have a unitary highest weight module of highest weight $\lambda \equiv (\lambda_f, \lambda_\ell, \lambda_r, \lambda_\sigma)$ and we assume that there exists a real number f_0 such that

$$\forall A: \ \lambda_f(A) = f_0.$$

We set

$$L_i = \sum_{\#(A)=i} \lambda_\ell(A).$$

It follows easily, by induction, from (2.3) that

$$\forall i \in \mathbb{N}: L_i = L_0 - (1 + \Lambda + \cdots + \Lambda^{i-1}) = L_0 - \frac{\Lambda^i - 1}{\Lambda - 1} \cdot f_0.$$

Now, by unitarity,

$$A < B \Rightarrow \lambda_\ell(B) \leq \lambda_\ell(A),$$

in particular, $\lambda_\ell(B_i) \leq \lambda_\ell(A_{i-1})$ for any $\#(B_i) = i$ and $\#(A_{i-1}) = i - 1$. Also observe that $(\Lambda - 1)^i \leq B_i \leq 0^i$ for any B_i with $\#(B_i) = i$.

Denote by L_i^m and L_i^M the minimum and the maximum of $\{\lambda_\ell(B_i)\}$ over such B_i. Then, clearly,

$$\frac{1}{\Lambda} L_{i+1} \leq \Lambda^i \cdot L_i^m \leq L_i \leq \Lambda^i L_i^M \leq \Lambda \cdot L_{i-1}, \tag{4.7}$$

and hence

$$\frac{L_0(\Lambda - 1) + f_0}{(\Lambda - 1)\Lambda^i} \leq L_i^M + \frac{f_0}{\Lambda - 1} \leq \frac{L_0(\Lambda - 1) + f_0}{(\Lambda - 1)\Lambda^{i-1}}.$$

Thus, $\lim_{i\to\infty} L_i^M = -\frac{f_0}{\Lambda-1}$, and the same type of arguments give that $\lim_{i\to\infty} L_i^m = -\frac{f_0}{\Lambda-1}$.

Finally observe that e.g.

$$e = \ell_{0^i}^J, f = \ell_J^{0^i}, h = \ell_J^J - \ell_{0^i}^{0^i}$$

span an $su(2)$ subalgebra for any $J \neq 0^i$ with $\#(J) = i$. Thus, for i sufficiently big,

$$L_i^m = L_i^M.$$

This implies that $\ell^I_J v_\lambda = 0$ for all $I \neq J$ with $\#(I)$ and $\#(J)$ sufficiently big. But then (2.3) gives that the whole representation must be trivial as far as the operators ℓ^I_J are concerned.

The same line of reasoning of course applies to the operators r^I_J. Hence, there exist constants ℓ_0, r_0 such that

$$\forall A: \ \lambda_\ell(A) = \ell_0, \text{ and } \forall B: \ \lambda_r(B) = r_0.$$

Finally, the same strategy can be made to work on the algebra generated by the operators σ^I_J. Indeed, when computing commutators of the form

$$[\sigma^{a^i}_J, \sigma^J_{a^i}],$$

and, specifically, the projection of this commutator onto the Cartan subalgebra, one encounters expressions of the form from the list

- $\sigma^{a^k C}_{D a^k}$ with $\#(C), \#(D) \leq k$ and $a^k C = D a^k$
- $\sigma^{K a^k}_{a^k L}$ with $\#(K), \#(L) \leq k$ and $K a^k = a^k L$
- $\sigma^{a^k J}_{J a^k}$ with $k \leq \#(J)$ and $a^k J = J a^k$.

In either case, this forces $J = a^i$. So, this again implies the right inequalities for λ_σ and finally there is an $su(2)$ based in e.g. $I = 0^i$ and $J = (\Lambda - 1)^i$ than can be used to clinch the argument, just as before. □

5. Momentum p. General structure

Define the *momentum* p of an integer sequence $\dot{K}$ by the formulae

$$p(\emptyset) = 0 \ \text{ and } p(K) = \sum_{i=1}^{\#(K)=r} k_i \text{ if } \qquad K = k_1 k_2 \ldots k_r.$$

The momentum of a basis vector in $\mathcal{T}_o$ is then defined by

$$p(s^{\dot{K}}) = p(\dot{K}).$$

If a vector is the sum of basis vectors whose momenta all are p, then we say that this vector has a momentum p, too. Since $0 \leq k_i \leq \Lambda$, p is a non-negative integer. It is easy for the reader to verify by inspection that $\mathcal{T}_o$ is a momentum-graded representation for Ξ_o. Consider the subalgebra Ξ'_o of Ξ_o consisting of elements which map a vector with the momentum p to another vector with the same momentum. This subalgebra is then spanned by all $f^{\dot{I}}_{\dot{J}}$, $l^{\dot{I}}_{J}$, and $r^{\dot{I}}_{J}$, σ^{I}_{J} such that $p(\dot{I}) = p(\dot{J})$ or $p(I) = p(J)$.

Consider the subspace spanned by all vectors of momentum P. This subspace clearly forms a representation space for Ξ'_o. Take the quotient $\mathcal{B}_{\leq P}$ of Ξ'_o by the annihilator of this subspace. $\mathcal{B}_{\leq P}$ is spanned by all $f^{\dot{I}}_{\dot{J}}$, $l^{\dot{I}}_{J}$, $r^{\dot{I}}_{J}$, and σ^{I}_{J} such that

$$p(\dot{I}) = p(\dot{J}) \leq P \text{ or } p(I) = p(J) \leq P.$$

We call $\mathcal{B}_{\leq P}$ *the momentum-P algebra.* It contains the algebra $\mathcal{A}_P$ of all operators of exact momentum P.

Consider the ideal in $\mathcal{B}_{\leq P}$ spanned by all $f^{\dot{I}}_{\dot{I}} - f^{\dot{J}}_{\dot{J}}$ as well as all $f^{\dot{I}}_{\dot{J}}$ in $\mathcal{B}_{\leq P}$ such that $\dot{I} \neq \dot{J}$. This ideal is isomorphic to $sl(\infty)$. The *truly infinite momentum-P algebra* $\mathcal{B}^{\infty}_{\leq P}$ is then defined as the quotient of $\mathcal{B}_{\leq P}$ by this ideal.

The general structure is as follows:

$$0 \to \mathcal{A}_p \to \mathcal{B}_{\leq p} \to \mathcal{B}_{\leq p-1} \to 0 \qquad \text{(non-split)},$$

with a similar sequence for the truly infinite algebras.

The algebra $\mathcal{A}_p$ is itself built up of f, ℓ, r, σ's - all of momentum p. We now take a closer look at it:

We define

$$\begin{aligned}
F_p &= \{f^A_B \mid p(A) = p(B) = p\} \\
F^p_0 &= [F_p, F_p] \\
\tilde{L}_p &= \{f^A_B, \ell^C_D \mid p(A) = p(B) = p(C) = p(D) = p\} \\
\tilde{R}_p &= \{f^A_B, r^C_D \mid p(A) = p(B) = p(C) = p(D) = p\}.
\end{aligned}$$

Then $F_p, F^0_p, \tilde{L}_p$, and $\tilde{R}_p$ are ideals in $\mathcal{A}_p$.

Proposition 5.1. *Let $f^p_p = f^{1^p}_{1^p}, \ell^p_p = \ell^{1^p}_{1^p}$, and $r^p_p = r^{1^p}_{1^p}$. In a slight abuse of notation we also use these symbols to denote the equivalence classes of said elements in appropriate quotients. We have*

- $F_p \equiv gl(\infty)$

- $F_p^0 \equiv sl(\infty)$
- $\tilde{L}_p/F_P^0 \equiv \widetilde{\text{loop}}(gl(\infty))$ *with central charge* $c = f_p^p$*. More specifically,*

$$0 \to sl(\infty) \to \tilde{L}_p \to \widetilde{\text{loop}}(gl(\infty)) \to 0 \qquad \textit{(non split)}, \tag{5.8}$$

  where here, and elsewhere, $\widetilde{\text{loop}}(\mathfrak{g})$ *denotes the usual central extension of the loop algebra over the (reductive) Lie algebra* $\mathfrak{g}$*.*
- $\tilde{R}_p/F_P^0 \equiv \widetilde{\text{loop}}(gl(\infty))$ *with central charge* f_p^p*, i.e.*

$$0 \to sl(\infty) \to \tilde{F}_p \to \widetilde{\text{loop}}(gl(\infty)) \to 0 \qquad \textit{(non split)}.$$

- $S_p = \mathcal{A}_p/F_p^0$ *is a "double Kac-Moody algebra" whose spanning elements, as far as the elements* σ_B^A *go, may be denoted by* $t_R^n \otimes E_{A,B} \otimes t_L^r$ *with* $n, r \in \mathbb{Z}$ *and* $p(A) = p(B) = p$*, and where the commutator is given by*

$$\begin{aligned}
&[t_R^{n_1} \otimes E_{A,B} \otimes t_L^{r_1}, t_R^{n_2} \otimes E_{C,D} \otimes t_L^{r_2}] = \\
&t_R^{n_1+n_2} \otimes [E_{A,B}, E_{C,D}] \otimes t_L^{r_1+r_2} + \\
&(\delta_{B,C}\ \tilde{\ell}(A, D, n_1, n_2) + \delta_{A,D}\ \tilde{\ell}(B, C, n_1, n_2)) \otimes t_L^{r_1+r_2} + \\
&(\delta_{B,C}\ \tilde{r}(A, D, r_1, r_2) + \delta_{A,D}\ \tilde{r}(B, C, r_1, r_2)) \otimes t_R^{n_1+n_2} + \\
&\delta_{B,C}\delta_{A,D}\delta_{n_1+n_2,0}\delta_{r_1+r_2,0} \cdot (-n_1 r_1) \cdot \frac{(sgn(n_1) + sgn(r_1))}{2} f_p^p,
\end{aligned}$$

where the terms $\tilde{\ell}(A, D, n_1, n_2), \ldots, \tilde{r}(B, C, r_1, r_2)$ are certain elements in the $gl(\infty)$ corresponding to $\tilde{L}_p$ and $\tilde{R}_p$, respectively.

Thus we get non-split *towers of Kac-Moody algebras* (and more general algebras). We may also restrict by e.g. omitting the operators σ_J^I etc. The bottom layer ($p = 0$) is a little different - c.f. below.

Remark 5.2. *To get a genuine Kac-Moody algebra we would need also to adjoin the usual derivation* d_0*. Actually, there are elements of the Lie algebra* Ξ_o *which behave very much like such a derivation without exactly being equal to it. For instance,*

$$[f_{0^c10^d}^{0^a10^b}, \sigma_0^0] = (c + d - a - b) f_{0^c10^d}^{0^a10^b}, \ \textit{ and } \ [\ell_{0^c10^d}^{0^a10^b}, \sigma_0^0] = (c + d - a - b)\ell_{0^c10^d}^{0^a10^b}.$$

We will denote the corresponding derivation by d_{Ξ_o}. See below for the case $p = 1$.

Remark 5.3. *It is an interesting question, related of course to more general cohomological questions, to classify e.g. the* **central extensions** *of these algebras.*

Here is an example of how the $p = 1$ Kac-Moody of the ℓ^I_J's sits inside the one with $p \le 2$. Actually, it shows that it is only the $sl(\infty)$ that appears in the extension.

$$[t^n \otimes E^1_{a,c}, t_L^{-r} \otimes E^1_{b,d}]_{p\le 2} =$$

$$[t^n \otimes E^1_{a,c}, t_L^{-c} \otimes E^1_{b,d}]_{p=1} +$$

$$(\text{ only if } sgn(nr) > 0)$$

$$\delta_{a,d}(E^2_{(b,r-1),(c,n-1)} + E^2_{(b,r-2),(c,n-2)} + \cdots + E^2_{(b,r-n),(c,0)})$$

Above, we assumed that and $r \ge n$, $b \ne c$, and

$$E^2_{(x,y),(z,w)} \equiv \ell^{0^x 1 0^y 1}_{0^z 1 0^w 1} \otimes t^0_L.$$

Here and below, $0^x = \underbrace{0 \cdot 0 \cdot \cdots 0}_{x}$ and e.g. 1^p similarly.

6. $\underline{p = 0}$

This case for a general Λ is identical to the full case for $\Lambda = 1$. Moreover, $\forall x, y \in \mathbb{N}_0$: $\ell^{0^x}_{0^y} = r^{0^x}_{0^y}$. Let

$$\mathfrak{g}_{0,\infty} = \mathcal{A}_0 / F^0_0$$

Let, for $\alpha \ge 0$, $T_a = \ell^{0^a}_{\emptyset}$, $\sigma_a = \sigma^{0^{a+1}}_0 + \ell^{0^a}_{\emptyset}$, and $F_0 = E_{1,1}$. Define operators analogously for $a \le 0$ and use the same symbols to denote their classes in $\mathfrak{g}_{0,\infty}$.

We have the following commutation relations in $\mathfrak{g}_{0,\infty}$:

$$
\begin{aligned}
[\sigma_a, \sigma_b] &= (a-b)\sigma_{a+b} \text{ if } (\text{sgn } ab) = 1 \\
[\sigma_a, T_b] &= -b \cdot T_{a+b} \text{ if } a+b \neq 0 \\
[T_a, T_b] &= 0 \text{ if } a+b \neq 0 \\
[\sigma_a, \sigma_b] &= (a-b)\sigma_{a+b} - (\text{sgn } a)\min\{a^2, b^2\}T_{a+b} \\
&\quad \text{if } (\text{sgn } ab) = -1 \text{ and } a+b \neq 0,
\end{aligned}
\tag{6.9}
$$

$$
\begin{aligned}
[\sigma_{-a}, \sigma_a] &= -2a\sigma_0 + a^2 T_0 - \frac{(a-1)a(2a-1)}{6}F_0 \text{ for } a \geq 0 \\
&\quad - \left(\frac{1}{12}(a+1)a(a-1)Z_0\right) \\
[T_{-a}, \sigma_a] &= [\sigma_{-a}, T_a] = -aT_0 + \frac{a(a-1)}{2}F_0 \text{ for } a \geq 0 \\
[T_{-a}, T_a] &= -aF_0 \text{ for } a \geq 0 \\
[\sigma_a, F_0] &= [T_a, F_0] = 0.
\end{aligned}
$$

The term Z_0 has been added by hand to give, according to our computations, the most general central extension; $\mathfrak{g}_{0,\infty}(Z_0)$. Compare with ([16],[8]). We shall see below that it occurs naturally.

It is clear from the above that $\mathfrak{h}_\infty = \text{Span}\{T_i, F_0 \mid i \in \mathbb{Z}\}$ is an ideal equal to an infinite dimensional Heisenberg algebra. Modulo this ideal we get the Virasoro algebra Vir,

$$
0 \to \mathfrak{h}_\infty \to \mathfrak{g}_{0,\infty}(Z_0) \to Vir \to 0.
$$

Remark 6.1. *This* **is** *really non-split. Details will appear elsewhere.*

We consider a highest weight module V_Λ generated by a highest weight vector v_Λ satisfying

$$\begin{aligned}
\sigma_0 v_\Lambda &= s_0 v_\Lambda \\
T_0 v_\Lambda &= t_0 v_\Lambda \\
F_0 v_\Lambda &= f_0 v_\Lambda \\
Z_0 v_\Lambda &= z_0 v_\Lambda \text{ and} \\
\sigma_a v_\Lambda &= T_a v_\Lambda = 0 \text{ for } a > 0.
\end{aligned}$$

We consider

$$\begin{aligned}
\overline{\sigma}_{-1} &= (\sigma_{-1} - \frac{t_0}{f_0} T_{-1}) \\
\overline{\sigma}_{-2} &= (\sigma_{-2} - \frac{1}{2f_0} T_{-1}^2 - \frac{(2t_0 - f_0)}{2f_0} T_{-2}), \\
\overline{\sigma}_{1} &= (\sigma_1 - \frac{t_0}{f_0} T_1) \text{ and} \\
\overline{\sigma}_{2} &= (\sigma_2 - \frac{1}{2f_0} T_1^2 - \frac{(2t_0 - f_0)}{2f_0} T_2).
\end{aligned}$$

Lemma 6.2. *In a highest weight module of highest weight* (s_0, t_0, f_0, z_0)*,*

$$[T_1, \overline{\sigma}_{-1}] = 0 \; ; \; [T_2, \overline{\sigma}_{-1}] = 2T_1 \quad ; \quad [T_{-2}, \overline{\sigma}_1] = -2T_{-1} \; ; \; [T_{-1}, \overline{\sigma}_1] = 0 \tag{6.10}$$

$$[T_1, \overline{\sigma}_{-2}] = 0 \; ; \; [T_2, \overline{\sigma}_{-2}] = 0 \quad ; \quad [T_{-1}, \overline{\sigma}_2] = 0 \; ; \; [T_{-2}, \overline{\sigma}_2] = 0. \tag{6.11}$$

Moreover,

$$\begin{aligned}
[\overline{\sigma}_1, \overline{\sigma}_{-1}] &= 2s_0 - t_0 - \frac{t_0^2}{f_0} \\
[\overline{\sigma}_2, \overline{\sigma}_{-2}] &= 2(2s_0 - t_0 - \frac{t_0^2}{f_0}) + \frac{1}{2}(f_0 - 1 + z_0), \text{ and} \\
[\overline{\sigma}_1, \overline{\sigma}_{-2}] &= 3\overline{\sigma}_{-1}.
\end{aligned}$$

Thus, these operators behave as if they were in a representation of h.w. $(\frac{1}{2}(2s_0 - t_0 - \frac{t_0^2}{f_0}), 0, 0, f_0 - 1 + z_0)$ *where, by unitarity the operators* T_{-i} *act trivially. The latter is a representation* $V(c, h)$ *of the Virasoro algebra of data* $(c, h) = (f_0 - 1 + z_0, \frac{1}{2}(2s_0 - t_0 - \frac{t_0^2}{f_0}))$*.*

Observe that the condition for the representation space to be that of the module $\mathcal{H}(\mathfrak{h}_\infty)$ generated by the Heisenberg subalgebra acting on v_Λ is that

$$2s_0 - t_0 - \frac{t_0^2}{f_0} = 0 = 2(2s_0 - t_0 - \frac{t_0^2}{f_0}) + \frac{1}{2}(f_0 - 1 + z_0).$$

Finally observe that

$$\begin{aligned} & (\frac{1}{2}(2s_0 - t_0 - \frac{t_0^2}{f_0}), 0, 0, z_0 + f_0 - 1) \\ + & (\frac{1}{2}(t_0 + \frac{t_0^2}{f_0}), t_0, f_0, -(f_0 - 1)) \\ = & (s_0, t_0, f_0, z_0). \end{aligned}$$

Proposition 6.3. *An irreducible unitary highest weight representation (i.u.h.w.) U of is the tensor product of an i.u.h.w. which stays irreducible when restricted to* $\mathfrak{h}_\infty$ *with an i.u.h.w. obtained from the Virasoro quotient algebra.*

Remark 6.4. *The point of the equations (6.10-6.11) in connection with unitarity is that the span of the elements*

$$\bar{\sigma}_{-a_1}\bar{\sigma}_{-a_2}\dots\bar{\sigma}_{-a_r}v_\Lambda$$

is annihilated by all T_a *with* $a > 0$ *and for this purpose, the commutator* $[T_2, \overline{\sigma}_{-1}] = 2T_1$ *causes no problems. However, the following choice of operators (which make sense in any module in which the Heisenberg generators act as in a h.w. module) satisfy that* **all** *commutators with the* T_a*'s are zero (compare with* [18, Exercise 14.8]*):*

$$\forall a \in \mathbb{Z}: \qquad \bar{\sigma}_a \stackrel{Def.}{=} \sigma_a + \frac{|a|-1}{2}T_a - \frac{1}{2f_0}\sum_{n=-\infty}^{\infty} T_{-n}T_{n+a}.$$

Moreover,

$$\forall a, b \in \mathbb{Z}: \qquad [\bar{\sigma}_a, \bar{\sigma}_b] = (a-b)\bar{\sigma}_{a+b} - \left(\frac{1}{12}(a+1)a(a-1)(F_0 + Z_0)\right).$$

Indeed, even more general representations than highest weight representations of the Virasoro algebra may then be put together with a highest weight representation of the Heisenberg algebra.

Recall ([6]) that $V(c,h)$ is unitary if and only if either $c \geq 1, h \geq 0$ or $(c,h) = (c_m, h_m^{r,s})$, where m, r, s are integers such that $m \geq 0$ and $1 \leq s \leq r \leq m+1$, and where

$$\begin{aligned} c_m &= 1 - \frac{6}{(m+2)(m+3)}, \\ h_m^{r,s} &= \frac{((m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)}. \end{aligned}$$

Also recall that other unitary representations are known of the Witt algebra, hence of the Virasoro algebra. See e.g. ([2]).

6.1. **KP - hierarchy?** In the special representation which stays irreducible when restricted to the Heisenberg algebra, we have

$$\begin{aligned} \sigma_{-1} v_\Lambda &= \frac{t_0}{f_0} T_{-1} \qquad \text{and} \\ \sigma_{-2} v_\Lambda &= (\frac{t_0}{f_0} - \frac{1}{2}) T_{-2} + \frac{1}{2f_0} T_{-1}^2, \end{aligned}$$

and hence

$$\begin{aligned} \sigma_{-1} &= \frac{t_0}{f_0} x_{-1} + \sum_{j=1}^{\infty} j \cdot x_{-j-1} \frac{\partial}{\partial x_{-j}} \qquad \text{and} \\ \sigma_{-2} &= (\frac{t_0}{f_0} - \frac{1}{2}) x_{-2} + \frac{1}{2f_0} x_{-1}^2 + \sum_{k=1}^{\infty} k \cdot x_{-k-2} \frac{\partial}{\partial x_{-k}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \sigma_1 &= \sum_{j=2}^{\infty} j \cdot x_{-j+1} \frac{\partial}{\partial x_{-j}} + t_0 \frac{\partial^1}{\partial x_{-1}} \qquad \text{and} \\ \sigma_2 &= 2 \sum_{k=3}^{\infty} k \cdot x_{-k+2} \frac{\partial}{\partial x_{-k}} + (2t_0 - f_0) \frac{\partial^1}{\partial x_{-2}} + \frac{f_0{}^1}{2} \frac{\partial^2}{\partial x_{-1}^2}. \end{aligned}$$

In principle, the action of σ_i is then determined for all $i \in \mathbb{Z}$. It will be interesting to compute it more explicitly, as well as $\exp(t\sigma_i)$.

[1]These terms come about because σ_1 commutes with T_{-1}, T_{-2} (Lemma 6.2)

We briefly comment upon the tensor product of such a representation with itself. Introduce, as usual, coordinates and differential operators

$$z_{-i} = \frac{x^{(1)}_{-i} + x^{(2)}_{-i}}{2} \quad , \quad w_{-i} = \frac{x^{(1)}_{-i} - x^{(2)}_{-i}}{2}$$
$$\frac{\partial}{\partial z_{-i}} = \frac{\partial}{\partial x^{(1)}_{-i}} + \frac{\partial}{\partial x^{(2)}_{-i}} \quad , \quad \frac{\partial}{\partial w_{-i}} = \frac{\partial}{\partial x^{(1)}_{-i}} - \frac{\partial}{\partial x^{(2)}_{-i}}.$$

It is clear that the space of polynomials in the "diagonal elements" z_{-i} are part of the highest component of the tensor product. This follows by looking just at the Heisenberg algebra. We compute the actions of the remaining generators in the tensor product:

$$\begin{aligned}
\sigma_{-1} \otimes 1 + 1 \otimes \sigma_{-1} &= \sum_{j=1}^{\infty} j \cdot w_{-j-1} \frac{\partial}{\partial w_{-j}} + \ E_{-1}\left(z, \frac{\partial}{\partial z}\right), \\
\sigma_{-2} \otimes 1 + 1 \otimes \sigma_{-2} &= \frac{1}{f_0} w_{-1}^2 + \sum_{k=1}^{\infty} k \cdot w_{-k-2} \frac{\partial}{\partial w_{-k}} + \ E_{-2}\left(z, \frac{\partial}{\partial z}\right), \\
\sigma_{1} \otimes 1 + 1 \otimes \sigma_{1} &= \sum_{j=2}^{\infty} j \cdot w_{-j+1} \frac{\partial}{\partial w_{-j}} + \ E_{1}\left(z, \frac{\partial}{\partial z}\right), \\
\sigma_{2} \otimes 1 + 1 \otimes \sigma_{2} &= \frac{f_0}{4} \frac{\partial^2}{\partial w_{-1}^2} + \sum_{k=3}^{\infty} k \cdot w_{-k+2} \frac{\partial}{\partial w_{-k}} + \ E_{2}\left(z, \frac{\partial}{\partial z}\right),
\end{aligned}$$

where the terms $E_i\left(z, \frac{\partial}{\partial z}\right)$ $i = -2, -1, 1, 2$ are expressions involving only (diagonal) Heisenberg generators. Thus, the space of polynomials in the "off-diagonal" elements w_{-i} transform, modulo the diagonal elements, as a representation of the Virasoro algebra of data $(c, h) = (1, 0)$ (because of the $\frac{1}{12}$).

Notice that w_{-1} is not in the highest component and is orthogonal to z_{-1} - the only other element of that weight. Thus, the lowest component (or at least some part of it) seems to be generated by this element, which is a highest weight vector of data $(c, h) = (1, 1)$.

7. $\widetilde{\text{loop}}(gl(\infty))$ - ORDERINGS, DERIVATIONS, UNITARITY

With an eye to later applications, we here digress to a study of a number of properties of $\widetilde{\text{loop}}(gl(\infty))$

Since it is natural in our setup to use "one-sided" infinity, we start by giving the definitions of the background structure:

Let $e_1, e_2, \dots$ denote the standard basis vectors in ℓ^∞ and let

$$\mathbb{C}^\infty = \mathrm{Span}\{e_i \mid i = 1, 2, \dots\}. \tag{7.12}$$

set

$$gl(\infty) = \mathrm{Span}\{E_{i,j} \mid i, j = 1, 2, \dots\}, \tag{7.13}$$

where the $E_{i,j}$ are the usual matrix units corresponding to (7.12). We view $gl(\infty)$ as a Lie algebra in the usual way and equip it with the usual (trace) Killing form. This is non-degenerate. Let $gl(\infty)^{(1)}$ denote the Kac-moody algebra obtained as the central extension of the loop algebra based on $gl(\infty)$. Occasionally we shall also consider $sl(\infty)^{(1)}$ which is defined analogously.

Recall that a subset $\triangle_+ \subset \triangle$ is *a set of positive roots* if the following three properties hold (see e.g. ([14, 15])):

- If $\alpha, \beta \in \triangle_+$ and $\alpha + \beta \in \triangle$ then $\alpha + \beta \in \triangle_+$
- Id $\alpha \in \triangle$ then either α or $-\alpha$ lie in $\triangle_+$
- If $\alpha \in \triangle_+$ then $-\alpha \notin \triangle_+$

We now want to classify all positive root systems.

Especially the **standard** ones. These are root systems for which no positive root α satisfies that $\alpha + n\delta$ is positive for all n.

The roots look like $\begin{bmatrix} \lambda \\ n\delta \end{bmatrix}$ with α a root corresponding to $sl(\infty)$ and $n \in \mathbb{Z}$. The Weyl group acts on the λ through the usual Weyl group W_0 for $sl(\infty)$ (leaving the δ piece invariant) and by

$$\begin{bmatrix} \lambda \\ n\delta \end{bmatrix} \mapsto \begin{bmatrix} \lambda \\ (n + \langle \lambda, \hat{\gamma} \rangle)\delta \end{bmatrix},$$

where $\hat{\gamma}$ is any (finite) co-root. This displays the affine Weyl group W in the usual fashion as a semi-direct product.

Observe that $gl(\infty)^{(1)}$ in a natural way is the limit as $n \to \infty$ of $gl(n)^{(1)}$.

Any standard positive root system is equivalent to the usual standard on any $gl(n)^{(1)}$. We shall see below that the classification of unitary highest weight modules then follows

More generally, we then have the following structure theorem:

Theorem 7.1. *Any standard set of positive roots $\mathfrak{n}_s^+$ is equivalent to one of the following form: Let $\overline{\lambda} = (\lambda_i)_{i=1}^{\infty}$ be a sequence of integers whose tail is non-zero (we count 2 sequences as equal if their difference is a constant sequence). Then either $\mathfrak{n}_s^+ = \mathfrak{n}_{KM}^+$, or $\mathfrak{n}_s^+$ is the image of $\mathfrak{n}_{KM}^+$ under the map F which, on the level of weights, is given by*

$$(\alpha_{a,b}, n) \mapsto ((\alpha_{a,b}, n + \lambda_b - \lambda_a).$$

Specifically, F is the map $E_{a,b} \otimes t^n \mapsto E_{a,b} \otimes t^{n+\lambda_b-\lambda_a}$ and $c \mapsto c$. Moreover, if $X = E_{a_1,b_1} \otimes t^{n_1}$ and $Y = E_{a_2,b_2} \otimes t^{n_2}$, we get

$$[F(X), F(Y)] = F([X, Y]) + \omega(X, Y) \cdot c,$$

where $\omega(X, Y) = \delta_{n_1+n_2,0} \cdot B(E_{a_1,b_1}, E_{a_2,b_2}) \cdot (\lambda_{b_1} - \lambda_{a_1})$.

Observe that F in any finite-dimensional subalgebra is given by a Weyl group element, i.e. there exists a sequence of Weyl group elements $\{w_i\}_{i=1}^{\infty}$ such that

$$F = \lim_{i \to \infty} w_i.$$

Suppose now that v_Λ defines a (unitary) highest weight module of highest weight Λ with respect to $\mathfrak{n}_{KM}^+$ and set the value of c in this representation equal to c_0.

Since the ω term is in the center it follows easily that if v_{Λ_F} defines a highest weight module of highest weight Λ_F with respect to $\mathfrak{n}_o^+$ where $\Lambda_F(h_{ii} - h_{jj}) = \Lambda(h_{ii} - h_{jj}) + (\lambda_j - \lambda_i) \cdot c_0$, then F induces a map (also denoted by F) by

$$u_1^- \cdots u_r^- \cdot v_\Lambda \mapsto F(u_1^-) \cdots F(u_r^-) \cdot v_{\Lambda_F}$$

which preserves the canonical hermitian forms. If one of the spaces is unitarizable, then so is the other, and F is a unitary map.

Regarding the full Kac-Moody algebra based on e.g. the above, notice that the d_{Ξ_o} considered in Remark 5.2 results in the derivation

$$d_{\Xi_o}(f^{0^a10^b}_{0^c10^d}) = (c+d-a-b)f^{0^a10^b}_{0^c10^d} \text{ and } d_{\Xi_o}(E_{a,b}\otimes t^n) = (b-a+n)E_{a,b}\otimes t^n.$$

Let $d_0 = t\frac{\partial}{\partial t}$ denote the standard derivation as before. Observe that the two Kac-Moody algebras

$$\hat{L}(gl(\infty)) = \widetilde{\text{loop}}(gl(\infty)) \oplus \mathbb{C}\cdot d_0 \text{ and } \hat{L}_{\Xi_o}(gl(\infty)) = \widetilde{\text{loop}}(gl(\infty)) \oplus \mathbb{C}\cdot d_{\Xi_o}$$

based on the two derivations are non-isomorphic Lie algebras since the only possibility for an isomorphism would be the identity on $\widetilde{\text{loop}}(gl(\infty))$ together with $d_0 \mapsto d_0 - \sum_i i\cdot E_{i,i}\otimes t^0$, and this is clearly not admissible.

More generally we have the following

Proposition 7.2. *Let*

$$d: E_{i,j}\otimes t^k \mapsto \lambda_{ijk}E_{i,j}\otimes t^{k+d_{ijk}}$$

be a derivation of $\widetilde{\text{loop}}(gl(\infty))$ *which commutes with* $\text{ad}(h)$ *for any* h *in the diagonal of* $gl(\infty)$*. Then* $d = \gamma\cdot d_0 + \text{ad}(\tilde{h})$ *where* $\gamma\in\mathbb{C}$ *and where* $\tilde{h} = \sum_i c_i\cdot E_{i,i}$ *for some sequence* $\{\lambda_i\}_{i=1}^{\infty}$ *of complex numbers.*

Proof: It follows by looking at the components in $\mathbb{C}\cdot c$ that the integer exponents d_{ijk} must all be zero. It then follows easily from the derivation property that $\lambda_{ijk} = \lambda_i - \lambda_j + \gamma\cdot k$. (A fact that of course also could have been established by considering $\widetilde{\text{loop}}(gl(\infty))$ as a limit of $\widetilde{\text{loop}}(gl(n))$'s.) □

Remark 7.3. *Two such derivations, defined by two sequences* $\{\lambda_i\}_{i=1}^{\infty}$ *and* $\{\lambda_i'\}_{i=1}^{\infty}$ *and the same multiple of* d_0 *are clearly equivalent if and only if the two sequences differ at at most finitely many places.*

Also notice that if $\forall i: c_i\in\mathbb{R}$*, the corresponding derivation, even for* $\gamma = 0$*, defines a set of positive roots*

$$\triangle_+^d = \triangle_+^{1,d}\cup\triangle_+^{2,d}$$

by setting $\triangle_+^{1,d}$ *equal to the set of roots corresponding to the elements in the positive eigenspaces of* d *and where* $\triangle_+^{2,d}$ *is any ordering of the kernel of* d *(if it is non trivial).*

Observe that $\gamma = 0$ opens up for the "natural" Borel subalgebras. Indeed, if the kernel of d is nontrivial, it will consist of a collection of subalgebras of the form $\widetilde{\text{loop}}(gl(n))$ and/or $\widetilde{\text{loop}}(gl(\infty))$. These, for $n < \infty$, may be ordered according to the classification by Jakobsen-Kac ([14, 15]) whereas the procedure from above may be repeated if there are components of the form $\widetilde{\text{loop}}(gl(\infty))$.

We may for the sake of defining these sets replace each λ_i by its integer part, $[\lambda_i]$, but the price one may have to pay is that the kernel increases in size.

Finally notice that all standard orderings in view of earlier comments give equivalent unitarizable h.w. modules.

7.1. **Relation to a_∞.** For the sake of completeness, we here show that the above constructions are not covered by the usual vertex algebra constructions:

Consider the map

$$T : \mathbb{Z} \mapsto \mathbb{N} : \qquad T(j) = \begin{cases} 2 \cdot j & j > 0 \\ -2 \cdot j + 1 & \text{else} \end{cases}$$

with inverse

$$T^{-1} : \mathbb{N} \mapsto \mathbb{Z} : \qquad T^{-1}(j) = \begin{cases} \frac{j}{2} & j \text{ even} \\ \frac{-(j-1)}{2} & j \text{ odd} \end{cases}$$

This map induces map, also denoted by T, from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{N})$, and it induces a map from matrices over $\mathbb{Z}$ to matrices over $\mathbb{N}$ given by matrix units as

$$\forall i, j \in \mathbb{Z} : E_{i,j} \mapsto E_{T(i),T(j)}$$

with inverse

$$\forall i, j \in \mathbb{N} : E_{i,j} \mapsto E_{T^{-1}(i),T^{-1}(j)}.$$

According to this, if n is odd, the element $\sum_{\alpha=1}^{\infty} E_{\alpha,n+\alpha} \in a_\infty^+$ is mapped to

$$\sum_{j=1}^{\infty} E_{j,-j+\frac{-n+1}{2}} + \sum_{k=0}^{\infty} E_{k,-k+\frac{n+1}{2}}.$$

Hence, the image is not in a_∞.

8. Interlude: Extensions of Kac-Moody algebras

The extensions considered above have natural analogues as regards, in the first instance, $A_n^{(1)}$. We digress here to a sketch of this case which, as a pay off, reveals that there may be other natural involutions besides the standard (3.4) that should be considered.

Let $\mathfrak{g}$ be a Lie subalgebra of $sl(N)$ for some $N \in \mathbb{N}$. Set

$$\widetilde{\mathrm{loop}}_\infty(\mathfrak{g}) = \{x \otimes t^r \mid x \in \mathfrak{g}\} \cup \mathrm{Span}\{y \otimes E_{i,j} \mid y \in gl(N)\}. \tag{8.14}$$

We then have an extension $\widetilde{\mathrm{loop}}_\infty(\mathfrak{g})$ of (e.g.) $\widetilde{\mathrm{loop}}(\mathfrak{g})$ by $sl(\infty)$

$$0 \to sl(\infty) \to \widetilde{\mathrm{loop}}_\infty(\mathfrak{g}) \to \widetilde{\mathrm{loop}}(\mathfrak{g}) \to 0$$

Specifically, we view $x \otimes t^r$ as $x \otimes (\sum_{\alpha=1}^{\infty} E_{r+\alpha,\alpha})$ when $r \geq 0$ and with a similar definition for $s < 0$.

We then have

$$[x \otimes t^r, y \otimes t^s] = [x,y] \otimes t^{r+s} - xy \otimes T_{r+1}^{-s+1} \text{ for } r \geq s,$$

where

$$T_a^b = \begin{cases} \sum_{i=1}^{a-1} E_{i,b-a+i} & b \geq a \geq 1 \\ \sum_{j=1}^{b-1} E_{j+a-b,j} & a \geq b \geq 1 \\ 0 & \text{for all other combinations of } a,b \in \mathbb{Z} \end{cases}.$$

We may also throw in the Virasoro but only as $I \otimes \sigma_i$.

If $s = -r$ above, we get that

$$-xy \otimes T_{r+1}^{r+1} = -r\frac{1}{n}\,\mathrm{tr}(xy) \cdot I_n \otimes E_{1,1} \mod sl(\infty)$$

- and the term $I_n \otimes E_{1,1}$ commutes with all the elements $x \otimes t^r$ **modulo** the finite rank ones and hence in the quotient plays the rôle of the central charge. Observe that above, the $sl(\infty)$ really is the $sl(\infty)$ of the right hand side in the following identification (with a mild abuse of notation) of $gl(N) \otimes gl(\infty)$ with $gl(\infty)$:

$$gl(N) \otimes gl(\infty) \ni E_{a,b} \otimes E_{i,j} \leftrightarrow E_{a+i\cdot n, b+j\cdot n} \in gl(\infty). \tag{8.15}$$

We define the derivation d_0 as follows:

$$d_0(x \otimes E_{i,j}) \overset{Def.}{=} (i-j) \cdot x \otimes E_{i,j}$$

and extend it to $x \otimes t^r$ in the natural way so that, with $d_0(x \otimes t^r) = r \cdot x \otimes t^r$, it is recognized as (an extension of) the usual derivation. Notice also that according to this definition, $d_0(I_n \otimes E_{1,1}) = 0$. Then we also have the following non-split extension:

$$0 \to sl(\infty) \to (\widetilde{\mathrm{loop}}_\infty(\mathfrak{g}) \oplus \mathbb{C} \cdot d_0) \to \mathfrak{g}^{(1)} \to 0 \; ,$$

where $\mathfrak{g}^{(1)}$ is the usual Kac-Moody algebra based on $\mathfrak{g}$.

8.1. **Unitary representations of $\widetilde{\mathrm{loop}}_\infty(\mathfrak{g})$.** In Theorem 4.2, unitary representations based on the standard (compact) involution on $gl(\infty)$ were extended to our algebras. The idea now is to use unitary representations of $gl(\infty)$ defined via other involutions to construct, in the same manner, unitary representations of $\widetilde{\mathrm{loop}}_\infty(\mathfrak{g})$. Specifically, an involution ω may mix our Kac-Moody as in (8.14) with the finite rank ones.

This phenomenon is in some sense a contrast to $sl(\infty)^{(1)}$ where the only involution that leads to unitary highest weight modules that are of non-zero central charge, is the standard one.

The most directly applicable new situation is $u(n, \infty)$: This is defined by means of a real form of $gl(\infty)$, i.e. by means of an anti-linear involution ω_n. We may use the involution for this algebra to define an involution (also denoted ω_n on the extension $\widetilde{\mathrm{loop}}_\infty(\mathfrak{g})$ as follows

$$\omega_n(x \otimes t^r) = x^t \otimes t^{-r} - 2x^t \otimes E_{1,r+1} \qquad (\text{ if } r \geq 0).$$

Let us call a unitary highest weight representation of $u(n, \infty)$ *essentially finite* if it comes as a trivial extension of a unitary highest weight representation of $u(n, m)$ for some m.

Using, again, the identification (8.15), it is straightforward to obtain

Theorem 8.1. *With the involution ω_n, any essentially finite unitary highest weight representation of $u(n, \infty)$ may be extended to a unitary representation of $\widetilde{\mathrm{loop}}_\infty(\mathfrak{g})$.*

Remark 8.2. *The set of unitary highest weight representations of $u(n,m)$ was determined in ([11]), proving thereby a conjecture of Kashiwara and Vergne ([19]). Recall that they are all, except the trivial representation, infinite dimensional.*

9. $p = 1$

We now return to the situation considered in Proposition 5.1, in particular, equation (5.8). A similar analysis might have been given for other cases.

We identify $\ell_{0^b1}^{0^a10^r} \leftrightarrow E_{a,b} \otimes t^{-r}$ and, likewise, $\ell_{0^b10^s}^{0^a1} \leftrightarrow E_{a,b} \otimes t^s$.

We then have, modulo terms of momentum $p \geq 2$,

$$\begin{aligned}
[\ell_{0^b1}^{0^a10^r}, \ell_{0^d10^s}^{0^c1}] &= \delta_{b,c}\ell_{0^d1}^{0^a10^{r-s}} - \delta_{a,d}\ell_{0^b1}^{0^c10^{r-s}} - \delta_{b,c}\left(f_{0^d10^{s-1}}^{0^a10^{r-1}} - \cdots - f_{0^d1}^{0^a10^{r-s}}\right) \text{ (if } r \geq s),\\
[\ell_{0^b1}^{0^a10^r}, \ell_{0^d1}^{0^c10^s}] &= \delta_{b,c}\ell_{0^d1}^{0^a10^{r+s}} - \delta_{a,d}\ell_{0^b1}^{0^c10^{r+s}}.
\end{aligned}$$

Analogous formulas hold in the remaining cases. If we further mod out by the commutator algebra of the elements f_Y^X of momentum 1 we obtain, in the new notation,

$$[E_{a,b} \otimes t^r, E_{c,d} \otimes t^s] = ([E_{a,b}, E_{c,d}])t^{s+r} + \delta_{a,d}\delta_{b,c}\delta_{s+r,0} \cdot r \cdot (f_1^1).$$

This is just $\widetilde{\text{loop}}(gl(\infty))$ with central charge f_1^1.

Our ordering from before here implies that an element $E_{a,b} \otimes t^n$ with the ordering of $\ell_{0^b10^n}^{0^a1}$ if $n \geq 0$ and of $\ell_{0^b10^{-n}}^{0^a1}$ if $n \leq 0$, is positive exactly if n satisfies:

If $a < b$: $n \geq a - b$.

If $a \geq b$: $n \geq a - b + 1$.

We recognize this as the special case of the general ones considered in Section 7.1 in which

$$\forall a \in \mathbb{N} : \quad \lambda_a = a.$$

10. $p = 1$ and $p \leq 1$; unitarity

We have the following result (sketchy).

Proposition 10.1. *The only situation that may lead to unitarity for $p \leq 1$ is for the algebra $S_{\leq 1}$ generated by "σ-operators" L_m and $L_{a,b}$ of momentum $p = 0, 1$, respectively:*

$$[L_m, L_{ab}] = -aL_{a+m,b} - bL_{a,b+m}$$

$$[L_{ab}, L_{de}] = 0,$$

$$[L_a, L_b] = (a-b)L_{a+b}$$

$$-max\{-sgn(ab), 0\} \cdot min(a^2, b^2)\frac{1}{2}(L_{a+b,0} + L_{0,a+b}).$$

We clearly have

Lemma 10.2. *In a unitary highest weight module, all operators $L_{a,b}$ must act trivially, and we are thus left with the Witt algebra*

$$[L_a, L_b] = (a-b)L_{a+b}.$$

In view of this result, we now search for central extensions of this algebra. Let

$$[L_{a,b}, L_{c,d}] = \omega(a, b, c, d)$$

then it follows that

$$\begin{aligned} \forall m \in \mathbb{Z} \quad &: \quad a\omega(a+m, b, c, d) + b\omega(a, b+m, c, d) \\ &+ \quad c\omega(a, b, c+m, d) + d\omega(a, b, c, d+m) = 0 \\ \omega(a, b, c, d) \quad &= \quad -\omega(c, d, a, b) \end{aligned}$$

In particular, $(a+b+c+d)\omega(a, b, c, d) = 0$.

Set

$$\omega_n(a, b, c, d) = (a+b)\delta_{a+b+c+d,0}.$$

Lemma 10.3. *Positive multiples of ω_n are the only non-singular cocycles which may lead to unitarity with respect to the given order.*

The classification of extensions, of which the above is just one piece, will appear elsewhere; for here we just want to describe one such (perhaps the most interesting)

10.1. **An example of a central extension.** The following defines a central extension of the algebra given by the commutators (10.16):

$$
\begin{aligned}
[L_m, L_{ab}] &= -aL_{a+m,b} - bL_{a,b+m} \\
&- (a+b)\delta_{a+b+m,0}\cdot\frac{F_0}{2} \\
&+ sgn(a+b))(a+b)^2)\delta_{a+b+m,0}\cdot\frac{F_0}{2} \\
[L_{ab}, L_{cd}] &= (a+b)\delta_{a+b+c+d,0}\cdot F_0 \\
[L_a, L_b] &= (a-b)L_{a+b} \\
&+ \frac{(-sgn(a)+sgn(b))}{2}\cdot min(a^2,b^2)\frac{(L_{a+b,0}+L_{0,a+b})}{2} \\
&+ \delta_{a+b,0}\left(a^3\cdot(\frac{F_0}{3}+\frac{Z_0}{12}) - a|a|\frac{F_0}{2} + a\cdot(\frac{F_0}{6}-\frac{Z_0}{12})\right).
\end{aligned}
\tag{10.16}
$$

Theorem 10.4. *In the algebra defined by the commutators (10.16), the following subspace is an ideal;*

$$\mathrm{Span}\{L_{a,b} - L_{c,d} \mid a+b = c+d\}.$$

Let T_{a+b} denote the equivalence class $[L_{a,b}]$, and let σ_a denote the image of L_a in the quotient. Then the quotient algebra is isomorphic to the algebra given by the commutators (6.9).

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Mathematics Institute, Universitetsparken 5, DK–2100 Copenhagen Ø, Denmark

E-mail address: `jakobsen@math.ku.dk, h11lee@hopper.math.uwaterloo.ca`