CYCLIC REPRESENTATIONS OF
THE QUANTUM MATRIX ALGEBRAS

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ABSTRACT. In this paper we give a complete classification of the minimal cyclic $M_q(n)$-modules and construct them explicitly. Also, we give a complete classification of the minimal cyclic modules of the so-called Dipper-Donkin quantum matrix algebra as well as of two other natural quantized matrix algebras. In the last part of the paper we relate the results to the De Concini – Procesi conjecture.

1. INTRODUCTION

At present, much progress has been made in the representation theory of quantum groups both in the generic case where the quantum parameter q is not a root of unity and in the special cases where q is a root of unity. In the latter case, a completely new class of representations, cyclic representations, have been constructed and studied by different authors (e.g. [1], [6]). The name *cyclic* actually is a little unfortunate since it does not mean "having a generating vector" but we shall keep the notation since the terminology seems to have become widely accepted. See Definition 2.1 below for the precise definition.

The cyclic representations of the quantum enveloping algebra $U_q(\mathfrak{g})$ at a root of unity have been studied by De Concini and Kac [2] for an arbitrary finite dimensional Lie algebra $\mathfrak{g}$. Some cyclic representations of $U_q(sl(n+1))$ were constructed in [6]. In [1], the minimal cyclic representations of $U_q(\mathfrak{g})$ were constructed explicitly in case $\mathfrak{g}$ is of type A_n, B_n, or C_n.

The quantum matrix algebra $M_q(n)$ is an associative algebra over $\mathbb{C}$ generated by $Z_{i,j}$ for $i, j = 1, 2, \cdots, n$ subject to the following relations:

$$
\begin{aligned}
(1) \qquad Z_{i,j}Z_{i,k} &= qZ_{i,k}Z_{i,j} \text{ if } j < k, \\
Z_{i,j}Z_{k,j} &= qZ_{k,j}Z_{i,j} \text{ if } i < k, \\
Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} \text{ if } i < s \text{ and } t < j, \\
Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} + (q - q^{-1})Z_{i,t}Z_{s,j} \text{ if } i < s \text{ and } j < t,
\end{aligned}
$$

where $i, j, k, s, t = 1, 2, \cdots, n$, and $q \in \mathbb{C}^*$ is the quantum parameter.

The representation theory of $M_q(n)$ is related to the representation theory of the algebra of functions on the quantized $SL(n)$, roughly speaking by setting the quantum determinant equal to 1. Thus, the investigation in [12] of the representation theory of

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the algebra of functions on a compact quantum group is of importance. Even more so are the articles by De Concini and Lyubashenko [3], and by De Concini and Procesi [4], [5] which culminate in a complete classification of the irreducible representations of the quantum function algebras, corresponding to a simple complex Lie group, at a root of unity.

In this paper we study the cyclic representations of the quantum matrix algebra $M_q(n)$, the so-called Dipper-Donkin quantum matrix algebra, and two more quantized matrix algebras, $J_q^0(n)$ and $J_q^Z(n)$. These algebras all belong to a natural family of quantized matrix algebras that share many properties, e.g. they are all quadratic algebras with the same Hilbert series (see [10] for more details). We classify completely the minimal cyclic modules. Furthermore, we state the De Concini – Procesi conjecture and relate our results to that by computing the dimensions of the symplectic leaves that correspond to the representations. The most difficult case by far is that of $M_q(n)$. In all cases, the points of the leaves are matrices of rank 1 or 2, but only for $M_q(n)$ do the rank 2 points occur.

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2. Basic facts and definitions

First a general definition:

Definition 2.1. *Let $\mathcal{A}$ be an algebra with a fixed set of generators $a_1, \ldots, a_N$. An $\mathcal{A}$-module V is called cyclic if every generator a_i, $i = 1, \ldots, N$, is invertible on V. If V is cyclic and if the dimension of V is minimal then V is called minimal cyclic.*

Evidently, a minimal cyclic module is irreducible.

We now begin an investigation of $M_q(n)$. In the following we always assume that the quantum parameter q is a primitive mth root of unity for some odd positive integer m.

Using induction, it is easy to prove that

Lemma 2.2. *If $i < k$ and $j < l$, then*

$$(2) \qquad Z_{k,l}Z_{i,j}^s = Z_{i,j}^s Z_{k,l} + (q^{-1} - q^{2s-1})Z_{i,l}Z_{k,j}Z_{i,j}^{s-1}$$

and

$$(3) \qquad Z_{i,j}Z_{k,l}^s = Z_{k,l}^s Z_{i,j} + (q - q^{1-2s})Z_{i,l}Z_{k,j}Z_{k,l}^{s-1}.$$

Corollary 2.3. *If q is an mth root of unity then $Z_{i,j}^m$ is a central element for all $i, j = 1, 2, \cdots, n$.*

Remark 2.4. *Let V be an irreducible cyclic module. Since $Z_{i,j}^m$ is central there exists an $a_{i,j} \in \mathbb{C}^*$ such that $Z_{i,j}^m = a_{i,j}$ on V.*

For any $\chi = (\lambda_1, \lambda_2, \cdots, \lambda_n, \mu_1, \mu_2, \cdots, \mu_n) \in (\mathbb{C}^*)^{2n}$ we define an automorphism χ of $M_q(n)$ by

$$(4) \qquad \chi(Z_{i,j}) = \lambda_i \mu_j Z_{i,j} \text{ for all } i, j = 1, 2, \cdots, n.$$

These automorphisms χ generate a subgroup $\mathcal{S}$ of the group of automorphisms of $M_q(n)$. $\mathcal{S}$ is clearly isomorphic with $(\mathbb{C}^*)^{2n}$, but there is a 1 dimensional subgroup which acts trivially.

By the action of $\mathcal{S} \cong (\mathbb{C}^*)^{2n}$ we can assume that

$$(5) \qquad Z_{n+1-i,i}^m = 1 \text{ for all } i$$

in an irreducible cyclic $M_q(n)$-module V. As part of a more general assumption, to be stated later, we assume from now on and throughout the article that where $M_q(n)$ is considered, the generators have been scaled such that (5) is satisfied.

Let $\mathcal{I} = \{(x_1, x_2, \cdots, x_n) \mid x_i^m = 1 \text{ for } i = 1, 2, \cdots, n\}$. Define

$$(6) \qquad \tau_i : \mathcal{I} \ \mapsto \ \mathcal{I},$$
$$(x_1, x_2, \cdots, x_n) \ \mapsto \ (x_1, \cdots, x_{i-1}, qx_i, x_{i+1}, \cdots, x_n).$$

Let G be the automorphism group of $\mathcal{I}$ generated by τ_i for all $i = 1, 2, \cdots, n$. Let T be the subgroup of G generated by $\eta_i = \tau_i \tau_{i+1}$ for all $i = 1, 2, \cdots, n-1$.

The cyclic module V admits a weight space decomposition with respect to the commutative subalgebra H generated by $Z_{n+1-i,i}$ for all i:

$$(7) \qquad V = \oplus_{\mathcal{I}} V(x_1, x_2, \cdots, x_n),$$

where

$$(8) \qquad V(x_1, x_2, \cdots, x_n) = \{v \in V \mid Z_{n+1-i,i}v = x_i v \text{ for all } i\}.$$

If $V(x_1, x_2, \cdots, x_n) \neq 0$, $(x_1, x_2, \cdots, x_n)$ is called a weight of V and $V(x_1, x_2, \cdots, x_n)$ is called a weight space, a non-zero element from $V(x_1, x_2, \cdots, x_n)$ is called a weight vector of weight $(x_1, x_2, \cdots, x_n)$ and $\dim V(x_1, x_2, \cdots, x_n)$ is called the multiplicity of the weight $(x_1, x_2, \cdots, x_n)$. We denote by $P(V)$ the set of weights of the module V. Clearly, by the action of $\mathcal{S}$, we can always assume that $(1, 1, \cdots, 1)$ is a weight of V.

Theorem 2.5. *Let V be an irreducible cyclic $M_q(n)$-module. Then $\dim V = sm^{n-1}$ for some positive integer s.*

Proof. For any weight space $V(x_1, x_2, \cdots, x_n)$ we have

$$(9) \qquad Z_{n-i,i}V(x_1, x_2, \cdots, x_n) \subset V(\eta_i^{-1}(x_1, x_2, \cdots, x_n)) \text{ for } i = 1, 2, \cdots, n-1,$$

and

$$(10) \quad Z_{n-i+1,i+1}V(x_1, x_2, \cdots, x_n) \subset V(\eta_i(x_1, x_2, \cdots, x_n)) \text{ for } i = 1, 2, \cdots, n-1.$$

Since both $Z_{n-i,i}$ and $Z_{n-i+1,i+1}$ are invertible we have

$$(11) \qquad \dim V(x_1, x_2, \cdots, x_n) = \dim V(\gamma(x_1, x_2, \cdots, x_n)) \text{ for all } \gamma \in T.$$

So the weight set $P(V)$ is T-invariant and the weight multiplicities are also T-invariant. Obviously, each T-orbit of $P(V)$ consists of m^{n-1} elements. This completes the proof. $\square$

We shall now construct some basic minimal cyclic $M_q(n)$-modules. Let $\mathcal{H}$ be the quasipolynomial algebra generated by $x_1, \cdots, x_{n-1}; z; y_2, \cdots, y_n$ subject to the following relations:

$$(12) \qquad \begin{aligned} x_i x_j &= q x_j x_i, \text{ if } i < j, \\ x_i z &= q z x_i, \\ y_s y_t &= q y_t y_s, \text{ if } s < t, \\ z y_s &= q y_s z, \\ x_i y_s &= q^2 y_s x_i \text{ for all } i, s. \end{aligned}$$

Theorem 2.6. *The following map is an algebra homomorphism:*

$$(13) \qquad \begin{aligned} \phi : M_q(n) &\longrightarrow \mathcal{H}[z^{-1}], \\ Z_{1,n} &\mapsto z, \\ Z_{1,j} &\mapsto x_j \text{ for } j = 1, 2, \cdots, n-1, \\ Z_{i,n} &\mapsto y_i \text{ for } i = 2, 3, \cdots, n, \\ Z_{i,j} &\mapsto q^{-1} z^{-1} x_j y_i \text{ for } i = 2, 3, \cdots, n, j = 1, 2, \cdots, n-1. \end{aligned}$$

Proof. Direct calculation. $\square$

It is well-known that a quasipolynomial algebra is an iterated Ore extension and so it is a domain. Let Z be the center of the quasipolynomial algebra $\mathcal{H}$. Let $Q(Z)$ be the fraction field of Z and $\overline{Q(Z)}$ be the algebraic closure of $Q(Z)$. Then $Q(\mathcal{H}) := \mathcal{H} \otimes_Z \overline{Q(Z)}$ is a full matrix algebra $M_d(\overline{Q(Z)})$. The number d is uniquely determined and is called the degree of $\mathcal{H}$ (Cf. [4]). Hence $\dim V \leq d$ for all irreducible $\mathcal{H}$-modules V.

Let V be an irreducible $\mathcal{H}$-module such that each x_i, z, y_j is invertible for $i = 1, 2, \cdots, n-1$ and $j = 2, 3, \cdots, n$. Then $\dim V = d$. By [4], the degree is less than or equal to m^x, where x is half the rank of (here) a $(2n-1) \times (2n-1)$ skew symmetric matrix, so $d \leq m^{n-1}$. By Theorem 2.6, V can be viewed as a cyclic $M_q(n)$-module. By Theorem 2.5 $\dim V \geq m^{n-1}$. So $\dim V = m^{n-1}$ and V is a minimal cyclic $M_q(n)$-module.

Corollary 2.7. *If V is a minimal cyclic module then $\dim V = m^{n-1}$.*

If V is a minimal cyclic module then clearly $P(V) = T(1, 1, \cdots, 1)$ and each weight is of multiplicity one. Obviously we have

Lemma 2.8. *Let V be a minimal cyclic module and let $v \in V(x_1, x_2, \cdots, x_n)$. Then*

$$(14) \qquad \{\Pi_{i=1}^{n-1} Z_{n-i+1,i+1}^{s_i} v \mid s_i = 0, 1, \cdots, m-1, i = 1, 2, \cdots, n-1\}$$

is a basis of V.

Let $\sigma, D \in \text{End}(\mathbb{C}^m)$ be defined such that with respect to the standard basis $v_0, v_1, \cdots, v_{m-1}$ of $\mathbb{C}^m$,

$$(15) \qquad \sigma(v_{\mathbf{j}}) = v_{\mathbf{j+1}}, \text{ and } D(v_{\mathbf{j}}) = q^j v_{\mathbf{j}} \text{ for all } \mathbf{j} = \mathbf{0}, \ldots, \mathbf{m-1} \in \mathbb{Z}/m \cdot \mathbb{Z}.$$

We denote by σ_i and D_i, for $i = 1, 2, \cdots, r$, the operators $1 \otimes 1 \otimes \cdots \otimes \sigma \otimes 1 \cdots \otimes 1$ and $1 \otimes 1 \otimes \cdots \otimes D \otimes 1 \cdots \otimes 1$ on $(\mathbb{C}^m)^r$ with σ and D, respectively, in the ith position.

Let V be an $M_q(n)$-module of dimension m^r for some positive integer r. We identify V with $(\mathbb{C}^m)^r$. Note that $\dim \operatorname{End} V = m^{2r}$ which is equal to the dimension of the subalgebra of $\operatorname{End} V$ generated by $\sigma_1, \sigma_2, \cdots, \sigma_r$ and $D_1, D_2, \cdots, D_r$. Hence, each $Z_{i,j} \in M_q(n)$ acts in the module by a polynomial in the σ_i's and D_i's.

Let

(16) $$M(n-1) = \langle Z_{ij} \mid i = 2, 3, \cdots, n, j = 1, 2, \cdots, n-1 \rangle.$$

(This is clearly equivalent to $M_q(n-1)$.)

Proposition 2.9. *Viewed as an $M(n-1)$-module the minimal cyclic $M_q(n)$-module V is completely reducible and each irreducible summand is a minimal cyclic $M(n-1)$-module.*

Proof. Choose $0 \neq v \in V(x_1, x_2, \cdots, x_n)$ and let

(17) $$V(0) = M(n-1)v.$$

As a submodule of a cyclic module, $V(0)$ is evidently cyclic over $M(n-1)$. Moreover, $V(0)$ is invariant under each $Z_{n+1-i,i}$, $i = 1, \ldots, n$. Hence $V(0)$ is a direct sum of some weight spaces. By comparing the weight set of $V(0)$ and V we get

(18) $$V(0) = \operatorname{Span}\{\Pi_{i=1}^{n-2} Z_{n+1-i,i+1}^{n_i} v \mid n_i = 1, 2, \cdots, m\}$$

which is a cyclic module over $M(n-1)$ and of dimension m^{n-2}. So $V(0)$ is a minimal cyclic module over $M(n-1)$. Let

(19) $$V(i) = Z_{2,n}^i V(0) \text{ for } i = 0, 1, \cdots, m-1.$$

Then we have

(20) $$V = \sum_i V(i).$$

Since $\dim V = m^{n-1} = \sum_i \dim V(i)$ we get

(21) $$V = \oplus_i V(i).$$

Clearly, each $V(i)$ is a minimal cyclic $M(n-1)$-module. This completes the proof. $\square$

Let V be a minimal cyclic $M_q(n)$-module. Choose a weight vector v of weight $(x_1, x_2, \cdots, x_n)$ of V. One can then identify V with $(\mathbb{C}^m)^{\otimes n-1}$ by the following linear map:

(22) $$L : V \longrightarrow (\mathbb{C}^m)^{\otimes n-1},$$

(23) $$\Pi_{s=1}^{n-1} Z_{n-s+1,s+1}^{t_s} v \mapsto \Pi_{s=1}^{n-1} \lambda_s^{t_s} v_{t_1} \otimes v_{t_2} \otimes \cdots \otimes v_{t_{n-1}},$$

where $\lambda_s \in \mathbb{C}^*$ for $s = 1, 2, \cdots, n-1$.

It is easy to see that on V, $Z_{n,1} = x_1 D_1$, $Z_{n-i+1,i} = x_i D_{i-1} D_i$ for $i = 2, 3, \cdots, n-1$, $Z_{1,n} = x_n D_{n-1}$, and $Z_{n+1-i,i+1} = \lambda_i \sigma_i$ for $i = 1, 2, \cdots, n-1$. It is then obvious that we can use the action of $\mathcal{S}$ to set $x_1 = \cdots = x_n = \lambda_1 = \cdots = \lambda_{n-1} = 1$. *We do this and maintain the following assumptions throughout the rest of the paper: Any*

minimal cyclic module is identified as a vector space with $(\mathbb{C}^m)^{n-1}$ and is normalized such that

$$
\begin{aligned}
(24) \qquad Z_{n,1} &= D_1 \\
Z_{n-i+1,i} &= D_{i-1}D_i \text{ for } i = 2, 3, \cdots, n-1, \\
Z_{1,n} &= D_{n-1}, \\
Z_{n+1-i,i+1} &= \sigma_i \text{ for } i = 1, 2, \cdots, n-1.
\end{aligned}
$$

Let $Z_1 = Z_{i,j}$, $Z_2 = Z_{i,j+a}$, $Z_3 = Z_{i+b,j}$, and $Z_4 = Z_{i+b,j+a}$ be elements of $M_q(n)$. It is then easy to see that in a representation obtained from the homomorphism ϕ,

$$
(25) \qquad Z_1 Z_4 = q Z_2 Z_3.
$$

Moreover, it follows from (24) combined with (25) that

$$
(26) \qquad \forall i, j = 1, \ldots, n : Z_{i,j}^m = 1.
$$

Definition 2.10. *A matrix $A = \{a_{i,j}\} \in M(n, \mathbb{C})$ is called quasi regular if all entries are non-zero. An irreducible representation V of $M_q(n)$ for which the matrix $\{Z_{i,j}^m\}$ is quasi regular and of rank 1 is called type I.*

The following lemma can be found in [11], where also the formula for the quantum determinant $\det_q$ may be found:

$$
(27) \qquad \det_q = \Sigma_{\sigma \in S_n} (-q)^{l(\sigma)} Z_{1,\sigma(1)} Z_{2,\sigma(2)} \cdots Z_{n,\sigma(n)}.
$$

Lemma 2.11. *Let μ, Δ and ϵ denote multiplication, coproduct, and counit, respectively.*

- *Let $A(i,j)$ be the quantum minor associated to (i,j). Then there is an algebra antihomomorphism*

$$
\begin{aligned}
(28) \qquad S : M_q(n) &\longrightarrow M_q(n) \\
Z_{i,j} &\mapsto (-q)^{i-j} A(j,i)
\end{aligned}
$$

- *Let $Y \in M_q(n)$ be a homogeneous element of degree r. Then*

$$
(29) \qquad \mu(1 \otimes S)\Delta(Y) = \epsilon(Y)\det_q^r = \mu(S \otimes 1)\Delta(Y).
$$

Now we can prove a formula which is useful in studying the representation theory.

Proposition 2.12. *Let q be a primitive mth root of unity. Then*

$$
(30) \qquad det_q^m = det(Z_{i,j}^m).
$$

More generally, such a formula holds for any $r \times r$ minor, $r = 1, \ldots, n$.

Proof. We use induction on n. Of course the formula is true for $n = 1$.

It is easy to prove that

$$(31) \qquad \Delta(Z_{i,j}^m) = \sum_k Z_{i,k}^m \otimes Z_{k,j}^m.$$

Now consider the homogeneous element $Z_{i,i}^m$, we have

$$(32) \qquad \begin{aligned} \epsilon(Z_{i,i}^m)det_q^m &= \mu(1 \otimes S)\Delta(Z_{i,i}^m) \\ det_q^m &= \sum_j Z_{i,j}^m S(Z_{j,i}^m) \\ det_q^m &= \sum_j (-q)^{m(i-j)} Z_{i,j}^m A(i,j)^m. \end{aligned}$$

Hence we get our assertion by Laplace expansion. The final claim follows because the computation for an arbitrary $r \times r$ minor can be chosen to take place inside an appropriate copy of $M_q(r)$. $\quad\square$

Lemma 2.13. *Suppose that V is a module of type I. Then it comes from an irreducible cyclic representation of $\mathcal{H}$ by the homomorphism ϕ.*

Proof. Observe that the action of $\mathcal{S}$ does not change the rank of the matrix $\{Z_{i,j}^m\}$. So we are allowed to impose our standard assumption (24). According to this, the matrix becomes $E = \sum_{i,j=1}^n E_{i,j}$, where $E_{i,j}$ is the matrix with 1 in the i,jth place and zeros elsewhere. Actually, we shall only use that for a rank 1 matrix, any 2×2 subdeterminant is zero. Consider a 2×2 submatrix Z_1, Z_2, Z_3, Z_4 with the entries defined as in (25). Then it follows from Proposition 2.12 that $(Z_1 Z_4 - q Z_2 Z_3)^m = 0$. We cannot conclude directly that $(Z_1 Z_4 - q Z_2 Z_3) = 0$, though this is what we want. But in one case, namely the 2×2 matrix $Z_{1,n-1}, Z_{1,n}, Z_{2,n-1}, Z_{2,n}$, we *can* make the deduction, since in this case, the quantum determinant is covariant ([9]). But, given this information it follows that the 2×2 matrix $Z_{1,n-2}, Z_{1,n}, Z_{2,n-2}, Z_{2,n}$ has a quantum determinant which is covariant, and hence is zero. This kind of bootstrapping can then be continued to yield that $(Z_1 Z_4 - q Z_2 Z_3) = 0$ for any 2×2 submatrix. Finally, by looking at "squares" Z_1, Z_2, Z_3, Z_4 where 3 generators are already known and then using $Z_1 Z_4 = q Z_2 Z_3$ (and cyclicity) to determine the remaining, one can, starting from (24), write down all remaining generators. $\quad\square$

Observe that the homomorphism ϕ gives the existence of a module of type I. Actually, it is easy, as mentioned in the proof of Lemma 2.13, to write the action of any generator $Z_{i,j}$ on a minimal cyclic module of type I. In particular, due to our normalization, we have uniqueness. These generators will from now on be denoted by $W_{i,j}$.

$$(33) \qquad Z_{i,j} := W_{i,j} =$$
$$q^{i+j-n-2} D_{n+1-i}^{-1} D_{n+2-i}^{-2} D_{n+3-i}^{-2} \cdots D_{j-2}^{-2} D_{j-1}^{-1} \sigma_{n+1-i} \sigma_{n+2-i} \cdots \sigma_{j-1} \text{ if } i+j \geq n+3.$$

This can be proved by induction. For $i + j \leq n$ it is easy to see that

$$(34) \qquad Z_{n-1,1} = q D_1^2 D_2 \sigma_1^{-1}, Z_{1,n-1} = q D_{n-2} D_{n-1}^2 \sigma_{n-1}^{-1}$$

and

$$(35) \qquad Z_{n-i,i} := W_{n-i,i} = qD_{i-1}D_i^2D_{i+1}\sigma_i^{-1} \text{ for } i = 2, 3, \cdots, n-2.$$

Then we have

$$(36) \qquad Z_{i,j} = W_{i,j} := q^{n+1-i-j}D_{j-1}D_j^2\cdots D_{n-i}^2D_{n-i+1}\sigma_j^{-1}\cdots\sigma_{n-i}^{-1}$$

for $i + j \leq n$ and $j > 1$. Also we have

$$(37) \qquad Z_{i,1} = q^{n-i}D_1^2D_2^2\cdots D_{n-i}^2D_{n-i+1}^2\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-i}^{-1}$$

for $i < n$. The remaining generators are given by (24).

3.   THE CLASSIFICATION OF MINIMAL CYCLIC MODULES OF $M_q(n)$

In section 2 we have constructed the type I minimal cyclic modules. In this section we will construct some other types of minimal cyclic modules which, up to $(\mathbb{C}^*)^{2n}$ action, together with type I minimal cyclic modules, will exhaust all of the minimal cyclic $M_q(n)$-modules. At first we need the following result.

Let $LM_q(n) = M_q(n)[Z_{i,j}^{-1} \mid i, j = 1, 2, \cdots, n]$ be the Laurent quantum matrix algebra. Then we have

Theorem 3.1. *The linear map*

$$
\begin{aligned}
(38) \qquad \psi_n : LM_q(n) &\longrightarrow LM_q(n) \\
Z_{i,j} &\mapsto Z_{i,j} \text{ for } i = 2, 3, \cdots, n \text{ and } j = 1, 2, \cdots, n-1 \\
Z_{2,n} &\mapsto Z_{2,n} \\
Z_{1,n} &\mapsto Z_{1,n} \\
Z_{i,n} &\mapsto Z_{i,n-1}Z_{2,n-1}^{-1}Z_{2,n} \text{ for } i = 3, 4, \cdots, n \\
Z_{1,j} &\mapsto qZ_{2,j}Z_{1,n}Z_{2,n}^{-1} \text{ for } j = 1, 2, \cdots, n-1.
\end{aligned}
$$

is an algebra homomorphism.

Proof. Direct verification. □

Note that the image of ψ_n is given as $\psi_n(LM_q(n)) = LM(n-1) \times \overline{A_2}$, where $LM(n-1)$ is the subalgebra of $LM_q(n)$ generated by

$$(39) \qquad \{Z_{i,j}, Z_{i,j}^{-1} \mid i = 2, \cdots, j = 1, 2, \cdots, n-1.\}$$

and $\overline{A_2}$ is the subalgebra generated by $Z_{1,n}, Z_{2,n}$ and $Z_{1,n}^{-1}, Z_{2,n}^{-1}$. Clearly

$$(40) \qquad LM(n-1) \cong LM_q(n-1).$$

Given any minimal cyclic $M_q(n-1)$-module $V(0)$ we can then construct a minimal cyclic $M_q(n)$-module $V = \Psi_{n-1}^n(V(0))$ such that

$$(41) \qquad V = V(0) \otimes \mathbb{C}^m,$$

simply by letting

(42)
$$Z_{i,j} \;\to\; Z_{i,j} \otimes 1 \text{ for } i = 2, \ldots, n, \text{ and } j = 1, \ldots, n-1,$$
$$Z_{1,n} \;\to\; 1 \otimes D \quad (= D_{n-1}),$$
$$Z_{2,n} \;\to\; 1 \otimes \sigma \quad (= \sigma_{n-1}),$$

and then defining the remaining generators as in (38).

Now we can construct more minimal cyclic $M_q(n)$-modules:

Consider a minimal cyclic module V of type I. For an even integer i and an integer $j > i$ let

(43)
$$O_{i,j} = D_{n-j+1}^{-1} D_{n-j+2}^{-2} \cdots D_{n-i}^{-2} D_{n-i+1}^{-1} \sigma_{n-j+1} \sigma_{n-j+2} \cdots \sigma_{n-i}.$$

For any $a \in \mathbb{C}$ let

(44)
$$Z_{i,n} = W_{i,n} + a D_{n-i+1}^{-1} D_{n-1}^{-1} \sigma_{n-i+1} \sigma_{n-i+2}^{-1} \sigma_{n-i+3} \cdots \sigma_{n-2}^{-1} \sigma_{n-1},$$

and

(45)
$$Z_{j,n} = q^{j-i} O_{i,j} Z_{i,n},$$

and keep the other generators $Z_{i,j}$ unchanged, i.e. $Z_{i,j} = W_{i,j}$. Then we have

Lemma 3.2. *The above generators $Z_{i,j}$ for all $i, j = 1, 2, \cdots, n$ give an irreducible $M_q(n)$-module which is a minimal cyclic module provided that $a^m + 1 \neq 0$.*

Proof. We need to verify the defining relations of the algebra $M_q(n)$. This can be done by direct calculation. $\square$

By symmetry we can also construct modules based on a change of the operators in the first row. Together, the two kinds of modules are called minimal cyclic $M_q(n)$-modules of type II.

The extension of a type II minimal cyclic $M_q(s)$-module by using the extension functor $\Psi_{n-1}^n \cdots \Psi_s^{s+1}$ is called a minimal cyclic $M_q(n)$-module of type III.

Now we can state the main result.

Theorem 3.3. *Let V be a minimal cyclic $M_q(n)$-module. Then up to $(\mathbb{C}^*)^{2n}$ action, V is either of type I, type II or type III.*

Proof. We use induction on n. The case $n = 1$ is trivial. The case $n = 2$ has been done in [8].

Let V be a minimal cyclic module. We then have (21), (24), and

(46)
$$V(i) = Z_{2,n}^i V(0) = \sigma_{n-1}^i V(0) \text{ for } i = 0, 1, 2, \cdots, m-1.$$

By Proposition 2.9, the modules $V(i)$ are minimal cyclic $LM(n-1) \cong LM_q(n-1)$-modules, where $LM(n-1)$ is given in the beginning of this section.

At first we assume that $V(0)$ is a minimal cyclic $M_q(n-1)$-module of type I, and that V is not of type I. We examine which generator is not given by the algebra homomorphism ϕ. Without loss of generality we can assume that the generator which is different from the corresponding $W_{i,j}$ is $Z_{i,n}$ and that i is minimal with respect to this property. Observe that by our standard assumption, $i \geq 3$. We now wish to establish the following

Claim. $Z_{i,n}$ *can only differ from* $W_{i,n}$ *by a multiple of the following monomial:*

(47) $$X_i = D_{n-i+1}^{-1} D_{n-1}^{-1} \sigma_{n-i+1} \sigma_{n-i+2}^{-1} \sigma_{n-i+3} \cdots \sigma_{n-2}^{-1} \sigma_{n-1}$$

and this is only possible when i is even.

Proof of claim. Observe that the difference X_i between $Z_{i,n}$ and $W_{i,n}$ is a polynomial in the σ_j's and D_j's. It follows easily from the defining relations that

(48) $$D_j X_i = X_i D_j \text{ for } j = 1, \cdots, n-i,$$

(49) $$D_k D_{k+1} X_i = X_i D_k D_{k+1} \text{ for } k = n-i+1, \cdots, n-2,$$

(50) $$D_{n-i} D_{n-i+1} X_i = q X_i D_{n-i} D_{n-i+1},$$

(51) $$D_{n-1} X_i = q X_i D_{n-1},$$

(52) $$\sigma_j X_i = X_i \sigma_j \text{ for } j \neq n-i+1, n-1,$$

and

(53) $$\sigma_{n-i+1} X_i = q X_i \sigma_{n-i+1}, \sigma_{n-1} X_i = q X_i \sigma_{n-1}.$$

Hence X_i is a multiple of

(54) $$D_{n-i+1}^{-1} D_{n-1}^{-1} \sigma_{n-i+1} \sigma_{n-i+2}^{-1} \sigma_{n-i+3} \cdots \sigma_{n-2}^{-1} \sigma_{n-1}$$

and i must be even. This establishes the claim.

Assume then that

(55) $$Z_{i,n} = W_{i,n} + a_i X_i \text{ for some } a_i \in \mathbb{C}^*.$$

We have proved that

(56) $$Z_{j,n} = q^{j-i} O_{i,j} Z_{i,n} \text{ for } j = i+1, \cdots, n$$

together with the other generators give a minimal cyclic structure. We claim that this is the unique choice. Assume that $Z_{j,n}$ is not given by the above equation for some $j > i$. Then j must be even, and for similar reasons, $Z_{j,n}$ can only differ from the term given by the equation above by a multiple of

(57) $$X_j := D_{n-j+1}^{-1} D_{n-1}^{-1} \sigma_{n-j+1} \sigma_{n-j+2}^{-1} \sigma_{n-j+3} \cdots \sigma_{n-2}^{-1} \sigma_{n-1}.$$

But $X_i X_j = q^{-1} X_j X_i$ while $X_i X_j = q X_j X_i$ is required by the defining relations of the quantum matrix algebra. Hence the choice of $Z_{j,n}$ is unique for $j > i$.

Similarly, if $Z_{i,n}$ has been modified, then the generator at the $(1, j)$th place must be given by $W_{1,j}$. Indeed, it is easy to see that it can at most differ from $W_{1,j}$ by a multiple of

(58) $$Y_j := D_{j-1} \sigma_j^{-1} \sigma_{j+1} \sigma_{j+2}^{-1} \cdots \sigma_{n-2} \sigma_{n-1}^{-1}$$

and this is only possible when $n - j$ is odd. Note that $n - i + 1$ and $n - 1$ have the same parity, and $j - 1$ and $n - 2$ have the same parity. Now observe that

(59) $$X_i Y_j = q^2 Y_j X_i$$

which follows by elementary calculation by distinguishing the two cases $n - i + 1 \geq j$ and $n - i + 1 < j$. But it is required by the defining relations that X_i commute with Y_j. Hence the choice of $Z_{1,j}$ is unique. So V is a minimal cyclic $M_q(n)$-module of type II.

Assume now that V is a minimal cyclic $M_q(n)$-module of type II. Specifically we assume, without loss of generality, that we have the type II module above. We then use the extension map Ψ_n^{n+1} and get a module with generators $Z_{i,j}$; $i, j = 1, \ldots, n+1$. Notice that the term previously given as $Z_{i,n}$ with expression given by (55), in the numeration of the extended module is $Z_{i+1,n}$.

Now suppose there is a generator, say $\tilde{Z}_{j,n+1}$, which is different from $Z_{j,n+1}$ for some $j \leq i$ and j is minimal with respect to this property (observe that $j > 2$). Then j must be even and $\tilde{Z}_{j,n+1}$ can only differ from $Z_{j,n+1}$ by a multiple of

$$(60) \qquad C_j := D_{n-j}^{-1} D_n^{-1} \sigma_{n-j} \sigma_{n-j+1}^{-1} \sigma_{n-j+2} \cdots \sigma_{n-1}^{-1} \sigma_n.$$

Let

$$(61) \qquad \tilde{Z}_{j,n+1} = Z_{j,n+1} + d_j C_j \text{ for some } d_j \in \mathbb{C}^*.$$

We know that the generators $\tilde{Z}_{k,n+1}$ are uniquely given by

$$(62) \qquad \tilde{Z}_{k,n+1} = \tilde{O}_{jk} \tilde{Z}_{j,n+1} \text{ for } k = j+1, \cdots, n+1$$

where

$$(63) \qquad \tilde{O}_{jk} = D_{n-k+2}^{-1} D_{n-k+3}^{-2} \cdots D_{n-j+1}^{-2} D_{n-j+2}^{-1} \sigma_{n-k+2} \sigma_{n-k+3} \cdots \sigma_{n-j+1}.$$

If we now consider that $Z_{i+1,n}$ must q-commute with $\tilde{Z}_{i+1,n+1}$, a contradiction is easily reached. So the $(n+1)$th column must be given by Ψ_n^{n+1}, at least for $j \leq i$. In the case $j > i+1$ a contradiction is reached by computing $Z_{j,n}$ versus $Z_{j,n+1}$ and the case $i + 1 = j$ is ruled out for parity reasons. Finally it follows, by computing the relation between X_i and Y_j just as in the previous case, that one cannot change the first row. Hence V must be of type III. In the same way one can prove that if $V(0)$ is a minimal cyclic $M_q(n-1)$-module of type III, then V is a minimal cyclic $M_q(n)$-module which is also of type III. This completes the proof. $\square$

Lemma 3.4. *Let $\alpha \in \mathbb{Z}$. Assume that $AB = q^\alpha BA$. Then $(AB)^m = A^m B^m$. Moreover, if $\alpha \not\equiv 0 \mod m$, then $(A + B)^m = A^m + B^m$.*

Proof. The first equation is trivial, the other follows from the quantum binomial expression. $\square$

Using Lemma 3.4, that $\forall r, s : W_{r,s}^m = 1$, and that $W_{i,n} X_i = q^2 X_i W_{i,n}$, it follows easily that

Proposition 3.5. *The matrix $\{Z_{i,j}^m\}$ of a minimal cyclic module of $M_q(n)$, subject to the standard assumption, is given by*

$$(64) \qquad \{Z_{i,j}^m\} = E + \sum_{i \geq i_0} \sum_{j \geq j_0} a^m E_{i,j} \text{ or } \{Z_{i,j}^m\} = E + \sum_{i \leq i_0} \sum_{j \leq j_0} a^m E_{i,j},$$

where $a^m \neq -1$, and where $(i_0, j_0) \in \{1, \ldots, n\}^2$ must satisfy the following restrictions: In the first sum, $i_0 + j_0 = n + 2r$ for some positive integer r and, in the second sum, $i_0 + j_0 = n - 2r$ for some positive integer r. The case $a = 0$ corresponds to type I, type II is represented by the configurations $(j_0 = n, i_0 \geq 4)$ in the first sum and by $(i_0 = 1, j_0 \leq n - 1)$ in the second sum. The remaining cases are type III.

In particular, for types II and III, $\{Z_{i,j}^m\}$ has rank 2.

4. THE MINIMAL CYCLIC REPRESENTATIONS OF THE DIPPER-DONKIN QUANTUM MATRIX ALGEBRA

We first need to introduce the so-called coordinate ring of quantum n-space which is an associative algebra A_n generated by $x_1, x_2, \cdots, x_n$ subject to the following relations:

$$(65) \qquad\qquad x_i x_j = q x_i x_j \text{ if } i < j.$$

It is well known that the degree of A_n is $m^{[\frac{n}{2}]}$ if q is a primitive mth root of unity (c.f. [4]), where $[\frac{n}{2}]$ is the integral part of $\frac{n}{2}$. Let V be an irreducible A_n-module. Then $\dim V$ coincides with the degree of A_n if every x_i is invertible on V. We use $\overline{A_n}$ to denote the Laurent quasipolynomial algebra $A_n[x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}]$. Then every irreducible $\overline{A_n}$-module is of dimension $m^{[\frac{n}{2}]}$.

The so-called Dipper-Donkin quantum matrix algebra $D_q(n)$ is an associative algebra over the complex numbers $\mathbb{C}$ generated by elements $Z_{i,j}, i, j = 1, 2, \cdots, n$ subject to the following relations:

$$
\begin{aligned}
(66) \qquad Z_{i,j} Z_{s,t} &= q Z_{s,t} Z_{i,j} \text{ if } i > s \text{ and } j \leq t, \\
Z_{i,j} Z_{s,t} &= Z_{s,t} Z_{i,j} + (q-1) Z_{s,j} Z_{i,t}, \text{ if } i > s \text{ and } j > t, \\
Z_{i,j} Z_{i,k} &= Z_{i,k} Z_{i,j} \text{ for all } i, j, k
\end{aligned}
$$

where $q \in \mathbb{C}^*$. We assume from now on that q is a primitive mth root of unity.

Proposition 4.1. *Let V be a cyclic module over the Dipper-Donkin quantum matrix algebra. Then*

$$(67) \qquad\qquad dim V \geq m^{[\frac{n}{2}]},$$

where $[\frac{n}{2}]$ is the integral part of $\frac{n}{2}$.

Proof. Note that the subalgebra of $D_q(n)$ generated by $Z_{i,1}$ for $i = 1, 2, \cdots, n$ is isomorphic with A_n. Hence V can be viewed as a cyclic A_n-module. Hence

$$(68) \qquad\qquad dim V \geq m^{[\frac{n}{2}]}.$$

This completes the proof. $\square$

The following lemma gives us some minimal cyclic modules.

Lemma 4.2. *The following linear map*

(69)
$$\psi : D_q(n) \longrightarrow A_n,$$
$$Z_{i,j} \mapsto x_i \text{ for all } i, j = 1, 2, \cdots, n.$$

is an algebra homomorphism. In particular, the dimension of a minimal cyclic module is $m^{[\frac{n}{2}]}$.

Theorem 4.3. *Up to $(\mathbb{C}^*)^{2n}$ action, all of the minimal $D_q(n)$-modules are given by the above algebra homomorphism.*

Proof. Let V be a minimal cyclic $D_q(n)$-module. Then viewed as a module over the subalgebra which isomorphic with A_n generated by the first column, V is a cyclic module and since $\dim V = m^{[\frac{n}{2}]}$, it is irreducible. Hence, by the Jacobson Density Theorem, every generator $Z_{i,j}$ is a polynomial of the first column. Then it is easy by induction on i to show that there is a scalar $c_j \in \mathbb{C}^*$ such that on V

(70)
$$Z_{i,j} = c_j Z_{i,1} \text{ for all } i.$$

This completes the proof. $\square$

5. THE MINIMAL CYCLIC MODULES OF $J_q^0(n)$ AND $J_q^Z(n)$

In this section we briefly treat two more matrix algebras which belong to the same family as the previous two. See [10] for more details.

Let $J_q^0(n)$ be the associative algebra generated by $\tilde{Z}_{i,j}$ for all $i, j = 1, \ldots, n$ with defining relations:

(71)
$$\tilde{Z}_{i,j}\tilde{Z}_{s,t} = q^{s+t-i-j}\tilde{Z}_{s,t}\tilde{Z}_{i,j}, \text{ if } (s-i)(t-j) \le 0,$$

(72)
$$q^{1-t+j}\tilde{Z}_{i,j}\tilde{Z}_{s,t} = q^{s-i-1}\tilde{Z}_{s,t}\tilde{Z}_{i,j} + (q-q^{-1})\tilde{Z}_{i,t}\tilde{Z}_{s,j} \text{ if } s > i, t > j.$$

Lemma 5.1. *There is an algebra homomorphism*

(73)
$$\psi : J_q^0(n) \longrightarrow A_2$$
$$\tilde{Z}_{i,j} \mapsto y^{i+j}x.$$

Hence there is an irreducible cyclic $J_q^0(n)$-module of dimension m.

The following is then clear:

Corollary 5.2. *A minimal cyclic $J_q^0(n)$-module V has $\dim V = m$.*

Note that if $n = 2$ the algebra $J_q^0(2)$ coincides with the algebra $M_q(2)$. Hence we can assume that $n \ge 3$ in the following text.

Theorem 5.3. *Up to $(\mathbb{C}^*)^{2n}$-action, the minimal cyclic $J_q^0(n)$-modules are given be the algebra homomorphism ψ.*

Proof. Let V be a minimal cyclic $J_q^0(n)$-module. By choosing the basis of V properly we can assume that

(74) $$\tilde{Z}_{1,1} = a_{1,1}\sigma \text{ and } \tilde{Z}_{2,1} = a_{2,1}D$$

on V for some $a_{1,1}, a_{2,1} \in \mathbb{C}^*$. Each generator $\tilde{Z}_{i,j}$ is a polynomial of σ and D by the Jacobson Density Theorem. Invoking the assumption $n \geq 3$ it follows that there are at least two monomials of σ and D which q-commute with $\tilde{Z}_{i,j}$. It then follows easily that each generator $\tilde{Z}_{i,j}$ is a monomial in σ and D. More precisely we have

(75) $$\tilde{Z}_{i,j} = a_{i,j}\sigma^{j-1}D^i \text{ for some } a_{i,j} \in \mathbb{C}^*,$$

where any (s,t) such that $i < s$ and $j < t$

(76) $$a_{i,j}a_{s,t} = a_{i,t}a_{s,j}.$$

This completes the proof. $\square$

Let $J_q^Z(n)$ be the associative algebra generated by $\tilde{Z}_{i,j}$ for all $i, j = 1, \ldots, n$ with defining relations:

(77) $$\check{Z}_{i,j}\check{Z}_{s,t} = \check{Z}_{s,t}\check{Z}_{i,j}, \text{ if } (s-i)(t-j) \leq 0,$$

(78) $$q\check{Z}_{i,j}\check{Z}_{s,t} = q^{-1}\check{Z}_{s,t}\check{Z}_{i,j} + (q - q^{-1})\check{Z}_{i,t}\check{Z}_{s,j} \text{ if } s > i, t > j.$$

Theorem 5.4. *A minimal cyclic $J_q^Z(n)$-module V has $dim V = 1$. The scalars $\check{Z}_{i,j} = a_{i,j}$ satisfy*

(79) $$a_{i,j}a_{s,t} = a_{i,t}a_{s,j}$$

for all $i < s$ and $j < t$.

Proof. Follows directly from the defining equations. $\square$

6. A Conjecture by De Concini and Procesi

Let q be an indeterminate and let A be an algebra over $\mathbb{C}[q, q^{-1}]$ generated by $x_1, x_2, \cdots, x_n$ and satisfying the following relations:

(80) $$x_i x_j = q^{h_{ij}} x_j x_i + p_{ij} \text{ if } i > j,$$

where the matrix (h_{ij}) is a skew-symmetric matrix over $\mathbb{Z}$ and $p_{ij} \in \mathbb{C}[x_1, x_2, \cdots, x_{i-1}]$. Let $m > 1$ be an integer relatively prime to all of the elementary divisors of the matrix (h_{ij}) and let ϵ be a primitive mth root of unity. Let $A_\epsilon = A/(q - \epsilon)$ and assume that all the elements x_i^m's are central. Let $Z_0 = \mathbb{C}[x_1^m, x_2^m, \cdots, x_n^m]$. Then Z_0 has a canonical Poisson structure defined by

(81) $$\{x_i^m, x_j^m\} = \frac{[x_i^m, x_j^m]}{q - \epsilon} \mid_{q=\epsilon}.$$

With respect to this Poisson structure, Spec Z_0 becomes a Poisson manifold ([4]).

DP-Conjecture. In [4], the following equation (82) is conjectured under the following assumptions: π is an irreducible representation of A_ϵ over the point p in Spec Z_0 and $\mathcal{O}_\pi$ is the symplectic leaf containing p.

$$(82) \qquad \dim \pi = m^{\frac{1}{2} \dim \mathcal{O}_\pi}.$$

For convenience, we shall in the following use the following Poisson structure on $M(n, \mathbb{C})$ defined by (c.f. [3])

$$(83) \qquad \{x_i^m, x_j^m\} = \frac{[x_i^m, x_j^m]}{m \cdot (q^m - 1)} \mid_{q=1} .$$

This structure is equivalent to the former in the sense that the Poisson bracket in (83) is a constant multiple of that in (81). Indeed, it follows easily that the Poisson bracket coming from $M_q(n)$ is given by

$$(84) \qquad \begin{aligned} \{a_{i,j}, a_{i,k}\} &= a_{i,j}a_{i,k} \text{ if } j < k, \\ \{a_{i,j}, a_{k,j}\} &= a_{i,j}a_{k,j} \text{ if } i < k, \\ \{a_{i,j}, a_{s,t}\} &= 0 \text{ if } i < s \text{ and } t < j, \\ \{a_{i,j}, a_{s,t}\} &= 2a_{i,t}a_{s,j} \text{ if } i < s \text{ and } j < t. \end{aligned}$$

Let $a_1 = a_{i,j}$, $a_2 = a_{i,j+a}$, $a_3 = a_{i+b,j}$, and $a_4 = a_{i+b,j+a}$ be standard coordinate functions on $M(n, \mathbb{C})$ as in (84). Identify these, when convenient, with the coordinates. It is then easy to see that the flow of the Hamiltonian vector field corresponding to a_1, restricted to a_1, a_2, a_3, a_4, is given by

$$(85) \qquad a_1(t) = a_1(0), \; a_2(t) = a_2(0)e^{a_1(0)\cdot t}, \; a_3(t) = a_3(0)e^{a_1(0)\cdot t}, \text{ and}$$
$$a_4(t) = \frac{a_2(0)a_3(0)}{a_1(0)}e^{2a_1(0)\cdot t} + \left(a_4(0) - \frac{a_2(0)a_3(0)}{a_1(0)}\right).$$

Similarly, the flow of the Hamiltonian vector field corresponding to a_4, restricted to a_1, a_2, a_3, a_4 is given by

$$(86) \qquad a_4(t) = a_1(0), \; a_2(t) = a_2(0)e^{-a_4(0)\cdot t}, \; a_3(t) = a_3(0)e^{-a_4(0)\cdot t}, \text{ and}$$
$$a_1(t) = \frac{a_2(0)a_3(0)}{a_4(0)}e^{-2a_4(0)\cdot t} + \left(a_1(0) - \frac{a_2(0)a_3(0)}{a_4(0)}\right).$$

Consider now the $n \times n$ matrix

$$(87) \qquad A = E + (a - 1) \cdot E_{n,n}.$$

For later use, the following observation is useful:

Proposition 6.1. *At the point $A \in M(n, \mathbb{C})$, the Poisson structure coming from $M_q(n)$ is given by an $n^2 \times n^2$ matrix Ω which satisfies*

$$(88) \qquad \text{rank}\,\Omega = \begin{cases} 2n - 2 \text{ if } n \text{ is even} \\ 2n - 2 \text{ if } n \text{ is odd and } a = 1 \\ 2n \text{ if } n \text{ is odd and } a \neq 1 \end{cases}.$$

Proof. Let $B = \sum_{i>j} E_{i,j}$, $B_s = B - B^t$, $I = \sum_i E_{i,i}$, $H = I + 2B$, $F = \sum_{j=1}^{n-1} E_{n,j}$, and $F_s = F - F^t$. The Poisson structure is then defined through the matrix Ω whose entries are $\Omega_{(i,j),(s,t)} = \{Z_{i,j}, Z_{s,t}\}$. It is easy to see that in an appropriate ordering of the matrices,

(89)
$$\Omega =$$
$$\left\{ \begin{matrix} B_s & -H^t & -H^t & \ldots & -H^t & -H^t - (a-1) \cdot E_{n,n} \\ H & B_s & -H^t & \ldots & -H^t & -H^t - (a-1) \cdot E_{n,n} \\ H & H & B_s & \ldots & -H^t & -H^t - (a-1) \cdot E_{n,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ H & H & H & \ldots & B_s & -H^t - (a-1) \cdot E_{n,n} \\ H + (a-1) \cdot E_{n,n} & H + (a-1) \cdot E_{n,n} & H + (a-1) \cdot E_{n,n} & \ldots & H + (a-1) \cdot E_{n,n} & B_s + (a-1) \cdot F_s \end{matrix} \right\}.$$

If we subtract (in block form) the second row from the first row and then the second column from the first column we get an equivalent matrix

(90)
$$\Omega_1 = \left\{ \begin{matrix} 0 & -E & \ldots & 0 \\ E & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \ldots & * \end{matrix} \right\},$$

where the blocks at the places of the $*$'s precisely are the blocks of Ω, i.e. Ω_1 differs only in the first block row and column from Ω. If $n > 3$, this procedure may be continued. After the next step we then obtain a matrix Ω_2 given by

(91)
$$\left\{ \begin{matrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{matrix} \right\}.$$

Observe that E has rank 1. So, after 2 steps we have a rank 2 matrix split off. If n is even we finally end up with a "matrix of $*$'s" of the form

(92)
$$\left\{ \begin{matrix} B_s & -H^t - (a-1) \cdot E_{n,n} \\ H + (a-1) \cdot E_{n,n} & B_s + (a-1) \cdot F_s \end{matrix} \right\},$$

and using the same kind of strategy together with the observation that

(93)
$$\operatorname{rank} B_s = \left\{ \begin{matrix} n \text{ if } n \text{ is even} \\ n - 1 \text{ if } n \text{ is odd} \end{matrix} \right.$$

it follows easily that the last matrix has rank n always. In case n is odd we end up with a "matrix of $*$'s" of the form

(94)
$$\left\{ \begin{matrix} 0 & -E & 0 \\ E & B_s & -H^t - (a-1) \cdot E_{n,n} \\ 0 & H + (a-1) \cdot E_{n,n} & B_s + (a-1) \cdot F_s \end{matrix} \right\},$$

and using the previous observation about the rank of B_s it is easy to see that this matrix has rank $n + 1$ if $a = 1$ and rank $n + 3$ if $a \neq 1$. Putting this together with the rank 2 matrices split off in the preceding steps we finally get the result. $\square$

Proposition 6.2. *In the Poisson structure from $M_q(n)$, the symplectic leaf through a point of the form $\{a_{i,j}\} = \{Z_{i,j}^m\}$, with $\{Z_{i,j}^m\}$ as in Proposition 3.5, has dimension $2n - 2$.*

Proof. This can be proved by induction in much the same way that the classification was obtained. To conform with that case, let us consider $M(n + 1, \mathbb{C})$. Then we can decompose the Poisson structure at a point $E + \sum_{i \geq i_0, j \geq j_0} c \cdot E_{i,j} \in M(n + 1, \mathbb{C})$ by changing the coordinate functions into

$$
\begin{align}
(95) \qquad a_{i,j} &\mapsto a_{i,j} \text{ for } i = 2, 3, \cdots, n + 1 \text{ and } j = 1, 2, \cdots, n \\
a_{2,n+1} &\mapsto a_{2,n+1} a_{1,n+1} a_{2,n}^{-1} \\
a_{1,n+1} &\mapsto a_{1,n+1} \\
a_{1,n} &\mapsto a_{1,n} a_{1,n+1} a_{2,n}^{-1} \\
a_{i,n+1} &\mapsto a_{i,n+1} a_{i,n}^{-1} a_{2,n} \text{ for } i = 3, 4, \cdots, n + 1 \\
a_{1,j} &\mapsto a_{1,j} a_{2,j}^{-1} a_{2,n+1} a_{1,n+1}^{-1} \text{ for } j = 1, 2, \cdots, n - 1
\end{align}
$$

In terms of these variables the Poisson structure decomposes into a direct sum of the Poisson structure of the variables $a_{i,j}$, for $i = 2, 3, \cdots, n + 1$ and $j = 1, 2, \cdots, n$ together with a rank 2 matrix. Thus, we can reduce dimensions, and using a symmetry of the problem we can finally get to a configuration as in Proposition 6.1. $\qquad\square$

For the algebras $D_q(n)$, $J_q^0(n)$, and $J_q^Z(n)$ there is only one kind of minimal cyclic modules, and they correspond to the case where $\{Z_{i,j}^m\}$ is a rank 1 matrix.

It is straightforward to see that the Poisson structures arising from these algebras have rank $2 \cdot [\frac{n}{2}]$, 2, and 0, respectively, at the given point. Hence we get

Proposition 6.3. *The symplectic leaf through E is of dimension $2 \cdot [\frac{n}{2}]$, 2, and 0, in the Poisson structures arising from $D_q(n)$, $J_q^0(n)$, and $J_q^Z(n)$, respectively $(n \geq 3)$.*

In particular, the conjecture is true in all cases considered here.

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