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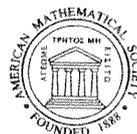
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The Full Set of Unitarizable  
Highest Weight Modules  
of Basic Classical Lie Superalgebras

Hans Plesner Jakobsen



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## ABSTRACT

This manuscript contains a complete description of the set of all unitarizable highest weight modules of classical Lie superalgebras, unitarity being defined in the superalgebraic sense. The algebras are over  $\mathbb{C}$ , and it is a part of the classification to determine which real forms, defined by anti-linear anti-involutions, may occur.

*Key words and phrases.* Lie superalgebras, classical superalgebras, unitarity, highest weight representations, classification.

We would like to use this opportunity to thank Victor Kac for many helpful conversations in the early stages of this investigation. We are also grateful to Michel Duflo for useful questions. Finally, we thank Johan van de Leur for drawing our attention to his dissertation [20] from which we learned about the existence of the cases II, III, and IV for  $G(3)$  as well as case V for  $F(4)$ .

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## CHAPTER 1

### INTRODUCTION

This article contains a complete description of the set of all unitarizable highest weight modules of classical Lie superalgebras.

The algebras are over  $\mathbb{C}$ , and it is a part of the classification to determine which real forms, defined by anti-linear anti-involutions, may occur.

As far as we know, this is the first systematic study of the problem but there have been many investigations for some special superalgebras, [2], [3], [4], [6], [7], [8], [10]. For a recent contribution, see [21].

Superalgebras and supergroups have now for some time been attracting a lot of attention for various reasons. Our investigation only deals with superalgebras, and our main references are the articles of V. Kac ([16], [17]). We try to give all the relevant background information in the first two chapters. For further background we refer to the articles by C. Fronsdal, M. Flato, and T. Hirai, collected in [5]. This latter, as well as references cited therein, also provides an introduction to the physical theory.

One of the main reasons for our own interest in the subject was our general interest in unitarity, in particular of highest weight modules. Several years ago we completely solved the analogous question for simple Lie algebras ([14]), after previously having solved the question for special series of algebras (cf. [12] for  $su(p, q)$ , and [13] for  $sp(n, \mathbb{R})$ ). Superalgebras are a natural setting for both applying and extending these results. In fact, it has for long been known to us that the main idea behind the proof for the case of simple Lie algebras, "unitarity at the last possible place" ([13]), once properly extended, again would be the key that could open up to the whole classification.

Probably the best known finite-dimensional Lie superalgebra is

$$(1.1) \quad sl(m, n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \in sl(m), d \in sl(n), b, c \text{ arbitrary, and } \operatorname{tr} a = \operatorname{tr} d \right\}.$$

For  $m \neq n$ , this is in fact a basic classical Lie superalgebra which we, to conform to the notation of V. Kac ([16]), will denote by  $A(m, n)$  ( $A(n, n)$  is the

quotient of  $sl(n, n)$  by its one-dimensional center). Moreover, it turns out that the most crucial case is the determination of the set of unitary finite dimensional representations of the superalgebra  $A(m, n)$  ( $m \neq n$ ). Once this has been settled, the other cases follow by using this result repeatedly, together with the old results for the simple Lie algebras.

We have that

$$(1.2) \quad \begin{aligned} A(m, n) &= G_{-1} \oplus G_0 \oplus G_1, \text{ and} \\ G_0 &= su(m+1)^{\mathbb{C}} \oplus su(n+1)^{\mathbb{C}} \oplus \mathbb{C}. \end{aligned}$$

Any finite dimensional irreducible module  $L$  is a quotient of some induced module

$$(1.3) \quad M = \bigwedge(G_{-1}) \otimes V,$$

where  $V = V_{(\alpha, \tau_1, \tau_2)}$  is an irreducible  $su(m+1) \times su(n+1)$  module on which the center of  $G_0$  acts by the (real) value  $\alpha$ , and where  $su(m+1) \times su(n+1)$  acts by  $(\tau_1, \tau_2)$ . Our first goal is thus to describe, for each pair  $(\tau_1, \tau_2)$ , the set  $\{\alpha \mid L_\alpha = L_{(\alpha, \tau_1, \tau_2)} \text{ unitarizable}\}$ . (Since  $(\tau_1, \tau_2)$  is fixed, we will often suppress it.) Equivalently, since for any real value of  $\alpha$  there is a canonical contravariant hermitian form  $H_\alpha$  on  $\bigwedge(G_{-1}) \otimes V_\alpha$ , we wish to describe the set  $\{\alpha \mid H_\alpha \text{ is positive semi-definite}\}$ . For this purpose it turns out to be of critical importance to examine the restriction of  $H_\alpha$  to  $\bigwedge^1(G_{-1}) \otimes V_\alpha = G_{-1} \otimes V_\alpha$ . Write  $G_{-1} \otimes V_\alpha = \oplus_i V_i^1$ , where each  $V_i^1$  is an irreducible  $su(m+1) \times su(n+1)$  module, and the restriction  $H_{i, \alpha}^1$  of the hermitian form to  $V_i^1$  has the form

$$(1.4) \quad H_{i, \alpha}^1 = (\alpha \cdot \tilde{H}_i^1) \oplus H_{i, 0}^1,$$

where  $\tilde{H}_i^1$  and  $H_{i, 0}^1$  do not depend on  $\alpha$  (due to the irreducibility of  $V_i^1$  they are of course equal up to multiplication by a real constant). There are now two choices of unitarity, one being automatically satisfied for  $\alpha$  sufficiently positive and the other for  $\alpha$  sufficiently negative. Let us for simplicity look at the latter. For each  $i$  we then get from (1.4) a value  $\alpha_i$  at which the corresponding hermitian form  $H_{i, \alpha}^1 \equiv 0$ . The smallest of these  $\alpha_i$ 's is then the last possible place of unitarity. It is then a crucial step to prove that the module indeed is unitary at this value. After this the whole description follows by a tensoring argument.

Let us finally again use  $A(m, n)$  to illustrate our remarks about the central role of these finite dimensional modules. Now, however, write

$$(1.5) \quad G_0 = su(p, q)^{\mathbb{C}} \oplus su(n+1)^{\mathbb{C}} \oplus \mathbb{C}, \quad (p+q = m+1)$$

to indicate the real form we have in mind. The Lie algebra  $K$  corresponding to the maximal compact subgroup of  $SU(p, q)$  is  $K = su(p) \times su(q) \times \mathbb{R}$ , and

$$(1.6) \quad su(p, q)^{\mathbb{C}} = W^- \oplus K^{\mathbb{C}} \oplus Z^+,$$

where  $W^-$  and  $Z^+$  are complex  $\text{ad}(K)$ -invariant abelian subalgebras. In a similar way,  $G_{-1}$  decomposes further into two irreducible  $K \times \text{su}(n+1)$  modules:

$$(1.7) \quad G_{-1} = G_{-1}^1 \oplus G_{-1}^2.$$

(Actually, there is more than one possible choice of positive (resp. negative) part of the “odd” part of  $G$ , but for simplicity we choose  $G_{-1}$  as above.) Then, as before, a unitarizable highest weight module  $M$  may be written as a quotient of some  $M = \bigwedge(G_{-1}) \otimes V$ , but where now  $V = V_\alpha = V_{\alpha,\lambda}$  is an irreducible  $\text{su}(p, q)$  module, and as such, besides being an irreducible  $\text{su}(p) \times \text{su}(q)$  module, depends on an extra parameter,  $\lambda$ . In fact,  $V_{\alpha,\lambda}$  is a quotient of a module of the form  $U(W^-) \cdot \hat{V}_{(\alpha,\lambda,\tau_1,\tau_2)}$ , where, suppressing  $(\tau_1, \tau_2)$  as usual,  $\hat{V}_{(\alpha,\lambda)}$  is a finite dimensional module for  $K$  as well as for the center of  $G_{\bar{0}}$ . One has

$$(1.8) \quad \bigwedge(G_{-1}) \otimes V_{\alpha,\lambda} = U(W^-) \cdot \bigwedge(G_{-1}^2) \otimes \bigwedge(G_{-1}^1) \otimes \hat{V}_{(\alpha,\lambda)}.$$

The investigation of unitarity in this case starts with picking a value of  $\lambda$  for which  $V_{\alpha,\lambda}$  is unitarizable. One then investigates first  $\bigwedge(G_{-1}^1) \otimes \hat{V}_{(\alpha,\lambda)}$  as a finite dimensional  $A(q-1, n)$  module ( $A(q-1, n)$  being defined by the  $\text{su}(q) \times \text{su}(n+1)$ ). This can be done with the help of the previous results. Then one examines  $\bigwedge(G_{-1}^2) \otimes \left( \bigwedge(G_{-1}^1) \otimes \hat{V}_{(\alpha,\lambda)} \right)$  as a finite dimensional  $A(p-1, n)$  module. In both cases, the critical values of  $\alpha$  depend on  $\lambda$ . Finally, one needs to examine  $U(W^-) \cdot \bigwedge(G_{-1}) \otimes \hat{V}_{(\alpha,\lambda)}$  as a collection of  $\text{su}(p, q)$  modules to see what extra conditions, if any, are imposed on the allowed values of  $\lambda$ . Especially around some special values where some pieces of  $\bigwedge(G_{-1}) \otimes \hat{V}_{(\alpha,\lambda)}$  may be in the radical of the hermitian form, this is somewhat delicate since then  $\alpha$  depends on  $\lambda$ , and, in some sense, the allowed values of  $\lambda$  depend on  $\alpha$ . However, with some care, the problem becomes manageable.

The above examples are treated in full in chapters 4 and 5. More generally, chapters 4 through 12 deal with all the pertinent real forms of the various algebras that constitute the set of basic classical Lie superalgebras. Finally, in chapter 13 we lay the ground for future investigations into covariant differential operators, singular representations, and infinite dimensional groups.

## CLASSICAL LIE SUPERALGEBRAS

We present here the basic definitions and results concerning classical Lie superalgebras and their highest weight representations. For further details, see the articles [16] and [17]. Though a good deal of the following carries over to infinite dimensional Lie superalgebras, we will here only consider finite dimensional ones.

A *Lie superalgebra*  $G$  is a complex vector space which is a direct sum of two subspaces  $G_{\bar{0}}$  and  $G_{\bar{1}}$ , called the *even* and the *odd* part, respectively, endowed with a super Lie bracket  $[\cdot, \cdot] : G \times G \rightarrow G$  satisfying

$$(2.1) \quad \begin{aligned} [x, y] &= -(-1)^{(\deg x)(\deg y)}[y, x], \\ [x, [y, z]] &= [[x, y], z] + (-1)^{(\deg x)(\deg y)}[y, [x, z]]. \end{aligned}$$

Here,  $\deg x = 0$  (1) if and only if  $x$  is even (odd).

A bilinear form  $B$  on  $G$  is called *invariant* if

$$(2.2.1) \quad (i) \quad B(x, y) = (-1)^{(\deg x)(\deg y)}B(y, x).$$

$$(2.2.2) \quad (ii) \quad B(x, y) = 0 \quad \text{for } x \in G_{\bar{0}} \text{ and } y \in G_{\bar{1}}.$$

$$(2.2.3) \quad (iii) \quad B([x, y], z) = B(x, [y, z]).$$

The Lie superalgebra is then called (*basic*) *classical* if

2.3.a)  $G$  is simple.

2.3.b)  $G_{\bar{0}}$  is reductive.

2.3.c) There exists a non-degenerate invariant bilinear form  $B$  on  $G$ .

The list of (basic) classical Lie superalgebras is the following as determined by Kac :

2.4.1) Simple Lie algebras.

2.4.2)  $A(m, n), B(m, n), C(n), D(m, n), D(2, 1, \alpha), F(4)$ , and  $G(3)$ .

We will describe the elements of the list in 2.4.2) in greater detail as we proceed.

From now on we assume that  $G$  is a (basic) classical Lie superalgebra. We let  $H$  denote a fixed Cartan subalgebra of  $G_{\bar{0}}$ , we let, as usual,  $\Delta$  denote the set of roots of  $H$  in  $G$ , and for  $\gamma \in \Delta$  we let  $h_\gamma \in H$  be determined by  $\gamma(h) = B(h_\gamma, h), h \in$

$H$ . Furthermore we let  $G_\gamma$  denote the space of root vectors corresponding to  $\gamma$ . Unless  $G = A(1, 1)$ , this space is 1 dimensional.

Given a Borel subalgebra  $B_{\bar{0}}$  of  $G_{\bar{0}}$ , containing  $H$ , fix a Borel subalgebra  $B$  of  $G$  containing  $B_{\bar{0}}$ . We then have  $B = H \oplus N^+$ , and

$$(2.5.) \quad G = N^- \oplus H \oplus N^+,$$

where  $N^\pm$  are  $\text{ad}(H)$ -invariant subalgebras. We write  $N^\pm = \sum_{\gamma \in \Delta^\pm} G_\gamma$  and define

$\Delta^\pm$  to be the set of positive and negative roots, respectively. For later use we let

$$(2.6) \quad \rho = \frac{1}{2} \sum_{\gamma \in \Delta_{\bar{0}}^+} \gamma - \frac{1}{2} \sum_{\gamma \in \Delta_{\bar{1}}^+} \gamma,$$

where the subscript on  $\Delta^+$  refers to the decomposition of  $G$  into even and odd. If we define a positive root  $\gamma$  to be *simple* if it cannot be decomposed into a sum of two positive roots, then the crucial property satisfied by  $\rho$  is

$$(2.7) \quad (\rho, \gamma) = \frac{1}{2} B(\gamma, \gamma) \quad \text{for all } \gamma \text{ simple.}$$

(The form  $(\cdot, \cdot)$  on  $H^*$  is defined by  $(\gamma, \beta) = B(h_\gamma, h_\beta)$  for  $\gamma, \beta \in \Delta$ .)

Finally, we mention that the Poincaré-Birkhoff-Witt Theorem in the present context states that the enveloping algebra  $U(G)$  satisfies

$$(2.8) \quad U(G) = \bigwedge (G_{\bar{1}}) \otimes U(G_{\bar{0}}).$$

Naturally, the definition of  $U(G)$  is as the quotient

$$(2.9) \quad T(G)/R$$

of the tensor algebra  $T(G)$  by the 2-sided ideal  $R$  generated by the elements

$$(2.10) \quad [x, y] - x \otimes y + (-1)^{(\text{deg } x)(\text{deg } y)} y \otimes x.$$

One of the main ingredients in what follows is that of an *anti-linear anti-involution*  $\omega$  of  $G$ . This is, by definition an anti-linear map

$$\omega : G \rightarrow G$$

satisfying  $\omega^2 = 1$  and

$$(2.11) \quad \forall x, y \in G : \quad \omega([x, y]) = [\omega(y), \omega(x)].$$

Furthermore we will always assume that  $\omega$  is *consistent*, that is,

$$(2.12) \quad \forall \gamma \in \Delta : \omega(G_\gamma) = G_{-\gamma}.$$

We will always extend  $\omega$  to  $U(G)$  in the obvious way.

It follows that one can choose a basis  $\{e_\gamma\}_{\gamma \in \Delta}$  of  $N^- \oplus N^+$  such that

$$(2.13) \quad \forall \gamma \in \Delta : \omega(e_\gamma) = \varepsilon_\gamma \cdot e_{-\gamma},$$

where  $\varepsilon_\gamma = \pm 1$  for all  $\gamma \in \Delta$ . We will say that  $\omega$  is *compact* if for all even roots,  $\varepsilon_\gamma = 1$ . We then have the following somewhat surprising fact.

PROPOSITION 2.1. *If  $G_{\bar{1}}$  is an irreducible  $G_{\bar{0}}$  module, then there are no compact consistent anti-linear anti-involutions.*

*Proof (sketched).* First assume that there is an even root  $\gamma_0$  and an odd root  $\gamma_1$  such that

$$(2.14) \quad [e_{\gamma_0}, e_{\gamma_1}] = e_{-\gamma_1}.$$

Let  $h_0 = [e_{\gamma_0}, e_{-\gamma_0}]$ , then, letting  $h_0$  act on both sides of (2.14), it follows that

$$(2.15) \quad [h_0, e_{\gamma_1}] = -e_{\gamma_1}.$$

If we then apply  $\omega$  to both sides of (2.14) it follows that

$$(2.16) \quad \varepsilon_{\gamma_0} \cdot \varepsilon_{\gamma_1} \cdot [e_{-\gamma_1}, e_{-\gamma_0}] = \varepsilon_{\gamma_1} \cdot e_{\gamma_1},$$

from which it follows, using (2.14) and (2.15), that  $\varepsilon_{\gamma_0} = -1$ .

In the general case, given an odd root vector  $e_{\gamma}^{odd}$ , one can always, because of the irreducibility assumption, find even root vectors  $e_{\gamma_1}, \dots, e_{\gamma_r}$  such that

$$[e_{\gamma_1}, [e_{\gamma_2}, [\dots, [e_{\gamma_r}, e_{\gamma}^{odd}] \dots]]] = e_{-\gamma}^{odd}.$$

Furthermore, it follows as above that  $\varepsilon_{\gamma_1} \cdot \varepsilon_{\gamma_2} \cdots \varepsilon_{\gamma_r} = (-1)^r$ . The proof is then completed if we can show that there always is a configuration as above, where, furthermore,  $r$  is **odd**. This follows by an easy examination of the various cases.  $\square$

DEFINITION 2.2. *Let  $\Lambda \in H^*$ . A highest weight module  $V_{\Lambda}$  for  $U(G)$  of highest weight  $\Lambda$  is then a left  $U(G)$  module containing a (non-zero) vector  $v_{\Lambda}$  such that*

$$\begin{aligned} N^+ \cdot v_{\Lambda} &= 0, \\ \forall h \in H : h \cdot v_{\Lambda} &= \Lambda(h) \cdot v_{\Lambda}, \\ V_{\Lambda} &= U(G) \cdot v_{\Lambda}. \end{aligned}$$

DEFINITION 2.3. *A representation  $\pi$  of  $U(G)$  on a complex vector space  $V$  will be called unitarizable with respect to a given anti-linear anti-involution  $\omega$  if there exists a hermitian form  $\mathcal{H}$  on  $V$  such that*

- (1)  $\forall v_1, v_2 \in V, \forall x \in G : \mathcal{H}(\pi(x)v_1, v_2) = \mathcal{H}(v_1, \pi(\omega(x))v_2)$ .
- (2)  $\mathcal{H}$  is positive (semi-)definite.

We remark in this connection that the representation space  $V$  above is assumed to be graded,  $V = V_0 \oplus V_1$ , and  $\pi(x_i)V_j \subseteq V_{i+j}$ .

In the case of a highest weight module  $V_{\Lambda}$  we have  $V_{\Lambda} = U(N^-) \cdot v_{\Lambda}$ . Hence, for a given  $\omega$  (always assumed consistent), it follows from Poincaré-Birkoff-Witt ([16]) that there is a hermitian form  $\mathcal{H}$  satisfying (1), and it is unique up to multiplication by a non-zero real number.

The following definition of unitarity is perhaps more standard:

DEFINITION 2.3'. Let  $\tilde{\omega}$  be an anti-linear involution and let  $\pi$  be a representation on  $V$ . A form  $\tilde{H}$  is then called *contravariant* if

$$\begin{aligned} \forall x \in G, \forall v_a, v_b \in V : \\ \tilde{H}(\pi(x)v_a, v_b) &= -(-1)^{(\deg x)(\deg v_a)} \tilde{H}(v_a, \pi(\omega(x))v_b), \\ \tilde{H}(v_a, v_b) &= (-1)^{(\deg v_a)(\deg v_b)} \overline{\tilde{H}(v_b, v_a)}, \text{ and} \\ \tilde{H}(V_0, V_1) &= 0. \end{aligned}$$

A contravariant form  $\tilde{H}$  is called (*super-*)*positive* if it is positive definite on  $V_0$  and  $i \cdot$  (positive definite) (or  $-i \cdot$  (positive definite)) on  $V_1$ . In this case  $\pi$  is called *unitarizable*.

Actually, it is easy to see that unitarity in the sense of Definition 2.3' implies unitarity in the sense of Definition 2.3:

Let us write vectors  $v_a \in V$  as  $v_a = \begin{pmatrix} v_a^0 \\ v_a^1 \end{pmatrix}$ . Then the pair  $H, \omega$ , where

$$(2.17) \quad \begin{aligned} H(v_a, v_b) &= \tilde{H} \left( \begin{pmatrix} v_a^0 \\ v_a^1 \end{pmatrix}, \begin{pmatrix} v_b^0 \\ i \cdot v_b^1 \end{pmatrix} \right), \\ \omega \left( \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \right) &= \begin{pmatrix} -\omega(x_0) \\ -i\omega(x_1) \end{pmatrix}, \end{aligned}$$

makes  $\pi$  unitary in the sense of Definition 2.3.

In the case of highest weight modules, the converse follows easily, but in general it seems that one must add the condition that  $V_0 \perp V_1$  to Definition 2.3 to get equivalence with Definition 2.3'.

Still following Kac, we now introduce the fundamental bilinear form

$$A : U(G) \otimes U(G) \rightarrow U(H) :$$

Let  $\sigma$  be a consistent anti-linear anti-automorphism. As for simple Lie algebras we have a decomposition

$$(2.18) \quad U(G) = U(H) \oplus (N^-U(G) + U(G)N^+).$$

We let  $\beta$  denote the projection

$$(2.19) \quad \beta : U(G) \rightarrow U(H)$$

defined by this, and we set

$$(2.20) \quad A(x, y) = \beta(\sigma(x)y) \quad x, y \in U(G).$$

We have a weight decomposition of  $U(N^-)$  with respect to  $H$ :

$$(2.21) \quad U(N^-) = \bigoplus_{\eta \in H^*} U(N^-)^{-\eta}.$$

Let  $A_\eta$  be the restriction of  $A$  to  $U(N^-)^{-\eta}$ .

The following formula for  $\det A_\eta$  has been communicated to us by V. Kac. It is slightly different from the one in [16]. Actually, it is quite straightforward to deduce the present formula in analogy with [18]. One simply has to define the concept of the partition function  $P(\gamma)$  appropriately. Namely,  $P(\gamma)$  is the number of ways that  $\gamma$  can be written as a sum

$$\gamma = \gamma_1 + \cdots + \gamma_n$$

of positive roots  $\gamma_i$  subject to the constraint that if  $\gamma_i \in \Delta_1^+$  then  $\gamma_j \neq \gamma_i$  for all  $j \neq i$ . That this constraint is necessary and sufficient follows immediately from the anti-commutativity of  $\Lambda(G_1^-)$  inside  $U(G)$ .

**THEOREM 2.4 (KAC).** *Let  $\overline{\Delta}_1 = \{\gamma \in \Delta_1 \mid 2\gamma \notin \Delta_0\}$ .*

$\det A_\eta = A \cdot B \cdot C$ , where

$$\begin{aligned} A &= \prod_{n \in \mathbb{N}} \prod_{\gamma \in \Delta_0^+} (h_\gamma + (\rho, \gamma) - n(\gamma, \gamma)/2)^{P(\eta - n\gamma)}, \\ B &= \prod_{n \in 1+2\mathbb{Z}_+} \prod_{\gamma \in \Delta_1^+ \setminus \overline{\Delta}_1^+} (h_\gamma + (\rho, \gamma) - n(\gamma, \gamma)/2)^{P_\gamma(\eta - n\gamma)}, \text{ and} \\ C &= \prod_{\gamma \in \overline{\Delta}_1^+} (h_\gamma + (\rho, \gamma))^{P_\gamma(\eta - \gamma)}. \end{aligned}$$

Here,  $P_\gamma(\gamma)$  is the number of partitions of  $\gamma$ , as above, which furthermore do not involve  $\gamma$ .

The following follows analogously.

**THEOREM 2.5.** *If a highest weight module  $V_\Lambda$  has a subquotient isomorphic to the unique irreducible highest weight module  $V(\Lambda^1)$  of weight  $\Lambda^1$  then there exists a chain*

$$\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_k = \Lambda^1$$

such that  $\Lambda_{i+1} = \Lambda_i - n_i \beta_i$ , where  $\beta_i \in \Delta^+$ ,  $n_i$  is a positive integer, and  $2(\Lambda_i + \rho, \beta_i) = n_i(\beta_i, \beta_i)$ . Furthermore, if  $\beta_i \in \Delta_1^+$  then  $\forall j \neq i : \beta_j \neq \beta_i$ , and if  $\beta_i \in \overline{\Delta}_1^+$ , then  $n_i = 0$ .

In most cases that follow there will be a compact subalgebra  $K_0$  of  $G_{\bar{0}}$  and a corresponding decomposition

$$\Delta_0 = \Delta_0^{K_0} \cup \Delta_0^P,$$

where  $P$  denotes the so-called non-compact roots. Moreover, the highest weight representations will be defined by highest weights which define, when restricted to  $K_0$ , finite dimensional representations. In this case, and in analogy with [14], Theorem 2.5 can be expanded further:

**COROLLARY 2.6.** *If a highest weight module  $V_\Lambda$  has a subquotient isomorphic to the unique irreducible highest weight module  $V(\Lambda^1)$  of weight  $\Lambda^1$  and if  $\Lambda^1$  defines a finite dimensional  $K_0$  representation, then, in the chain*

$$\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_k = \Lambda^1,$$

we may assume that  $\forall_i : \beta_i \in (\Delta_0^P)^+ \cup \Delta_1^+$ .

*Proof.* This follows just as in the case of Lie algebras, in fact we will use the result of [14] and only consider the following two cases in which  $\beta_i = \gamma$  is compact and  $\beta_{i+1} = \beta$  is odd. First we assume that  $2\beta \notin \Delta_0$ . Then

$$(2.22) \quad \Lambda_{i+2} = \Lambda_{i+1} - \beta = \left( \Lambda_i - \frac{2(\gamma, \Lambda_i)}{(\gamma, \gamma)} \Lambda_i \right) - \beta,$$

where  $(\Lambda_{i+1}, \beta) = 0$  and  $n_i = \frac{2(\gamma, \Lambda_i)}{(\gamma, \gamma)}$ .

We wish to replace the pair  $(\gamma, \beta)$  by a pair  $(\beta^1, \gamma^1)$  where  $\beta^1$  is odd and  $\gamma^1$  is compact.

Since

$$(2.23) \quad \Lambda_{i+2} = (S_\gamma \Lambda_i) - \beta = S_\gamma (\Lambda_i - S_\gamma \beta),$$

it is natural to try  $(\beta^1, \gamma^1) = (S_\gamma \beta, \gamma)$ , and, indeed, this choice works: We need to show that  $S_\gamma \beta$  still is positive and odd. This follows by a case by case analysis that we omit. Further,

$$(2.24) \quad (\Lambda_i, S_\gamma \beta) = (S_\gamma \Lambda_i, \beta) = 0$$

and

$$(2.25) \quad \frac{2(\gamma, \Lambda_i - S_\gamma \beta)}{(\gamma, \gamma)} = n_i + \frac{2(\gamma, \beta)}{(\gamma, \gamma)},$$

and again by a case by case analysis,  $2(\gamma, \beta)/(\gamma, \gamma)$  in absolute value is at most 1 so the above quantity is in  $\mathbb{N} \cup \{0\}$ . Naturally, in case  $2(\gamma, \beta)/(\gamma, \gamma) = -1$  we simply delete the  $\gamma^1 = \gamma$  from the list.

In the case where  $2\beta \in \Delta_0$ , we are dealing with the Lie superalgebra  $G = B(m, n)$ . In this case it is straightforward to see how to replace  $(\gamma, \beta)$  by either a pair  $(\beta^1, \gamma^1)$  with  $\beta^1$  odd and  $\gamma^1$  compact or  $\beta^1$  in  $(\Delta_0^P)^+$  and  $\gamma^1$  odd. We omit the details.

The conclusion of this analysis is that it is possible to push the  $\beta_i$ 's corresponding to  $(\Delta_0^{K_0})^+$  towards the end of the sequence, or to replace them by other  $\beta$ 's from  $(\Delta_0^P)^+ \cup \Delta_1^+$ . However, it is easy to see that in any chain  $\beta_0, \dots, \beta_{k-1}$  satisfying the positivity condition,  $\beta_{k-1}$  cannot be compact. To wit,

$$(2.26) \quad S_{\beta_{k-1}} \Lambda_{k-1} = \Lambda_k = \Lambda^1$$

and if  $(\beta_{k-1}, \Lambda_{k-1}) > 0$  then evidently

$$(2.27) \quad (\beta_{k-1}, \Lambda_k) < 0$$

and hence  $\Lambda^1$  cannot define a finite dimensional representation. Thus, the rearrangement of the chain by pushing the compact roots towards the end will eventually result in a chain where all compact roots have been removed.

One of the most important cases is when  $G = A(m, n)$  and  $K_0 = su(m+1)^{\mathbb{C}} \oplus su(n+1)^{\mathbb{C}} \oplus \mathbb{C}$ . In this case we will be interested in sequences  $\beta_0, \dots, \beta_{k-1}$  that define chains

$$(2.28) \quad \Lambda = \Lambda_0, \dots, \Lambda_k = \Lambda^1$$

between highest weights that both define  $K_0$  dominant modules (i.e. such that the representation defined by  $U(K_0) \cdot v_{\Lambda}$  is finite dimensional).

**COROLLARY 2.7.** *In the above case for  $G = A(m, n)$  we may assume that all the  $\beta_i$ 's are odd (and pairwise different).*

We shall see that almost everything follows from the description of  $A(m, n)$ . For this purpose, Corollary 2.7 is the most important tool.

## BACKGROUND RESULTS

We present here, without proofs, two sets of results which will be of the utmost importance. The first deals with the classification of the unitarizable highest weight representations of the simple Lie algebras  $A_n, B_n, C_n$ , and  $D_n$ . Here, the reader may consult [14] for the proofs of the full classification (including  $E_6$  and  $E_7$ ). The other is the so-called Littlewood-Richardson rule which describes the decomposition of tensor products of  $SU(n)$  representations into irreducibles.

**I. Unitarizable highest weight representations of  $A_n, B_n, C_n$ , and  $D_n$  corresponding to Hermitian symmetric spaces of the non-compact type.**

We give here the list of such representations for the algebras  $su(p, q)$ ,  $sp(n, \mathbb{R})$ ,  $so^*(2n)$ ,  $so(2n-1, 2)$ , and  $so(2n-2, 2)$ . We denote by  $\Delta_c$  and  $\Delta_n$  the sets of compact and non-compact roots, respectively. For a fixed representation of the semi-simple part of a fixed maximal compact subalgebra, we list the allowed range of the "non-compact parameter".

$su(p, q)$ . Let  $e_1, \dots, e_{p+q}$  be the standard basis of  $\mathbb{R}^{p+q}$ . Then

$$(3.1) \quad \begin{aligned} \Delta_c^+ &= \{e_i - e_j \mid 1 \leq i < j \leq p \text{ or } p+1 \leq i < j \leq p+q\}. \\ \Delta_n^+ &= \{e_i - e_j \mid 1 \leq i \leq p, p+1 \leq j \leq p+q\}. \\ \rho &= \left( \frac{p+q-1}{2}, \frac{p+q-3}{2}, \dots, -\frac{(p+q-1)}{2} \right). \end{aligned}$$

Consider a highest weight representation  $V_\Lambda$  of highest weight

$$(3.2) \quad \Lambda = (\lambda/2, \lambda/2 - a_2, \dots, \lambda/2 - a_p, -\lambda/2 + b_1, \dots, -\lambda/2 + b_q),$$

where the  $a_i$ 's are positive integers satisfying

$$(3.3) \quad 0 \leq a_2 \leq \dots \leq a_p,$$

and the  $b_j$ 's are integers satisfying

$$(3.4) \quad b_1 \geq b_2 \geq \dots \geq b_{q-1} \geq b_q = 0.$$

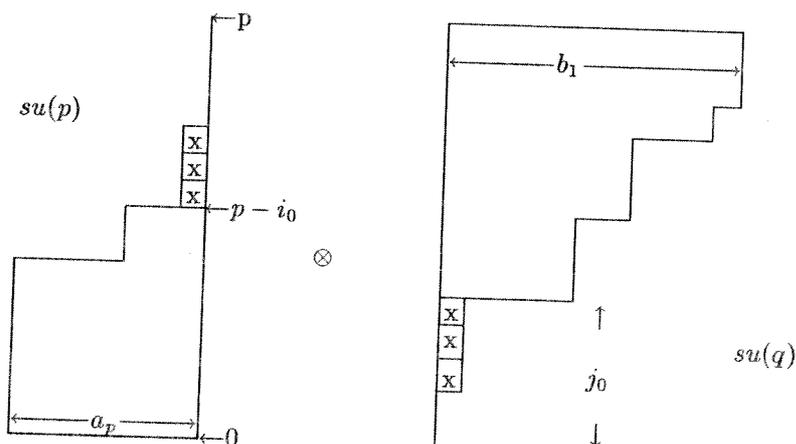


Fig. 3.5

Let  $i_0$  be the biggest index for which  $a_i = 0$  and let  $j_0$  be the smallest index for which  $b_{q-j_0} \neq 0$  (cf. Fig. 3.5). (If  $b_1 = 0$  then let  $j_0 = q$ .)

(The configurations in Fig. 3.5 are examples of Young diagrams, see II later in this chapter for further details.)

Then there is unitarity for

$$\begin{aligned}
 (3.6) \quad & \lambda = -(p+q) + (i_0 + j_0), \\
 & \lambda = -(p+q) + (i_0 + j_0) - 1, \\
 & \vdots \\
 & \lambda = -(p+q) + (i_0 + j_0) - (\tau - 1), \quad \tau = \min(i_0, j_0), \\
 & \text{and for } \lambda < -(p+q) + (i_0 + j_0) - (\tau - 1).
 \end{aligned}$$

At the discrete point  $\lambda = -(p+q) + (i_0 + j_0) - i$  there is a non-trivial submodule contained in the kernel of the hermitian form generated by an  $(i+1)$ th order polynomial. We have indicated, for  $i = 2$ , the 3rd order polynomial that vanishes (i.e. is in the kernel). At general  $i$ 's the picture is similar, the only difference is that the columns each contain  $i+1$  boxes. These diagrams also describe the homomorphisms into the given representations, resulting from the degeneracy of the Hermitian form. See [14] and [15] for further details.

$sp(n, \mathbb{R})$ . Let  $e_1, \dots, e_n$  be the standard bases of  $\mathbb{R}^n$ . Then

$$(3.7) \quad \begin{aligned} \Delta_c^+ &= \{e_i - e_j \mid 1 \leq i < j \leq n\}, \\ \Delta_n^+ &= \{e_i + e_j \mid 1 \leq i \leq j \leq n\}, \\ \rho &= (n, n-1, \dots, 1). \end{aligned}$$

Consider a highest weight representation  $V_\Lambda$  of highest weight

$$(3.8) \quad \Lambda = (\lambda, \lambda - a_2, \dots, \lambda - a_n),$$

where the  $a_i$ 's are integers satisfying  $0 \leq a_2 \leq \dots \leq a_n$ .

Let  $i_0$  be the smallest index for which  $a_{n-i_0+1} = 0$ . If all the  $a_i$ 's are zero, let  $i_0 = 1$ . If furthermore  $a_{n-i_0} = 1$ , let  $j_0$  be the smallest index for which  $a_{n-j_0+1} = 1$ . Otherwise let  $j_0 = i_0$ . Then there is unitarity for

$$(3.9) \quad \begin{aligned} \lambda &= \frac{1}{2}(2 - i_0 - j_0), \\ \lambda &= \frac{1}{2}(2 - i_0 - j_0) - \frac{1}{2}, \\ &\vdots \\ \lambda &= \frac{1}{2}(2 - i_0 - j_0) - \frac{1}{2}(n - i_0), \end{aligned}$$

and for  $\lambda < \frac{1}{2}(2 - i_0 - j_0) - \frac{1}{2}(n - i_0)$ .

At the discrete point  $\lambda = \frac{1}{2}(1 - i_0 - j_0) - \frac{\pi}{2}$  there is a non-trivial submodule contained in the kernel of the hermitian form generated by and  $(r+1)$ th order polynomial. Cf. Fig. 3.10:

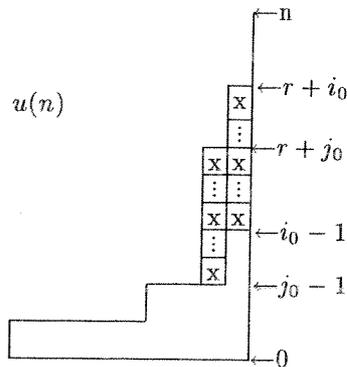


Fig. 3.10

so\*(2n). Let  $e_1, \dots, e_n$  be the standard basis of  $\mathbb{R}^n$ . Then

$$(3.11) \quad \begin{aligned} \Delta_c^+ &= \{e_i - e_j \mid 1 \leq i < j \leq n\}, \\ \Delta_n^+ &= \{e_i + e_j \mid 1 \leq i < j \leq n\}, \\ \rho &= (n-1, n-2, \dots, 0). \end{aligned}$$

Consider a highest weight representation  $V_\Lambda$  of highest weight (a little non-standard)

$$(3.12) \quad \Lambda = (\mu, \mu - a_2, \dots, \mu - a_n),$$

where the  $a_i$ 's are integers satisfying

$$(3.13) \quad 0 \leq a_2 \leq \dots \leq a_n.$$

Set  $a_1 = 1$  and let  $i_0$  be the smallest index for which  $a_{n-i_0+1} = 0$ . If  $i_0 = n$  (then  $a_2 \neq 0$ ) let  $j_0$  be the smallest index for which  $a_{n-j_0+1} = a_2$ . Then there is unitarity

a) If  $i_0 = n$  for

$$(3.14) \quad \mu \leq \frac{3 + a_2 - n - j_0}{2}.$$

b) If  $i_0 < n$  for

$$(3.15) \quad \begin{aligned} \mu &= 1 - i_0, \\ \mu &= 1 - i_0 - 1, \\ &\dots \\ \mu &= 1 - i_0 - r, \quad r = \max\{i \in \mathbb{N} \mid i_0 + 2i + 1 \leq n\}. \end{aligned}$$

At the discrete point  $\mu = 1 - i_0 - 2i$  there is a non-trivial submodule contained in the kernel of the hermitian form generated by an  $(i + 1)$ th order polynomial, cf. Fig. 3.16:

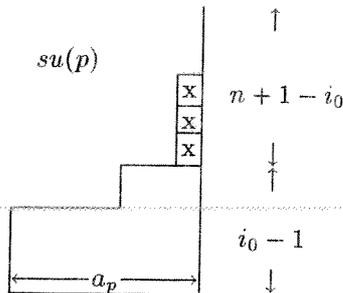


Fig. 3.16

$so(2n-1, 2)$ . Let  $e_1, \dots, e_n$  be the standard basis of  $\mathbb{R}^n$ . Then

$$(3.17) \quad \begin{aligned} \Delta_c^+ &= \{e_i \pm e_j \mid 2 \leq i < j \leq n\} \cup \{e_i \mid 2 \leq i \leq n\}, \\ \Delta_n^+ &= \{e_1 \pm e_j \mid 2 \leq j \leq n\} \cup \{e_1\}, \\ \rho &= \left( \frac{2n-1}{2}, \frac{2n-3}{2}, \dots, \frac{1}{2} \right). \end{aligned}$$

Consider a highest weight representation  $V_\Lambda$  of highest weight

$$(3.18) \quad \Lambda = (\lambda - a_2, a_2, \dots, a_n),$$

where  $(a_2, \dots, a_n) \subseteq \mathbb{Z}$  or  $(a_2, \dots, a_n) \subseteq \mathbb{Z} + 1/2$  satisfy

$$(3.19) \quad a_2 \geq a_3 \geq \dots \geq a_n \geq 0.$$

Let  $i_0$  be the smallest index for which  $a_{i_0} \neq a_2$ .

(3.20.a) If  $i_0 < n$  or, if  $i_0 = n$  and  $a_n \geq 1$ , there is unitarity for

$$\lambda \leq -2n + i_0.$$

(3.20.b) If  $i_0 = n$  and  $a_n = 1/2$  there is unitarity for

$$\lambda \leq 3/2 - n.$$

(3.20.c) If  $i_0 = n$  and  $a_2 = 0$  there is unitarity for

$$\lambda \leq -n + 3/2.$$

In the cases a) and b) the endpoint of the interval is a place where a first order polynomial is vanishing. In case c, at the endpoint a second order polynomial is vanishing. In this case, actually the place where a first order polynomial is vanishing is also unitary since it corresponds to the trivial representation,  $\lambda = 0$ .

$so(2n-2, 2)$ . Let  $e_1, \dots, e_n$  be the standard basis of  $\mathbb{R}^n$ . Then

$$(3.21) \quad \begin{aligned} \Delta_c^+ &= \{e_i \pm e_j \mid 2 \leq i < j \leq n\}, \\ \Delta_n^+ &= \{e_1 \pm e_j \mid 2 \leq j \leq n\}, \\ \rho &= (n-1, n-2, \dots, 0). \end{aligned}$$

Consider a highest weight representation  $V_\Lambda$  of weight

$$(3.22) \quad \Lambda = (\lambda - a_2, a_2, \dots, a_n),$$

where  $(a_2, \dots, a_n) \subseteq \mathbb{Z}$  and

$$(3.23) \quad a_2 \geq a_3 \geq \dots \geq |a_n|.$$

Let  $i_0$  be the smallest index for which  $|a_{i_0}| \neq a_2$ . If  $a_2 = |a_n|$  put  $i_0 = n + 1$ .  
 (3.24.a) If  $a_2 \neq 0$  there is unitarity for

$$\lambda \leq -2n + i_0 + 1.$$

(3.24.b) If  $a_2 = 0$  there is unitarity for

$$\lambda \leq 2 - n.$$

In case a) the endpoint corresponds to the vanishing of a first order polynomial and in b) the endpoint corresponds to the vanishing of a second order polynomial. As for  $so(2n - 1, 2)$ , the trivial representation  $\lambda = a_2 = 0$  corresponds to the vanishing of a first order polynomial.

**II. The Littlewood—Richardson rule.** The Littlewood-Richardson rule (henceforth the “LR-rule”) describes the decomposition of the  $\otimes$ -product of two irreducible unitary representations of  $SU(n)$ . For further details, see [11]. Let  $\tau$  be such a representation of  $SU(n)$ , and let  $\Lambda_\tau$  denote its highest weight. Then we may represent  $\Lambda_\tau$  as

$$(3.25) \quad \Lambda_\tau = (a_1, \dots, a_n),$$

where the  $a_i$ 's are integers satisfying

$$(3.26) \quad a_1 \geq a_2 \geq \dots \geq a_n.$$

Let  $c$  be an arbitrary integer. Then  $\tilde{\Lambda}_\tau = (a_1 + c, \dots, a_n + c)$  and  $\Lambda_\tau$  describe the same representation of  $SU(n)$ . For this reason we will frequently assume that  $a_n = 0$ . However, in what follows there will appear, quite naturally, representations where in a convenient parametrization,  $a_n \neq 0$ . Furthermore, it often pays to treat the representations as representations of  $U(n)$ . In this case, the various  $(c, \dots, c)$ 's of course are inequivalent and correspond to the representations  $u \rightarrow (\det u)^c$ .

Let now  $\Lambda = \Lambda_\tau = (a_1, \dots, a_n)$  be given as above and assume that  $a_n \geq 0$ . Out of this data, the *Young diagram* of  $\tau$  is constructed as a configuration of boxes making up  $n$  rows on top of each other in such a way that row number  $i$  (counting from the top) contains  $a_i$  boxes, and such that the left-most boxes in each row are right on top of each other.

**Example 3.1.**

$$(6, 4, 4, 3, 1, 1) \equiv \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & & \\ \hline 3 & 3 & 3 & 3 & & \\ \hline 4 & 4 & 4 & & & \\ \hline 5 & & & & & \\ \hline 6 & & & & & \\ \hline \end{array} ; \quad (4, 2, 2, 1, 0, 0) \equiv \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & & \\ \hline 3 & 3 & & \\ \hline 4 & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

Fig. 3.27

Later on we shall be concerned with sequences made up of  $a_1$  1's,  $a_2$  2's,  $\dots$ , and  $a_n$  n's. For this reason we have filled up the diagrams above appropriately.

Let  $\Lambda_a = (a_1, \dots, a_n)$  and  $\Lambda_b = (b_1, \dots, b_n)$  be given highest weights, and assume that  $a_n \geq 0$  and  $b_n \geq 0$ . We will say that  $\Lambda_b$  contains  $\Lambda_a$ , and write

$$(3.28) \quad \Lambda_b \geq \Lambda_a,$$

if

$$\forall i = 1, \dots, n : b_i \geq a_i.$$

This of course is exactly saying that the diagram of  $\Lambda_b$  contains the diagram of  $\Lambda_a$ .

**DEFINITION 3.2.** Given  $\Lambda_\tau = (a_1, \dots, a_n)$  with  $a_n \geq 0$ , let

$$(3.29) \quad N_\tau = \sum_{i=1}^n a_i.$$

Further, let  $\{i_\alpha\}_{\alpha=1}^{N_\tau}$  be a sequence made up of  $a_1$  1's,  $a_2$  2's,  $\dots$ , and  $a_n$  n's. We will say that this sequence is *good* if all the terms are good. We will say that a term  $i_\alpha$  is *good* if either  $i_\alpha = 1$  or, if  $i_\alpha \neq 1$ , the number of preceding good  $(i_\alpha - 1)$ 's is strictly greater than the number of preceding good  $i_\alpha$ 's. If not then we call it *bad*. A good sequence always starts with a 1.

**Example 3.3.**

$(1, 1, 2, 2, 3, 3)$  is *good*

$(1, 1, 2, 2, 3, 3, 3)$  is *bad* (because of the last 3)

Now let  $\Lambda_{\tau_1} = (a_1, \dots, a_n)$  and  $\Lambda_{\tau_2} = (b_1, \dots, b_n)$  be highest weights and assume that

$$(3.30) \quad a_n = b_n = 0.$$

**THEOREM 3.4. (The LR-Rule).** In the  $\otimes$ -product

$$\tau_1 \otimes \tau_2,$$

an irreducible representation  $\tau_3$ , represented by  $\Lambda_{\tau_3} = (c_1, \dots, c_n)$  ( $c_n \geq 0$ ) occurs if and only if:

- i)  $\Lambda_{\tau_3} \geq \Lambda_{\tau_1}$ ,  $\Lambda_{\tau_3} \geq \Lambda_{\tau_2}$ , and  $\Lambda_{\tau_3}$  contains  $N_{\tau_1} + N_{\tau_2}$  boxes.
- ii) Let  $\Lambda_{\tau_3} \setminus \Lambda_{\tau_1}$  be the part of the Young diagram of  $\Lambda_{\tau_3}$  which is outside that of  $\Lambda_{\tau_1}$ . Then it is possible to fill these  $N_{\tau_2}$  boxes by integers such that:
  - a) There are  $b_1$  1's,  $b_2$  2's,  $\dots$ , and  $b_n$  n's.
  - b) The numbers are non-decreasing along rows.
  - c) The numbers are strictly increasing down columns.
  - d) When reading the sequence from **right to left** in successive rows, the sequence is **good**.

The multiplicity of  $\tau_3$  in  $\tau_1 \otimes \tau_2$  is equal to the number of different such sequences.

**Example 3.5.**

$$\begin{array}{|c|c|c|} \hline x & x & x \\ \hline x & & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} =$$

$$\begin{array}{|c|c|c|c|c|} \hline x & x & x & 1 & 1 \\ \hline x & 2 & & & \\ \hline 2 & & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline x & x & x & 1 & 1 \\ \hline x & 2 & 2 & & \\ \hline & & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline x & x & x & 1 \\ \hline x & 1 & 2 & \\ \hline 2 & & & \\ \hline \end{array} +$$

$$\begin{array}{|c|c|c|c|} \hline x & x & x & 1 \\ \hline x & 2 & & \\ \hline 1 & & & \\ \hline 2 & & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline x & x & x \\ \hline x & 1 & 1 \\ \hline 2 & 2 & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline x & x & x \\ \hline x & 1 & \\ \hline 1 & 2 & \\ \hline 2 & & \\ \hline \end{array} +$$

$$\begin{array}{|c|c|c|c|} \hline x & x & x & 1 \\ \hline x & 1 & & \\ \hline 2 & 2 & & \\ \hline \end{array}$$

Observe that if we view example 3.5 as an example for  $SU(3)$  then two of these diagrams are illegal.

**THE UNITARIZABLE HIGHEST  
WEIGHT MODULES OF  $A(n, m)$ ,  $m \neq n$**

Recall the definition of  $A(m, n)$  for  $m \neq n$  ([16]):

$$(4.1) \quad A(m, n) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M(m+n+2, \mathbb{C}) \mid \begin{array}{l} \alpha \text{ is an } (m+1) \times (m+1), \\ \beta \text{ an } (m+1) \times (n+1), \gamma \text{ an } (n+1) \times (m+1), \\ \delta \text{ an } (n+1) \times (n+1) \text{ matrix, and } \operatorname{tr} \alpha = \operatorname{tr} \delta \end{array} \right\}.$$

We let  $G_0$  denote the subspace of  $A(m, n)$  consisting of all matrices as above with  $\gamma = \beta = 0$ .  $G_{-1}$  is defined analogously to be all matrices where at most  $\gamma \neq 0$ , and  $G_1$  likewise to be all matrices where at most  $\beta \neq 0$ . Thus

$$(4.2) \quad A(m, n) = G_{-1} \oplus G_0 \oplus G_1.$$

The indices on the  $G$ 's define the grading, and the graded Lie bracket is defined to be

$$(4.3) \quad [a, b] = ab - (-1)^{(\deg a)(\deg b)}ba.$$

As a Cartan subalgebra we choose the set of diagonal matrices in  $A(m, n)$  and we choose the standard ordering in which the positive root vectors are the upper triangular matrices.

In the following,  $\omega_c$  will always denote the standard anti-linear anti-involution corresponding to mapping a matrix into its adjoint. Define two anti-linear anti-involutions,  $\omega_+$  and  $\omega_-$ , on  $A(m, n)$  by

$$(4.4) \quad \omega_{\pm} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \omega_c(\alpha) & \pm\omega_c(\gamma) \\ \pm\omega_c(\beta) & \omega_c(\delta) \end{pmatrix}.$$

If a finite dimensional representation of  $A(m, n)$  is to be unitarizable, it must clearly be with respect to either  $\omega_+$  or  $\omega_-$ . In the sequel we shall only treat the case of  $\omega_+$  since the case of  $\omega_-$  is entirely similar.

Let  $v_0$  be the highest weight vector of a finite dimensional representation of  $A(m, n)$ . Then, since  $G_{-1}$  is a  $G_0$  module, the full module has the form  $U(G_0) \cdot \bigwedge(G_{-1}) \cdot v_0$ . To study unitarity it is then enough to examine the inner product on  $\bigwedge(G_{-1}) \cdot v_0$ , but we stress that this is only sufficient because  $G_0$  is required to act by finite dimensional representations.

The action of  $G_0$  on  $G_{-1}$ , is clearly the action inherited from the action of  $SU(m+1) \times SU(n+1)$  on the space  $M(n+1, m+1)$  of all complex  $(n+1) \times (m+1)$  matrices given by

$$(4.5) \quad (u_1, u_2)M = u_2 M u_1^{-1}$$

for  $u_1 \in SU(m+1)$ ,  $u_2 \in SU(n+1)$ , and  $M \in M(n+1, m+1)$ .

Let us from now on assume that  $m > n$ .

PROPOSITION 4.1. *As a representation space of  $SU(m+1) \times SU(n+1)$ ,*

$$\sum_{i=0}^{(n+1) \cdot (m+1)} \bigwedge^i(M(n+1, m+1)) = \sum_{\tau \in \widehat{SU}(n+1)} (\tilde{\tau})' \otimes \tau,$$

where  $\tilde{\tau}$  denotes the representation of  $SU(m+1)$  whose Young diagram is obtained from that of  $\tau$  by taking the mirror image in the diagonal, where we set  $(\tilde{\tau})' \otimes \tau$  to zero if  $\tilde{\tau}$  contains more than  $m+1$  rows (i.e. when  $\tilde{\tau}$  is too big for  $SU(m+1)$ ), and where  $(\tilde{\tau})'$  denotes the contragredient representation to  $\tilde{\tau}$ . As usual,  $\widehat{SU}(n+1)$  denotes the unitary dual of  $SU(n+1)$ .

*Proof.* Let  $\tau$  be a unitary representation of  $SU(n+1)$ . Let  $e_1, \dots, e_{n+1}$  denote the standard basis of  $\mathbb{C}^{n+1}$ . Then  $\tau$  occurs inside a suitable number of tensor products of the defining representation and the highest weight vector is a suitable sum of permutations (with signs) of the vector  $e_1^{p_1} \otimes e_2^{p_2} \otimes e_3^{p_3} \otimes \dots$ , where  $e_i^{p_i} = e_i \otimes \dots \otimes e_i$  ( $p_i$  copies) and  $p_1 \geq p_2 \geq \dots \geq p_{n+1} \geq 0$ . Let  $\{\gamma_{s,t}\}_{1,1}^{n+1, m+1}$  denote the standard basis of  $M(n+1, m+1)$ . Just to have something corresponding to  $e_1^p$  occur inside  $\bigwedge(G_{-1})$  we must clearly use elements of the form  $\gamma_{1, i_1} \wedge \gamma_{1, i_2} \wedge \dots \wedge \gamma_{1, i_p}$ , and if this is also to correspond to a highest weight vector of  $SU(m+1)$  under the action (4.5), then  $i_p = m+1$ ,  $i_{p-1} = m, \dots$ , and  $i_1 = m+2-p$ . If we bear in mind how the upper triangular matrices of  $SU(n+1)$  act on the  $\gamma_{i,j}$ 's, together with the defining relation  $x \wedge x = 0$ , it is in fact clear that the vector

$$(4.6) \quad \gamma_{1, m+1} \wedge \gamma_{1, m} \wedge \dots \wedge \gamma_{1, m+2-p_1} \wedge \dots \wedge \gamma_{n+1, m+1} \wedge \dots \wedge \gamma_{n+1, m+2-p_{n+1}}$$

is a highest weight vector for  $SU(n+1)$ . Moreover, after rearranging the indices, it is equally obvious that it is also a highest weight vector for  $SU(m+1)$ .

The result then follows directly from (4.6) by using the well-known relationship between  $\tau$  and  $\tau'$ .  $\square$

We now consider a highest weight representation of  $A(n, m)$  defined by a highest weight  $\Lambda$  of the form

$$(4.7) \quad \Lambda = \left( \frac{\alpha}{2}, \frac{\alpha}{2} - a_2, \dots, \frac{\alpha}{2} - a_{m+1}, \frac{\alpha}{2} + b_1, \dots, \frac{\alpha}{2} + b_n, \frac{\alpha}{2} \right),$$

where the  $a_i$ 's and the  $b_j$ 's are integers satisfying  $0 \geq -a_2 \geq \dots \geq -a_{m+1}$  and  $b_1 \geq \dots \geq b_n \geq 0$ . (Here, and in the sequel, we view  $\Lambda$  as a linear functional on the set of all diagonal matrices. The value  $\Lambda(h)$  is then the euclidean inner product between  $\Lambda$  as above and the vector whose coefficients are the diagonal entries of  $h$ .)

**THEOREM 4.2.** *For fixed  $a_2, \dots, a_{m+1}, b_1, \dots, b_n$  as above, let  $j_0$  be the smallest index for which the corresponding  $b_{j_0}$  is zero. If all  $b$ 's are non-zero, set  $j_0 = n + 1$ . Then  $\Lambda = \Lambda(\alpha)$  as above defines a unitarizable finite dimensional representation of  $(A(m, n), \omega_+)$  exactly for*

$$\alpha \in \{ \alpha_0 = a_{m+1} + j_0 - 1, \alpha_0 + 1, \dots, \alpha_0 + n + 1 - j_0 \} \cup ]\alpha_0 + n + 1 - j_0, \infty[.$$

*Proof.* Let  $v_\Lambda$  denote that highest weight vector and let  $V_\Lambda^0$  denote the  $G_0$  module generated by  $v_\Lambda$ . By construction, the hermitian form on  $\bigwedge^i(G_{-1}) \otimes V_\Lambda^0$  is an  $i$ th order polynomial in  $\alpha$  and it is easy to see that the coefficient to  $\alpha^i$  is strictly positive. Thus there is unitarity for  $\alpha$  sufficiently large. Hence, there is a first possible point of non-unitarity (a first "reduction point") which, it should be stressed, need not be the first place (in terms of  $\alpha$  coming from  $+\infty$ ) where the Kac determinant (Theorem 2.1) vanishes, but is the first place at which a  $G_0$  type is missing from  $\bigwedge(G_{-1}) \otimes V_\Lambda^0$ . With this in mind, we now turn to another interesting value of  $\alpha$ , a value we will denote by  $\alpha_0$  and which we shall call "the last possible place of unitarity". Consider  $G_{-1} \otimes V_\Lambda^0$ . Under  $G_0$ , this decomposes into a finite sum of irreducibles, and on each piece the hermitian form is a first order polynomial. Clearly then, the first value at which one of these mutually orthogonal spaces,  $\mu_0^1$ , vanishes, is the last possible place of unitarity in the sense that, below that, the hermitian form is definitely not positive semi-definite. (We remark here, that as  $\alpha$  goes to  $-\infty$ , there will occur a situation in which one can change all signs on the odd spaces and thereby obtain unitarity. However, logically this belongs to the situation for the anti-involution  $\omega_-$ .) Our first objective will be to prove unitarity at the last possible place. For this purpose it suffices to consider the cases in which either all  $a_i$ 's are zero or all  $b_j$ 's are zero. This is so because the general case then follows by tensoring the two mentioned cases together. (Of course, we must still prove that  $\alpha_0 = a_{m+1} + j_0 - 1$  indeed

is the so-called last possible place, but this will be easy). The two cases are analogous, so we will only consider the case in which all the  $a_i$ 's are zero: To begin with, we consider the Kac determinant. The only places where  $\alpha$  occurs are in the factors

$$(4.8) \quad (h_\beta + (\rho, \beta) - n \frac{(\beta, \beta)}{2}), \quad \beta \in \Delta_1^+, \quad n \in 1 + 2\mathbb{Z}_+.$$

Let us, to be specific, choose the inner product  $(\cdot, \cdot)$  such that it is the usual when restricted to  $su(m+1)$  and is minus the usual on  $su(n+1)$ . In the present situation  $(\beta, \beta) = 0$  and if  $h_{\beta_{ij}}$  corresponds to the upper triangular matrix whose only non-zero entry is at the  $(i, m+1+j)$ th place, then  $h_{\beta_{ij}}$  is the diagonal matrix with ones at the  $(i, i)$ th and  $(m+1+j, m+1+j)$ th places. It follows easily from this that the value at  $\Lambda$  of  $(h_\beta + (\rho, \beta))$  is obtained by applying the following  $\Lambda_\rho$  to  $h_\beta$ ,

$$(4.9) \quad \Lambda_\rho = \left( \frac{\alpha}{2} - \frac{(n+1)}{2} + \frac{m}{2}, \frac{\alpha}{2} - \frac{(n+1)}{2} + \frac{m}{2} - 1 - a_2, \dots, \right. \\ \left. \frac{\alpha}{2} - \frac{(n+1)}{2} - \frac{m}{2} - a_{m+1}, \frac{\alpha}{2} + \frac{(m+1)}{2} + \frac{n}{2} + b_1, \right. \\ \left. \frac{\alpha}{2} + \frac{(m+1)}{2} + \frac{n}{2} - 1 + b_2, \dots, \frac{\alpha}{2} + \frac{(m+1)}{2} - \frac{n}{2} \right).$$

The general situation is shown in Fig. 4.10 (in which the number of details as usual is at the absolute minimum):

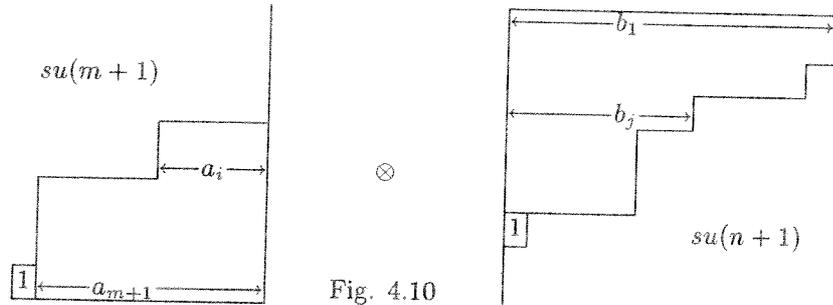


Fig. 4.10

We have indicated by 1's the boxes corresponding to  $\mu_0^1$ , or, equivalently, the  $h_{\beta_{ij}}$  which gives the last possible place of unitarity. It is clear that any other choice of location of the boxes either gives a smaller  $\alpha$  or a diagram which does not correspond to a finite dimensional  $G_0$  representation.

Returning to the special situation where all the  $a_i$ 's are zero, we need to examine those  $G_0$  types in  $\Lambda^i(G_{-1}) \otimes V_\Lambda^0 (i \geq 2)$  which may have vanished above  $\alpha_0$ . Let  $\mu$  be a such. We may assume, with no loss of generality, that  $\mu$  is not in the ideal generated by some other vanishing  $G_0$  type. We want, namely, to prove that  $\mu$  is in the ideal generated by the  $G_0$  type that vanishes at  $\alpha_0$  and hence it suffices to consider the mentioned  $G_0$  types. It follows that  $\mu$  defines a subquotient and hence we can apply Corollaries 2.3 and 2.4 to yield a sequence of roots  $\beta_1, \dots, \beta_r$  from  $\Delta_1^+$  such that if, for  $i = 0, 1, \dots, r$ ,

$$(4.11) \quad \mu_i = \Lambda - \beta_1 - \dots - \beta_i,$$

then  $\mu_r = \mu$  and

$$(4.12) \quad \forall i = 0, \dots, r-1 : (\mu_i + \rho, \beta_{i+1}) = 0.$$

Moreover, the  $\beta_i$ 's are all different. This puts some severe limitations on  $\mu$ : in fact, a few tries are enough to make it clear that any  $\mu$  satisfying the requirements must be of the form as in Fig. 4.13,

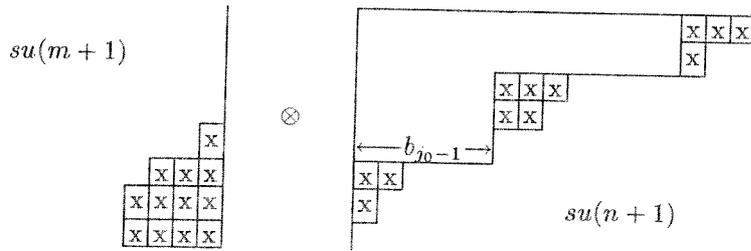


Fig. 4.13

where, at least formally, the corresponding double diagram contains the diagram of  $\mu_0^1$ . Let us also mention that there are indeed some solutions to the problem and these solutions will in fact later be seen to correspond to *bona fide* subquotients:

EXAMPLE 4.3.  $\alpha = \alpha_0 + 3$ :

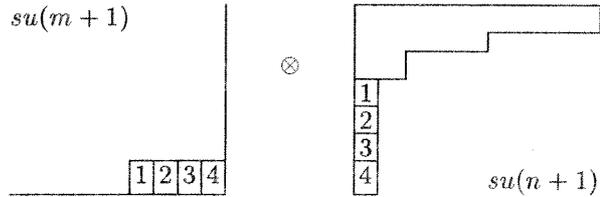


Fig. 4.14

To complete the proof of the theorem, we need to prove the following lemma (in which  $\Lambda$  still is assumed to be of the particular form just considered):

LEMMA 4.4. Any  $G_0$  type  $\mu$  from  $\wedge^i(G_{-1}) \otimes V_\Lambda^0$  whose double diagram contains that of  $\mu_0^1$  is in the ideal generated by  $\mu_0^1$ ,

$$\mu \subset \wedge^{i-1}(G_{-1}) \wedge \mu_0^1.$$

*Proof.* It suffices to prove the lemma for  $\mu$ 's of the form as in Fig. 4.15:

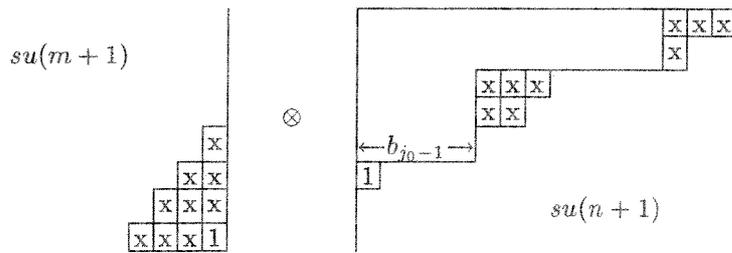


Fig. 4.15

Namely, we may think of a more general  $\mu$  as being obtained from  $V_\Lambda^0$  by wedging a number of times by  $G_{-1}$ . On our way to  $\mu$ , we must then pass through a  $G_0$  type of the simpler form.

Let us then consider the multiplicity inside  $\wedge(G_{-1}) \otimes V_\Lambda^0$  of a  $\mu$  as in Fig. 4.15:

The  $G_0$  module  $V_\Lambda^0$  is of course always an irreducible  $su(m+1) \times su(n+1)$  module and can, as such, be written as

$$(4.16) \quad V_\Lambda^0 = V_{\Lambda_{m+1}}^0 \otimes V_{\Lambda_{n+1}}^0,$$

where  $\Lambda_{m+1}$  and  $\Lambda_{n+1}$  are dominant integral weights for  $su(m+1)$  and  $su(n+1)$ , respectively. We identify these with the corresponding irreducible unitary representations. In the present situation,  $\Lambda_{m+1}$  is the trivial representation 1. It thus follows from Proposition 4.1 that there is a unique  $\tau \in \widehat{su}(n+1)$  such that

$$(4.17) \quad \mu \subseteq (\tilde{\tau}' \otimes 1) \otimes (\tau \otimes \Lambda_{n+1}).$$

Naturally, we can also write  $\mu$  as an  $su(m+1) \times su(n+1)$  module,

$$(4.18) \quad \mu = \Lambda_{m+1}(\mu) \otimes \Lambda_{n+1}(\mu).$$

It follows that the multiplicity of  $\mu$  is equal to the multiplicity of  $\Lambda_{n+1}(\mu)$  in  $\tau \otimes \Lambda_{n+1}$ . This multiplicity can be computed by means of the Littlewood-Richardson rule. Let  $\Lambda_{n+1}^{-1}(\mu)$  denote the diagram that is obtained from that of  $\Lambda_{n+1}(\mu)$  by removing the box labelled 1 in Fig. 4.15. Let, furthermore,  $\tau_1, \dots, \tau_N$  denote the totality of Young-diagrams obtainable from that of  $\tau$  by the removal of a box. (In other words,  $\tau_1, \dots, \tau_N$  are those representations  $\tilde{\tau}$  for which  $\tau \subset \tilde{\tau} \otimes \mathbb{C}^{n+1}$  where  $\mathbb{C}^{n+1}$  symbolizes the defining representation of  $su(n+1)$ ). Then

$$(4.19) \quad m((\tau \otimes \Lambda_{n+1}) : \Lambda_{n+1}(\mu)) = \sum_{j=1}^N m(\tau_j \otimes \Lambda_{n+1} : \Lambda_{n+1}^{-1}(\mu)),$$

where  $m(a : b)$  denotes the multiplicity of  $b$  in  $a$ . Let us now consider the multiplicity of  $\mu$  in  $\Lambda^{i-1}(G_{-1}) \wedge \mu_0^1$ . First observe that because of Proposition 4.1 and because there are no multiplicities in  $\tilde{\tau} \otimes \mathbb{C}^{m+1}$ , evidently

$$(4.20) \quad m(\Lambda^{i-1}(G_0) \otimes \mu_0^1 : \mu) = m(\tau \otimes \Lambda_{n+1} : \Lambda_{n+1}(\mu)).$$

The only problem is that it might be that

$$(4.21) \quad m(\Lambda^{i-1}(G_0) \wedge \mu_0^1 : \mu) < m(\Lambda^{i-1}(G_0) \otimes \mu_0^1 : \mu).$$

To prove that this is *not* the case, consider the highest weight vector

$$(4.22) \quad v_{\mu_0^1} = \gamma_{j_0, m+1} v_{\Lambda} + \sum_{s=1}^{j_0-1} \gamma_{j_0-s, m+1} v_{\Lambda}(s)$$

of  $\mu_0^1$ , where the vectors  $v_{\Lambda}(s)$ ,  $s = 1, \dots, j_0 - 1$ , belong to  $V_{\Lambda}^0$  and are weight vectors of  $su(n+1)$  weight strictly less than that of  $v_{\Lambda}$ . That  $v_{\mu_0^1}$  must have this form follows from the facts that it must contain  $\gamma_{j_0, m+1} v_{\Lambda}$  as a summand and it

must be annihilated by the positive root vectors in  $su(n+1)$  (and  $su(m+1)$ ). Similarly, if  $q$  denotes a  $G_0$  type in  $\bigwedge^{i-1}(G_0) \otimes \mu_0^1$ , then  $q$  must have the form

$$(4.23) \quad q = q_0 \cdot v_{\mu_0^1} + \sum_s (k_s^+ q_0)(k_s^- v_{\mu_0^1}),$$

where  $k_s^-$  denotes a (product of) negative root vector(s) and  $k_s^+$  denotes the analogous (product of) positive root vector(s). The precise form is not important since the only thing we need to ascertain is that the  $v_\Lambda$ -coefficient of  $q$  is

$$(4.24) \quad q|_{v_\Lambda} = q_0 \gamma_{j_0, m+1} v_\Lambda,$$

and this follows immediately. Finally, observe that due to the particular form of  $\Lambda_{n+1}(\mu)$ ,  $q_0$  cannot contain a factor of  $\gamma_{j_0, m+1}$ . Hence  $q$  survives the passage to  $\bigwedge^{i-1}(G_0) \wedge \mu_0^1$ .  $\square$

REMARK 4.5. *The analogous lemma for the case where both  $\Lambda_{n+1}$  and  $\Lambda_{m+1}$  are non-trivial is actually false as can be seen already at the level  $\bigwedge^2(G_{-1}) \otimes V_\Lambda^0$ .*

We can now easily establish the unitarity at the last possible place: By Lemma 4.4, any  $G_0$  type in  $\bigwedge(G_{-1}) \otimes V_\Lambda^0$  that may have changed sign at an  $\alpha > \alpha_0$ , is in the ideal generated by  $\mu_0^1$  and hence vanishes at  $\alpha_0$ .

With this part of the proof finished, let us now turn to the case where both  $\Lambda_{n+1}$  and  $\Lambda_{m+1}$  vanish. We will call this case the “scalar” case. Clearly, at the last possible place of unitarity,  $\alpha_0 = 0$ , and this corresponds to the trivial representation. A straightforward computation now shows that

$$(4.25) \quad \begin{aligned} \mathcal{H}_\alpha(\gamma_{1, m+1} \wedge \cdots \wedge \gamma_{i, m+1}, \gamma_{1, m+1} \wedge \cdots \wedge \gamma_{i, m+1}) = \\ \alpha(\alpha-1) \cdots (\alpha-i), \end{aligned}$$

where the hermitian form is denoted by  $\mathcal{H}_\alpha$ . Similarly,

$$(4.26) \quad \begin{aligned} \mathcal{H}_\alpha(\gamma_{1, m+1} \wedge \cdots \wedge \gamma_{1, m+1-j}, \gamma_{1, m+1} \wedge \cdots \wedge \gamma_{1, m+1-j}) = \\ \alpha(\alpha+1) \cdots (\alpha+j). \end{aligned}$$

If we keep in mind the explicit description of the highest weight vectors in  $\bigwedge(G_{-1})$  as given in the proof of Proposition 4.1, it is easy to see that at  $\alpha = 1$ , there is unitarity in  $\bigwedge(G_{-1})$  and that the  $G_0$  types that are non-zero exactly are those types whose highest weight vectors are of the form

$$(4.27) \quad \gamma_{1, m+1} \wedge \cdots \wedge \gamma_{1, m+1-j}.$$

In diagram form this is shown in Fig. 4.28:

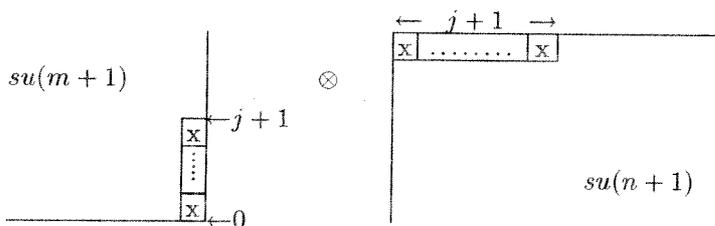


Fig. 4.28

More generally, it follows that at  $\alpha = k \in \mathbb{N}$ , the non-zero  $G_0$  types are as described in Fig. 4.29:

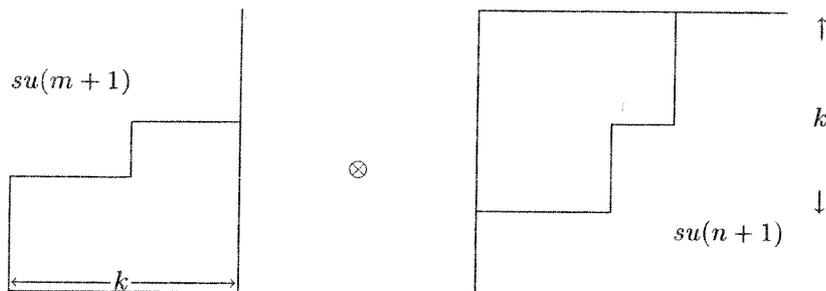


Fig. 4.29

Let  $S_k$  denote the scalar representation corresponding to  $\alpha = k$ . If we form the  $\otimes$ -product between  $\Lambda(\alpha_0)$  and  $S_k$ , where  $\Lambda(\alpha_0)$  denotes an arbitrary representation at its last possible place of unitarity, then  $\Lambda(\alpha_0 + k) \subset \Lambda(\alpha_0) \otimes S_k$  and is unitary. Furthermore, it is clear that for  $k = 1, \dots, n+1-j_0$  (void if  $j_0 = n+1$ ), there are some  $G_0$  types missing from the representation. Specifically, anything of the form given in Fig. 4.30

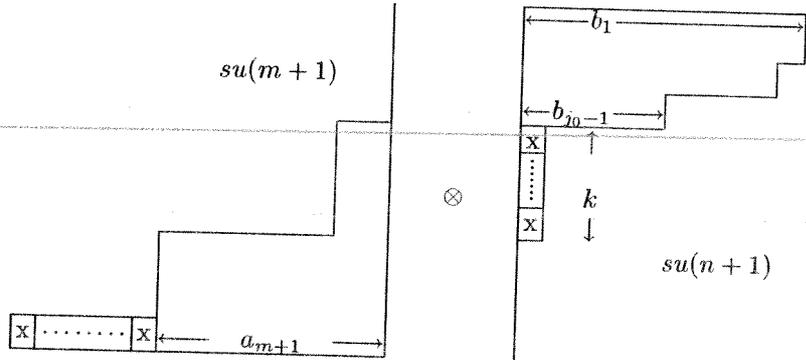


Fig. 4.30

cannot be reached from  $\Lambda(\alpha_0) \otimes S_k$ .

This then, finally, shows that there cannot be unitarity in-between before we reach  $\alpha_0 + n + 1 - j_0$ , because at  $\alpha_0 + (k - 1)$  less than or equal to this, there is a  $k$ th order polynomial which vanishes at  $\alpha_0, \alpha_0 + 1, \dots, \alpha_0 + (k - 1)$ . The value  $\alpha_0 + n + 1 - j_0$  is clearly the first possible place of unitarity, above that nothing can change sign. This, then, completes the proof of Theorem 4.2.

By analogy we obtain the following

**THEOREM 4.6.** For fixed  $a_2, \dots, a_{m+1}, b_1, \dots, b_n$ , let  $i_0$  be the smallest integer for which  $a_{m+2-i_0}$  is zero. If all the  $a_i$ 's are non-zero set  $i_0 = m + 1$ . Set  $\check{\alpha}_0 = -b_1 - i_0 + 1$ . Then  $\Lambda = \Lambda(\alpha)$  defines a unitarizable finite dimensional representation of  $(A(m, n), \omega_-)$  exactly for

$$\alpha \in ] -\infty, \check{\alpha}_0 - (m + 1 - i_0)[ \cup \{ \check{\alpha}_0 - (m + 1 - i_0), \dots, \check{\alpha}_0 - 1, \check{\alpha}_0 \}.$$

**INFINITE DIMENSIONAL UNITARY  
REPRESENTATIONS OF  $A(n, m)$ ,  $m \neq n$**

(i)  $su(p, q) \times su(n + 1)$ .

The most general real form of  $A_m$  having unitarizable highest weight representations is  $su(p, q)$  with  $p = 0, 1, 2, \dots$ ;  $q = 0, 1, 2, \dots$ , and  $p + q = m + 1$ . If both  $p$  and  $q$  are non-zero, the representations are all (apart from the trivial representation) infinite dimensional. The full set of such representations was found in [12], based on the work in [19]. We shall in this chapter examine highest weight representations of  $A(m, n)$  based on this real form on  $A_m$  but with the compact form on  $A_n$ . The case with both  $A_m$  and  $A_n$  non-compact will be examined later.

Let us in this section agree to write elements of  $A(m, n)$  as

$$(5.1) \quad \begin{bmatrix} \alpha_1 & z^+ & p_1 \\ w^- & \alpha_2 & p_2 \\ q_1 & q_2 & \delta \end{bmatrix}.$$

LEMMA 5.1. *There are exactly two anti-linear anti-involutions of  $A(m, n)$ , compatible with the standard ordering, which produce the real form  $su(p, q) \times su(n + 1)$  on the semi-simple part of  $G_0$ , namely*

$$\omega(+, -) \equiv \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}, \quad \omega(-, +) \equiv \begin{pmatrix} + & - & - \\ - & + & + \\ - & + & + \end{pmatrix},$$

where the involution is that of taking adjoints followed by multiplication by  $\pm 1$  as indicated.

*Proof.* If we let  $\varepsilon_1$  be the sign on  $q_1$  and  $\varepsilon_2$  the sign on  $q_2$  then, in somewhat abbreviated form,  $\varepsilon_1 q_1 = \omega(p_1) = \omega([z^+, p_2]) = \varepsilon_2 \cdot (-1)[q_2, w^-] = -\varepsilon_2 q_1$ . Finally, it is easy to see that both  $\omega(+, -)$  and  $\omega(-, +)$  indeed are involutions of  $A(m, n)$ .  $\square$

Let  $Q_1$  and  $Q_2$  denote the spaces in  $G$  corresponding to the variables  $q_1$  and  $q_2$ ; respectively (compare with (0.7)). Let  $P_1$  and  $P_2$  be defined analogously.

There are essentially three choices for the part of  $N^+$  that comes from the odd part, namely

$$(5.2) \quad \begin{aligned} N_{st}^+ &= P_1 \oplus P_2, \\ \tilde{N}_{st}^+ &= Q_1 \oplus Q_2, \text{ and} \\ N_{nst}^+ &= P_1 \oplus Q_2. \end{aligned}$$

Of these, the description of  $\tilde{N}_{st}^+$  follows from that of  $N_{st}^+$  and will not be considered separately.

We now proceed to examine the two remaining cases:

**Subcase  $N_{st}^+$ .**

As we did in Chapter 4, we will examine the positivity of the hermitian form by considering  $\Lambda(G_{-1}) \otimes V_\Lambda^0$ . To begin with, we will consider this module as an  $su(p) \times su(q) \times su(n+1)$  module, but if, and when, we have found that the hermitian form is non-negative on all types, we still need to check that the result defines a unitary representation of  $su(p, q)$ . In fact, in the present situation there are two parameters that can be varied without changing the restriction of  $V_\Lambda^0$  to  $su(p) \times su(q) \times su(n+1)$ , the  $\alpha$  from before, and a  $\lambda$  from  $su(p, q)$ . Our strategy then is to keep  $\lambda$  fixed (and corresponding to a value for which  $V_\Lambda^0$  is unitarizable), and see what restrictions positivity on  $\Lambda(G_{-1}) \otimes V_\Lambda^0$  gives on  $\alpha$ , go back and see if  $\lambda$  makes all types from  $\Lambda(G_{-1}) \otimes V_\Lambda^0$  unitarizable, and, if not, then change  $\lambda$  appropriately, then return to  $\alpha$  for a second look, etc.. This process will eventually stop.

Using the decomposition above we get

$$(5.3) \quad \Lambda(G_{-1}) = \Lambda(Q_1) \wedge \Lambda(Q_2),$$

and the idea now is to use the results of Chapter 4, first on  $\Lambda(Q_2) \otimes V_\Lambda^0$ , and then on  $\Lambda(Q_1) \otimes (\Lambda(Q_2) \otimes V_\Lambda^0)$ . If  $Z^+ = S(z^+)$  denotes the subspace of  $A(m, n)$  generated by the matrices  $z^+$ , then  $[p_1, q_2] \in Z^+$  and, moreover,  $[z^+, q_2] = 0$  so the second application of Chapter 4 also makes sense. (Later on, it also becomes important that  $[z^+, q_1] \in Q_2^+$ .)

For the present purpose it is most practical to write the highest weight  $\Lambda$  as

$$(5.4) \quad \Lambda = (\alpha/2 + \lambda/2 - a_1, \dots, \alpha/2 + \lambda/2 - a_p, \\ \alpha/2 - \lambda/2 - a_{p+1}, \dots, \alpha/2 - \lambda/2 - a_{p+q}, \\ \alpha/2 + b_1, \dots, \alpha/2 + b_{n+1}),$$

where the  $a_i$ 's and the  $b_j$ 's are integers satisfying  $0 = -a_1 \geq \dots \geq -a_p$ ,  $-a_{p+1} \geq \dots \geq -a_{p+q} = 0$ , and  $b_1 \geq \dots \geq b_{n+1} = 0$ . The weight  $\Lambda_\rho$  which represents

$\Lambda + \rho$  through the standard action on the diagonal in  $A(m, n)$  is (cf. Chapter 4)

$$(5.5) \quad \Lambda_\rho = (\alpha/2 + \lambda/2 - a_1 - (n+1)/2 + (p+q-1)/2, \dots, \\ \alpha/2 - \lambda/2 - a_{p+q} - (n+1)/2 + (p+q-1)/2 - (p+q) + 1, \\ \alpha/2 + n/2 + (p+q)/2 + b_1, \dots, \alpha/2 + n/2 + (p+q)/2 + b_{n+1} - n).$$

**THEOREM 5.2.** For fixed  $a_1, \dots, a_{p+q}, b_1, \dots, b_{n+1}$  as above, let  $i_0$  be the biggest index for which  $a_1 = \dots = a_{i_0} = 0$  ( $i_0 \leq p$ ), let  $j_0$  be the biggest integer for which  $a_{p+q} = \dots = a_{p+q-j_0+1} = 0$  ( $j_0 \in \{1, \dots, q\}$ ), and let  $k_0$  be the smallest integer for which  $b_{k_0} = 0$ . Then  $\Lambda = \Lambda(\alpha, \lambda)$  defines a unitarizable highest weight representation of  $A(m, n, \omega(-, +))$ , based on  $N_{st}^+$ , exactly for the following values of the parameters  $\alpha$  and  $\lambda$ :

**Case a:** Not all the  $b_i$ 's are zero ( $k_0 > 1$ ):

Set

$$\begin{aligned} N_1 &= (i_0 - 1) - (b_1 + q), \\ N_2 &= n + 1 - k_0, \\ \alpha_2^0(+) &= \lambda/2 + k_0 - 1, \\ \alpha_2^r(+) &= \lambda/2 + n, \\ \alpha_1^0(-) &= -\lambda/2 - b_1 - (p+q) + i_0, \quad \text{and} \\ \alpha_1^r(-) &= -\lambda/2 - b_1 - (p+q) + 1. \end{aligned}$$

**Subcase a<sub>1</sub>:**  $b_1 + q \leq i_0 - 1$  and  $q > 1$ .

Set

$$\begin{aligned} \lambda_0 &= -b_1 - k_0 - p - q + i_0 + 1, \\ \lambda_c &= \lambda_0 - q + 1. \end{aligned}$$

Let  $\lambda = \lambda_c - N_1 - R$  for  $R = 0, 1, \dots, N_2$ . Then for

$$\alpha = \alpha_2^0(+), \alpha_2^0(+) + 1, \dots, \alpha_2^0(+) + R.$$

Here, and everywhere,  $\alpha_2^0(+)$  is computed for the given  $\lambda$ . Let  $\lambda = \lambda_c - N_2 - R$  for  $R = 0, 1, \dots, N_1$ . Then for

$$\alpha = \alpha_1^0(-), \alpha_1^0(-) - 1, \dots, \alpha_1^0(-) - R.$$

Let  $\lambda = \lambda_c - N_1 - N_2 - R - 1$  for  $R = 0, 1, \dots, b_1$ . Then for

$$\alpha = \alpha_2^0(+), \alpha_2^0(+) + 1, \dots, \alpha_1^0(-).$$

Finally, let  $\lambda < \lambda_c - N_1 - N_2 - 1 - b_1$ . Then for

$$\alpha \in \{\alpha_2^0(+), \alpha_2^0(+)+1, \dots, \alpha_2^0(+)+n+1-k_0 = \alpha_2^r(+)\} \\ \cup [\alpha_2^r(+), \alpha_1^r(-)] \cup \{\alpha_1^r(-), \alpha_1^r(-)+1, \dots, \alpha_1^r(-)+(i_0-1) = \alpha_1^0(-)\}.$$

**Subcase a<sub>2</sub>:**  $b_1 + q \leq i_0 - 1$  and  $q = 1$ .

Let  $\lambda_0$  and  $\lambda_c$  be defined as above.

Let  $\lambda = \lambda_c - R$  for  $R = 0, 1, \dots, N_1 + N_2 + 1 + b_1$ . Then for

$$\alpha = \alpha_2^0(+), \alpha_2^0(+)+1, \dots, \alpha_1^0(-).$$

Let  $\lambda < \lambda_c - N_1 - N_2 - 1 - b_1$ . Then for

$$\alpha \in \{\alpha_2^0(+), \alpha_2^0(+)+1, \dots, \alpha_2^0(+)+n+1-k_0 = \alpha_2^r(+)\} \\ \cup [\alpha_2^r(+), \alpha_1^r(-)] \cup \{\alpha_1^r(-), \alpha_1^r(-)+1, \dots, \alpha_1^r(-)+(i_0-1) = \alpha_1^0(-)\}.$$

**Subcase a<sub>3</sub>:**  $b_1 + q > i_0 - 1$  and  $b_1 \leq i_0 - 2$ .

Set

$$\lambda_0 = -b_1 - k_0 - p - q + i_0 + 1, \\ \lambda_c = \lambda_0 + b_1 - (i_0 - 2).$$

Let  $\lambda = \lambda_c - R$  for  $R = 0, 1, \dots, N_2$ . Then for

$$\alpha = \alpha_2^0(+), \alpha_2^0(+)+1, \dots, \alpha_2^0(+)+R.$$

Let  $\lambda = \lambda_c - N_2 - R - 1$  for  $R = 0, 1, \dots, +b_1$ . Then for

$$\alpha = \alpha_2^0(+), \alpha_2^0(+)+1, \dots, \alpha_1^0(-).$$

Finally, let  $\lambda < \lambda_c - N_2 - 1 - b_1$ . Then for

$$\alpha \in \{\alpha_2^0(+), \alpha_2^0(+)+1, \dots, \alpha_2^0(+)+n+1-k_0 = \alpha_2^r(+)\} \\ \cup [\alpha_2^r(+), \alpha_1^r(-)] \cup \{\alpha_1^r(-), \alpha_1^r(-)+1, \dots, \alpha_1^r(-)+(i_0-1) = \alpha_1^0(-)\}.$$

**Subcase a<sub>4</sub>:**  $b_1 + q > i_0 - 1$  and  $b_1 > i_0 - 2$ .

Let  $\lambda_0$  and  $\lambda_c$  be as in the previous case.

Let  $\lambda = \lambda_0 - R$  for  $R = 0, 1, \dots, N_2 + i_0 - 1$ . Then for

$$\alpha = \alpha_2^0(+), \alpha_2^0(+)+1, \dots, \alpha_1^0(-).$$

Let  $\lambda < \lambda_c - N_2 - 1 - b_1$ . Then for

$$\alpha \in \{\alpha_2^0(+), \alpha_2^0(+)+1, \dots, \alpha_2^0(+)+n+1-k_0 = \alpha_2^r(+)\} \\ \cup [\alpha_2^r(+), \alpha_1^r(-)] \cup \{\alpha_1^r(-), \alpha_1^r(-)+1, \dots, \alpha_1^r(-)+(i_0-1) = \alpha_1^0(-)\}.$$

**Case b:** All the  $b_i$ 's are zero ( $k_0 = 1$ ): All cases obtained from the above by setting  $b_1 = 0$ , with the exception of those pairs that are of the form  $(\lambda, \alpha_2^0(+))$ . These latter pairs are to be replaced by the following, where we only list the  $\lambda$ 's:

Let  $\lambda_s = -p - q - n + j_0 + 1$ . Then there is always unitarity for  $\lambda \leq \lambda_s$ . Furthermore, if  $q > j_0$  then for

$$\lambda = \lambda_x = j_0 + 1 - (p + q), \lambda_x - 1, \dots, \lambda_s.$$

Finally, if  $q = j_0$ , then for

$$\lambda = \lambda_y = i_0 - p, \lambda_y - 1, \dots, \lambda_s.$$

*Proof.* First we consider

**Subcase a<sub>1</sub>:** By viewing the highest weight vector  $v_\Lambda$  as the highest weight vector of an  $su(p, q)$  representation, we get the first restriction on  $\lambda$ , namely (cf. Chapter 3) that it must be smaller than or equal to the last possible place of unitarity for this  $su(p, q)$  representation,

$$(5.6) \quad \lambda \leq j_0 + i_0 - (p + q).$$

Now consider the space  $\Lambda(Q_2) \otimes V_\Lambda^0$ : This, with the chosen involution, corresponds exactly to the situation for  $A(q-1, n, \omega_+)$  as described in Chapter 4—but, of course, with the values of  $\alpha$  and  $\lambda$  as given above. It follows that there is positivity for

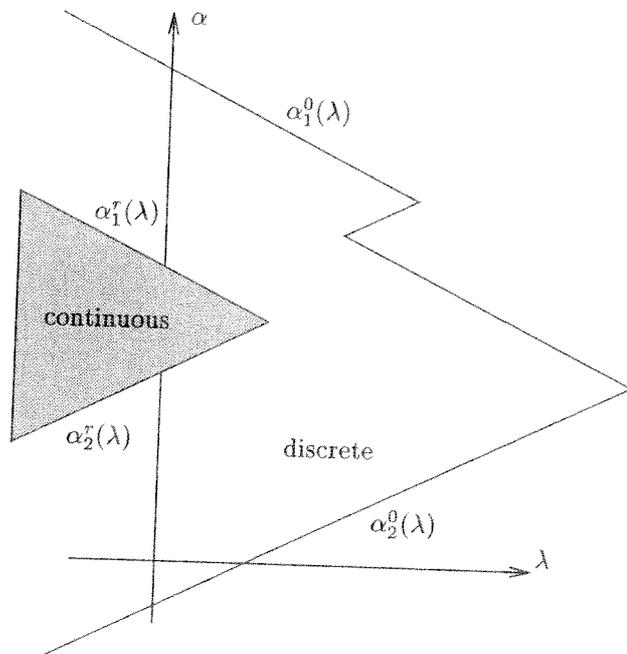
$$(5.7) \quad \alpha \in \{\alpha_2^0(+), \dots, \alpha_2^r(+)=\alpha_2^0(+)+n+1-k_0\} \cup [\alpha_2^r(+), \infty[ ,$$

where

$$\alpha_2^0(+)=\lambda/2+k_0-1.$$

Let us then consider  $\Lambda(Q_1) \otimes \Lambda(Q_2) \otimes V_\Lambda^0$ : Again, we can use the results from Chapter 4, this time on  $A(p-1, n, \omega_-)$ . However, there is one complication, namely that  $\Lambda(Q_2) \otimes V_\Lambda^0$  is not an irreducible  $su(p) \times su(n+1)$  module. However, it is clear that the type whose highest weight vector is

$$(5.8) \quad (q_2)_{11} \wedge (q_2)_{12} \wedge \dots \wedge (q_2)_{1q} \cdot v_\Lambda$$

Figure of case  $a_1$

is the one that will give the smallest last possible place of unitarity for  $\Lambda(Q_1)$ . Since not all the  $b_i$ 's are zero, this element will be non-zero in the range of  $\alpha$  that we are interested in,  $\alpha \geq \alpha_2^0(+)$ . The last possible place of unitarity for the above mentioned  $su(p) \times su(n+1)$  type, as seen from  $A(p-1, n, \omega_-)$  is the value of  $\alpha$  that is denoted by  $\alpha_1^0(-)$  in the theorem (with  $\lambda = \lambda_R$ —but of course,  $\lambda_R$  has not been specified yet). This latter place is marked by 1's in Fig. 5.9, whereas the type that gives  $\alpha_2^0(+)$  is marked by 2's. (Notice that  $\alpha_1^0(-)$  cannot be computed directly from (5.5) since the " $\rho$ " is not the relevant one).

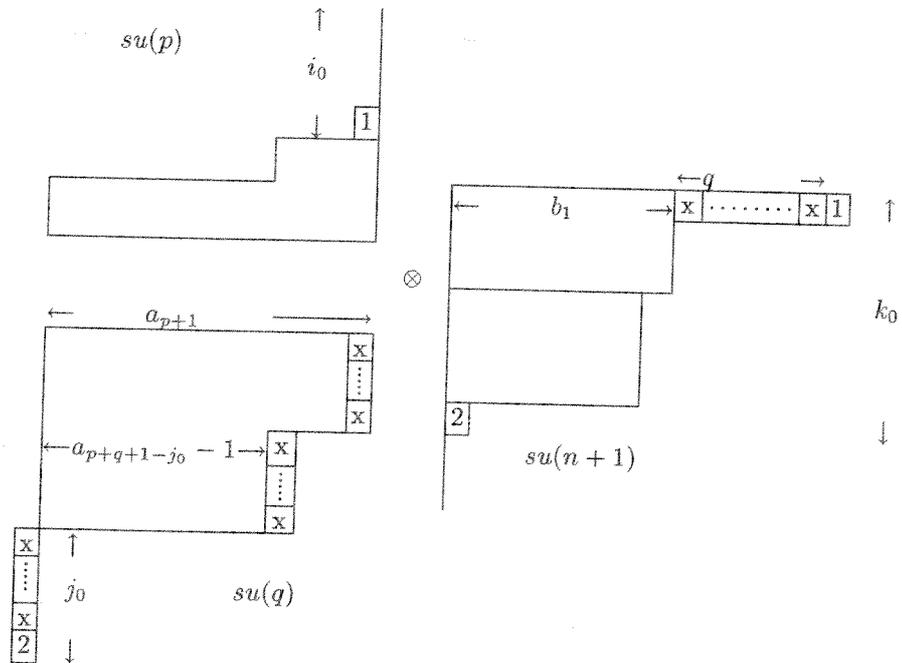


Fig. 5.9

We have also indicated, by x's, the type responsible for  $\alpha_1^0(-)$  (except that there should also have been an x in the box containing the 2 in the  $su(q)$ -diagram). Since  $A(m, n, \omega_+)$  is unitary for  $\alpha \rightarrow \infty$ , and  $A(m, n, \omega_-)$  is unitary for  $\alpha \rightarrow -\infty$ , it is then clear that we must have

$$(5.10) \quad \alpha_2^0(+) \leq \alpha_1^0(-).$$

Or, equivalently,

$$(5.11) \quad \lambda \leq \lambda_0 = -b_1 - k_0 - p - q + i_0 + 1,$$

where the equality sign corresponds to  $\alpha_2^0(+)=\alpha_1^0(-)$ . This point is then a point of positivity. More generally, the whole description of the positivity of the hermitian form on  $\Lambda(G_{-1})\otimes V_\Lambda$  then follows from Chapter 5. We can see that as  $\lambda$  decreases, the spacing between  $\alpha_2^0(+)$  and  $\alpha_1^0(-)$  increases. To begin with, the distance must be integer valued until we reach the point at which

$$(5.12) \quad \alpha_2^r(+)\leq\alpha_1^r(-),$$

where the  $r$  indicates a first possible place of non-unitarity, a "reduction point". After that we have the discrete values of  $\alpha$  together with the interval from  $\alpha_2^r(+)$  to  $\alpha_1^r(-)$ . Of course, one needs to check that there are no other types in  $\Lambda(Q_1)\otimes V_\Lambda$  that make this picture more complicated, but this is easily seen to be the case.

Finally, we need to check if there are further restrictions on  $\lambda$  imposed by the requirement that all the  $su(p,q)$  highest weights in  $\Lambda(G_{-1})\otimes V_\Lambda$  should correspond to unitary modules. This, indeed, turns out to be the case. Let us first look at the problem when  $\lambda=\lambda_0$  (cf. (5.11)) and  $\alpha=\alpha_2^0(+)$ .

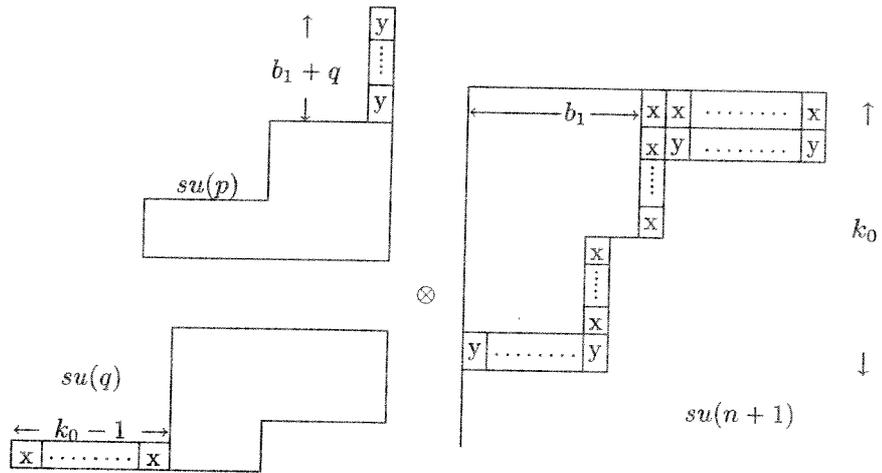


Fig. 5.13

The diagram in Fig. 5.13 is easily seen to be the one that gives the most serious restriction on  $\lambda$  (in terms of pushing the allowed range of  $\lambda$  towards  $-\infty$ ). We have indicated by x's the term from  $\Lambda(Q_2)\otimes V_\Lambda$ , and by y's the one from  $\Lambda(Q_1)\otimes V_\Lambda$ . The latter, of course, is only interesting if  $i_0\geq b_1+q+1$ , which

is exactly the case we are currently considering. That the former can have a row of at most  $k_0 - 1$  boxes follows because otherwise we would have to have a column of at least  $k_0$  boxes in the  $su(n+1)$  diagram, and hence fall inside the ideal generated by the vanishing term from  $Q_2 \otimes V_\Lambda$ . We can now compute the last possible place of unitarity for the indicated  $su(p, q)$  module:

$$(5.14) \quad \lambda/2 - (i_0 - (b_1 + q)) + 1 + \lambda/2 + (p + q) - 1 + (k_0 - 1) = 1 \quad \Leftrightarrow \\ \lambda = \lambda_c = -(p + q) - k_0 + 2 + i_0 - b_1 - q.$$

Since  $\lambda_0 - \lambda_c = q - 1$ , it follows that we must at least have  $\lambda \leq \lambda_c$  to have unitarity at  $\alpha = \alpha_2^0(+)$ . But, if  $q - 1 > 0$  then we are properly below  $\alpha_1^0(-)$ , and hence we have more types in  $\Lambda(Q_2) \otimes V_\Lambda^0$ , specifically, we can increase the column of  $y$ 's in the  $su(p)$  part of Fig. 5.13. This then gives more restrictions on  $\lambda$ , and it is easy to see that at  $\alpha = \alpha_2^0(+)$  we are forced to have  $\lambda \leq \lambda_c - N_1 = -(p + q) + 3 - k_0$  which is the ultimate demand on  $\lambda$  coming from the  $su(p)$  part in Fig. 5.13. Moving on to  $\alpha = \alpha_2^0(+) + R$  we begin to get further demands on  $\lambda$  from the  $su(q)$  piece of Fig. 5.13 because we can increase the row of  $x$ 's by  $R$ . This, then, explains the first series. The second comes about via the same sort of argument, by symmetrically beginning the analysis at  $\alpha = \alpha_1^0(-)$  and then letting  $\alpha$  decrease. Finally, when we reach  $\lambda = \lambda_c - N_1 - N_2 - 1$ ,  $\lambda$  is so small that all types in  $\Lambda(Q_2) \otimes V_\Lambda^0$  are unitary. However, due to the requirements on  $\alpha$  coming from Chapter 4 we must still stay at integer points until we reach the first  $\lambda$  where  $\alpha_2^r(+) \leq \alpha_1^r(-)$ . After this, the picture is clear.

**Subcase a<sub>2</sub>:** Proceeding as in subcase a<sub>1</sub>, we do not get any demand on  $\lambda$  from the  $su(p)$  diagrams because at those values of  $\alpha$  where a demand otherwise could be felt, namely those where  $\lambda$  has been pushed as much as possible by the  $su(q)$  diagram. The reason for this is that in the current situation,  $\lambda_c = \lambda_0$  and hence  $\alpha_2^0(+) = \alpha_1^0(-)$ .

**Subcase a<sub>3</sub>:** This is very much like a mixture of the two previous cases. The  $su(p)$  diagram is already used to its fullest extent in pushing  $\lambda$  towards  $-\infty$  whereas we still get some requirements from the  $su(q)$  diagram. In the current situation,  $\lambda_c$  (determined in analogy with the previous cases) satisfies

$$\lambda_0 - \lambda_c = (i_0 - 2) - b_1 \geq 0,$$

hence, what corresponds to the first series in subcase a<sub>1</sub> comes about by the same mechanism, as does the others.

**Subcase a<sub>4</sub>:** Here,  $\lambda_0 < \lambda_c$  and hence we cannot start at  $\lambda = \lambda_c$ . Rather, we must proceed to  $\lambda = \lambda_0$ . At this point, there is only one possible value for  $\alpha$ , namely  $\alpha = \alpha_2^0(+) = \alpha_1^0(-)$ . After this observation the discussion follows in analogy with the previous.

Finally, the exceptional points in Case b) come about as follows: The  $su(n+1)$  diagram in Fig. 5.9 with a row of  $q$  boxes that was used to compute the value

of  $\alpha_1^0(-)$  cannot be used exactly at the point where  $\alpha = \alpha_2^0(+)$  because here, when all the  $b_i$ 's are zero, this type is inside the ideal generated by the vanishing  $su(q) \times su(n+1)$  type from  $Q_2 \otimes V_\Lambda$ . Instead, we can only make a row of  $(q - j_0)$  boxes, since this is the length of the column we can make in the  $su(q)$  diagram. At this particular point, the equation for  $\lambda$  then becomes

$$(5.15) \quad \lambda \leq -(p+q) + i_0 + j_0,$$

which is the requirement that  $\alpha_2^0(+)$   $\leq$   $\alpha_1^0(-)$ . (Observe that  $\alpha_1^0(-)$  and  $\alpha_2^0(+)$  here are computed for quite different diagrams compared to the previous cases. Hence, their values have changed significantly.) However, if  $q - j_0 > 0$ , by using  $\wedge Q_1$  we can make an  $su(p)$  diagram with a column of  $q - j_0$  boxes on top of the original diagram, and this then forces  $\lambda$  to be this amount smaller. (This is only interesting if  $q - j_0 \leq i_0 - 1$ , otherwise the ultimate requirement on  $\lambda$  is  $\lambda \leq 1 + j_0 - (p+q)$ .) But now, if we insist that  $\alpha = \alpha_2^0(+)$ , then  $\alpha_1^0(-) - \alpha > 0$ , and hence we can make the column even bigger, thus forcing  $\lambda$  to be even smaller until we reach a column of height  $i_0 - 1$ . We must then use the discrete values as indicated until we reach the  $\lambda = \lambda_s$  where  $\alpha_2^r(+)$   $\leq$   $\alpha_1^r(-)$ . The discrete values of  $\lambda$  are obtained in analogy with the previous cases. We only have to pay special attention to the value  $\alpha = \alpha_2^0(+)$  because this particular phenomenon only takes place here. Indeed, at  $\alpha = \alpha_2^0(+)$  + 1, the  $su(q) \times su(n+1)$  type that vanishes from  $\wedge^2(Q_2) \otimes V_\Lambda$  has the form shown in Fig. 5.16,

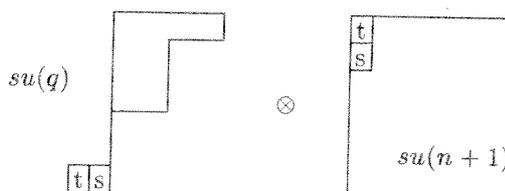


Fig. 5.16

and this, and the analogues for bigger values of  $\alpha$ , cannot prevent the diagram in  $su(q)$  with a column of  $q$  boxes from occurring.  $\square$

We now turn to the other involution,  $\omega(+, -)$ . For this, the situation is remarkably different from that of the previous:

**THEOREM 5.3.** For fixed  $a_1, \dots, a_p, a_{p+1}, \dots, a_{p+q}, b_1, \dots, b_{n+1}$  as before,  $\Lambda = \Lambda(\lambda, \alpha)$  defines a unitarizable highest weight module of  $A(m, n, \omega(+, -))$ , based on  $N_{st}^+$ , only when  $\Lambda$  is trivial.

*Proof.* Let us first consider the case in which  $b_1 > 0$ . Proceeding in analogy with the previous proof we choose a fixed value  $\lambda$  for which the  $su(p, q)$  module

$V_\Lambda$  is unitarizable. Thus,  $\lambda \leq -(p+q) + i_0 + j_0$ . Let  $\tilde{j}_0$  be the biggest integer for which  $a_{p+1} = \dots = a_{p+\tilde{j}_0}$ . Then it is easy to see that

$$(5.17) \quad \alpha_2(-) = \lambda/2 + a_{p+\tilde{j}_0} - b_1 - q + \tilde{j}_0$$

is the smallest value at which an  $su(q) \times su(n+1)$  type vanishes from  $\Lambda^1(Q_2) \otimes V_\Lambda$ . Likewise,

$$(5.18) \quad \alpha_1(+) = -\lambda/2 + a_p + n$$

is a value (not always the biggest) at which an  $su(p) \times su(n+1)$  type vanishes from  $\Lambda^1(Q_1) \otimes \Lambda(Q_2) \otimes V_\Lambda$ . Since we must have that

$$(5.19) \quad \alpha_1(+) \leq \alpha_2(-)$$

it follows that

$$(5.20) \quad \lambda \geq n + a_p - a_{p+\tilde{j}_0} + b_1 - \tilde{j}_0 + q,$$

and since  $-a_{p+\tilde{j}_0}$  is positive and since  $\lambda \leq -(p+q) + i_0 + j_0$ , we see that there can be no unitarity here since  $\lambda$  cannot satisfy both inequalities at the same time. The same argument can be used to do away with the case  $b_1 = 0$  and  $a_{p+1} \neq 0$ . The  $\lambda$ -inequalities are still disjoint.

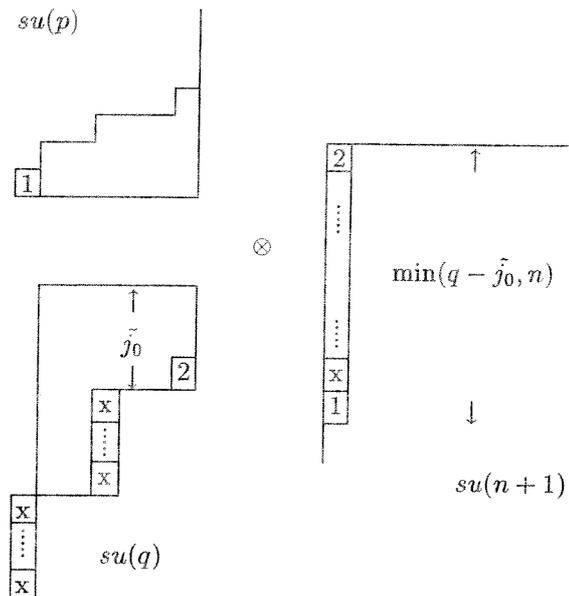


Fig. 5.21

Let us finally look at the case where  $a_{p+1} = a_{p+q} = b_1 = \dots = b_{n+1} = 0$ . We get here that  $\alpha_2(-) = \lambda/2$  and  $\alpha_1(+) = -\lambda/2 + a_p$ , and since again we must have  $\alpha_2(-) \geq \alpha_1(+)$ ,

$$(5.22) \quad a_p \leq \lambda.$$

This inequality clearly implies that  $\Lambda$  must be trivial.  $\square$

**Subcase  $N_{nst}^+$ .**

**THEOREM 5.4.** *For fixed  $a_1, \dots, a_{p+q}, b_1, \dots, b_{n+1}$ , as above, let  $i_0$  be the biggest index for which  $a_1 = \dots = a_{i_0} = 0$  ( $i_0 \leq p$ ), let  $j_0$  be the biggest integer for which  $a_{p+q} = \dots = a_{p+q-j_0+1} = 0$  ( $j_0 \in \{1, \dots, q\}$ ), and let  $k_0$  be the smallest integer for which  $b_{k_0} = 0$ . Then  $\Lambda = \Lambda(\alpha, \lambda)$  defines a unitarizable highest weight representation of  $A(m, n, \omega(-, +))$ , based on  $N_{nst}^+$ , exactly for the following values of the parameters  $\alpha$  and  $\lambda$ :*

*Case a: Either  $b_1 \neq 0$ ,  $(p - i_0) \neq 0$ , or  $(q - j_0) \neq 0$ :*

*Set*

$$\begin{aligned} \alpha_2^0(+) &= \lambda/2 + q - j_0, \\ \alpha_2^r(+) &= \lambda/2 + q - 1, \\ \alpha_1^0(-) &= -\lambda/2 - b_1 - p + i_0, \quad \text{and} \\ \alpha_1^r(-) &= -\lambda/2 - b_1 - p + 1. \end{aligned}$$

**Subcase  $a_1$ :**  $b_1 + p < j_0 + 1$ .

*Set*

$$\lambda_1 = -b_1 + j_0 - 2p - q + 1.$$

*Let  $\lambda = \lambda_1 - R$  for  $R = 0, 1, \dots, j_0 - (b_1 + p - 1)$ . Then for*

$$\alpha = \alpha_2^0(+), \alpha_2^0(+) + 1, \dots, \alpha_2^0(+) + R.$$

*Here, and everywhere,  $\alpha_2^0(+)$  is computed for the given  $\lambda$ .*

**Subcase  $a_2$ :**  $b_1 + q < i_0 + 1$ .

*Set*

$$\lambda_1 = -b_1 + i_0 - p - 2q + 1.$$

*Let  $\lambda = \lambda_1 - R$  for  $R = 0, 1, \dots, i_0 - (b_1 + q - 1)$ . Then for*

$$\alpha = \alpha_2^0(+), \alpha_2^0(+) + 1, \dots, \alpha_2^0(+) + R.$$

**Subcase  $a_3$ :**  $\lambda < 2 - (p + q)$ .

Then for

$$\alpha \in \{\alpha_2^0(+), \alpha_2^0(+) + 1, \dots, \alpha_2^r(+)\} \\ \cup [\alpha_2^r(+), \alpha_1^r(-)] \cup \{\alpha_1^r(-), \alpha_1^r(-) + 1, \dots, \alpha_1^0(-)\}.$$

**Case b:**  $b_1 = 0, i_0 = p,$  and  $j_0 = q.$

Let  $\lambda = -R$  for  $R = 0, 1, \dots, p + q - 2.$  Then for

$$\alpha = \alpha_2^0(+), \alpha_2^0(+) + 1, \dots, \alpha_1^0(-).$$

Let  $\lambda < p + q - 2.$  Then for

$$\alpha \in \{\alpha_2^0(+), \alpha_2^0(+) + 1, \dots, \alpha_2^r(+)\} \\ \cup [\alpha_2^r(+), \alpha_1^r(-)] \cup \{\alpha_1^r(-), \alpha_1^r(-) + 1, \dots, \alpha_1^0(-)\}.$$

*Proof.* This follows along the same lines as the proof of Theorem 5.2. We omit the details.

Likewise one obtains

**THEOREM 5.5.** For fixed  $a_1, \dots, a_p, a_{p+1}, \dots, a_{p+q}, b_1, \dots, b_{n+1}$  as before,  $\Lambda = \Lambda(\lambda, \alpha)$  defines a unitarizable highest weight module of  $A(m, n, \omega(+, -))$ , based on  $N_{nst}^+$ , only when  $\Lambda$  is trivial.

(ii)  $su(p_1, q_1) \times su(p_2, q_2).$

In this section we examine the case in which both  $A_m$  and  $A_n$  correspond to hermitian symmetric spaces of the non-compact type. Thus, we write elements of our superalgebra as

$$\begin{bmatrix} \alpha_1 & z_1 & p_{11} & p_{12} \\ w_1 & \alpha_2 & p_{21} & p_{22} \\ q_{11} & q_{21} & \alpha_3 & z_2 \\ q_{12} & q_{22} & w_2 & \alpha_4 \end{bmatrix},$$

where  $\alpha_1, z_1, w_1,$  and  $\alpha_2$  correspond to  $su(p_1, q_1)$ , and where  $\alpha_2, z_2, w_2,$  and  $\alpha_2$  correspond to  $su(p_2, q_2).$  In particular,  $\alpha_1 \in u(p_1), \alpha_2 \in u(q_1), \alpha_3 \in u(p_2),$  and  $\alpha_4 \in u(q_2).$  It follows that there are two possible anti-linear anti-involutions possible,

$$\omega_1 \equiv \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}, \text{ and } \omega_2 \equiv \begin{bmatrix} + & - & - & + \\ - & + & + & - \\ + & - & - & + \end{bmatrix}.$$

PROPOSITION 5.6. *Neither  $\omega_1$  nor  $\omega_2$  lead to any non-trivial unitarizable modules.*

*Proof.* The two cases are similar, and for this reason we only treat  $\omega_2$ :

To begin with we observe that there are several choices for the part of  $N^+$  that comes from the odd part:

$$N_1^+ = P_{11} \oplus P_{12} \oplus P_{21} \oplus P_{22},$$

$$N_2^+ = P_{11} \oplus P_{12} \oplus Q_{21} \oplus Q_{22},$$

$$N_1^+ = P_{11} \oplus P_{21} \oplus Q_{12} \oplus Q_{22},$$

$$N_1^+ = P_{12} \oplus P_{22} \oplus Q_{11} \oplus Q_{21}, \text{ and}$$

$$N_1^+ = P_{11} \oplus P_{12} \oplus P_{22} \oplus Q_{21}.$$

In all these cases, one can use either Theorem 5.3 or Theorem 5.5 appropriately to exclude a particular case. We illustrate the phenomenon with the case  $N_1^+$ :

By looking at the sub-algebra whose elements are

$$\begin{bmatrix} \alpha_1 & z_1 & p_{11} & 0 \\ w_1 & \alpha_2 & p_{21} & 0 \\ q_{11} & q_{21} & \alpha_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

it follows from Theorem 5.3 that the highest weight must be trivial on  $su(q_1)$  and  $su(p_2)$ . Then observe that

$$[p_{11}, w_2] = [p_{21}, w_2] = [z_1, w_2] = 0$$

and

$$[\alpha_3, w_2] \subseteq w_2.$$

It then follows that  $Q_3 \otimes V_\Lambda^0$  must vanish since otherwise this space will define highest weight representations of the above mentioned subalgebra that will not be trivial on  $su(p_2)$ . This, however, is only possible if the restriction of the highest weight to  $su(p_2, q_2)$  is trivial. By looking at  $Q_1$  it follows similarly that the restriction of the highest weight to  $su(p_1, q_1)$  is trivial. But then also  $\alpha$  must be zero (cf. Theorem 5.2).

$A(n, n)$ 

Recall that

$$(6.1) \quad A(n, n) = sl(n+1, n+1)/\{\lambda I_{2n+2}\}, \quad n > 0.$$

Any highest weight representation of this superalgebra can be viewed as being obtained by passing to the quotient of a highest weight representation of  $sl(n+1, n+1)$ . Of course, this highest weight representation of  $sl(n+1, n+1)$  must behave correctly on  $\{\lambda I_{2n+2}\}$ , it must vanish. Now, the unitarity of any highest weight module of  $sl(n+1, n+1)$  is settled by Chapter 4 and Chapter 5 merely by setting  $m = n$ . This is the case because it was nowhere used in those chapters that  $n \neq m$ . Thus, the unitarity of any highest weight representation of  $A(n, n)$  can be determined by the following procedure: a) Adjust the free parameter  $\alpha$  in the analogous case for  $A(n, m)$  ( $m = n$ ) so that the representation passes to the quotient. b) Examine whether this  $\alpha$  belongs to the set of values that lead to unitarity.

We begin by considering the two cases corresponding to Chapter 4. We shall call these the finite dimensional representations. Consider then, as in Chapter 4, a highest weight  $\Lambda$  of the form

$$(6.2) \quad \Lambda = \left( \frac{\alpha}{2}, \frac{\alpha}{2} - a_2, \dots, \frac{\alpha}{2} - a_{m+1}, \frac{\alpha}{2} + b_1, \dots, \frac{\alpha}{2} + b_n, \frac{\alpha}{2} \right).$$

Clearly, the requirement that this  $\Lambda$  should pass to the quotient is that

$$(6.3) \quad \alpha = \frac{1}{n+1}((a_2 + \dots + a_{n+1}) - (b_1 + \dots + b_n))$$

(recall that  $m = n$ ). We observe that

$$(6.4) \quad -\frac{n}{n+1}b_1 \leq \alpha \leq \frac{n}{n+1}a_{n+1}$$

and it then follows easily that

**PROPOSITION 6.1.** *There are no non-trivial unitarizable finite-dimensional representations of  $A(n, n)$ .*

We now turn to the two cases treated in Chapter 5: In this case, the critical value of  $\alpha$  is

$$(6.5) \quad \alpha = \frac{1}{n+1}((a_1 - \lambda/2) + \cdots + (a_p - \lambda/2) \\ + (a_{p+1} + \lambda/2) + \cdots + (a_{p+q} + \lambda/2) - b_1 - \cdots - b_n).$$

The simplest case is, naturally, the one which corresponds to Theorem 5.3 and Theorem 5.5:

**PROPOSITION 6.2.** *There are no non-trivial unitarizable highest weight modules of  $A(n, n)$  corresponding to the involution  $\omega(+, -)$ .*

In the cases which correspond to Theorem 5.2 and Theorem 5.4 there may be unitarity for certain values of the  $a_i$ 's and the  $b_j$ 's whereas others may lead to non-unitarity. We have found no simpler description than the following.

**PROPOSITION 6.3.** *There are unitarizable representations of  $A(n, n)$  corresponding to  $\omega(-, +)$  and based on either  $N_{st}^+$  or  $N_{nst}^+$  exactly when the pair*

$$(\lambda, \alpha) = \left( \lambda, \frac{1}{n+1}((a_1 - \lambda/2) + \cdots + (a_p - \lambda/2) \\ + (a_{p+1} + \lambda/2) + \cdots + (a_{p+q} + \lambda/2) - b_1 - \cdots - b_n) \right)$$

*satisfies the requirements either in Theorem 5.2 or in Theorem 5.4.*

**Example.** If  $p = q$ ,  $\alpha$  does not depend on  $\lambda$  and it is then clear, since  $a_1, \dots, a_p$  are positive whereas  $a_{p+1}, \dots, a_{p+q}$  are negative, that for some values of the  $a_i$ 's there will be unitarity, whereas for others, there will not be unitarity.



We need to investigate which kinds of anti-linear anti-involutions may occur. Since both  $B_m$  and  $C_n$  have non-compact unitarizable highest weight modules, we shall consider matrices of the following form

$$(7.4) \quad \begin{bmatrix} \epsilon_1 & & \dots & y_{1i} & \dots & & 0 & p_{11} & p_{12} \\ & \vdots & & a & & & 0 & \vdots & p_{21} & p_{22} \\ & x_{j1} & & & 0 & & & -y_{1i} & * & p_{32} \\ & \vdots & & 0 & & & & \vdots & * & * \\ & 0 & & \dots & -x_{j1} & \dots & -\epsilon_1 & * & * & * \\ q_{11} & q_{21} & & * & * & & * & \beta & z^+ \\ q_{12} & q_{22} & & q_{32} & * & & * & w^- & -\beta^r \end{bmatrix},$$

where the \*'s denote appropriate matrices determined completely by the  $p$ 's and the  $q$ 's. In  $y_{1i}$ , the  $i$  runs from 2 to  $2m$  and the  $-y_{1i}$  is located as the  $(2m+2-i, 2m+1)$  entry. For the  $x_{j1}$ 's the situation is analogous.

Let  $\epsilon_y, \epsilon_z, \epsilon_{11}, \epsilon_{12}, \epsilon_{21}, \epsilon_{22}$ , and  $\epsilon_{32}$  be  $\pm 1$  and let  $\omega(\epsilon_y, \epsilon_z, \epsilon_{11}, \epsilon_{21}, \epsilon_{12}, \epsilon_{22}, \epsilon_{32})$  be the involution which is complex conjugation followed by the appropriate sign on the appropriate piece, i.e.

$$(7.5) \quad \begin{aligned} a &\rightarrow a^*, \\ \beta &\rightarrow \beta^*, \\ y_{1j} &\rightarrow \epsilon_y x_{j1}, \\ z &\rightarrow \epsilon_z (w^- = z^*), \text{ and} \\ p_{ij} &\rightarrow \epsilon_{ij} (q_{ij} = p_{ij}^*). \end{aligned}$$

First we observe that due to the fact that the sign of the involution is constant on  $a, \beta, y$ , and  $z$ , it follows easily, by letting the even part act on the odd part, that the signs must be constant on each of the remaining pieces, as indicated.

LEMMA 7.1.

$$\begin{aligned} \epsilon_{21} = -\epsilon_{22} = -\epsilon_{32} = \epsilon_y \epsilon_{11} = -\epsilon_y \epsilon_{12}, \quad \text{and} \\ \epsilon_z = \epsilon_{21} \epsilon_{22} = -1. \end{aligned}$$

*Proof.* The first set of equations follows easily by letting  $so(2m+1)$  act on the odd piece. The second equation follows from a computation which, in matrix notation is

$$(7.6) \quad \omega(z^+) = \omega([q_{22}^+, q_{21}^-]) = \varepsilon_{21} \cdot \varepsilon_{22} [q_{21}^+, q_{22}^-] = \varepsilon_{21} \varepsilon_{22} w^-.$$

The proof of the fact that an  $\omega$ , subject to these constraints, actually does define an involution is elementary, and will be omitted. As a consequence of this lemma, we are forced into a situation where the  $C_n$  corresponds to  $sp(n, R)$ . The unitarizable highest weight representation of this Lie algebra were described in Chapter 3. Let us now turn our attention to another major complication, namely that there are two non-conjugate sets of positive roots. We shall refer to these as case #A and case #B, respectively.

Case #A.

$$(7.7) \quad \begin{aligned} \Delta_1^+ = & \{ \delta_i + \varepsilon_j \mid i = 1, \dots, n \text{ and } j = 1, \dots, m \} \\ & \cup \{ \delta_i - \varepsilon_j \mid i = 1, \dots, n \text{ and } j = 1, \dots, m \} \\ & \cup \{ \delta_i \mid i = 1, \dots, n \}. \end{aligned}$$

In the notation from before, the space of positive root vectors in  $G_{\bar{1}}$  is

$$(7.8) \quad N_A^+ = \text{span}\{p_{12}, p_{22}, p_{32}, q_{21}, q_{11}\}.$$

Case #B.

$$(7.9) \quad \begin{aligned} \Delta_1^+ = & \{ \varepsilon_i - \delta_j \mid i = 1, \dots, m \text{ and } j = 1, \dots, n \} \\ & \cup \{ \varepsilon_i + \delta_j \mid i = 1, \dots, m \text{ and } j = 1, \dots, n \} \\ & \cup \{ \delta_i \mid i = 1, \dots, n \}. \end{aligned}$$

In this case, the positive odd vectors are

$$(7.10) \quad N_B^+ = \text{span}\{p_{11}, p_{12}, p_{21}, p_{22}, p_{32}\}.$$

It is easy to see that these are the only possibilities. Before examining the two cases separately, we make the following observation:

LEMMA 7.2. *If there are to be unitarizable highest weight representations other than the trivial, then  $\varepsilon_{32} = 1$ .*

*Proof.* In either case #A or case #B, the space spanned by the matrices  $q_{32}$  is inside the negative root space. If we let  $v_0$  denote the highest weight vector and let the hermitian form be denoted by  $\langle \cdot, \cdot \rangle$ , then

$$(7.11) \quad \langle q_{32}v_0, q_{32}v_0 \rangle = \varepsilon_{32} \left\langle v_0, \begin{pmatrix} \beta & 0 \\ 0 & -\beta^r \end{pmatrix} \cdot v_0 \right\rangle,$$

where  $\beta$  denotes the element

$$(7.12) \quad -(q_{32}^r)^*(q_{32}^r)$$

and  $\begin{pmatrix} \beta & 0 \\ 0 & -\beta^r \end{pmatrix}$  denotes an element of  $sp(n, R)^{\mathbb{C}}$  or, rather, the image of this under the natural imbedding into  $B(m, n)$ . If this inner product is to be non-negative, and  $\varepsilon_{32} = -1$ , then  $sp(n, R)$  must act trivially on  $v_0$ , in which case the inner product is zero. In case #A,  $p_{32}p_{21} \cdot v_0 = 0$ , hence it follows that also  $q_{32}p_{21} \cdot v_0$  must be zero, hence the representation must also be trivial on  $B_m$ . In case #B the same conclusion is obtained by looking at  $p_{32}q_{21} \cdot v_0$ .  $\square$

Let us then turn to

**Case #A:**

By looking at  $\langle p_{11}v_0, p_{11}v_0 \rangle$  it follows easily, using lemma 7.1 and lemma 7.2, that

LEMMA 7.3. *If there are to be unitarizable highest weight representations other than the trivial, in case #A, then  $\varepsilon_y = 1$ .*

This means that the involution is completely fixed: On  $G_{\bar{1}}$ , it has to be  $-1$  times complex conjugation on  $\text{span}\{p_{11}, p_{21}\}$  and  $+1$  times complex conjugation on  $\text{span}\{p_{12}, p_{22}, p_{32}\}$ . On  $B_m$  it is just complex conjugation and on  $C_n$  it is the anti-involution whose  $-1$  eigenspace is  $sp(n, \mathbb{R})$ . Since there is no reason here to split off  $p_{11}$  separately, etc., we let

$$(7.13) \quad \begin{aligned} P_1 &= \text{span}\{p_{11}, p_{21}\}, \\ P_2 &= \text{span}\{p_{12}, p_{22}\}, \\ P_3 &= \text{span}\{p_{32}\}, \\ Q_1 &= \text{span}\{q_{11}, q_{12}\}, \\ Q_2 &= \text{span}\{q_{12}, q_{22}\}, \text{ and} \\ Q_3 &= \text{span}\{q_{32}\}. \end{aligned}$$

Thus, a general element of  $B(m, n)$  has the form

$$(7.14) \quad \begin{bmatrix} \alpha_1 & * & * & p_1 & p_2 \\ * & 0 & * & * & p_3 \\ * & * & \alpha_4 & * & * \\ q_1 & * & * & \beta & z^+ \\ q_2 & q_3 & * & w^- & -\beta^r \end{bmatrix},$$

where  $\alpha_1 \in Gl(m, \mathbb{C})$ , and  $\alpha_4 = -\alpha_1^t$ . The space  $N_A^-$  of negative odd root vectors satisfies

$$(7.15) \quad N_A^- = \text{span}\{p_1, q_2, q_3\}.$$

Let  $n^+(B_m)$  denote the space of positive root vectors in  $so(2m+1)$ . Then

$$(7.16) \quad \begin{aligned} [p_3, p_1] &\in n^+(B_m), \\ [p_2, q_3] &\in n^+(B_m), \text{ and} \\ [p_2, p_1] &\in n^+(B_m). \end{aligned}$$

$N_A^-$  is a  $G_{\bar{0}}$  module, and

$$(7.17) \quad \bigwedge(N_A^-) = \bigwedge(Q_2) \otimes \bigwedge(Q_3) \otimes \bigwedge(P_1),$$

where a capital letter indicates the space spanned by the matrices named by the corresponding small letter.

The maximal compact subalgebra of the real form of  $B(m, n)_0$  under investigation is  $so(2m+1) \times u(n)$ . Inside  $B(m, n)_0$  we have the subalgebra

$$(7.18) \quad \left\{ \begin{bmatrix} u_1 & & & & \\ & 0 & & & \\ & & -u_1^r & & \\ & & & u_2 & \\ & & & & -u_2^r \end{bmatrix} \mid u_1 \in u(m), u_2 \in u(n) \right\}$$

which is isomorphic to  $u(m) \times u(n)$ .

Let us now consider a fixed highest weight representation  $V_\Lambda$  of highest weight  $\Lambda \in h^*$ , where

$$(7.19) \quad \Lambda = (\mu_1, \dots, \mu_m, \lambda, \lambda - a_2, \dots, \lambda - a_n)$$

satisfies that the  $\mu$ 's are integers or half-integers satisfying  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m \geq 0$  (to make the restriction to  $so(2m+1)$  unitary), the  $a$ 's are integers such that  $a_n \geq \dots \geq a_2 \geq 0$ , and  $\lambda$  is chosen such that the  $sp(n, \mathbb{R})$  module of highest weight  $(\lambda, \lambda - a_2, \dots, \lambda - a_n)$  is unitarizable.

Let  $V_\Lambda^0$  denote the irreducible representation of  $so(2m+1) \times u(n)$  generated by the action of this subalgebra on the highest weight vector  $v_\Lambda$ . The idea now is to proceed as in the case of  $A(s, t)$ , but one complication is here that, in general,  $V_\Lambda^0$  is not irreducible with respect to the  $u(m) \times u(n)$  subalgebra. We let

$$(7.20) \quad V_\Lambda^0 \upharpoonright_{u(m) \times u(n)} = \sum_{i=1}^{M(\Lambda)} V_\Lambda^0(i)$$

be the decomposition into irreducibles.

PROPOSITION 7.4. *In case # A, there are non-trivial integrable highest weight representations of  $B((m, n), \omega)$  only when*

$$\omega = \omega(1, -1, -1, -1, 1, 1, 1).$$

*In this case, consider, for  $a_2, \dots, a_n, \mu_1, \dots, \mu_m$  fixed, a one-parameter family  $\Lambda = \Lambda(\lambda)$  of highest weights satisfying the conditions above. Let  $i_0$  be the smallest integer for which  $a_{n-i_0+1} = 0$ . If  $a_2 \neq 0$ , let  $i_0 = n$ . The highest weight representation corresponding to  $\Lambda(\lambda)$  is then unitarizable exactly in the following cases:*

- (I)  $\lambda \leq 1 - n - \mu_1$ .
- (II) If  $\mu_1 = 0$  then, in addition to the above,  $\lambda = -i_0 + 1 - R$  for an  $R \in \{0, \dots, n - i_0\}$ .

*Proof.* Let us choose the inner product on  $B(m, n)$  to be positive definite on  $B_m$  and negative definite on  $C_n$ . It is easy to see that the elements of the spaces  $Q_2, Q_3$ , and  $P_1$  satisfy that if  $x$  is a root vector in any of these, of weight  $-\beta$ , then

$$(7.21) \quad \begin{aligned} x \in P_1 &\Rightarrow -\beta = \varepsilon_i - \delta_j \quad \text{and} \\ &\quad (h_\beta)_{kk} = -\delta_{k,i} + \delta_{k,2m+2-i} - \delta_{k,2m+1+j} + \delta_{k,2m+2n+2-j}, \\ x \in Q_2 &\Rightarrow -\beta = -\varepsilon_i - \delta_j \quad \text{and} \\ &\quad (h_\beta)_{kk} = \delta_{k,i} - \delta_{k,2m+2-i} - \delta_{k,2m+1-j} + \delta_{k,2m+2n+2-j}, \\ x \in Q_3 &\Rightarrow -\beta = -\delta_j \quad \text{and} \\ &\quad (h_\beta)_{kk} = -\delta_{k,2m+1+j} + \delta_{k,2m+2n+2-j}, \end{aligned}$$

where  $i = 1, \dots, j = 1, \dots, n$ , and  $h_\beta$  is the diagonal matrix which takes part in the Kac determinant.

The element  $\Lambda_\rho$  is then represented by

$$(7.22) \quad \begin{aligned} \Lambda_\rho = &(\mu_1 + m - \frac{1}{2}, \dots, \mu_i + m - i + \frac{1}{2}, \dots, \mu_m + \frac{1}{2}, \\ &\lambda + n - m - \frac{1}{2}, \lambda - a_2 + (n - 1) - m - \frac{1}{2}, \dots, \lambda - a_n + 1 - m - \frac{1}{2}) \end{aligned}$$

which then acts euclideanly on the  $h_\beta$ 's. In case  $h_\beta$  corresponds to an element of either  $P_1$  or  $Q_2$ , then  $\Lambda_\rho(h_\beta)$  is what occurs in the Kac determinant. In case  $h_\beta$  corresponds to an element of  $Q_3$ , the corresponding term is obtained as

$\tilde{\Lambda}_\rho(h_\beta)$  where  $\tilde{\Lambda}_\rho$  is  $\Lambda_\rho$  with  $+\frac{x}{2}$  subtracted on the last  $n$  places, for some integer  $x = 1, \dots$  (since  $(\beta, \beta) \neq 0$  here).

Let us now consider the decomposition of  $V_\Lambda^0$ : Bearing in mind that  $so(2m+1)$  is self-dual, we see that there are two obvious  $u(m)$  types in  $V_\Lambda^0$ , as described in Fig. 7.23:

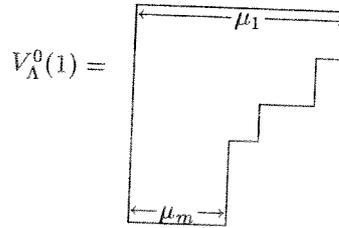


Fig. 7.23 a

and

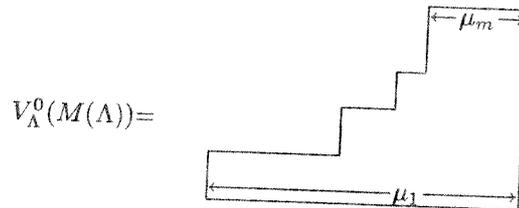


Fig. 7.23 b

Moreover, if we write

$$(7.24) \quad so(2m+1) = N_0^- \oplus u(m) \oplus N_0^+,$$

where  $N_0^-$  and  $N_0^+$  are the span of those lower and upper triangular matrices, respectively, which are not in the  $(u(m), -u(m)^\tau)$  subalgebra, then clearly,

$$(7.25) \quad \begin{aligned} V_\Lambda^0 &= U(N_0^+) \cdot V_\Lambda^0(M(\Lambda)) \\ &= U(N_0^-) \cdot V_\Lambda^0(1). \end{aligned}$$

(Also observe that the  $u(m)$  representation occurring, in general only are integrable to a double covering of  $u(m)$ .)

The way we now proceed is similar to what we did in Chapter 5: We assume first that we have a unitarizable module and get some condition on  $\lambda$ . These

conditions again turn out to be so good-natured that one can directly see that they also are sufficient.

We have already written  $\Lambda(N_A^-)$  in a way that indicates more precisely our way of attack ((7.17)): First we consider  $\Lambda(P_1) \otimes V_A^0$  and make sure that this part is positive definite. Then we consider this as a fixed module  $V_1$  and consider  $V_{13} = \Lambda(Q_3) \otimes V_1$ . Finally we then consider  $\Lambda(Q_2) \otimes V_{13}$ . According to the formulas ((7.16)) it is indeed possible to view the modules  $V_1$  and  $V_{13}$  in this manner, from  $Q_3$  and  $Q_2$ , respectively.

Let us then consider  $\Lambda(P_1) \otimes V_A^0$ . By inspection we see that the computation of the inner product on each  $\Lambda(P_1) \otimes V_A^0(i)$ ,  $i = 1, \dots, M(\Lambda)$ , really takes place inside  $A(n-1, m-1)$ , and then the method and results of Chapter 4 apply. In the current situation we are studying the unitarity of  $\Lambda = \Lambda(\lambda)$  for fixed  $\mu_1, \dots, \mu_m, a_2, \dots, a_n$ . Already, we have made the assumption that  $\lambda$  is sufficiently negative that the "top"  $sp(n, \mathbb{R})$  representation is unitarizable. Now, for each  $i = 1, \dots, M(\Lambda)$ , we get a last possible place of unitarity  $\lambda(P_1, i)$  for  $\lambda$  coming from  $-\infty$ . The element of  $P_1 \otimes V_A^0(i)$  that vanishes at  $\lambda(P_1, i)$  evidently is the one marked by  $i$ 's in Fig. 7.26,

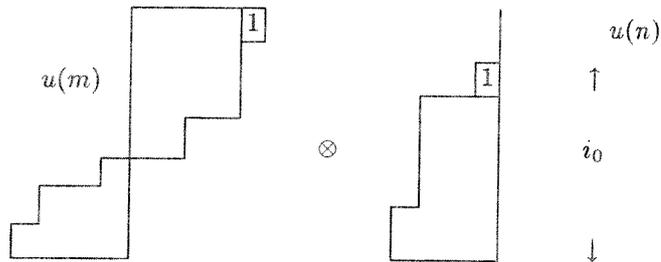


Fig. 7.26

and since  $V_A^0 = U(N_0^-) \cdot V_A^0(1)$  it is clear that

$$(7.27) \quad \lambda(P_1, 1) = -\mu_1 - i_0 + 1 \leq \min\{\lambda(P_1, i) \mid i = 2, \dots, M(\Lambda)\}.$$

Naturally, if we are to have unitarity,  $\lambda$  must be below  $\lambda(P_1, 1)$ ,  $\lambda = \lambda(P_1, 1) - R$ , and for  $R \leq n - i_0$ ,  $R$  must be integer valued,  $R = 0, 1, \dots, n - i_0$ .

Let us now consider  $\Lambda(Q_3)$ . If we are to have a zero in the Kac determinant at  $\lambda = \lambda(P_1, 1) - R$  then it must be in a "high"  $\tilde{i}_0$  (cf. the figure below) for which

$$(7.28) \quad \lambda(P_1, 1) - R + \tilde{i}_0 - m - 1 - \frac{x-1}{2} = 0 \quad (x \geq 0) \quad \Leftrightarrow \\ \tilde{i}_0 = \mu_1 + i_0 + R.$$

This means that at  $\lambda(P_1, 1) - R$  we have the following type in  $\Lambda(Q_3) \otimes V_\Lambda^0$  (Fig. 7.29):

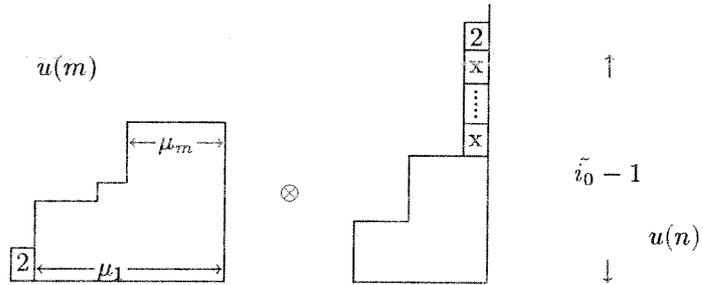


Fig. 7.29

It is clear that we cannot, by throwing in  $\Lambda(P_1)$ , make this  $\tilde{i}_0$ , defined by  $Q_3$ , any bigger. Naturally, this construction is only relevant as long as  $\tilde{i}_0 \leq n$ , i.e.

$$(7.30) \quad \mu_1 + i_0 + R \leq n.$$

In the end we are going to look at last possible places of unitarity for  $\Lambda(Q_2) \otimes V_{13}$ , hence the desire to find the types in  $\Lambda(Q_3) \otimes \Lambda(P_1) \otimes V_\Lambda^0$  whose height is as big as possible. The ultimate height is not necessarily attained at the type described in Fig. 7.29. Consider, namely,  $\Lambda(P_1) \otimes V_\Lambda^0(M(\Lambda))$ . The value  $\lambda(P_1, M(\Lambda))$  is obtained at the diagram below (Fig. 7.31):

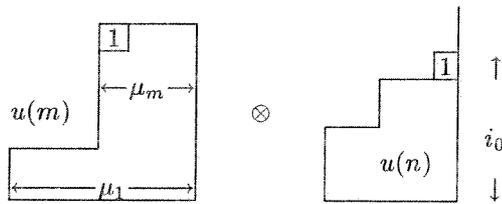


Fig. 7.31

and

$$(7.32) \quad \lambda(P_1, \Lambda(M)) = \mu_m - i_0 + 1.$$

Thus,

$$(7.33) \quad \lambda(P_1, 1) - R = \lambda(P_1, \Lambda(M)) - (R + (\mu_1 + \mu_m))$$

i.e., at  $\lambda = \lambda(P_1, 1) - R$  we are  $R + (\mu_1 + \mu_m)$  units below  $\lambda(P_1, \Lambda(M))$ , and hence the following type (Fig. 7.34) is present in  $\Lambda(P_1) \otimes V_A^0$  at this  $\lambda$ :

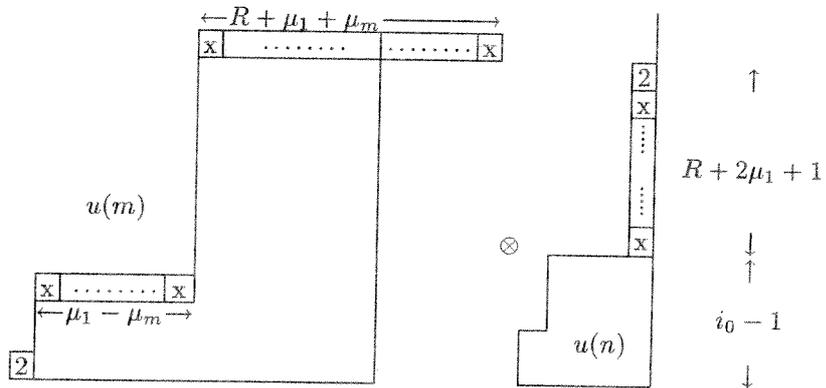


Fig. 7.34

Again, this is only relevant when

$$(7.35) \quad R + 2\mu_1 + i_0 \leq n.$$

(Other diagrams may also give a column of height  $R + 2\mu_1$ , but clearly none will give something strictly greater.)

In case  $R + 2\mu_1 + i_0 \leq n$  and  $m + \mu_1 + i_0 + R \leq n$  we get a last possible place of unitarity,  $\lambda_{2,a}$ , for  $\Lambda(Q_2)$ , as indicated by the 2's above. An easy computation gives

$$(7.36) \quad \lambda_{2,a} = -i_0 - \mu_1 + 1 - R - 2\mu_1.$$

It is clear that if  $\mu_1 \neq 0$  then  $\lambda_{2,a} < \lambda(P_1, 1) - R$  and hence there can be no unitarity, whereas if  $\mu_1 = 0$  then  $\lambda_{2,a} = \lambda_{2,b} + m$  and  $\lambda_{2,b} = \lambda(P_1, 1) - R$ , and this is a possible place of unitarity. We shall need to examine the modules as  $sp(n, \mathbb{R})$  modules. However, when  $\mu_1 = 0$  it is easy to see that the strongest requirement on  $\lambda$  is  $\lambda \leq 1 - i_0 - R$  since we cannot get to any height greater than  $i_0 + R$  in the  $u(n)$  diagram. Hence there is unitarity in this case.

Let us now turn to the case where  $R + 2\mu_1 + i_0 > n$ . Here, the ultimate requirement imposed by  $\bigwedge(Q_2)$  is a  $\lambda_2$  satisfying

$$(7.37) \quad \begin{aligned} \lambda_2 + n - \frac{1}{2} - (-\mu_1 + \frac{1}{2}) &= 0 \Leftrightarrow \\ \lambda_2 &= -\mu_1 - n + 1. \end{aligned}$$

Observe that the signs on the involutions together with the formulas for the inner products (cf. (7.21)) imply that we have unitarity for  $\lambda \rightarrow -\infty$  on all three pieces  $P_1$ ,  $Q_2$ , and  $Q_3$ . To have unitarity, we must thus have

$$(7.38) \quad \lambda(P_1, 1) - R \leq \lambda_2.$$

However, there is a requirement imposed by  $sp(n, \mathbb{R})$ , and it is easy to see that this requirement is that

$$(7.39) \quad \lambda(P_1, 1) - R \leq 1 - n.$$

Thus,

$$(7.40) \quad R \geq n - i_0,$$

and for such an  $R$  we are indeed in the assumed case. Conversely, due to the one-sidedness of the unitarity of  $\lambda(Q_2)$  at  $\lambda_2$ , it is easy to see that these conditions suffice.

This takes care of the situation  $\lambda = \lambda(P_1, 1) - R$  with  $R = 0, \dots, n - i_0$ . Suppose now that  $\lambda < 1 - n - \mu_1$ . Then, all of  $\bigwedge(P_1) \otimes V_\Lambda^0$  is positive definite, and such a  $\lambda$  clearly satisfies the ultimate  $sp(n, \mathbb{R})$  requirement and is less than the ultimate  $\lambda_2$ . Hence, in this range there is unitarity.

We now turn to

**Case #B:**

As in the previous case we see that  $\varepsilon_y = -1$  forces  $\varepsilon_{11} = 1$  and then, by looking at  $\langle q_{11}v_0, q_{11}v_0 \rangle$ , it follows that this possibility is ruled out. Thus

**LEMMA 7.5.** *If there are to be non-trivial unitarizable modules in case #B, then  $\varepsilon_y = 1$ .*

In the present situation,

$$(7.41) \quad N_B^- = \text{span} \{Q_1, Q_2, Q_3\},$$

where the  $Q$ 's are defined as before.

One of the first obstacles is that this is not an  $so(2m+1)$  module.

Let  $N_0^+$  and  $N_0^-$  be as in (7.24), and observe that we have the following relations between commutators

$$(7.42) \quad \begin{aligned} [P_2, Q_1] &\subset Z^+, \\ [Z^+, Q_3] &\subset P_3, \\ [P_2, Q_3] &\subset N_0^+, \text{ and} \\ [P_1, Q_3] &= 0. \end{aligned}$$

(As before, capital letters indicate the spans of the sets of the lower case letters). Let  $\Lambda$  be a given highest weight as in (7.19), satisfying the same condition on the  $a_i$ 's and the  $\mu_j$ 's as before. Let  $V_\Lambda^0(1)$  be as before (Fig. 7.23). Observe that

$$(7.43) \quad \begin{aligned} &N_0^+ U^d(W^-) \otimes \wedge^a(Q_2) \otimes \wedge^b(Q_1) \otimes \wedge^c(Q_3) \otimes V_\Lambda^0(1) \\ &\subseteq U^{d+1}(W^-) \otimes \wedge^{a-2}(Q_2) \otimes \wedge^b(Q_1) \otimes \wedge^c(Q_3) \otimes V_\Lambda^0(1) \\ &+ U^d(W^-) \otimes \wedge^{a-1}(Q_2) \otimes \wedge^{b-1}(Q_1) \otimes \wedge^c(Q_3) \otimes V_\Lambda^0(1) \\ &+ U^d(W^-) \otimes \wedge^{a-1}(Q_2) \otimes \wedge^b(Q_1) \otimes \wedge^{c+1}(Q_3) \otimes V_\Lambda^0(1) \\ &+ U^d(W^-) \otimes \wedge^a(Q_2) \otimes \wedge^{b-1}(Q_1) \otimes \wedge^{c+1}(Q_3) \otimes V_\Lambda^0(1), \end{aligned}$$

where the superscripts  $a, b, c, d$  denote degree. Thus it makes sense to view our highest weight module as a submodule of

$$(7.44) \quad U(N_0^-) \cdot U(W^-) \wedge(Q_2) \otimes \wedge(Q_1) \otimes \wedge(Q_3) \otimes V_\Lambda^0(1),$$

in fact, the submodule is the space generated by the  $so(2m+1)$  finite vectors (if there are any) in  $\wedge(Q_2) \otimes \wedge(Q_1) \otimes \wedge(Q_3) \otimes V_\Lambda^0(1)$ .

Thus, our analysis may begin with examining

$$(7.45) \quad \wedge(Q_2) \otimes \wedge(Q_1) \otimes \wedge(Q_3) \otimes V_\Lambda^0(1)$$

and we can use the same procedure as before. (Also observe that the last space is  $sp(n, \mathbb{R})$  invariant though the individual pieces, of course, are not.)

**PROPOSITION 7.6.** *In case #B there are only non-trivial unitarizable highest weight modules of  $B((m, n), \omega)$  when*

$$\omega = \omega(1, -1, -1, -1, 1, 1, 1).$$

*In this case consider, for  $a_2, \dots, a_n, \mu_1, \dots, \mu_m$  fixed, a one-parameter family  $\Lambda = \Lambda(\lambda)$  of highest weights satisfying the conditions above. Let  $j$  be the largest integer for which  $\mu_j = \mu_1$  and let  $i_0$  be defined for the  $a$ 's as previously.*

The highest weight representation corresponding to  $\Lambda(\lambda)$  then has a unitarizable quotient exactly when

$\mu_m \geq 2n$ . Then

$$\lambda \in \{-\mu_1 - (m - j), \dots, -\mu_1 - m + 1\} \cup ]-\infty, -\mu_1 - m + 1[.$$

*Proof.* Let us again choose the inner product on  $B(m, n)$  to be positive definite on  $B_m$  and negative definite on  $C_n$ . If  $x$  is a root vector in any of the spaces  $Q_1, Q_2, Q_3$ , of weight  $-\beta$ , then it follows easily (cf. the proof of Proposition 7.4) that

(7.46)

$$x \in Q_1 \Rightarrow -\beta = -\varepsilon_i + \delta_j \text{ and}$$

$$(h_\beta)_{kk} = \delta_{k,i} - \delta_{k,2m+2-i} + \delta_{k,2m+1+j} - \delta_{k,2m+2n+2-j},$$

$$x \in Q_2 \Rightarrow -\beta = -\varepsilon_i - \delta_j \text{ and}$$

$$(h_\beta)_{kk} = \delta_{k,i} - \delta_{k,2m+2-i} - \delta_{k,2m+1-j} + \delta_{k,2m+2n+2-j},$$

$$x \in Q_3 \Rightarrow -\beta = -\delta_j \text{ and}$$

$$(h_\beta)_{kk} = -\delta_{k,2m+1+j} + \delta_{k,2m+2n+2-j},$$

for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . The element  $\Lambda + \rho$  is here represented by

$$(7.47) \quad \Lambda_\rho = (\mu_1 - n + m - 1/2, \dots, \mu_i - n + m - (\frac{2i-1}{2}), \dots, \mu_m - n + 1/2, \\ \lambda + n - 1/2, \lambda + n - 1 - 1/2 - a_2, \dots, \lambda + 1 - 1/2 - a_n)$$

which then acts euclideanly on the  $h_\beta$ 's. In case  $h_\beta$  corresponds to an element of either  $Q_1$  or  $Q_2$ , then  $\Lambda_\rho(h_\beta)$  is what occurs in the Kac determinant. In case  $h_\beta$  corresponds to an element of  $Q_3$ , the corresponding term is obtained as  $\tilde{\Lambda}_\rho(h_\beta)$  where  $\tilde{\Lambda}_\rho$  is  $\Lambda_\rho$  with  $x/2$  subtracted (i.e.  $-x/2$  added),  $x = 1, \dots$ , from the last  $n$  places.

We now continue where we were before by looking at  $\Lambda(Q_1) \otimes \Lambda(Q_3) \otimes V_\Lambda^0(1)$ . The last possible place of unitarity,  $\lambda_1(Q_1)$ , of  $\Lambda(Q_1) \otimes V_\Lambda^0(1)$  is obtained for the boxes marked by 1's in the diagram below, and satisfies

$$(7.48) \quad (\lambda_1(Q_1) + n - 1/2) + (\mu_j - n + m - j + 1/2) = 0 \Leftrightarrow \\ \lambda_1(Q_1) = -\mu_j - (m - j).$$

Unitarity then implies that  $\lambda_1(Q_1) \leq \lambda_2(Q_2)$ , i.e.

$$(7.64) \quad 2\mu_1 \geq +i_0,$$

and we must have  $\mu_1 \geq i_0 - 1$ . But then in fact  $\mu_1 \geq 2(i_0 - 1)$  since we can have a column of length  $i_0 - 1$  both from  $\bigwedge(Q_2) \otimes \bigwedge(Q_3) \otimes V_\Lambda^0(1)$  and from  $\bigwedge(Q_1) \otimes \bigwedge(Q_2) \otimes \bigwedge(Q_3) \otimes V_\Lambda^0(1)$ . Now, unless  $i_0 = 1$  this implies that  $\lambda < -i_0 + 1$ , in which case we get more non-zero elements in  $\bigwedge(Q_3) \otimes V_\Lambda^0(1)$  which forces  $\mu_1$  to be even greater, forcing more non-zero elements of  $\bigwedge(Q_3) \otimes V_\Lambda^0(1)$  until, finally,  $\mu_1$  is forced above  $n - 1$ . The only remaining case is  $i_0 = 1$  and  $\mu_1 = 0$ . This is the trivial representation.

The whole discussion of unitarity would of course not be complete without an examination of the  $sp(n, \mathbb{R})$  modules defined by the part of  $\bigwedge(Q_2) \otimes \bigwedge(Q_1) \otimes \bigwedge(Q_3) \otimes V_\Lambda^0(1)$  which at a given  $\lambda$  is non-zero (and positive definite). At a value  $\lambda = -\mu_1 - (m - j) - R$  in the discrete range, the most serious demand on  $\lambda$  to make all these modules unitary is  $\lambda \leq 1 - n - (m - j) - R$ . Hence this is fulfilled. Unitarity follows in a similar manner in the continuous range of  $\lambda$ .

We have now proved that  $\bigwedge(Q) \otimes V_\Lambda$ , with  $Q = Q_2 \oplus Q_1 \oplus Q_3$ , defines finite dimensional  $so(2m + 1)$  modules. More precisely, there is an  $so(2m + 1)$  invariant ideal  $I$ ,

$$(7.65) \quad I \subseteq U(N_0^-) \otimes \bigwedge(Q) \cdot v_\Lambda$$

such that

$$(7.66) \quad U(N_0^-) \otimes \bigwedge(Q) \cdot v_\Lambda / I$$

is a finite sum of  $so(2m + 1)$  modules. To finish the construction we must then consider the left  $U(G)$  ideal  $J$  generated by  $I$  inside  $U(G) \cdot v_\Lambda$  since the space which carries our unitarizable module is

$$(7.67) \quad U(G) \cdot v_\Lambda / J.$$

It thus remains to make sure that  $J$  is not actually equal to the whole space.

It is easy to see that the last eventuality will occur exactly if

$$(7.68) \quad v_\Lambda \in \bigwedge(P) \cdot I,$$

where

$$(7.69) \quad P = P_1 \oplus P_2 \oplus P_3.$$

However, the last statement is, by duality, tantamount to having

$$(7.70) \quad \bigwedge(Q) \cdot v_\Lambda \cap I \neq \{0\}$$

and, more precisely, to have a vector  $\hat{v}$

$$(7.71) \quad \hat{v} \in \bigwedge(Q) \cdot v_\Lambda$$

such that  $\hat{v}$  is a highest weight vector for  $so(2m+1)$  ( $I$  is generated by such vectors) and whose weight is not dominant (cf. the Bernstein–Gelfand–Gelfand Theorem [1]). But we have exactly made sure that the parameter  $\Lambda$  is such that *all the representations in  $\bigwedge(Q) \cdot v_\Lambda$  are dominant for  $so(2m+1)$* , hence this eventuality does not materialize.  $\square$

**THE UNITARIZABLE HIGHEST  
WEIGHT MODULES OF  $D(m, n)$**

By definition,

$$(8.1) \quad D(m, n) = osp(2m, 2n), \quad m \geq 2, \quad n > 0.$$

We thus can, and will, take over the notation and conventions already adopted for  $B(m, n)$ . It might also be thought that the results more or less could be taken over but this is not quite the case. For one thing, there is one more series of unitarizable modules, corresponding to another involution, under case #B. Moreover, since the even part satisfies

$$(8.2) \quad D(m, n)_0 = D_m \oplus C_n,$$

we need to consider  $D_m$  not only as the complex Lie algebra corresponding to the hermitian symmetric space based on  $so(2m-2, 2)$  but also as corresponding to that based on  $so^*(2m)$ .

We begin by the cases that correspond to #A and #B for  $B(m, n)$ . Thus we consider matrices of the form (7.4) with the rows and columns corresponding to  $p_{32}$  and  $q_{32}$  deleted. It is easily seen that Lemma 7.1 is still true, once the  $-\varepsilon_{32}$  has been deleted. For clarity we write explicitly what we now mean by the cases #A and #B:

**Case #A.**

$$(8.3) \quad \Delta_1^+ = \{\delta_i + \varepsilon_j \mid i = 1, \dots, n \text{ and } j = i, \dots, m\} \\ \cup \{\delta_i - \varepsilon_j \mid i = 1, \dots, n \text{ and } j = 1, \dots, m\}.$$

The space of positive root vectors in  $G_{\bar{1}}$  is

$$(8.4) \quad N_A^+ = \text{span} \{p_{12}, p_{22}, q_{21}, q_{11}\}.$$

**Case #B.**

$$(8.5) \quad \Delta_1^+ = \{\varepsilon_j - \delta_i \mid i = 1, \dots, n \text{ and } j = 1, \dots, m\} \\ \cup \{\varepsilon_j + \delta_i \mid i = 1, \dots, n \text{ and } j = 1, \dots, m\}.$$

The space of positive root vectors in  $G_{\bar{1}}$  is

$$(8.6) \quad N_B^+ = \text{span} \{p_{11}, p_{12}, p_{21}, p_{22}\}.$$

Let us then begin with

**Case #A:**

By looking at  $\langle p_{11}v_0, p_{11}v_0 \rangle$  it follows easily that  $\varepsilon_{11} = -1$  if  $\varepsilon_{11} = -1$ . (The conclusion of Lemma 7.3 was based on Lemma 7.2 whose analogue does not exist here.)

As in the case of  $B(m, n)$  we consider a highest weight representation  $V_\Lambda$  of highest weight

$$(8.7) \quad \Lambda = (\mu_1, \dots, \mu_m, \lambda, \lambda - a_2, \dots, \lambda - a_n)$$

and it follows that the element  $\Lambda + \rho$  is represented by

$$(8.8) \quad \Lambda_\rho = (\mu_1 + m - 1, \mu_2 + m - 2, \dots, \mu_m, \\ \lambda + n - m, \lambda - a_2 + n - 1 - m, \dots, \lambda - a_n + 1 - m)$$

(cf. the proof of Proposition 7.4).

Let us begin by assuming that  $\varepsilon_y = -1$ .

It then follows from Lemma 7.1 that  $\varepsilon_{21} = \varepsilon_{12} = +1$ . This leads to no unitarity:

**LEMMA 8.1.** *In case #A there are no unitarizable modules, other than the trivial, corresponding to an involution with  $\varepsilon_y = -1$ .*

*Proof.* Consider  $P_{21} \wedge V_\Lambda^0$  as an  $u(m-1) \times su(n)$  module. The decomposition of this into irreducibles was discussed in Chapter 7. Let us for convenience assume that  $m-1$  is even and that  $\mu_r \geq 0$ . The general case follows analogously. Then we have the following type (Fig. 8.9) inside  $P_{21} \wedge V_\Lambda^0$ ,

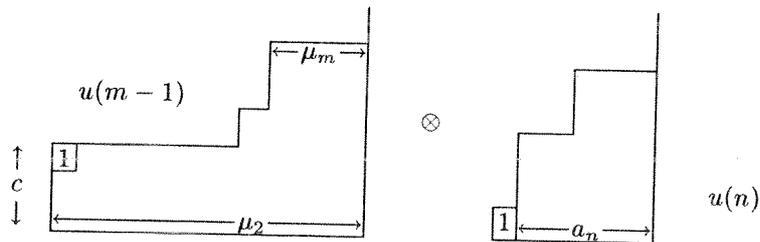


Fig. 8.9

where the “1”’s indicate the contribution from  $P_{21}$ . This type vanishes at a  $\tilde{\lambda}_{21}$  satisfying (see Fig. 8.9 for the definition of  $c$ )

$$(8.10) \quad \tilde{\lambda}_{21} - a_n + 1 - m + (-\mu_2) + c - 1 = 0 \quad \Leftrightarrow \\ \tilde{\lambda}_{21} = \mu_2 + a_n + m - c.$$

Since this is clearly non-negative and since we can only have unitarity for  $\lambda$  greater than or equal to this value (since  $\varepsilon_{21} = 1$ ), we get at most the trivial representation.  $\square$

We will now assume that  $\varepsilon_y = 1$  and then there is no reason to distinguish between  $p_{11}$  and  $p_{12}$ , so we switch to the notation with  $P_1, P_2, Q_1$  and  $Q_2$  as in Chapter 7.

Observe that the proof of Lemma 8.1 actually shows that we must always have  $\varepsilon_{21} = -1$ . Thus, we are back to the situation that was treated in case #A in Chapter 7. We assume from now on that  $\mu_1, \dots, \mu_m, a_2, \dots, a_n$  are integers satisfying  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{m-1} \geq |\mu_m|$  and  $a_n \geq a_{n-1} \geq \dots \geq a_2 \geq 0$ .

**PROPOSITION 8.2.** *In case #A there are only non-trivial unitarizable highest weight representations of  $D((m, n), w)$  when, in the involution  $w$ ,  $\varepsilon_y = 1$  and  $\varepsilon_{21} = -1$ . In this case consider, for  $\mu_1, \dots, \mu_m, a_2, \dots, a_n$  fixed and as above, a one-parameter family  $\Lambda = \Lambda(\lambda)$  of highest weights. Let  $i_0$  be the smallest integer for which  $a_{n-i_0+1} = 0$ . If  $a_2 \neq 0$ , let  $i_0 = n$ . The highest weight representation corresponding to  $\Lambda(\lambda)$  is then unitarizable exactly in the following cases:*

- (I) If  $\mu_1 \neq 0$  then for  $\lambda \leq 1 - n - \mu_1$ .
- (II) If  $\mu_1 = 0$  then for  $\lambda \in \{-i_0 + 1, \dots, -n + 1\} \cup ]-\infty, -n + 1[$ .

*Proof.* This proceeds in analogy with the proof of Proposition 7.4 with the simplification that “ $Q_3$ ” is not present. First, the smallest of the last possible places of unitarity in  $\Lambda(P_1) \otimes V_\Lambda^0$  is taken at the same diagram as for Proposition 7.4 and the value is the same,

$$(8.11) \quad \lambda(P_1, 1) = -\mu_1 - i_0 + 1.$$

Observe that what corresponds to  $V_\Lambda^0(M(\lambda))$  in the present situation is Fig. 8.12:

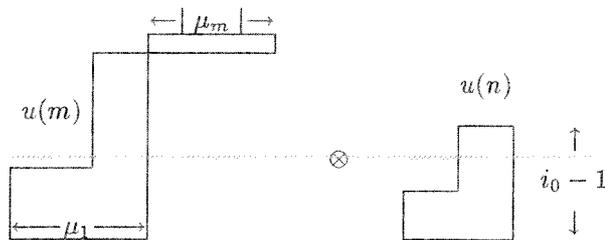


Fig. 8.12

This follows from the fact that when  $m$  is odd then  $o(2m)$  is not self dual. The case where  $n \geq i_0 + R + 2\mu_1$  again (with one exception) does not lead to unitarity. This follows since the ultimate demand on  $\lambda$  coming from  $sp(n, \mathbb{R})$  is

$$(8.13) \quad \lambda = \lambda(P_1, 1) - R \leq 1 - i_0 - R - 2\mu_1 \Leftrightarrow$$

$$2\mu_1 \leq \mu_1.$$

Hence only the trivial  $o(2m)$  representation survives this demand. However, for it, we get a  $\lambda_2$  from  $\Lambda(Q_2)$  of value

$$(8.14) \quad \lambda_2 = 1 - i_0 - R$$

and this is greater than  $\lambda_1 - R$ , hence all requirements are met.

The case  $n < i_0 + R + 2\mu_1$  gives an ultimate  $\lambda_2 = \lambda_2(Q_2)$  from  $\Lambda(Q_2)$  of value

$$(8.15) \quad \lambda_2(Q_2) = -n + 1 - \mu_1,$$

hence the requirement becomes  $R \geq n - i_0$  whereas the  $sp(n, \mathbb{R})$  requirement as before gives  $R \geq n - i_0 - \mu_1$ . Of these, the first requirement is of course the strongest.  $\square$

#### Case #B:

We consider a highest weight module  $V_\Lambda$  of highest weight

$$(8.16) \quad \Lambda = (\mu_1, \dots, \mu_m, \lambda, \lambda - a_2, \dots, \lambda - a_n).$$

The element  $\Lambda + \rho$  is then seen to be represented by

$$(8.17) \quad \Lambda_\rho = (\mu_1 - n + m - 1, \mu_2 - n + m - 2, \dots, \mu_m - n, \\ \lambda + n, \lambda + n - 1 - a_2, \dots, \lambda + 1 - a_n).$$

By looking at  $\langle q_{11}v_0, q_{11}v_0 \rangle$  it follows again that if  $\varepsilon_y = -1$  then  $\varepsilon_{11} = -1$ .

LEMMA 8.3. *In case #B there are no unitarizable modules, other than the trivial, corresponding to an involution with  $\varepsilon_y = -1$ .*

*Proof.* Assume  $\varepsilon_y = -1$ . Then  $\varepsilon_{11} = -1$  and  $\varepsilon_{21} = 1$ . We consider the module  $Q_{21} \wedge V_\Lambda^0$  as an  $u(m-1) \times su(n)$  module. The biggest value of  $\lambda$ ,  $\lambda_1$ , for which a type is vanishing clearly satisfies

$$(8.18) \quad \lambda_1 + s_0 - a_n + \mu_m - n = 0,$$

where  $s_0$  is the biggest positive integer for which  $a_n = a_{n+1-s_0}$ . Observe that if  $\mu_{m-1} = -\mu_m$  then we *must* be at this value in order to get finite dimensional  $u(m-1)$  modules (Fig. 8.19). Thus,

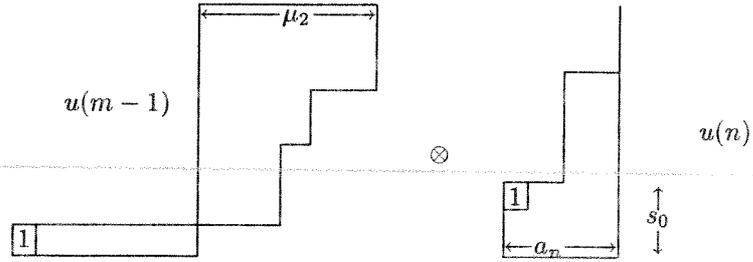


Fig. 8.19

$$(8.20) \quad \lambda_1 = a_n - s_0 - \mu_m + n.$$

Now consider  $Q_{12} \wedge \wedge(Q_{21}) \otimes V_\Lambda^0$ . Since  $P_{12} \wedge \wedge(Q_{21})V_\Lambda^0 = 0$  we can easily compute an appropriate value of  $\lambda, \lambda_2$ , for which a type is vanishing from this module (keeping  $\wedge(Q_{21}) \otimes V_\Lambda^0$  fixed): If  $\mu_2 \neq -\mu_m$  or if  $a_n > 0$  the following type is in  $Q_{21} \otimes V_\Lambda^0$  and is non-zero (indicated by 2's in Fig. 8.21):

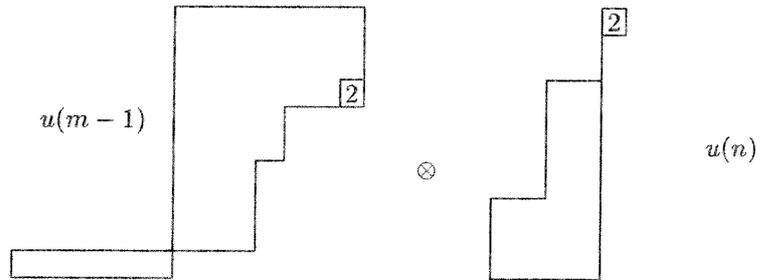


Fig. 8.21

Hence, using the "2" from  $su(n)$ ,

$$(8.22) \quad \lambda_2 + n - (\mu_1 + m - 1) = 0 \Leftrightarrow$$

$$\lambda_2 = \mu_1 - n + m - 1.$$

Now, evidently, to have unitarity we must have  $\lambda_1 \leq \lambda_2$ , that is,

$$(8.23) \quad a_n - s_0 - \mu_m + n \leq \mu_1 - n + m - 1 \quad \Leftrightarrow$$

$$\mu_1 + \mu_m \geq a_n + 2n + 1 - s_0 - m,$$

and this is in a range of  $\mu_1$  for which the  $so(2m-2, 2)$  module is *not* unitarizable (cf. Chapter 3). Finally, if  $a_n = 0$  and  $\mu_2 = -\mu_m$  then  $\lambda_1 = -\mu_m \geq 0$ , and this leads to the trivial  $sp(n, \mathbb{R})$  module. We must then have  $\lambda_2 \geq 0$ . In this case  $\lambda_2$  is easily computed to be  $\lambda_2 = \mu_1 + m - 1$ , and thus  $\mu_1 \geq 1 - m$ . This is still a possible place of unitarity for  $so(2m-2, 2)$ . However, one can here use  $\Lambda(Q_{12})$  or  $\Lambda(Q_{11})$  to produce some non-trivial  $sp(n, \mathbb{R})$  modules which is in conflict with  $\lambda = 0$ .  $\square$

We now assume that  $\varepsilon_y = 1$  and again switch to the notation with  $P_1, P_2, Q_1$ , and  $Q_2$  as in Chapter 7. Let us first consider the case in which  $\varepsilon_{21} = 1$ . As in the previous cases, this can be ruled out, however, it is a little more complicated. Let us assume that  $\mu_1, \dots, \mu_m, a_2, \dots, a_n$  are integers satisfying  $\mu_1 \geq \mu_2 \geq \dots \geq |\mu_m|$  and  $a_n \geq a_{n-1} \geq \dots \geq a_2 \geq 0$ .

**PROPOSITION 8.4.** *In case #B, there are no non-trivial unitarizable modules based on involutions in which  $\varepsilon_y = 1$  and  $\varepsilon_{21} = 1$ .*

*Proof.* As in Chapter 7 let  $N_0^+$  and  $N_0^-$  be defined as those positive and negative, respectively, root vectors in  $o(2m)$  which are not in the  $u(m)$  subalgebra. Again we have a decomposition of  $V_\Lambda^0$  into irreducible  $u(m) \times u(n)$  modules,

$$(8.24) \quad V_\Lambda^0 = \sum_{i=1}^{M(\Lambda)} V_\Lambda^0(i),$$

where  $V_\Lambda^0 = U(N_0^-) \cdot V_\Lambda^0(1) = U(N_0^+) \cdot V_\Lambda^0(M(\Lambda))$ . Again we shall then consider our highest weight module as a submodule of

$$(8.25) \quad U(N_0^-)U(W^-) \Lambda(Q_2) \otimes \Lambda(Q_1) \otimes V_\Lambda^0(1).$$

In this connection observe that to settle the issue of finite dimensionality of the  $o(2m)$  modules we must examine the full content of  $\Lambda(Q_2) \otimes \Lambda(Q_1) \otimes V_\Lambda^0(1)$ . In the present situation, however, most of the restrictions on the parameters follow by looking at  $\Lambda(Q_1) \otimes V_\Lambda^0(1)$  alone.

We first investigate  $Q_1 \wedge V_\Lambda^0$ . The biggest value at which a type is vanishing from  $Q_1 \wedge V_\Lambda^0$  is clearly obtained at the place indicated by 1's in Fig. 8.26 below. The corresponding value of  $\lambda$ ,  $\lambda_1$ , satisfies

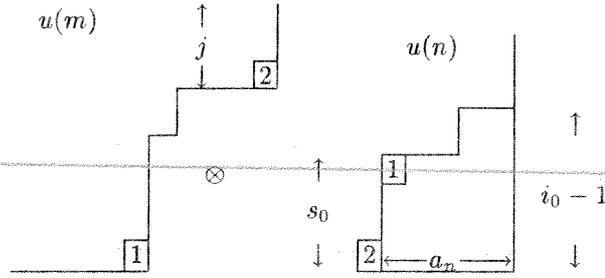


Fig. 8.26

$$(8.27) \quad \begin{aligned} \lambda_1 - a_n + s_0 + \mu_m - n &= 0 && \Leftrightarrow \\ \lambda_1 &= a_n - s_0 + n - \mu_m. \end{aligned}$$

Naturally, we must have  $\lambda \geq \lambda_1$ , and since we have the following element in  $Q_1 \wedge V_\Lambda^0$  (Fig. 8.28),

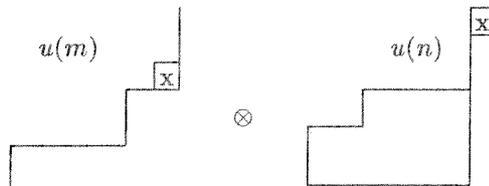


Fig. 8.28

the fact that this has got to be a unitary  $sp(n, \mathbb{R})$  module forces

$$(8.29) \quad \lambda \leq 1/2 - (i_0 + n)/2.$$

(Typically, there will be even stronger requirements, but the above suffices for here). Thus,

$$(8.30) \quad \mu_m \geq n + a_n - s_0 - 1/2 + (i_0 + n)/2.$$

The type in  $Q_2 \wedge \wedge(Q_1) \otimes V_\Lambda^0$  indicated by 2's in Fig. 8.26 vanishes at a  $\lambda_2$  satisfying

$$(8.31) \quad (\lambda_2 - a_n) - (\mu_1 + m - j) = 0.$$

(There may be others which vanish at even bigger values). Thus

$$(8.32) \quad \lambda_2 = \mu_1 + (m - j) + a_n.$$

Since both  $\lambda \leq 0$  and  $\lambda \geq \lambda_2$  are required by unitarity, only the trivial representation survives this.

PROPOSITION 8.5. In case #B there are unitarizable modules based on an involution in which  $\varepsilon_y = 1$  and  $\varepsilon_{21} = -1$  exactly in the cases described below: Let  $t$  be the biggest integer for which  $\mu_t \neq \mu_m$  ( $t \geq j$ ), let  $i_0$  be defined in terms of the  $a_i$ 's as before, and define  $j_0$  as follows: If  $a_{n-i_0+2} > 1$  or if  $i_0 = 1$ , then  $j_0 = i_0$ . Otherwise,  $j_0$  is the biggest integer less than  $i_0$  for which  $a_{n-j_0+2}$  is different from  $a_{n-i_0+2}$ , if such a  $j_0$  exists, if not, set  $j_0 = 1$ .

(I) Let  $R = 0, 1, \dots, j - 1$ , and assume that  $R + (m - j) \geq 2$ . Then for

$$\lambda = -\mu_1 - m + j - R, \quad \text{provided that}$$

$$|\mu_m - 2n| \leq \mu_t \quad (\text{and } |\mu_m| \leq \mu_t).$$

(II) For

$$\lambda < -\mu_1 - m + 1, \quad \text{provided that}$$

$$|\mu_m - 2n| \leq \mu_t \quad (\text{and } |\mu_m| \leq \mu_t).$$

*Proof.* Again we view our module as a submodule of

$$(8.50) \quad U(N_0^-)U(W^-) \wedge(Q_2) \otimes \wedge(Q_1) \otimes V_\Lambda^0(1).$$

Consider  $Q_1 \wedge V_\Lambda^0(1)$ . Here, the smallest value at which a representation vanishes is clearly at the  $\lambda_1$  satisfying

$$(8.51) \quad \lambda_1 + n + (\mu_1 - n + m - j) = 0 \quad \Leftrightarrow \quad \lambda_1 = -\mu_1 - (m - j),$$

and this is obtained at the representation marked by 1's below in Fig. 8.52:

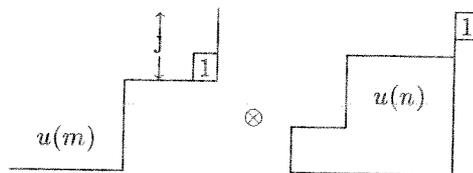


Fig. 8.52

Unitarity forces  $\lambda \leq \lambda_1$ . Let us then write  $\lambda = \lambda_1 - R$  for some non-negative integer  $R$  ( $R \leq j$ ). Assume first that  $R + (m - j) \geq 2$ . In this case, inside  $\wedge(Q_1) \otimes V_\Lambda^0(1)$  we have the following representation (Fig. 8.53),

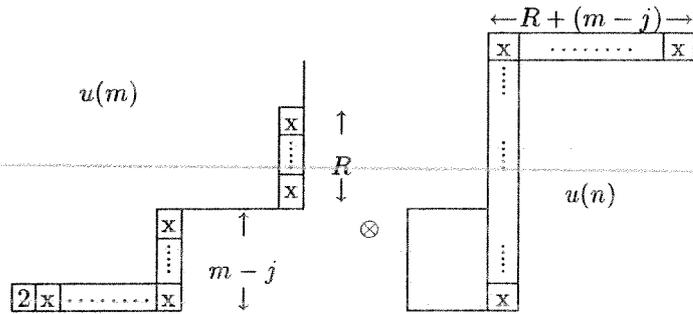


Fig. 8.53

which means that there is a type in  $Q_2 \wedge \Lambda(Q_1) \otimes V_\Lambda^0(1)$ , the one indicated by 2's above, which vanishes at a  $\lambda_2$  satisfying

$$(8.54) \quad \begin{aligned} \lambda_2 + n - 1 + R + (m - j) - (\mu_m - n) &= 0 & \Leftrightarrow \\ \lambda_2 &= \mu_m - 2n - R - (m - j) + 1. \end{aligned}$$

This is the last possible place of unitarity, in this range of  $\lambda$ , for  $Q_2 \wedge \Lambda(Q_1) \otimes V_\Lambda^0(1)$ .

Again unitarity implies that  $\lambda \leq \lambda_2$ , that is,

$$(8.55) \quad \mu_m - 2n + 1 - R - (m - j) \geq -\mu_1 - (m - j) - R \quad \Leftrightarrow \quad \mu_1 + \mu_m \geq 2n - 1.$$

With  $t$  defined as above, we have by assumption  $|\mu_m| \leq \mu_t$ . Since we must have that all types in  $\Lambda(Q_1) \otimes V_\Lambda^0(1)$  define finite dimensional  $so(2m)$  modules, we must also have that  $|\mu_m - n| \leq \mu_t$  as implied by e.g. the type in the diagram above. Furthermore, it is easy to see (since  $\lambda_2 \geq \lambda_1$ ) that there is a type in  $\Lambda(Q_2) \otimes \Lambda(Q_1) \otimes V_\Lambda^0(1)$  which subtracts  $2n$  from  $\mu_m$  and hence  $|\mu_m - 2n| \leq \mu_t$ . This condition, on the other hand, clearly implies that  $\mu_1 + \mu_m \geq 2n - 1$ . The ultimate requirement imposed on  $\lambda$  by viewing  $\Lambda(Q_2) \otimes \Lambda(Q_1) \otimes V_\Lambda^0(1)$  as an  $sp(n, \mathbb{R})$  module is  $\lambda \leq -R - (m - j) - n$ , and this is clearly met here.

In the case where  $R + (m - j) > m$  we can only push the  $\lambda_2$  from  $\Lambda(Q_2) \otimes (\Lambda(Q_1) \otimes V_\Lambda^0(1))$  to  $\lambda_2 = \mu_m - 2n - m + 1$ , and the condition becomes  $\mu_1 + \mu_m \geq 2n + 1 - R + j$ . Still at this level there are enough types in  $\Lambda(Q_2) \otimes \Lambda(Q_1) \otimes V_\Lambda^0(1)$  to restrict  $\mu_m$  as indicated. We observe, in passing, that we here are below the discrete part of the allowed values of  $\lambda$  where some types in  $\Lambda(Q_2) \wedge \Lambda(Q_1) \otimes V_\Lambda^0(1)$  are missing.

Now suppose that  $R = 0$  and  $m = j - 1$ . Then the following diagram gives the ultimate  $\lambda_2$  (Fig. 8.56):

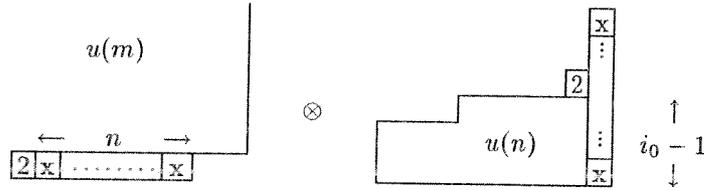


Fig. 8.56

That is,

$$(8.57) \quad \begin{aligned} \lambda_2 + (i_0 - 1) - (\mu_m - n) &= 0 \Leftrightarrow \\ \lambda_2 &= \mu_m - n - i_0 + 1. \end{aligned}$$

Since  $\lambda = \lambda_1$  we must have  $\lambda_1 \leq \lambda_2$ , hence

$$(8.58) \quad \begin{aligned} \mu_m - n - i_0 + 1 &\geq -\mu_1 - 1 \Leftrightarrow \\ \mu_1 + \mu_m &\geq n + i_0 - 2. \end{aligned}$$

Let us now assume that  $\lambda_2 = \lambda_1 + \Delta$  for some positive integer  $\Delta$ . Then we have a representation in  $\wedge(Q_2) \otimes \wedge(Q_1) \otimes V_\Lambda^0(1)$  whose  $so(2m)$  content is as in Fig. 8.59,

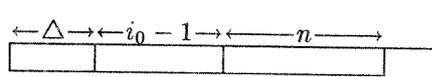


Fig. 8.59

and hence we must have

$$(8.60) \quad |\mu_m - n - (i_0 - 1) - \Delta| \leq \mu_1.$$

However, solving for  $\mu_1 + \mu_m$  we see that

$$(8.61) \quad \mu_1 + \mu_m = n + i_0 - 2 + \Delta$$

and hence, this requirement is not met.

The case  $(m - j) = 0$  and  $R = 1$  follows by the exact same analysis.

Finally, suppose that  $m - j = 0$  and  $R = 0$ . Let us for simplicity assume that  $i_0 = j_0$ , the other case can be treated analogously. It follows that

$$(8.62) \quad \lambda = \lambda_1 = -\mu_1$$

and

$$(8.63) \quad \lambda_2 = \mu_1 - 2i_0 + 2$$

(cf. Fig. 8.64 below).

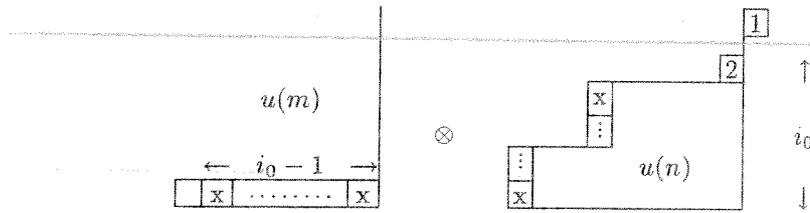


Fig. 8.64

Since  $\lambda_2 \geq \lambda_1$  is implied by unitarity, it follows that

$$(8.65) \quad 2\mu_1 \geq 2i_0 - 2.$$

In case  $\lambda_2 = \lambda_1 + \Delta$  for some positive integer  $\Delta$  we have diagrams 8.66 and 8.67 to extremize the  $so(2m)$  and  $sp(n, \mathbb{R})$  conditions, respectively.

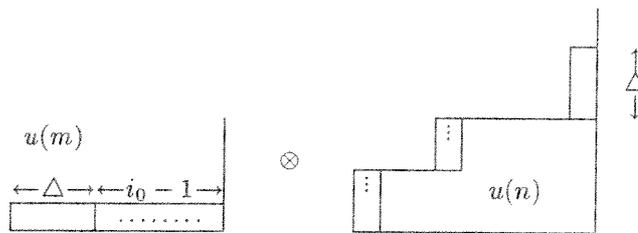


Fig. 8.66

and

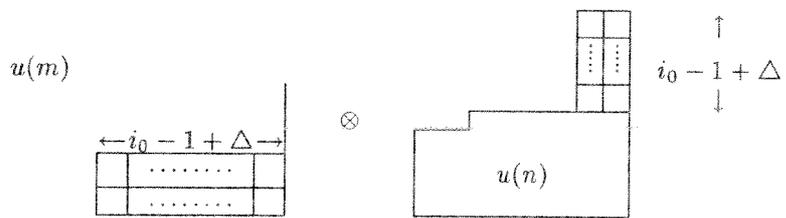


Fig. 8.67

From the first diagram we get

$$(8.68) \quad 2\mu_1 \geq i_0 - 1 + \Delta$$

and from the other

$$(8.69) \quad \lambda + 2(i_0 - 1) + \Delta + 1 \leq 1 \quad \Leftrightarrow \\ \mu_1 \geq 2i_0 - 2 + \Delta.$$

However, since  $\lambda_2 = \lambda_1 + \Delta$  we get that

$$(8.70) \quad 2\mu_1 = 2i_0 - 2 + \Delta$$

and hence, these conditions are not met.  $\square$

As remarked in the beginning of this chapter, there is another possibility for getting unitarizable highest weight modules of  $D_m$ , namely by viewing it as corresponding to  $so^*(2m)$ . This is the case to which we now turn.

We write elements of  $B(m, n)$  as

$$(8.71) \quad \begin{pmatrix} \alpha_1 & z_1^+ & p_1 & p_2 \\ w_1^- & -\alpha_1^r & q_2^r & -q_1^r \\ q_1 & -p_2^r & \beta & z_2^+ \\ q_2 & p_1^r & w_2^r & -\beta^r \end{pmatrix}$$

with  $(z_1^+)^r = -z_1^+$ ,  $(w_1^+)^r = -w_1^+$ ,  $(z_2^+)^r = z_2^+$ , and  $(w_2^+)^r = w_2^+$ . Let  $\varepsilon_{p_1}$ ,  $\varepsilon_{p_2}$ ,  $\varepsilon_{z_1}$ , and  $\varepsilon_{z_2}$  be  $\pm 1$  and let  $w$  be the anti-linear anti-involution which is complex conjugation followed by the appropriate sign, e.g.  $w(p_1) = \varepsilon_{p_1} p_1^*$ . By computing  $w([z_1^+, q_2])$  and  $w([z_2^+, p_1])$  in two different ways, we easily get that

$$(8.72) \quad \varepsilon_{p_1} = -\varepsilon_{p_2} \varepsilon_{z_1}, \text{ and} \\ \varepsilon_{p_1} = \varepsilon_{p_2} \varepsilon_{z_2}.$$

Thus, the  $so^*(2m)$  condition,  $\varepsilon_{z_1} = -1$ , forces the involution on  $C_n$  to be the compact one. This is the one we consider in the sequel.

To begin with, we observe that there still are two cases corresponding to different non-conjugate choices of positive root vectors in  $G_{\bar{1}}$ , and these we denote by  $A^*$  and  $B^*$  ( $A^*$  is the previous  $A$ , etc.) Thus,

Case #A\*:

$$(8.73) \quad N_A^+ = \text{span} \{P_2, Q_1\}.$$

Case #B\*:

$$N_B^+ = \text{span} \{P_1, P_2\}.$$

In each of these cases we then need to investigate the two cases  $\varepsilon_{p_1} = \pm 1$ . We shall then, in analogy with the previous cases, consider a one-parameter family of highest weights

$$(8.74) \quad \mathcal{M}(\mu) = (\mu, \mu - b_2, \dots, \mu - b_m, \lambda_1, \dots, \lambda_n),$$

where the  $b_i$ 's are integers satisfying  $b_m \geq \dots \geq b_2 \geq 0$ , and the  $\lambda_j$ 's are integers satisfying  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ .

It is, in fact, clear that we can proceed in analogy with the cases  $A$  and  $B$ , more or less just exchanging  $n$  and  $m$  (and exchanging  $A$  with  $B^*$ , etc.) Of course, one needs to change the parameters appropriately and to pay attention to the differences in the sets of allowed parameters for the  $sp(n, \mathbb{R})$  and  $so^*(2m)$  modules as well as for the  $sp_n$  and  $o(2m)$  modules. Furthermore, the case  $n = 1$  in this analogy corresponds to a case with  $m = 1$ , and this corresponds to a different series of superalgebras, namely  $C_n$ . (See Chapter 9.)

However, with all these concerns taken care of, one obtains the following results. We omit the details.

**PROPOSITION 8.6.** *In case #A\* there are no non-trivial unitarizable highest weight modules for which  $\varepsilon_{z_1} = -1$  and  $\varepsilon_{p_1} = \varepsilon_{p_2} = 1$ .*

In analogy with the  $sp(n, \mathbb{R})$  case, let  $\tilde{i}_0$  be the smallest integer for which  $b_{m-\tilde{i}_0+1} = 0$ . If  $b_2 \neq 0$ , let  $\tilde{i}_0 = m$ . If  $b_{m-\tilde{i}_0} > 1$  or if  $\tilde{i}_0 = 1$ , define  $\tilde{j}_0 = \tilde{i}_0$ . Otherwise,  $\tilde{j}_0$  is defined to be the biggest integer less than  $\tilde{i}_0$  for which  $b_{m-\tilde{j}_0+1} \neq b_{m-\tilde{i}_0}$  if there are solutions, if there are no solutions we set  $\tilde{j}_0 = 1$ . In addition, we extend the previous definition of  $i_0$  to be the smallest integer for which  $\lambda_1 = \lambda_{n-i_0+1}$ .

**PROPOSITION 8.7.** *In case #A\* there are unitarizable modules based on an involution in which  $\varepsilon_{z_1} = -1, \varepsilon_{p_1} = \varepsilon_{p_2} = -1$  only when  $\lambda_n \geq 2m$ . In this case, there is exactly for:*

- (I)  $\mu \in ] -\infty, 1 - \lambda_1 - n[$ .
- (II)  $\mu = 1 - i_0 - \lambda_1 - R$  for  $R = 0, 1, \dots, n - i_0$  provided that  $R + i_0 \geq 3$ .

**PROPOSITION 8.8.** *In cases #B\* there are no non-trivial unitarizable modules based on involutions with  $\varepsilon_{z_1} = -1$  and  $\varepsilon_{p_1} = \varepsilon_{p_2} = 1$ .*

**PROPOSITION 8.9.** *In case B\* there are non-trivial unitarizable modules based on the involution in which  $\varepsilon_{z_1} = -1$  and  $\varepsilon_{p_1} = \varepsilon_{p_2} = -1$  exactly in the following cases:*

- (I)  $\mu \leq -\lambda_1 - m + 1$ .

(II) If  $\lambda_1 = 0$  then, in addition to the above, for

$$\mu = -\lambda_1 - \tilde{i}_0 + 1 - R, R = 0, 1, \dots, n - \tilde{i}_0.$$

## BORDERLINE CASES

In this chapter we analyze the cases  $B(1, n)$ ,  $C_{n+1}$ , and  $B(0, n)$ .

I.  $B(1, n)$ .

To begin with, observe that  $B_1 = A_1$ , hence there is *a priori* the possibility of having a unitarizable module based on an involution which corresponds to  $sl(2, \mathbb{R})$  on  $B_1$ . However, this possibility is quickly ruled out again by Lemma 7.3 and Lemma 7.5. What remains is to go through the proofs of Proposition 7.4 and Proposition 7.6 and see what changes need to be made. We will be brief about this matter since we will attack similar questions for, in particular,  $C_n$  in great detail. It turns out that one gets the same results as previously, and that the only major modification needed in the proofs is in the proof of Propositions 7.4 and 7.6, where one must consider  $\Lambda(Q_2) \otimes \Lambda(Q_3) \otimes V_\Lambda^0$ , and not just  $\Lambda(Q_2) \otimes V_\Lambda^0$ , to get the appropriate  $sp(n, \mathbb{R})$  condition.

PROPOSITION 9.1. *For  $B(1, n)$  there are non-trivial unitarizable modules exactly in the relevant cases described by Proposition 7.4 and Proposition 7.6 with  $m = 1$ .*

II.  $C_{n+1}$ .

In the terminology of Chapter 8,  $C_{n+1} = D(1, n)$ . Specifically, an element has the form

$$(9.1) \quad \begin{bmatrix} a & 0 & p_1 & p_2 \\ 0 & -a & q_2^r & -q_1^r \\ q_1 & -p_2^r & \beta & z \\ q_2 & p_1^r & w & -\beta^r \end{bmatrix}$$

We consider, as previously, an anti-linear anti-involution  $w$  which is complex conjugation followed by the sign changes  $\varepsilon_z, \varepsilon_{p_1}$ , and  $\varepsilon_{p_2}$  on  $z, p_1$ , and  $p_2$ , respectively. It is easy to see that we still have

$$(9.2) \quad \varepsilon_z = \varepsilon_{p_1} \varepsilon_{p_2}.$$

Thus we need to go through the 8 cases corresponding to putting  $m = 1$  in Chapter 8.

PROPOSITION 9.2. *In case #A there are no non-trivial unitarizable modules with  $\varepsilon_z = -1$  and  $\varepsilon_{p_1} = 1$ .*

*Proof.* The biggest value,  $\lambda_1$ , for which a type vanishes from  $P_1 \otimes V_\Lambda^0$  is the  $\lambda_1$  satisfying

$$(9.3) \quad \begin{aligned} (\lambda_1 - a_n) + \mu_1 &= 0 \Leftrightarrow \\ \lambda_1 &= -\mu_1 + a_n. \end{aligned}$$

The biggest value,  $\lambda_2$ , for which a type vanishes from  $Q_2 \otimes V_\Lambda^0$  is the  $\lambda_2$  satisfying

$$(9.4) \quad \begin{aligned} (\lambda_2 - a_n) - \mu_1 &= 0 \Leftrightarrow \\ \lambda_2 &= \mu_1 + a_n. \end{aligned}$$

Since both  $\lambda_1$  and  $\lambda_2$  must be non-positive and  $a_n \geq 0$  it follows that only the trivial representation remains.

PROPOSITION 9.3. *In case #A there are non-trivial unitarizable modules with  $\varepsilon_z = -1$  and  $\varepsilon_{p_1} = -1$  exactly in the following cases:*

(I) *If  $\mu \geq 0$ , then for  $\lambda = -i_0 + 1 - \mu$ .*

(II) *If  $\mu$  is arbitrary and  $\lambda = -i_0 + 1 - \mu - R$ , with*

$$R \in \{1, \dots, n - i_0\} \cap \{x \mid x + 2\mu \geq n - i_0\}.$$

(III) *If  $\mu$  is arbitrary and  $\lambda = -i_0 + 1 - \mu - R$ , with*

$$R \geq \max(n - i_0, n - i_0 - \mu, -2\mu).$$

*Proof.* The smallest value,  $\lambda_1$ , at which a type is vanishing from  $P_1 \otimes V_\Lambda^0$  clearly satisfies

$$(9.5) \quad \begin{aligned} \lambda_1 + i_0 - 1 + \mu &= 0 \Leftrightarrow \\ \lambda_1 &= -\mu - i_0 + 1. \end{aligned}$$

If we assume that  $\lambda = \lambda_1 - R$  for an  $R \in \{0, \dots, n - i_0\} \cup \{x \mid x \geq n - i_0\}$  then it is easy to see that the smallest value,  $\lambda_2$ , at which a type is vanishing from  $Q_2 \wedge \wedge (P_1) \otimes V_\Lambda^0$ , due to cancellations, is independent of  $R$  and satisfies

$$(9.6) \quad \lambda_2 = \mu - i_0 + 1.$$

We must then have  $\lambda \leq \lambda_2$  i.e.

$$(9.7) \quad R + 2\mu \geq 0.$$

Observe that  $\lambda = \lambda_2 - R - 2\mu$ . Hence it is easy to see that there is a type in  $\Lambda(Q_2) \otimes \Lambda(P_1) \otimes V_\Lambda^0$  whose  $sp(n, \mathbb{R})$  content is as shown in Fig. 9.8:

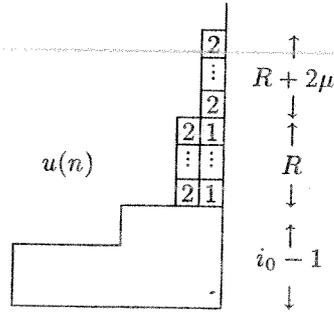


Fig. 9.8

(Of course this is only completely correct when  $2R + 2\mu + i_0 - 1 \leq n$ ). Due to  $sp(n, \mathbb{R})$ , when  $2R + 2\mu + i_0 - 1 \leq n - 1$  we must have

$$(9.9) \quad \begin{aligned} 2\lambda + 2R + 2\mu + i_0 + R + i_0 - 1 \leq 1 &\Leftrightarrow \\ R \leq 0, \end{aligned}$$

whereas when  $2R + 2\mu + i_0 - 1 > n - 1$  the condition is

$$(9.10) \quad \begin{aligned} 2\lambda + n + R + i_0 - 1 \leq 1 &\Leftrightarrow \\ R + 2\mu \geq n - i_0. \end{aligned}$$

(Of course, we still assume that  $R + i_0 - 1 \leq n - 1$ ).

Finally, when  $R + i_0 - 1 \geq n - 1$  the condition becomes

$$(9.11) \quad \begin{aligned} \lambda + n \leq 1 &\Leftrightarrow \\ R \geq n - i_0 - \mu. \end{aligned}$$

**PROPOSITION 9.4.** *In case #B there are no non-trivial unitarizable modules corresponding to  $\varepsilon_z = -1$  and  $\varepsilon_{p_i} = 1$ .*

*Proof.* The biggest value for which a type is vanishing from  $Q_1 \wedge V_\Lambda^0$ ,  $\lambda_1$ , satisfies (cf. Fig. 9.13)

$$(9.12) \quad \begin{aligned} (\lambda_1 - a_n + s_0) + (\mu - n) = 0 &\Leftrightarrow \\ \lambda_1 = a_n + n - s_0 - \mu. \end{aligned}$$

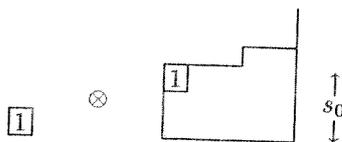


Fig. 9.13

Unitarity on the part of  $sp(n, \mathbb{R})$  forces  $\lambda_1 \leq \frac{1}{2} - \frac{(i_0 + j_0)}{2}$ . Hence,

$$(9.14) \quad \mu \geq a_n + n - s_0 - \frac{1}{2} + \frac{i_0 + j_0}{2}.$$

The biggest value,  $\lambda_2$ , at which a type is vanishing from  $Q_2 \wedge V_\Lambda^0$  then satisfies

$$(9.15) \quad \begin{aligned} \lambda_2 - a_n - (\mu) &= 0 \Rightarrow \\ \lambda_2 = \mu + a_n &\geq 2a_n - s_0 - \frac{3}{2} + \frac{(i_0 + j_0)}{2} + n + 1 > 0. \end{aligned}$$

This is clearly impossible.  $\square$

PROPOSITION 9.5. In case #B there are unitarizable modules corresponding to  $\varepsilon_z = -1$  and  $\varepsilon_{p_1} = 1$  exactly in the following cases

- (I) If  $\lambda \leq -n$  then for  $\lambda + n + i_0 \leq \mu < -\lambda$ .
- (II) If  $\lambda \leq \frac{2-n-i_0}{2}$  then for  $\mu = -\lambda$ .
- (III) If  $i_0 = 1$ , then, in addition, for  $(\lambda, \mu) = (0, 0), (-\frac{1}{2}, \frac{1}{2}), \dots, (\frac{1-n}{2}, \frac{n-1}{2})$ .

Proof. The smallest value,  $\lambda_1$ , for which a type is vanishing from  $Q_1 \wedge V_\Lambda^0$ , satisfies

$$(9.22) \quad \begin{aligned} \lambda_1 + n + \mu - n &= 0 \Leftrightarrow \\ \lambda_1 &= -\mu. \end{aligned}$$

Suppose that  $\lambda < \lambda_1$ . Then we have the following type in  $\wedge(Q_1) \otimes V_\Lambda^0$  (Fig. 9.23):

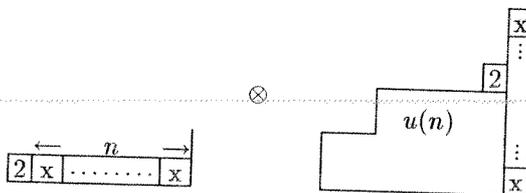


Fig. 9.23

This clearly leads to the smallest last possible place of unitarity,  $\lambda_2$ , for  $Q_2 \wedge \Lambda(Q_1) \otimes V_\Lambda^0$ :

$$(9.24) \quad \begin{aligned} \lambda_2 + i_0 - (\mu - n) &= 0 \Leftrightarrow \\ \lambda_2 &= \mu - n - i_0. \end{aligned}$$

Now, if  $\lambda_2 \geq \lambda_1 \Leftrightarrow 2n + i_0 + 1 - 2\mu \leq 0$  then everything is fine in  $\Lambda(Q_2) \otimes \Lambda(Q_1) \otimes V_\Lambda^0$ . However, if  $\lambda_2 < \lambda_1$  then we must insist that  $\lambda \leq \lambda_2$  also. In both cases it is easy to see that it follows that  $\lambda \leq -n$  because there are non-zero types in  $\Lambda(Q_2) \otimes \Lambda(Q_1) \otimes V_\Lambda^0$  which define  $sp(n, \mathbb{R})$  modules which only are unitarizable for  $\lambda \leq -n$ . When  $\lambda = \lambda_1$  it is easy to see that the  $\lambda_2$  from  $Q_2 \otimes \Lambda(Q_1) \otimes V_\Lambda^0$  satisfies

$$(9.25) \quad \begin{aligned} \lambda_2 + j_0 - 1 - (\mu_1 - (i_0 - 1)) &= 0, \quad \text{hence} \\ \lambda_2 &= \mu_1 - i_0 - j_0 + 2. \end{aligned}$$

In this case, the  $sp(n, \mathbb{R})$  condition is  $\lambda = -\mu \leq -(2 - n - i_0)/2$  because we are now missing a certain type in  $\Lambda(Q_1) \otimes V_\Lambda^0$  which previously was used to force  $\lambda$  towards  $-\infty$ . Still, unless  $i_0 = 1$  we have enough types in  $\Lambda(Q_2) \otimes \Lambda(Q_1) \otimes V_\Lambda^0$  to restrict  $\lambda$  as indicated. It is only when  $i_0 = 1$  that sufficiently many types are missing that we can use the  $\lambda$ 's as indicated in (III).  $\square$

From now on, let  $\lambda_1, \dots, \lambda_n$  be integers satisfying  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ , i.e. such that the  $n$ -tuple  $(\lambda_1, \dots, \lambda_n)$  defines a finite dimensional  $sp(n, \mathbb{R})$  module. Let  $i$  be the biggest integer for which  $\lambda_1 = \lambda_2 = \dots = \lambda_i$ .

**PROPOSITION 9.6.** *In case  $\#A^*$  (with  $\varepsilon_z = 1$ ) there are unitarizable highest weight modules corresponding to  $\varepsilon_{p_1} = \varepsilon_{p_2} = -1$  exactly if:*

(I)  $\lambda_n \geq 2$ , then for

$$\mu \in \{-\lambda_1 - n + i, \dots, -\lambda_1 - n + 1\} \cup ]-\infty, -\lambda - n + 1[.$$

(II) If  $\lambda_1 = \lambda_n = 1$ , then for  $\mu = -1$ .

*Proof.* The smallest value,  $\mu_1$ , for which a type is vanishing from  $P_1 \wedge V_\Lambda^0(1)$  clearly corresponds to the diagram in Fig. 9.27:

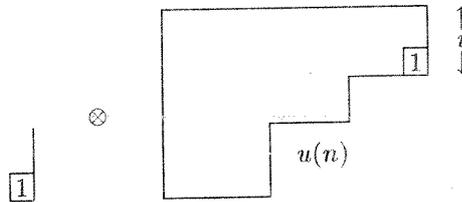


Fig. 9.27

Hence

$$(9.28) \quad \begin{aligned} \mu_1 + (\lambda_1 + n - i) = 0 &\Leftrightarrow \\ \mu_1 &= -\lambda_1 - n + i. \end{aligned}$$

We must have  $\mu = \mu_1 - R$  for some  $R \in \{0, \dots, i-1\} \cup \{x \mid x \geq i-1\}$ . Then we have the following type in  $\Lambda(P_1) \otimes V_{\Lambda}^0(1)$  (Fig. 9.29):

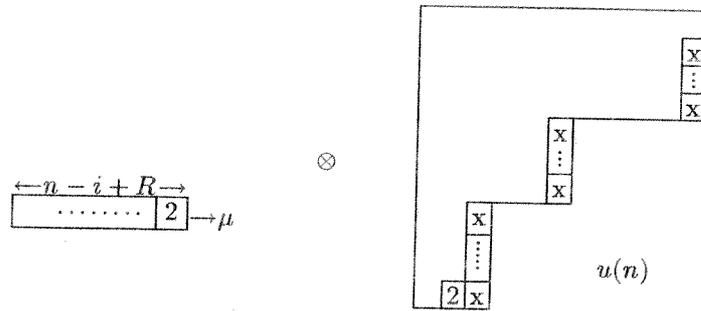


Fig. 9.29

Hence the smallest  $\mu_2$  for which a type is vanishing from  $Q_2 \wedge \Lambda(P_1) \otimes V_{\Lambda}^0(1)$  satisfies

$$(9.30) \quad \begin{aligned} \mu_2 + (n - i + R) - (\lambda_n - 1) = 0 &\Leftrightarrow \\ \mu_2 &= \lambda_n - 1 - R - (n - i). \end{aligned}$$

To have unitarity we must have

$$(9.31) \quad \mu_2 \geq \mu_1 - R \text{ i.e. } \lambda_1 + \lambda_n \geq 1.$$

It is also clear that the requirement that the modules in  $\Lambda(Q_2) \otimes \Lambda(P_1) \otimes V_{\Lambda}^0$  should generate finite dimensional  $sp(n, \mathbb{R})$  modules forces  $\lambda_n \geq 2$  unless  $\mu = \mu_1$  and  $\lambda_1 = \lambda_n$ , in which case it is sufficient to have  $\lambda_n \geq 1$ .  $\square$

**PROPOSITION 9.7.** *In case  $\#A^*$  (with  $\varepsilon_z = 1$ ) there are unitarizable highest weight modules corresponding to  $\varepsilon_{p_1} = \varepsilon_{p_2} = 1$  exactly if:*

(I)  $\lambda_n \geq 2$ , then for

$$\mu \in \{\lambda_1 + n - i, \dots, \lambda_1 + n - 1\} \cup [\lambda_1 + n - 1, \infty[.$$

(II)  $\lambda_1 = \lambda_n = 1$ , then for  $\mu = 1$ .

*Proof.* Similar to the previous, in fact, due to some cancellations, it is even easier.

PROPOSITION 9.8. In case  $\#B^*$  (with  $\varepsilon_z = 1$ ) there are unitarizable highest weight modules corresponding to  $\varepsilon_{p_1} = \varepsilon_{p_2} = 1$  exactly when

$$\mu \geq \lambda_1 + 2n.$$

*Proof.* The biggest value,  $\mu_1$ , at which something vanishes from  $Q_1 \wedge V_{\Lambda}^0(M(\Lambda))$  is clearly taken at the following diagram (Fig. 9.32):

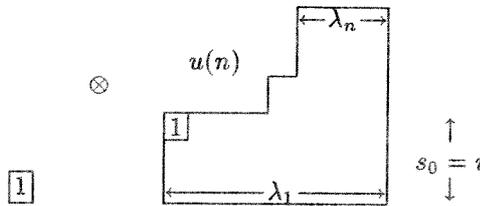


Fig. 9.32

Hence

$$(9.33) \quad (\mu_1 - n) + (-\lambda_1 + s_0) = 0 \Leftrightarrow \mu_1 = \lambda_1 + n - s_0.$$

Actually, there may be, for special configurations, even bigger  $\mu_1$ 's e.g.  $\check{\mu}_1$  in Fig. 9.34,

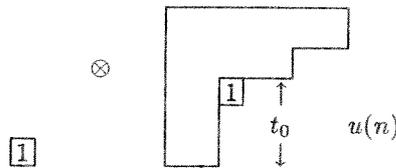


Fig. 9.34

where

$$(9.35) \quad (\check{\mu}_1 - n) + (\lambda_n + t_0) = 0 \Leftrightarrow \check{\mu}_1 = -\lambda_n + n - t_0.$$

However, the only important fact here is that the following type (Fig. 9.36) is non-zero in the range of unitarity (unless we are looking at the trivial representation), since it vanishes when  $\mu = \bar{\mu} = -\lambda_1$ :

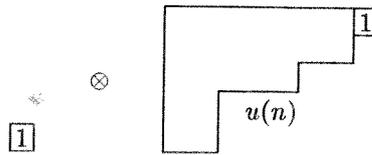


Fig. 9.36

Hence there is a type in  $Q_2 \wedge \Lambda(Q_1) \otimes V_\Lambda^0$  which vanishes for a  $\mu_2$  satisfying

$$(9.37) \quad (\mu_2 - 1) - (\lambda_1 + n - 1) = 0 \Leftrightarrow \mu_2 = \lambda_1 + n.$$

Since we must have that  $\mu \geq \mu_2$  it follows that the following type from  $\Lambda(Q_1) \otimes V_\Lambda^0$  (Fig. 9.38) occurs in our module:

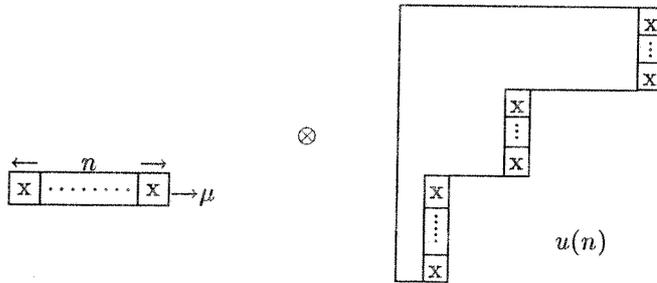


Fig. 9.38

Hence, the ultimate  $\mu_2$  satisfies

$$(9.39) \quad \mu_2 - n - (\lambda_1 + n) = 0 \Leftrightarrow \mu_2 = \lambda_1 + 2n.$$

Finally, it is completely straightforward to establish the following:

**PROPOSITION 9.9.** *In case  $\#B^*$  (with  $\varepsilon_z = 1$ ) there are unitarizable highest weight modules corresponding to  $\varepsilon_{p_1} = \varepsilon_{p_2} = -1$  exactly when*

$$\mu \leq -\lambda_1.$$

**III.**  $B(0, n)$ .

An element of  $B(0, n)$  has the form

$$(9.40) \quad \begin{bmatrix} 0 & q_3^r & p_3 \\ -p_3^r & \beta & z \\ q_3 & w & -\beta^r \end{bmatrix}.$$

As previously, we are forced to have  $\varepsilon_z = -1$ . Maintaining the notation of Chapter 7, the smallest value  $\lambda_s$  of  $\lambda$  for which a type is vanishing from  $Q_3 \wedge V_\Lambda^0$  is given by

$$\lambda_s = -i_0 + 1,$$

whereas the biggest value,  $\lambda_b$ , is given by

$$\lambda_b = a_n.$$

Thus, the following two propositions are immediate:

**PROPOSITION 9.10.** *In case  $\varepsilon_3 = 1$  (with  $\varepsilon_z = -1$ ) there are unitarizable highest weight modules exactly for*

$$\lambda \in \{-i_0 + 1, \dots, -n + 1\} \cup ]-\infty, -n + 1[.$$

**PROPOSITION 9.11.** *In case  $\varepsilon_3 = -1$  (with  $\varepsilon_z = -1$ ) there are no unitarizable highest weight modules except the trivial.*

CHAPTER 10

$F(4)$

We refer (again) to [16] and [17] for the details concerning the results and notation used here.

Let  $F(4)_0$  and  $F(4)_1$  denote the even and odd part, respectively. Then

$$(10.1) \quad \begin{aligned} F(4)_0 &= B_3 \oplus A_1, \text{ and} \\ F(4)_0 |_{F(4)_1} &= \text{spin}_7 \otimes \text{sl}_2. \end{aligned}$$

Let  $e_1, e_2, e_3$ , and  $\delta = e_4$  denote the usual basis of  $\mathbf{C}_4^*$ . Let  $\Delta_0$  and  $\Delta_1$  denote the even and the odd roots, respectively. Then

$$(10.2) \quad \begin{aligned} \Delta_0 &= \{\pm e_i \pm e_j, \pm e_i, \pm \delta\}, \text{ and} \\ \Delta_1 &= \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm \delta) \right\}. \end{aligned}$$

Up to  $W$  equivalence there are five sets of simple roots,  $\sum_1, \sum_2, \sum_3, \sum_4$ , and  $\sum_5$ .

Choose non-zero elements  $x_1, \dots, x_8, y_1, \dots, y_8$  in  $F(4)_1$  such that

$$(10.3) \quad \begin{aligned} x_1 \text{ has weight } \frac{1}{2}(-e_1 - e_2 - e_3 - \delta), & \quad y_1 \text{ has weight } \frac{1}{2}(-e_1 - e_2 - e_3 + \delta), \\ x_2 \text{ has weight } \frac{1}{2}(-e_1 - e_2 + e_3 - \delta), & \quad y_2 \text{ has weight } \frac{1}{2}(-e_1 - e_2 + e_3 + \delta), \\ x_3 \text{ has weight } \frac{1}{2}(-e_1 + e_3 - e_3 - \delta), & \quad y_3 \text{ has weight } \frac{1}{2}(-e_1 + e_2 - e_3 + \delta), \\ x_4 \text{ has weight } \frac{1}{2}(-e_1 + e_2 + e_3 - \delta), & \quad y_4 \text{ has weight } \frac{1}{2}(-e_1 + e_2 + e_3 + \delta), \\ x_5 \text{ has weight } \frac{1}{2}(+e_1 - e_2 - e_3 - \delta), & \quad y_5 \text{ has weight } \frac{1}{2}(+e_1 - e_2 - e_3 + \delta), \\ x_6 \text{ has weight } \frac{1}{2}(+e_1 - e_2 + e_3 - \delta), & \quad y_6 \text{ has weight } \frac{1}{2}(+e_1 - e_2 + e_3 + \delta), \\ x_7 \text{ has weight } \frac{1}{2}(+e_1 + e_2 - e_3 - \delta), & \quad y_7 \text{ has weight } \frac{1}{2}(+e_1 + e_2 - e_3 + \delta), \text{ and} \\ x_8 \text{ has weight } \frac{1}{2}(+e_1 + e_2 + e_3 - \delta), & \quad y_8 \text{ has weight } \frac{1}{2}(+e_1 + e_2 + e_3 + \delta). \end{aligned}$$

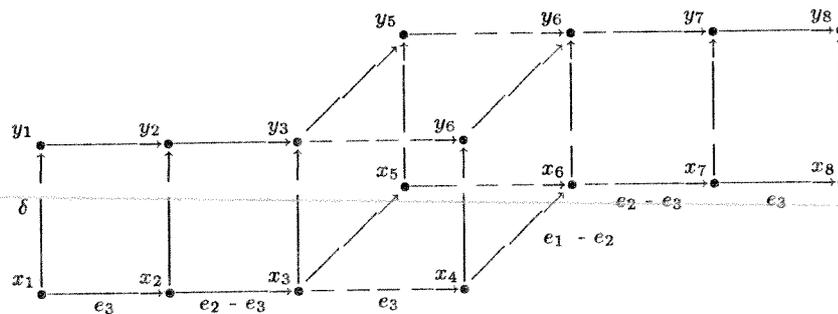


Fig. 10.4:  $F(4)_1$

We choose the standard ordering on  $B_3$  and  $A_1$  and proceed to describe the sets of positive and negative roots,  $F(4)_1^+$  and  $F(4)_1^-$ , in  $F(4)_1$ , as determined by  $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ , and  $\Sigma_5$ :

$$(I) \quad \Sigma_1 = \{e_1 - e_2, e_2 - e_3, e_3, \frac{1}{2}(-e_1 - e_2 - e_3 + \delta)\}.$$

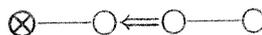


Fig. 10.5

$$F(4)_1^+ = \{y_1, y_2, \dots, y_8\}.$$

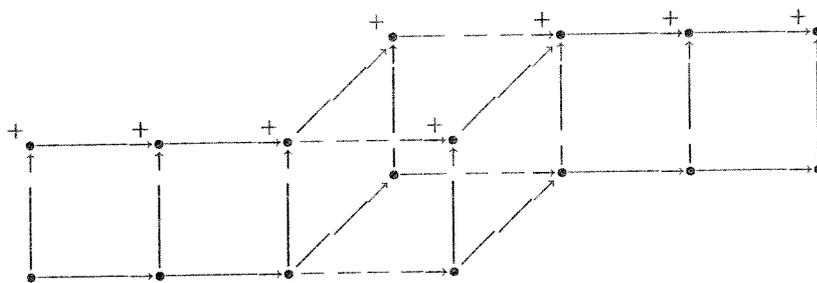


Fig. 10.6

$$(II) \quad \Sigma_2 = \{\delta, e_3, e_2 - e_3, \frac{1}{2}(e_1 - e_2 - e_3 - \delta)\}.$$



Fig. 10.7

$$F(4)_1^+ = \{x_5, x_6, x_7, x_8, y_5, y_6, y_7, y_8\}.$$

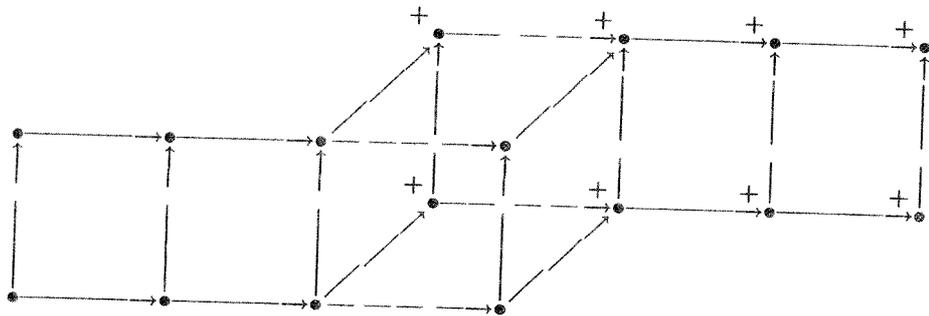


Fig. 10.8

$$(III) \sum_3 = \left\{ \frac{1}{2}(e_1 - e_2 - e_3 + \delta), \frac{1}{2}(e_1 - e_2 + e_3 - \delta), \frac{1}{2}(-e_1 + e_2 + e_3 + \delta), e_2 - e_3 \right\}.$$

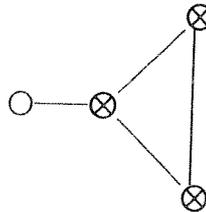


Fig. 10.9

$$F(4)_1^+ = \{y_4, y_5, y_6, y_7, y_8, x_6, x_7, x_8\}.$$

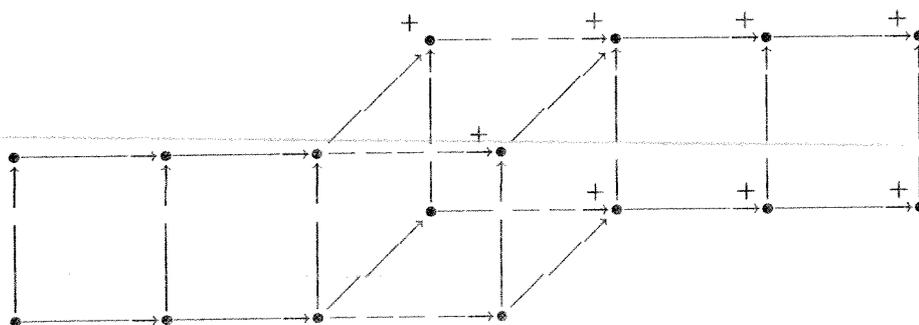


Fig. 10.10

$$(IV) \quad \sum_4 = \left\{ \frac{1}{2}(-e_1 - e_2 + e_3 + \delta), \frac{1}{2}(e_1 + e_2 + e_3 - \delta), e_2 - e_3, e_1 - e_2 \right\}.$$



Fig. 10.11

$$F(4)_1^+ = \{y_2, y_3, y_4, y_5, y_6, y_7, y_8, x_8\}.$$

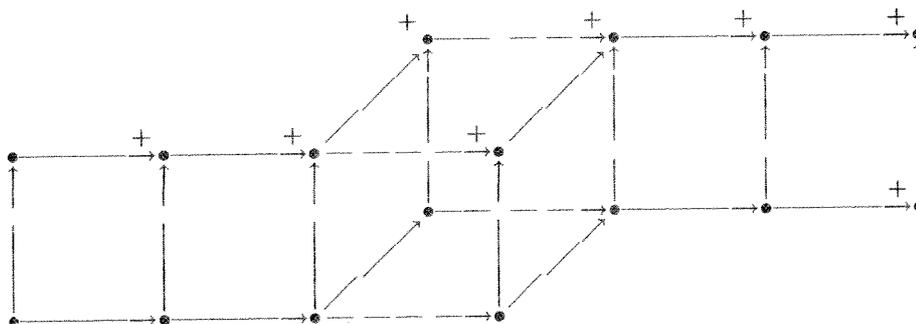


Fig. 10.12

$$(V) \quad \sum_5 = \left\{ \frac{1}{2}(-e_1 + e_2 - e_3 + \delta), \frac{1}{2}(e_1 + e_2 - e_3 - \delta), e_3, e_1 - e_2 \right\}.$$

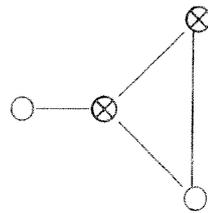


Fig. 10.11.a

$$F(4)_1^+ = \{y_3, y_4, y_5, y_6, y_7, y_8, x_7, x_8\}.$$

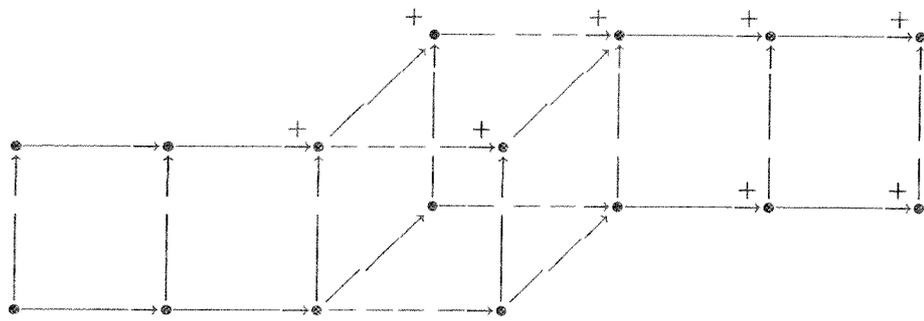


Fig. 10.12.a

(Observe that the situation

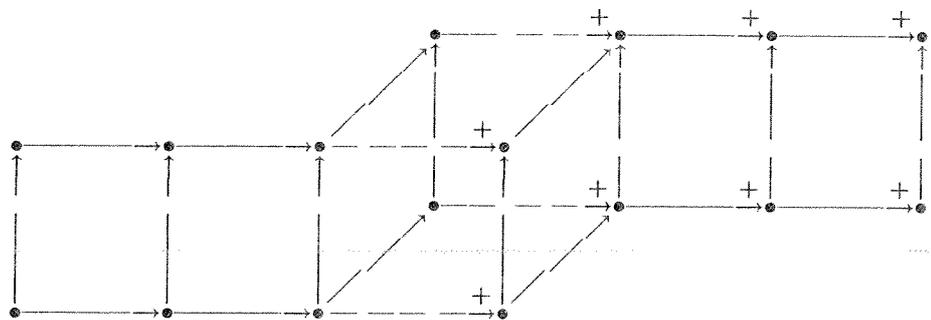
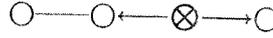


Fig. 10.13

corresponds to either



or

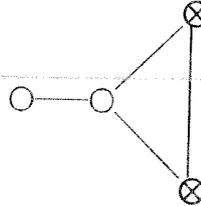


Fig. 10.14

and hence is covered by the treatment of the algebras  $D(m, n)$ .

We now choose the bilinear form to be positive definite on  $B_3$  and negative definite on  $A_1$ .

Choose vectors  $h_{e_1}, h_{e_2}, h_{e_3}$ , and  $h_\delta \in \mathbb{C}^4$  (identified with a Cartan subalgebra of  $F(4)$ ) such that  $e_i(h_{e_j}) = 2\delta_{i,j}$ ,  $\delta(h_{e_j}) = 0$ ,  $e_i(h_\delta) = 0$ , and  $\delta(h_\delta) = 2$ . We choose the  $x_1, \dots, y_8$  such that

$$\begin{aligned}
 (10.15) \quad [y_1, x_8] &= h_{-e_1} + h_{-e_2} + h_{-e_3} - 3h_\delta, \\
 &= -h_{e_1} - h_{e_2} - h_{e_3} - 3h_\delta, \\
 [y_2, x_7] &= h_{-e_1} + h_{-e_2} + h_{e_3} - 3h_\delta, \\
 &\vdots \\
 [y_8, x_1] &= h_{e_1} + h_{e_2} + h_{e_3} - 3h_\delta.
 \end{aligned}$$

We now analyze the conditions that must be fulfilled by an anti-linear anti-involution  $\omega$ . Let

$$(10.16) \quad \omega(y_1) = \varepsilon_0 x_8$$

for an  $\varepsilon_0 = \pm 1$ , and assume that the sign of  $\omega$  on the root spaces  $e_2 - e_3, e_1 - e_2, e_3$ , and  $\delta$  is  $\varepsilon_{2,3}, \varepsilon_{1,2}, \varepsilon_3$ , and  $\varepsilon_\delta$ , respectively.

LEMMA 10.1.

$$\begin{aligned}
 \omega(y_2) &= \varepsilon_0 \varepsilon_3 x_7, \\
 \omega(y_3) &= \varepsilon_0 \varepsilon_{2,3} \varepsilon_3 x_6, \\
 \omega(y_4) &= \varepsilon_0 \varepsilon_{2,3} x_5, \\
 \omega(y_5) &= \varepsilon_0 \varepsilon_{1,2} \varepsilon_{2,3} \varepsilon_3 x_4, \\
 \omega(y_6) &= \varepsilon_0 \varepsilon_{1,2} \varepsilon_{2,3} x_3, \\
 \omega(y_7) &= \varepsilon_0 \varepsilon_{1,2} x_2, \text{ and} \\
 \omega(y_8) &= \varepsilon_0 \varepsilon_{1,2} \varepsilon_3 x_1. \\
 \varepsilon_\delta &= -\varepsilon_3 \varepsilon_{1,2}.
 \end{aligned}$$

Moreover,

*Proof.* We compute  $\omega(y_2)$ . The general case of  $\omega(y_i)$  follows analogously. Let  $k_{e_3}^\pm$  be vectors in the root spaces  $\pm e_3$  such that  $[k_{e_3}^+, k_{e_3}^-] = h_{e_3}$ . Then

$$\begin{aligned}
 (10.17) \quad [[k_{e_3}^+, y_1], [x_1, k_{e_3}^-]] &= [[k_{e_3}^+, y_1], [x_1, k_{e_3}^-]] - [x_1, [[k_{e_3}^+, y_1], k_{e_3}^-]] = \\
 &= [[x_1, [k_{e_3}^+, y_1]], k_{e_3}^-] + [x_1, [k_{e_3}^-, [k_{e_3}^+, y_1]]] = \\
 &= [[k_{e_3}^+, -(h_{e_1} + h_{e_2} + h_{e_3} + 3h_\delta)], k_{e_3}^-] - [x_1, [h_{e_3}, y_1]] = \\
 &= 2h_{e_3} + [x_1, y_1] = \\
 &= (h_{-e_1} + h_{-e_2} + h_{e_3} - 3h_\delta).
 \end{aligned}$$

Thus, if  $y_2 = [k_{e_3}^+, y_1]$ , then  $x_7 = [x_8, k_{e_3}^-]$ . Hence

$$(10.18) \quad \omega(y_2) = \omega([k_{e_3}^+, y_1]) = \varepsilon_0 \varepsilon_3 [x_1, k_{e_3}^-] = \varepsilon_0 \varepsilon_3 x_7.$$

Not only the other  $y_i$ 's can be handled this way. By computing  $\omega(y_1)$  as  $\omega([k_\delta^+, x_1])$ , the claim concerning  $\varepsilon_\delta$  follows in the same manner.  $\square$

We now go through the four cases, one by one. Observe that always either  $\varepsilon_\delta = 1$  or  $\varepsilon_{1,2} = +1$ , but never both.

**Case I.** We write the highest weight as

$$(10.19) \quad \Lambda = (\lambda \cdot h_{e_1} + a \cdot h_{e_2} + b \cdot h_{e_3} + \mu/2 \cdot h_\delta)$$

or, simply, as

$$(10.20) \quad \Lambda = (\lambda, a, b, \mu/2).$$

The contribution to  $\rho$  from the even roots is  $(5/2, 3/2, 1/2, 1/2)$ , and it follows easily that half the sum of the positive odd roots is  $2\delta$ . (This must be subtracted). Thus

$$(10.21) \quad \Lambda + \rho = (\lambda + 5/2, a + 3/2, b + 1/2, \mu/2 - 3/2).$$

**PROPOSITION 10.2.** *In case I there are non-trivial unitarizable highest weight modules corresponding to an involution  $\omega$  in which  $\varepsilon_\delta = 1$  exactly when  $\varepsilon_0 = -1$ . In this case there is unitarity exactly in the following cases where, since  $\mu$  must be a non-negative integer, we have written  $\Lambda = (\lambda, a, b, n/2)$ :*

- i)  $n \geq 8$  and  $\lambda < -a - b - \frac{n}{2}$ .
- ii)  $n \geq 7, b \neq 0$  and  $\lambda = -a - b - \frac{n}{2}$ .
- iii)  $n \geq 6, b = 0, a \neq 0$ , and  $\lambda = -a - b - \frac{n}{2}$ .
- iv)  $n \geq 4, a = b = 0$ , and  $\lambda = -a - b - \frac{n}{2}$ .

*Proof.* Since  $\varepsilon_{1,2} = -1$ , we are considering the real form  $o(2,5)$  of  $B_3$ . Analogously to the previous chapters we let  $V_\Lambda^0$  denote the finite dimensional  $o(5)$  module defined by  $(a, b)$ , and we consider the vanishing of the various  $o(5)$  types in  $F(4)_1^- \wedge V_\Lambda^0$ . A priori we know that  $\lambda$  must be negative to have unitarity on  $o(2,5)$ . Moreover, it is easy to see that for all  $i = 1, \dots, 8$ ,

$$(10.22) \quad \langle x_i \wedge v_\Lambda^0, x_i \wedge v_\Lambda^0 \rangle = \varepsilon_0 \lambda + \text{lower order terms},$$

where  $v_\Lambda^0$  denotes the highest weight vector in  $V_\Lambda^0$ .

Finally,  $\langle x_4 \wedge v_\Lambda^0, x_4 \wedge v_\Lambda^0 \rangle$  is zero at a positive  $\lambda$  as follows easily by direct computation. Thus,  $\varepsilon_0 = -1$ . Now let  $\lambda_1, \dots, \lambda_8$  denote the values of  $\lambda$  for which the term corresponding to the root  $\alpha_{x_i}$  of  $x_i$  vanishes. That is, if  $h_{x_i} = [x_i, y_{9-i}]$  then

$$(10.23) \quad \begin{pmatrix} \lambda_i + 5/2 \\ a + 3/2 \\ b + 1/2 \\ n/2 - 3/2 \end{pmatrix} (h_{x_i}) = 0.$$

It follows that

$$(10.24) \quad \begin{aligned} \lambda_1 &= \frac{3n}{2} - a - b - 9, & \lambda_2 &= \frac{3n}{2} - a + b - 8, & \lambda_3 &= \frac{3n}{2} + a - b - 6, \\ \lambda_4 &= \frac{3n}{2} + a + b - 5, & \lambda_5 &= -\frac{3n}{2} + a + b - 4, & \lambda_6 &= -\frac{3n}{2} + a - b - 3, \\ \lambda_7 &= -\frac{3n}{2} - a + b - 1, & \lambda_8 &= -\frac{3n}{2} - a - b. \end{aligned}$$

Now, clearly we must have  $\lambda \leq \lambda_8$ , and unless  $\lambda = \lambda_8$  the vector  $x_8 \wedge x_7 \wedge x_6 \wedge x_5 \wedge v_\Lambda^0$  is non-zero and hence we must have  $n \geq 4$  to ensure finite dimensionality of the  $A_1 \equiv su(2)$  module corresponding to this. Thus, if  $\lambda < \lambda_8$  then to begin with  $n \geq 4$ . But for  $n \geq 2$ ,  $\lambda_8$  is the smallest of the  $\lambda_i$ 's, and only for  $n = 3$  does it equal  $\lambda_1$ , for  $n > 3$  it is strictly the smallest. But then, if  $\lambda < \lambda_8$  the vector  $x_1 \wedge x_2 \wedge \dots \wedge x_7 \wedge x_8 \wedge v_\Lambda^0$  is non-zero, and hence  $n \geq 8$ . It then also follows that  $\lambda_8 \leq -12 - a - b$ , and in this range, there is unitarity for the  $o(2,5)$  modules involved even if we take into account that  $x_7$  and  $x_8$  do make more restrictions on  $\lambda$  than those coming from  $v_\Lambda^0$ .

It remains to examine the cases where  $\lambda = \lambda_8$  and hence where  $x_8 \wedge v_\Lambda^0$  vanishes. Assume first that  $n = 3$ , i.e. that also  $x_1$  defines a zero in the Kac determinant. A direct computation then easily shows that the vector  $x_3 \wedge x_4 \wedge x_6 \wedge x_7 \wedge v_\Lambda^0$  is non-zero and hence  $n \geq 4$  is demanded after all. If  $b \neq 0$  then only  $x_8$  vanishes

whereas if  $b = 0$  and  $a \neq 0$  then also  $x_7$  vanishes. Finally, if both  $a$  and  $b$  are zero then  $x_8 \wedge v_\Lambda^0 = x_7 \wedge v_\Lambda^0 = x_6 \wedge v_\Lambda^0 = x_5 \wedge v_\Lambda^0 = 0$ .

In all cases it is easily seen that the  $o(2, 5)$  modules are unitarizable.

This establishes the unitarity as far as both  $F(4)$  and  $o(2, 5)$  are concerned. However, we still need to make the  $A_1 \cong su(2)$  action unitary and for this purpose we need to pass to yet another quotient. Let us be specific about this point:

Let  $\mathfrak{p}^+$  denote the subalgebra of  $o(2, 5)$  consisting of the root vectors corresponding to the roots  $e_1 \pm e_2, e_1 \pm e_3$ , and  $e_1$ , and let  $\mathfrak{p}^-$  be the corresponding negative root spaces. Let  $\mathfrak{k} = o(5)$ . Then

$$(10.25) \quad o(2, 5) = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$$

(in analogy with Chapter 6).

Due to the  $o(2, 5)$  invariance (here) of  $F(4)_1^+$  and  $F(4)_1^-$ , it is appropriate to write  $U(G)$  ( $G = F(4)$ ) as

$$(10.26) \quad U(G) = U(k_\delta^-)U(\mathfrak{p}^-)(\bigwedge(F(4)_1^-)U(\mathfrak{k}^-)U(H)(\bigwedge(F(4)_1^+))U(\mathfrak{k}^+)U(\mathfrak{p}^+)U(k_\delta^+),$$

where  $H$  is a fixed Cartan subalgebra corresponding to the roots  $e_1, e_2, e_3$ , and  $\delta$ , and  $\mathfrak{k}^\pm$  denote the positive and negative root spaces, respectively, of  $o(5)$ .

Let  $I_\Lambda$  denote the ideal in  $U(G)$  generated by  $\bigwedge(F(4)_1^+), \mathfrak{p}^+, \mathfrak{k}^+$ , and the expressions  $(h - \Lambda(h)), h \in H$ . The generalized Verma module is defined as

$$(10.27) \quad M_\Lambda = U(G)/I_\Lambda.$$

Let  $v_\Lambda$  denote the image of 1 in  $M_\Lambda$  under the natural projection of  $U(G)$  onto  $M_\Lambda$ . Let  $J_\Lambda(\mathfrak{k})$  denote the largest  $\mathfrak{k}$ -invariant subspace in  $U(G) \cdot v_\Lambda^0$  not containing  $v_\Lambda^0$ . Then

$$(10.28) \quad V_\Lambda^0 = U(\mathfrak{k})v_\Lambda^0/J_\Lambda(\mathfrak{k})$$

is the finite dimensional representation of  $\mathfrak{k}$  defined by the restriction of  $\Lambda$  to  $H \cap \mathfrak{k}$ . Let  $J_\Lambda(\mathfrak{k})$  denote the left ideal in  $U(G)$  generated by  $I_\Lambda$  and  $J_\Lambda(\mathfrak{k})$  (this is well defined since  $J_\Lambda(\mathfrak{k})$  is defined in  $U(G)$  modulo  $I_\Lambda$ ), and let

$$(10.29) \quad V_\Lambda = U(G)/J_\Lambda(\mathfrak{k}).$$

Then, as a  $\mathfrak{k}$  module,

$$(10.30) \quad V_\Lambda = U(k_\delta^-)U(\mathfrak{p}^-)(\bigwedge(F(4)_1^-) \otimes V_\Lambda^0).$$

Consider

$$(10.31) \quad (V_\Lambda)(0) = U(\mathfrak{p}^-) \left( \bigwedge (F(4)^-) \otimes V_\Lambda^0 \right).$$

It is clear that  $k_\delta^+$  acts nilpotently on this space, and in the course of the current proof we have made sure that the highest weights of  $A_1$  we get from this space, define finite dimensional representations of  $A_1$ . If our representation of  $G$  is unitary then we can write

$$(10.32) \quad \bigwedge (F(4)^-) \cdot v_\Lambda^0 = \bigoplus_{i=1}^R \mathbb{C} \cdot w_i,$$

where the sum is orthogonal and where each  $w^i$  is a non-zero highest weight vector for  $A_1$ , i.e.  $\forall i = 1, \dots, R : k_\delta^+ w_i = 0$ . Let  $J_\Lambda(\delta)$  denote the largest  $A_1$ -invariant subspace of  $U(k_\delta^-) \left( \bigoplus_{i=1}^R \mathbb{C} w_i \right)$  that does not contain any of the  $w_i$ 's, and let  $J_\Lambda(\delta)$  denote the left ideal in  $U(G)$  generated by  $J_\Lambda(\delta)$  and  $J_\Lambda(\mathfrak{k})$ . Then our highest weight module is

$$(10.33) \quad V_\Lambda = U(G) / J_\Lambda(\delta),$$

and again it is clear that the conditions for unitarity are sufficient. Our last problem is to show that  $V_\Lambda$  is non-zero. For this purpose it is sufficient to consider

$$(10.34) \quad \bigwedge (F(4)_1^+) J_\Lambda(\delta)$$

and to show that this cannot contain  $v_\Lambda^0$ . But, from the representation theory of  $A_1$  it follows easily that

$$(10.35) \quad J_\Lambda(\delta) = \bigoplus_{i=1}^R U(\mathfrak{k}) (k_\delta^-)^{n_i} (w_i),$$

where all the exponents  $n_i$  are strictly positive integers. Now, if  $f^+ \in F(4)_1^+$  then  $[k_\delta^-, f^+] = f^- \in F(4)_1^-$ . Hence, since  $k_\delta^-$  commutes with  $o(2, 5)$  and  $\bigwedge (F(4)_1^-)$ , we can never have  $J_\Lambda(\delta)$  equal all of  $U(G)$ .  $\square$

**PROPOSITION 10.3.** *In case I there are non-trivial unitarizable highest weight modules corresponding to an involution  $w$  in which  $\varepsilon_\delta = -1$  exactly when  $\varepsilon_0 = -1$ . In this case there is unitarity exactly when*

$$3\mu/2 \leq -(\lambda + a + b).$$

*Proof.* This is analogous to the previous case, and in fact easier since  $F(4)_1^-$ , is  $o(7)$ -invariant. We have, for each  $i = 1, \dots, 8$ ,

$$(10.36) \quad \langle x_i \wedge v_\Lambda^0, x_i \wedge v_\Lambda^0 \rangle = -\varepsilon_0 \cdot 3\mu/2 + \text{lower order terms},$$

and at least one of these vanishes at a positive real number. Hence  $\varepsilon_0 = -1$ , and we need only look at the place where  $x_8 \wedge v_\Lambda^0$  vanishes since it clearly corresponds to the smallest value of  $\mu$  where anything vanishes from  $F(4)_1^- \wedge v_\Lambda^0$ . This value of  $\mu$  exactly satisfies  $-\lambda - a - b - 3\mu/2 = 0$ .  $\square$

**Case II.**

$$(10.37) \quad \Lambda + \rho = (\lambda + 1/2, a + 3/2, b + 1/2, n/2 + 1/2).$$

**PROPOSITION 10.4.** *In case II there are non-trivial unitarizable highest weight modules corresponding to an involution  $w$  in which  $\varepsilon_\delta = 1$  exactly when  $\varepsilon_0 = -1$ . In this case there is unitarity exactly in the following cases, where we have written  $\Lambda = (\lambda, a, b, n/2)$ ,  $n = 0, 1, \dots$ :*

- i)  $b \neq 0$ ,  $n \geq 4$ , and  $\lambda \leq -a - b - 4 - 3n/2$ .
- ii)  $b = 0$ ,  $a \neq 0$ ,  $n \geq 4$  and  $\lambda \leq -a - 4 - \frac{3n}{2}$  or  $\lambda = -a - 3 - 3n/2$ .
- iii)  $b = a = 0$ ,  $n \geq 4$  and  $\lambda \leq -4 - \frac{3n}{2}$  or  $\lambda = -\frac{3n}{2}$ .

*Proof.* The discussion of the signs follows in analogy with the previous cases. In the current situation there are no additional restrictions on  $\lambda$  coming from  $F(4)_1^-$ , and the restrictions on  $n$  are only imposed by  $x_1, x_2, x_3, x_4$ . Moreover, it is easy to see that in all relevant cases  $x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge v_\Lambda^0$  is non-zero. When  $b \neq 0$  then  $y_1 \wedge v_\Lambda^0$  is the leading term of an  $o(5)$  type, and it is clear that this defines the last possible place of unitarity. If  $b = 0$  we need to move to either  $y_2$  (when  $a \neq 0$ ) or  $y_4$  (when  $a = 0$ ). The gap between the place where  $y_2$  (resp.  $y_4$ ) vanishes and the place where  $y_1$  vanishes occurs because a second order element has negative norm here. In the first case it is the  $o(5)$  type whose  $v_\Lambda^0$  coefficient is  $y_1 \wedge y_2$  and in the second case it is  $y_1 \wedge y_4 \pm y_2 \wedge y_3$  as follows either by direct computation or by looking at the Kac determinant. We omit the details. Finally, the unitarity on  $A_1$  follows once the appropriate quotients have been taken (cf. the proof of Proposition 10.2).  $\square$

**PROPOSITION 10.5.** *In case II there are non-trivial unitarizable highest weight representations corresponding to an involution  $\omega$  in which  $\varepsilon_\delta = -1$  exactly when  $\varepsilon_0 = -1$ . In this case there is unitarity exactly in the following cases:*

- i)  $0 \geq 3\mu/2 > a + b - \lambda$  and  $\lambda \geq a + b + 3$ .
- ii)  $3\mu/2 = a + b - \lambda$  and  $\lambda \geq a + b + 9/4$ .

*Proof.* The sign on the leading term in the  $\langle y_i \wedge v_\Lambda^0, y_i \wedge v_\Lambda^0 \rangle$ 's is the opposite of the sign on the  $\langle x_i \wedge v_\Lambda^0, x_i \wedge v_\Lambda^0 \rangle$ 's ( $i = 1, \dots, 4$ ). It then follows easily that  $\varepsilon_0 = -1$ . We know that  $\mu$  must be non-positive and it is easy to see that it must be greater than or equal to the value for which  $y_4 \wedge v_\Lambda^0$  vanishes, the value which satisfies  $3\mu/2 = a + b - \lambda$ . (Observe that  $y_4 \wedge v_\Lambda^0$  as well as  $x_4 \wedge v_\Lambda^0$  are always highest weight vectors.) If  $\mu$  is strictly greater then the value at which  $y_4 \wedge v_\Lambda^0$  vanishes then  $y_1 \wedge y_2 \wedge y_3 \wedge y_4 \wedge v_\Lambda^0$  is non-zero, and hence this forces  $\mu \leq -2$ , hence  $\lambda \geq a + b + 3$  to ensure a non-empty set of  $\mu$ 's. If  $y_4 \wedge v_\Lambda^0$  vanishes then  $y_1 \wedge y_2 \wedge y_3 \wedge v_\Lambda^0$  vanishes, hence  $\mu \leq -3/2$  and thus  $\lambda \geq a + b + 9/4$  is required. Of course, a major problem here is that  $F(4)_1^-$  is not  $o(7)$ -invariant but only  $o(5)$ -invariant. However, we can proceed in analogy with the similar cases for  $B(n, m)$  and  $D(n, m)$  and check that all  $o(7)$  representations obtained are finite dimensional. This is indeed the case as follows quite easily from the strong requirements on the size of  $\lambda$  compared to that of  $a$  (cf. the proof of Proposition 10.2 and (below) of Proposition 10.6).  $\square$

### Case III.

$$(10.38) \quad \Lambda + \rho = (\lambda + 1, a + 1, b, n/2).$$

PROPOSITION 10.6. *In case III the non-trivial unitarizable highest weight representations corresponding to an involution  $w$  in which  $\varepsilon_\delta = -1$  occur exactly when  $\varepsilon_0 = -1$ . In this case, for  $\Lambda = (\lambda, a, b, \mu/2)$ , there is unitarity exactly in the following cases:*

$$i) \lambda - a \geq 3, b \geq 5/2, \text{ and}$$

$$a - b - \lambda < 3\mu/2 \leq \min(-9/4, a + b - \lambda, \lambda - a - b).$$

$$ii) 3\mu/2 = a - b - \lambda, \text{ provided that}$$

$$a - b - \lambda \leq \min(-3/2, \lambda - a - b), \lambda - a \geq 2, b \geq 2, \text{ and } a > b.$$

$$iii) 3\mu/2 = a + b - \lambda, \text{ provided that}$$

$$a + b - \lambda \leq (-9/4), \lambda - a \geq 3, \text{ and } b \geq 2.$$

$$iv) 3\mu/2 = \lambda - a - b, \text{ provided that}$$

$$\lambda - a - b \leq (-9/4), \lambda - a \geq 2, \text{ and } b \geq 5/2.$$

*Proof.* It is easy to see that the sign of the first order elements in  $F(4)_1^- \wedge V_\Lambda^0$  corresponding to  $y_1, y_2$ , and  $y_3$  is the opposite of the sign corresponding to  $x_1, x_2, x_3, x_4$ , and  $x_5$ , and by looking at where they vanish it follows easily that  $\varepsilon_0 = -1$ . In the present situation  $F(4)_1^-$  is only invariant under the  $A_1$ -algebra generated by  $k_{\pm(e_2 - e_3)}$ . However, the philosophy of e.g. the proof of Proposition 10.2 can still be applied. We begin by proving that  $b > 0$ : If not, then we

must have  $x_5 \wedge v_\Lambda^0 = y_3 \wedge v_\Lambda^0 = 0$  since they both define illegal  $o(7)$  types. This implies that  $3\mu/2 = a - \lambda$ . Now consider  $x_3 \wedge v_\Lambda^0$ . This is non-zero as follows easily by direct computation (provided that  $\mu \neq 0$ ,  $\mu = a = \lambda = 0$  is the trivial representation). Thus there is a non-zero vector which is a highest weight vector for  $k_{e_2 - e_3}$  and which defines an illegal  $o(7)$  highest weight. Thus,  $b = 0$  is impossible. Now, it is clear that we must be in the  $\mu$ -interval between the value where  $y_3 \wedge v_\Lambda^0$  is zero and the value where either  $x_4 \wedge v_\Lambda^0$  or  $x_5 \wedge v_\Lambda^0$  vanishes (whatever happens first). We consider now the  $k_{\pm(e_2 - e_3)}$  module  $F(4)_1^- \wedge V_\Lambda^0$ . If we are at none of the endpoints of the above mentioned interval then, by using  $x_1, y_1, x_3, y_3$ , and  $x_5$ , we can lower the weight on  $e_3$  by  $-\frac{5}{2}e_3$ , hence  $b \geq 5/2$ . Likewise, by using  $x_3, y_3$ , and  $x_4$  we can see that  $\lambda - a \geq 3$ . We have that  $y_1 \wedge y_2 \wedge y_3 \wedge v_\Lambda^0$  is non-zero and this implies that  $\mu \leq -3/2$  or, equivalently, that  $3\mu/2 \leq -9/4$ . Observe that because  $b \neq 0$ , the place where  $y_3$  vanishes never coincides with the places where  $x_4$  and  $x_5$  vanish.

Now suppose that  $y_3$  vanishes. Then we can only conclude that  $\lambda - a \geq 2$  and  $b \geq 2$ . However, here  $y_2 \wedge v_\Lambda^0$  is a non-zero illegal highest weight vector for  $o(7)$  unless  $a > b$ .

If  $x_5$  vanishes then we get  $b \geq 2$  again, and if  $x_4$  vanishes, we only get  $\lambda - a \geq 2$ . The conditions on  $3\mu/2$  follow analogously.

Finally, we shall make some remarks about the quotients we have to take (cf. the proof of Proposition 10.2). The ideal we have to remove is generated by a subspace  $S$  of

$$(10.39) \quad U(\mathfrak{k}^-) \wedge (F(4)_1^-) \cdot v_\Lambda^0.$$

As previously, we have to make sure that we do not remove everything so that we end up with a zero dimensional vector space. Again, what we specifically need to investigate is whether  $\wedge(F(4)_1^+)$  applied to  $S$  could give us  $v_\Lambda^0$  back. However, if it could, this would imply that we inside  $\wedge(F(4)_1^-) \otimes V_\Lambda^0$  would have some highest weights for  $o(7)$  which do not define finite dimensional representations (as follows by duality), and these cases have already been carefully removed to ensure the unitarity (finite dimensionality) on  $o(7)$ .  $\square$

**PROPOSITION 10.7.** *In case III the non-trivial unitarizable highest weight representations corresponding to an involution  $w$  in which  $\varepsilon_\delta = 1$  occur exactly when  $\varepsilon_0 = -1$ . In this case there is unitarity precisely when:*

$$i) \ b \geq 5/2, \ n \geq 5 \text{ and } \lambda < -\frac{3n}{2} - a - b - 2.$$

$$ii) \ b \geq 2, \ n \geq 5 \text{ and } \lambda = -\frac{3n}{2} - a - b - 2.$$

*Proof.* All the leading coefficients in  $\langle x \wedge v_\Lambda^0, x \wedge v_\Lambda^0 \rangle$ ,  $x \in F(4)_1^-$ , have the same sign and it follows easily from this that  $\varepsilon_0 = -1$ . That we must have  $b \neq 0$

follows in analogy with the similar fact in the preceding proof. Clearly then, the last possible place of unitarity is the place where the highest weight vector whose  $v_\Lambda^0$  coefficient is  $y_1 \wedge v_\Lambda^0$ , vanishes. In this range,  $x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge v_\Lambda^0$  is non-zero and hence  $n \geq 5$ . This, on the other hand, also implies, as is easily seen, that all  $o(2,5)$  requirements are met. If we are below the place where the mentioned vector vanishes, it follows, by looking at  $x_1, y_1, x_3, y_3$ , and  $x_5$  in unison, that  $b \geq 5/2$ . Right at the point where the vector vanishes we can only conclude that  $b \geq 2$ .  $\square$

**Case IV.**

$$(10.40) \quad \Lambda + \rho = (\lambda + 2, a + 1, b, \frac{n}{2} - 1).$$

**PROPOSITION 10.8.** *In case IV the non-trivial unitarizable highest weight representations corresponding to an involution  $w$  in which  $\varepsilon_\delta = 1$  occur exactly when  $\varepsilon_0 = -1$ . In this case there is unitarity precisely when:*

- i)  $n \geq 7$ ,  $b \geq 5/2$ , and  $\lambda < -a - b - \frac{3n}{2}$ .
- ii)  $n \geq 7$ ,  $b \geq 2$ , and  $\lambda = -a - b - \frac{3n}{2}$ .

*Proof.* This follows in analogy with the previous cases. Again it is  $y_1$  which is important. Observe that  $y_1 \wedge v_\Lambda^0$  always is a highest weight vector (as is  $x_7 \wedge v_\Lambda^0$ ). It is also  $y_1$  which imposes conditions on  $n$  whereas  $x_1, y_1, x_7, x_5$ , and  $x_3$  can be used to impose restrictions on  $b$ .  $\square$

**PROPOSITION 10.9.** *In case IV the non-trivial unitarizable highest weight representations corresponding to an involution  $w$  in which  $\varepsilon_\delta = -1$  occur exactly when  $\varepsilon_0 = -1$ . In this case there is unitarity precisely when:*

- i)  $-\lambda - a - b < 3\mu/2 < -\lambda - a + b$ , and  $b \geq 5/2$ .
- ii)  $b \geq 2$ ,  $3\mu/2 = -\lambda - a - b$ .
- iii)  $b \geq 2$ ,  $3\mu/2 = -\lambda - a + b$ .

*Proof.* This also follows in analogy with previous cases. Here,  $y_1 \wedge v_\Lambda^0$  and  $x_7 \wedge v_\Lambda^0$  are always highest weight vectors, and the signs on the inner products are different. By inspection it follows that  $3\mu/2$  must be in the interval between the places where these vectors vanish. Observe that  $F(4)_1^-$  is invariant under the  $su(3)$  generated by the roots  $e_1 - e_2$  and  $e_2 - e_3$ . We omit further details.  $\square$

Finally, the following results are obtained analogously. We omit the proofs.

**Case V.**

$$(10.41) \quad \Lambda + \rho = (\lambda + \frac{3}{2}, a + \frac{1}{2}, b + \frac{1}{2}, \frac{n}{2} - \frac{1}{2}).$$

PROPOSITION 10.10. *In case V the non-trivial unitarizable highest weight representations corresponding to an involution  $w$  in which  $\varepsilon_\delta = 1$  occur exactly when  $\varepsilon_0 = -1$ . In this case there is unitarity precisely when:*

- i)  $n \geq 6$ ,  $b \geq 2$ , and  $\lambda = -a + b - \frac{3n}{2}$ .
- ii)  $n \geq 6$ ,  $b \geq \frac{3}{2}$ , and  $\lambda \leq -a - b - \frac{3n}{2} - 1$ .

PROPOSITION 10.11. *In case V the non-trivial unitarizable highest weight representations corresponding to an involution  $w$  in which  $\varepsilon_\delta = -1$  occur exactly when  $\varepsilon_0 = -1$ . In this case there is unitarity precisely when*

$$-\lambda - a + b \leq 3\mu/2 \leq -\lambda + a - b, \text{ and } b \geq 2.$$

CHAPTER 11

$G(3)$

We have

$$(11.1) \quad G_{\bar{0}} = G_2 \oplus A_1 \quad \text{and} \quad G_{\bar{0}} \upharpoonright G_{\bar{1}} = G_2 \otimes sl_2,$$

where the  $G_2$  in the tensor product denotes the irreducible 7 dimensional representation of  $G_2$ .

Let  $\varepsilon_1, \varepsilon_2, \varepsilon_3$ , and  $\delta$  denote vectors such that

$$(11.2) \quad \begin{aligned} (\varepsilon_1, \varepsilon_1) = (\varepsilon_2, \varepsilon_2) = (\varepsilon_3, \varepsilon_3) = 2 = -(\delta, \delta), \\ (\varepsilon_i, \varepsilon_j) = -1 \text{ if } i \neq j, \text{ and} \\ (\varepsilon_i, \delta) = 0 \text{ for all } i, \end{aligned}$$

and such that, moreover,  $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ . The sets of even and odd roots are

$$(11.3) \quad \Delta_0 = \{\varepsilon_i - \varepsilon_j, \pm\varepsilon_i, \pm 2\delta\}, \quad \Delta_1 = \{\pm\varepsilon_i \pm \delta, \pm\delta\}.$$

We represent  $G_{\bar{1}}$  as in Fig. 11.4:

$$\begin{array}{cccccccc} \delta + \varepsilon_1 & \rightarrow & \delta - \varepsilon_3 & \rightarrow & \delta - \varepsilon_2 & \rightarrow & \delta & \rightarrow & \delta + \varepsilon_2 & \rightarrow & \delta + \varepsilon_3 & \rightarrow & \delta - \varepsilon_1 \\ \uparrow & & \uparrow \\ -\delta + \varepsilon_1 & \rightarrow & -\delta - \varepsilon_3 & \rightarrow & -\delta - \varepsilon_2 & \rightarrow & -\delta & \rightarrow & -\delta + \varepsilon_2 & \rightarrow & -\delta + \varepsilon_3 & \rightarrow & -\delta - \varepsilon_1 \end{array}$$

Fig. 11.4

Up to  $W$  equivalence there are four sets of simple roots,  $\sum_1, \sum_2, \sum_3$ , and  $\sum_4$ . We choose the standard ordering on  $G_2$  and  $A_1$  and can then indicate the sets of positive and negative roots in  $G(3)_1$ , as determined by  $\sum_1, \sum_2, \sum_3$  and  $\sum_4$ , by two rows of signs corresponding to (11.4):

$$(I) \quad \sum_1 = \{\varepsilon_2, \varepsilon_3 - \varepsilon_2, \varepsilon_1 + \delta\}.$$

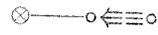


Fig. 11.5



Fig. 11.6

(II)  $\sum_2 = \{\varepsilon_3 - \varepsilon_2, -\varepsilon_3 + \delta, -\varepsilon_1 - \delta\}.$

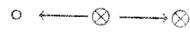


Fig. 11.7



Fig. 11.8

(III)  $\sum_3 = \{\varepsilon_2, -\varepsilon_2 + \delta, \varepsilon_3 - \delta\}.$

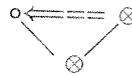


Fig. 11.9



Fig. 11.10

(IV)  $\sum_4 = \{\varepsilon_3 - \varepsilon_2, \delta, \varepsilon_2 - \delta\}.$



Fig. 11.11



Fig. 11.12

Identifying  $H$  with  $H^*$ , the co-roots become

$$(11.13) \quad h_{\varepsilon_2} = \varepsilon_2, \quad h_{\varepsilon_3 - \varepsilon_2} = \frac{1}{3}(\varepsilon_3 - \varepsilon_2), \quad \text{and} \quad h_{2\delta} = -\frac{1}{2}\delta.$$

Further we list the “ $\rho$ ”’s corresponding to the four cases:

$$(11.14) \quad \begin{aligned} \rho_I &= 2\varepsilon_2 + 3\varepsilon_3 - \frac{5}{2}\delta, \quad \rho_{II} = \varepsilon_2 + 2\varepsilon_3 - \frac{3}{2}\delta, \\ \rho_{III} &= \varepsilon_2 + \varepsilon_3 - \frac{1}{2}\delta, \quad \text{and} \quad \rho_{IV} = \varepsilon_3 + \frac{1}{2}\delta. \end{aligned}$$

Let  $z_{2\delta}$  be a non-zero root vector belonging to the root  $2\delta$  in  $A_1$  and let  $z_{-2\delta}$  be defined analogously and such that  $[z_{2\delta}, z_{-2\delta}] = h_{2\delta}$ .

It is clear that in order to have a unitarizable module with respect to some involution  $\omega$  of  $G(3)$ ,  $\omega$  restricted to  $G_2$  must be the compact involution. If we put

$$(11.15) \quad \begin{aligned} \omega(y_1) &= \varepsilon_0 x_7, \quad \text{and} \\ \omega(z_{2\delta}) &= \varepsilon_\delta z_{-2\delta}, \end{aligned}$$

so that  $\varepsilon_0^2 = \varepsilon_\delta^2 = 1$  for some appropriate non-zero root vectors  $y_1$  and  $x_7$ , then we get easily, in analogy with Chapter 10:

LEMMA 11.1. *In order to have non-trivial unitarity,  $\varepsilon_\delta = -1$ .*

(It is furthermore clear that the sign  $\varepsilon_0$  stays constant on the odd part.)

The four cases can be analyzed in completely analogy with the previous chapter. In fact, the analysis is quite simple since  $G_2$  has to be compact, so one just has to be careful in relation to the non-invariance of the sets of negative odd roots under  $G_2$  in the cases II, III, and IV. Moreover, it is in all four cases the root corresponding to either  $y_1$  or  $x_7$  (which in this connection amounts to the same) whose vanishing one has to consider. We omit further details.

In the following, let  $\Lambda = (a\varepsilon_2, b\varepsilon_3, \frac{1}{2}\mu\delta)$  be a highest weight for which  $(a\varepsilon_2, b\varepsilon_3)$  defines a finite dimensional representation of  $G_2$ , i.e. for which  $b - a \in \mathbb{N}_0$  and  $2a \in \mathbb{N}_0$ .

PROPOSITION 11.2. *There are non-trivial unitarizable modules for  $G(3)$  in case I exactly when  $\varepsilon_0 = 1$  (and  $\varepsilon_\delta = -1$ ). In this case there is unitarity exactly when:*

$$\mu \leq -3a - 3b - 9.$$

PROPOSITION 11.3. *There are non-trivial unitarizable modules for  $G(3)$  in case II exactly when  $\varepsilon_0 = 1$  (and  $\varepsilon_\delta = -1$ ). In this case there is unitarity exactly when:*

- (1)  $\mu < -3a - 3b - \frac{15}{2}$ ,  $b - a \geq 2$ , and  $2a \geq 5$ .
- (2)  $\mu = -3a - 3b - \frac{15}{2}$ ,  $b - a \geq 2$ , and  $2a \geq 4$ .

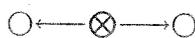
PROPOSITION 11.4. *There are non-trivial unitarizable modules for  $G(3)$  in case III exactly when  $\varepsilon_0 = 1$  (and  $\varepsilon_\delta = -1$ ). In this case there is unitarity exactly when:*

- (1)  $\mu < -3a - 3b - 5$ ,  $b - a \geq 3$ , and  $2a \geq 4$ .
- (2)  $\mu = -3a - 3b - 5$ ,  $b - a \geq 3$ , and  $2a \geq 3$ .

PROPOSITION 11.5. *There are non-trivial unitarizable modules for  $G(3)$  in case IV exactly when  $\varepsilon_0 = 1$  (and  $\varepsilon_\delta = -1$ ). In this case there is unitarity exactly when:*

- (1)  $\mu < -3a - 3b - 4$ ,  $b - a \geq 2$ , and  $2a \geq 6$ .
- (2)  $\mu = -3a - 3b - 4$ ,  $b - a \geq 2$ , and  $2a \geq 5$ .

There are two Dynkin diagrams of this superalgebra,



and

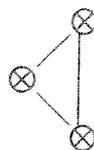


Fig. 12.1

The Cartan matrix is

$$(12.2) \quad D_\alpha = \begin{bmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

The roots are, expressed in terms of linear functions  $\varepsilon_1, \varepsilon_2$ , and  $\varepsilon_3$ ,

$$(12.3) \quad \Delta_0 = \{\pm 2\varepsilon_i, i = 1, 2, 3\}, \quad \Delta_1 = \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}.$$

Up to  $W$ -equivalence there are four systems of simple roots. However, three of these are equivalent by permutation of the indices, and for this reason we will only consider two systems, one of the three together with the fourth.

We choose non-zero root vectors  $e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4$ , such that

$$(12.4) \quad \begin{aligned} e_1 &\equiv -\varepsilon_1 + \varepsilon_2 - \varepsilon_3, & h_1 &= \frac{1}{2}h_{2\varepsilon_1} + \frac{\alpha+1}{2}h_{2\varepsilon_2} + \frac{\alpha}{2}h_{2\varepsilon_3}, \\ e_2 &\equiv \varepsilon_1 + \varepsilon_3 - \varepsilon_3, & h_2 &= -\frac{1}{2}h_{2\varepsilon_1} + \frac{\varepsilon+1}{2}h_{2\varepsilon_2} + \frac{\alpha}{2}h_{2\varepsilon_3}, \\ e_3 &\equiv -\varepsilon_1 + \varepsilon_2 + \varepsilon_3, & h_3 &= \frac{1}{2}h_{2\varepsilon_1} + \frac{\alpha+1}{2}h_{2\varepsilon_2} - \frac{\alpha}{2}h_{2\varepsilon_3}, \text{ and} \\ e_4 &\equiv \varepsilon_1 + \varepsilon_2 + \varepsilon_3, & h_4 &= -\frac{1}{2}h_{2\varepsilon_1} + \frac{\alpha+1}{2}h_{2\varepsilon_2} - \frac{\alpha}{2}h_{2\varepsilon_3}. \end{aligned}$$

We choose  $f_i$  to have weight equal to minus the weight of  $e_i$ , and we assume that for  $i = 1, 2, 3, 4$ ,

$$(12.5) \quad [e_i, f_i] = h_i.$$

(This is different from the convention in [17].)

We let  $k_{\pm 2\varepsilon_i}$  denote the usual root vectors for the three  $sl_2$ 's defined by the roots  $\pm 2\varepsilon_i$ ,  $i = 1, 2, 3$ . We consider now an anti-linear anti-involution  $\omega$  and we put

$$(12.6) \quad \begin{aligned} \omega(k_{2\varepsilon_i}) &= \bar{\varepsilon}_i k_{-2\varepsilon_i} & \text{for } i = 1, 2, 3, \\ \text{and } \omega(e_1) &= \bar{\varepsilon}_0 f_1. \end{aligned}$$

An easy computation, completely parallel to the one in Chapter 10, then gives

LEMMA 12.1.

$$\begin{aligned} \omega(e_2) &= \bar{\varepsilon}_0 \bar{\varepsilon}_1 f_2, \\ \omega(e_3) &= \bar{\varepsilon}_0 \bar{\varepsilon}_3 f_3, \text{ and} \\ \omega(e_4) &= \bar{\varepsilon}_0 \bar{\varepsilon}_1 \bar{\varepsilon}_3 f_4 = -\bar{\varepsilon}_0 \bar{\varepsilon}_2 f_4. \end{aligned}$$

In particular,  $\bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2 \cdot \bar{\varepsilon}_3 = -1$ .

We now return to the possible sets of positive root vectors in  $G_{\bar{\Gamma}}$ . As mentioned previously, there are essentially 2. We choose these such that the ordering on the three  $sl_2$ 's of  $G_{\bar{\Gamma}}$  is the usual.

**Case I.**

$$(12.7.a) \quad N_A^+ = \text{span} \{e_1, e_2, e_3, e_4\}.$$

**Case II.**

$$(12.7.b) \quad N_B^+ = \text{span} \{f_1, e_2, e_3, e_4\}.$$

For each of these there are four possibilities of the signs  $\bar{\varepsilon}_1, \bar{\varepsilon}_2$ , and  $\bar{\varepsilon}_3$  as indicated by a), b), c), and d) below:

$$(12.8) \quad \begin{array}{ccc} & \bar{\varepsilon}_1 & \bar{\varepsilon}_2 & \bar{\varepsilon}_3 \\ a) & -1 & -1 & -1 \\ b) & +1 & -1 & +1 \\ c) & +1 & +1 & -1 \\ d) & -1 & +1 & +1 \end{array}.$$

We have the following picture of  $G_{\bar{\Gamma}}$  (Fig. 12.9):

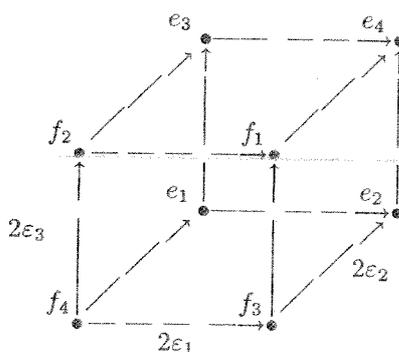


Fig. 12.9

It is easy to see that in both case (I) and (II), the cases c) and d) are equivalent in the sense that  $N_A^+$  and  $N_B^-$ , respectively, are symmetrical under a change of indices  $1 \leftrightarrow 3$ . Thus, we will omit case d) from further consideration.

One important consequence of Lemma 12.1 is that we always have at least one  $-1$  i.e. there is always at least one non-compact  $sl_2$  involved. It is then natural to concentrate on this. Thus, in case a) and c) we will assume that the representations corresponding to  $\bar{\epsilon}_1$  and  $\bar{\epsilon}_2$  are fixed and we will examine the unitarity as the last parameter varies. Likewise in case b) in which only the "middle" representation, the one corresponding to  $\bar{\epsilon}_2$ , will be allowed to vary.

In Case I, we have

$$(12.10) \quad (\Lambda + \rho) = ((n_1 + 1)h_{\epsilon_1} + (n_2 - 1)h_{\epsilon_2} + (n_3 + 1)h_{\epsilon_3}),$$

where  $\Lambda = (n_1, n_2, n_3)$  is the given highest weight. We compute the equations  $(\Lambda + \rho)(h_i) = 0$  for  $i = 1, 2, 3, 4$ :

$$(12.11.I) \quad \begin{aligned} h_1 : & (n_1 + 1) + (\alpha + 1)(n_2 - 1) + \alpha(n_3 + 1) = 0, \\ h_2 : & -(n_1 + 1) + (\alpha + 1)(n_2 - 1) + \alpha(n_3 + 1) = 0, \\ h_3 : & (n_1 + 1) + (\alpha + 1)(n_2 - 1) - \alpha(n_3 + 1) = 0, \text{ and} \\ h_4 : & -(n_1 + 1) + (\alpha + 1)(n_2 - 1) - \alpha(n_3 + 1) = 0. \end{aligned}$$

PROPOSITION 12.2. *In case I there are no non-trivial unitarizable modules corresponding to the signs in a). For the cases b) and c) there is unitarity precisely for the following values of the parameters:*

$$b) \quad \underline{(\alpha + 1)\bar{\epsilon}_0} < 0 \quad \text{and}$$

$$c) \quad n_2 < \min \left\{ \left( 1 + \frac{1}{\alpha + 1} (\pm(n_1 + 1) \pm \alpha(n_3 + 1)) \right), 0 \right\}.$$

- i) If  $\alpha > 0$  and  $n_2 = 1 + \frac{1}{\alpha+1}(-(n_1 + 1) - \alpha(n_3 + 1))$ .
- ii) If  $-1 < \alpha < 0$  and  $n_2 = 1 + \frac{1}{\alpha+1}(-(n_1 + 1) + \alpha(n_3 + 1))$ .
- iii) If  $\alpha < -1$  and  $n_2 = 1 + \frac{1}{\alpha+1}((n_1 + 1) - \alpha(n_3 + 1))$ .
- c)  $\alpha \cdot \bar{\varepsilon}_0 < 0$  and
- o)  $n_2 \geq 4$  and  $n_3 < \min \left\{ -1 + \frac{1}{\alpha}(\pm(n_1 + 1) \pm (\alpha + 1)(n_2 - 1)), -2 \right\}$ .
- i) If  $\alpha > 0$ ,  $n_2 \geq 3$ , and  $n_3 = -\frac{n_1+1}{\alpha} - \frac{\alpha+1}{\alpha}(n_2 - 1) - 1$ .
- ii) If  $-1 < \alpha < 0$ ,  $n_2 \geq 3$ , and  $n_3 = \frac{n_1+1}{\alpha} + \frac{(\alpha+1)}{\alpha}(n_2 - 1) - 1$ .
- iii) If  $\alpha < -1$ ,  $n_2 \geq 3$ , and  $n_3 = \frac{n_1+1}{\alpha} - \frac{(\alpha+1)}{\alpha}(n_2 - 1) - 1$ .

*Proof.* Consider  $\langle f_i \wedge v_\Lambda^0, f_i \wedge v_\Lambda^0 \rangle$  for  $i = 1, 2, 3, 4$ . In case a) and c) we look at these expressions as functions of  $n_3$ . It follows from Lemma 12.1 and (12.4) that in case a) the sign on the leading (first order) term is  $\bar{\varepsilon}_0 \cdot \alpha$  for  $i = 1, 3$  and  $-\bar{\varepsilon}_0 \alpha$  for  $i = 2, 4$ , whereas in c) the sign on all terms is  $\bar{\varepsilon}_0 \cdot \alpha$ . In case b) we look at the expressions as functions of  $n_2$  and it follows similarly that the leading terms all have sign  $\bar{\varepsilon}_0(\alpha + 1)$ . Since, in the equations (12.11.I), those corresponding to  $f_1$  and  $f_3$ , when solved for  $(n_3 + 1)$ , give opposite values, and similarly for  $f_2$  and  $f_4$ , it follows that there can be no unitarity in case a). (Observe that the triple  $(n_1, n_2, n_3) = (-1, 1, -1)$  is not allowed here, all three must be negative.) The analysis of case b) is very similar to that of c) and for this reason we only give the details for c): Let  $sl_2(i)$ ,  $i = 1, 2, 3$ , be the  $sl_2$ 's corresponding to the roots  $2\varepsilon_i$ ,  $i = 1, 2, 3$ . Then  $N_A^-$  is invariant under  $sl_2(1)$  and  $sl_2(3)$ . We have that  $n_3$  is a non-positive real number and it is clear that for

$$(12.12) \quad n_3 < \min \left\{ \frac{\pm(n_1 + 1) \pm (\alpha + 1)(n_2 - 1)}{\alpha} - 1 \right\}$$

there is unitarity as far as  $\wedge(N_A^-)$  is concerned. It is also easy to see that no matter what value  $\alpha$  has, the minimum is obtained in the equation corresponding to either  $h_1, h_2$ , or  $h_3$ . We observe that  $f_1 \wedge v_\Lambda^0$ ,  $f_2 \wedge f_1 \wedge v_\Lambda^0$ , and  $f_3 \wedge f_1 \wedge v_\Lambda^0$  always are highest weights, and the equations corresponding to  $i = 1, 2, 3$  in fact correspond to the vanishing of these three, respectively. It then follows easily that for  $n_3$  strictly greater than the minimum, there can be no unitarity. If  $n_3$  satisfies the inequality then, because of  $f_1 \wedge f_2 \wedge v_\Lambda^0$ ,  $n_3$  must be less than or equal to  $-2$  to insure unitarity on  $sl_2(3)$ .

It finally remains to check each of the three values of  $n_3$  for which one of the three mentioned highest weight vectors vanishes. This is straightforward, and one has only to make sure that the given value of  $n_3$  is sufficiently negative (at  $i = 1$  or  $i = 3$ ,  $n_3 \leq -1$  suffices, at  $i = 2$  we must still have  $n_3 \leq -2$ ).  $\square$

We now go to **Case II**. This is obtained by exchanging  $e_1$  and  $f_1$ . Here,

$$(12.13) \quad (\Lambda + \rho) = (n_{\varepsilon_1} + n_{\varepsilon_2} + n_{\varepsilon_3}),$$

and the equations  $(\Lambda + \rho)(h_i) = 0$ ,  $i = 1, 2, 3, 4$ , are

$$(12.14.II) \quad \begin{aligned} h_1 : \quad n_1 + (\alpha + 1)n_2 + \alpha n_3 &= 0, \\ h_2 : \quad n_1 + (\alpha + 1)n_2 + \alpha n_3 &= 0, \\ h_3 : \quad n_1 + (\alpha + 1)n_2 - \alpha n_3 &= 0, \text{ and} \\ h_4 : \quad n_1 + (\alpha + 1)n_2 - \alpha n_3 &= 0. \end{aligned}$$

The analysis now proceeds in complete analogy with case I. Indeed, several computations can be used again with obvious changes. Naturally, the conditions one obtains by demanding that  $\wedge(N_B^-) \otimes V_A^0$  should define unitarizable  $sl_2(i)$  modules for  $i = 1, 2, 3$  are different from the previous case. However, it is clear how these conditions are found. Thus, noticing the partial  $sl_2(i)$ -invariance of  $N_B^\pm$ , one easily obtains:

**PROPOSITION 12.3.** *In case II there are non-trivial unitarizable modules only corresponding to the cases b) and c). Here, there is unitarity precisely for the following values of the parameters*

$$\text{b) } \underline{(\alpha + 1)\bar{\varepsilon}_0 < 0} \quad \text{and}$$

$$\text{o) } n_1 \geq 2, n_3 \geq 2, \text{ and } n_2 < \min \left\{ \frac{1}{\alpha+1}(\pm n_1 \pm n_3), -1 \right\}.$$

$$\text{i) } \text{If } \alpha > 0, n_1 \geq 1, n_3 \geq 2, \text{ and } n_2 = \frac{1}{\alpha+1}(-n_1 - \alpha n_3).$$

$$\text{ii) } \text{If } -1 < \alpha < 0, n_1 \geq 2, n_3 \geq 1 \text{ and } n_2 = \frac{1}{\alpha+1}(-n_1 + \alpha n_3).$$

$$\text{iii) } \text{If } \alpha < -1, n_1 \geq 1, n_3 \geq 1, \text{ and } n_2 = \frac{1}{\alpha+1}(+n_1 - \alpha n_3).$$

$$\text{c) } \underline{\alpha \cdot \bar{\varepsilon}_0 < 0} \quad \text{and}$$

$$\text{o) } n_1 \geq 2, n_2 \geq 2, \text{ and } n_3 < \min \left\{ \frac{1}{\alpha}(\pm n_1 \pm (\alpha + 1)n_2), -1 \right\}.$$

$$\text{i) } \text{If } \alpha > 0, n_1 \geq 1, n_2 \geq 2, \text{ and } n_3 = -\frac{n_1}{\alpha} - \frac{\alpha+1}{\alpha}n_2.$$

- ii) If  $-1 < \alpha < 0$ ,  $n_1 \geq 2$ ,  $n_2 \geq 1$ , and  $n_3 = \frac{n_1}{\alpha} + \frac{\alpha+1}{\alpha}n_2$ .
- iii) If  $\alpha < -1$ ,  $n_1 \geq 1$ ,  $n_2 \geq 1$ , and  $n_3 = \frac{n_1}{\alpha} - \frac{(\alpha+1)}{\alpha}n_2$ .

## FURTHER DEVELOPMENTS

In this final chapter we return to the representations in Chapter 5 and use these to illustrate that there are still many things that need to be investigated about the representations we have treated in this paper.

One of the most pleasing aspects of our analysis of the unitarizable highest weight modules for simple (finite dimensional) Lie algebras was that it led directly, and was almost equivalent to, the analysis of covariant differential operators: The full description followed immediately once the singular unitary representations, i.e. those for which the hermitian form was positive semi-definite with a non-trivial radical, had been determined. Those radicals on the other hand gave rise to homomorphisms between generalized highest weight modules and then, by duality, to covariant differential operators. (See [12]; especially p. 71-73, [9], and [15]; notably p. 424-425, about these matters.)

In the present situation we have the chance of seeing again (some) of the singular representations from above since unitarizable highest weight modules of  $su(p, q)$ ,  $sp(n, \mathbb{R})$ , and  $o(n, 2)$  are among the building blocks for the modules here. Furthermore, by analogy there will be some (super-)covariant differential operators at the points where the hermitian forms become degenerate.

#### All singular representations.

It follows easily from the analysis in Chapter 5 that there are modules in which at least some of the occurring  $su(p, q)$  modules are singular. We will go one step further and call an  $A(m, n)$  representation **all-singular** if all the  $su(p, q)$ -modules occurring are singular.

**THEOREM 13.1.** *In the case corresponding to Theorem 5.2 there are only all-singular representations when  $b_1 = 0$  and  $q = j_0$ . In this case, where furthermore  $\alpha = \alpha_2^0(+)$ , there is when  $\lambda = i_0 - p, i_0 - p - 1, \dots, i_0 - p - \min\{i_0 - 1, q - 1\}$ . Furthermore, in what corresponds to Theorem 5.4, we must also be in case b). In the latter case the modules are all-singular when  $\lambda = 0, -1, \dots, \min\{i_0 - 1, j_0 - 1\}$ .*

*Proof.* It follows easily by looking at the possible values of  $\lambda$  that we must always be in "case b)". In case of Theorem 5.2, if  $q \neq j_0$  we can always make some diagrams which are non-singular. Exactly when  $q = j_0$  do we have a situation

in which, at the most singular value of  $\lambda$ ,  $\alpha = \alpha_1^0(-)$ . The full result then follows easily from this. Analogous arguments then yield the part corresponding to Theorem 5.4.

**Covariant differential operators.**

Let us specialize to the case treated in Theorem 5.2. In this case, the space of negative odd vectors is  $Q_1 \oplus Q_2$ . In complete analogy with the above mentioned situation for singular unitary representations of simple Lie algebras we then have:

**THEOREM 13.2.** *In the situation corresponding to Theorem 5.2, if for a given pair  $(\lambda, \alpha)$  which makes the corresponding module unitarizable the  $\alpha$  is either in the discrete set above  $\alpha_0^2(+)$  or in the discrete set below  $\alpha_0^1(-)$ , then there is a homomorphism between generalized highest weight modules for  $A(m, n)$  whose image is contained in the given module.*

Also, by duality, it follows that there is what perhaps should be called a covariant differential operator between the dual modules, but, since we always have

$$(13.1) \quad (\wedge(Q_1 \oplus Q_2))' \equiv (\wedge(Q_1 \oplus Q_2)),$$

this operator can also be viewed as a multiplication operator. In fact, we are somewhat uncertain about if there is a natural, say, geometrical interpretation of these operators.

The most complicated operators will be those corresponding to the discrete set below  $\alpha_0^1(-)$  since here, due to the commutation rules, the element which defines the homomorphism will consist of products of all the variables  $q_1, q_2$ , and  $w^-$ . Above  $\alpha_0^2(+)$  the analogous element will belong to  $\wedge(Q_2)$ .

**Infinite dimensional algebras.**

We finish with a brief look at the limits as some of the dimensions go to infinity. By this concept we mean that several of the series of classical Lie superalgebras also make sense in the limit where we allow  $\infty \times \infty$  matrices with only finitely many entries different from zero. Thus, in  $A(n, m)$  we can replace  $m$  or  $n$ , or both, by  $\infty$ . We can also go into the finer decomposition of  $A(p + q, n)$  as in Chapter 5 and let all, or some, of the integers  $p, q, n$  go to infinity. As an example we mention:

**THEOREM 13.3.** *In Theorem 5.2, there are unitarizable modules corresponding to formally replacing either i)  $n$  by  $\infty$  or ii)  $p$  by  $\infty$ , but not both at the same time. In case ii) it is a further requirement for the existence of unitarizable modules that the quantity  $p - i_0$  remains finite in the limit as  $p \rightarrow \infty$ .*

*Proof.*

Returning to the proof of Theorem 5.2, we have

$$\begin{aligned} \lambda_c - N_1 &= -k_0 - p - q + 3, \text{ and} \\ (13.2) \quad \lambda_c - N_2 &= -b_1 - n + 1 - 2q - (p - i_0). \end{aligned}$$

The allowed values of  $\lambda$  are, according to the proof of Theorem 5.2, smaller than either  $\lambda_c - N_1$  or  $\lambda_c - N_2$ , and if so, there is indeed unitarity. The claim follows immediately from this. Observe that  $p - i_0$  is the number of non-zero  $a_i$ 's.  $\square$

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