

# A CLASS OF QUADRATIC MATRIX ALGEBRAS ARISING FROM THE QUANTIZED ENVELOPING ALGEBRA $\mathcal{U}_q(A_{2n-1})$

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ABSTRACT. A natural family of quantized matrix algebras is introduced. It includes the two best studied such. Located inside  $\mathcal{U}_q(A_{2n-1})$ , it consists of quadratic algebras with the same Hilbert series as polynomials in  $n^2$  variables. We discuss their general properties and investigate some members of the family in great detail with respect to associated varieties, degrees, centers, and symplectic leaves. Finally, the space of rank  $r$  matrices becomes a Poisson submanifold, and there is an associated tensor category of rank  $\leq r$  matrices.

## 1. INTRODUCTION

Over the past few years many articles have constructed and investigated multiparameter quantum groups, [1], [3], [6], [7], [8], [9], [12], [14], [17], [18], [24], [27]. Most of the time this has been done from the point of view of quantum function algebras. A central feature has always been that the algebra in question should be a Hopf algebra; indeed, many may feel that this is a requirement for using the terminology ‘quantum group’. Nevertheless, we now introduce yet another multiparameter family for which the following hopefully will serve as arguments in favor of including them among the objects of ‘quantized mathematics’ – even though they need not even be bialgebras. They are all, however, subalgebras of a fundamental bialgebra. Our point of view will be that the underlying classical space should be a Hermitian symmetric space rather than a (reductive) Lie group. In the present context we will only consider the Hermitian symmetric space corresponding to  $SU(p, q)$  and thus end up by quantized  $p \times q$  matrices. Actually, we will only consider  $p = q = n$  though it is a strength of this approach that  $p$  and  $q$  may be different. All members of the family are quadratic algebras with the same Hilbert series as polynomials in  $n^2$  variables.

Our family is contained inside the quantized enveloping algebra of  $su(n, n)$ . It includes the standard (or ‘official’) quantum matrix algebra  $M_q(n)$  as well as the so-called Dipper Donkin

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algebra  $D_q(n)$ , and has indeed a sizable overlap with all previous families. But the way they appear is new. Actually, all members are cross sections of a semidirect product of any one of them with the abelian algebra  $\mathbb{C}[L_1, \dots, L_{2n-1}]$ , where  $L_1, \dots, L_{2n-1}$  are the generators of the quantum enveloping algebra corresponding to the fundamental weights.

The inclusion of the mentioned algebras in our family shows that some members may be closely related to Hopf algebras, but this is by far true for all of them. But there may be other ingratiating features such as ‘nice varieties’, ‘nice representations’, or, simply, ‘nice relations’. Along with the two mentioned, we pay special attention to 3 more, explicitly defined, quadratic algebras:  $J_q^n(n)$  (which like  $M_q(n)$  and  $D_q(n)$  define a Poisson Lie group structure on  $GL(n, \mathbb{C})$ ),  $J_q^z(n)$  (which, through its Poisson structure, is related to  $D_q(n)$ ), and  $J_q^0(n)$  (which is related to  $J_q^n(n)$ ).

For these specific algebras, we determine the varieties, degrees, centers, and discuss the dimensions of the symplectic leaves. For the general members we discuss the symplectic structures and the relation to a symplectic structure on  $M(n, \mathbb{C}) \times T^{2n-1}$ . Specifically, the projections of the symplectic leaves in  $M(n, \mathbb{C}) \times T^{2n-1}$  onto the first factor (according to some splitting) gives what we call the symplectic loaves; orbits of symplectic leaves under a  $2n - 1$  dimensional scaling group. Also quantum determinants are investigated, and some representation theory is included. Finally, we discuss the rank  $r$  matrices.

More specifically: in Section 2 we introduce the algebras and prove that they are iterated Ore extensions. In Section 3 we list briefly some major results of De Concini and Procesi about the degree of an algebra. In Section 4 we discuss the quantum determinants and Laplace expansions and in Section 5 we study the Poisson structures. For use, among other things, in determining degrees, we study some modules in Section 6. We have affixed the name Verma to these (but they are defined in terms of the opposite diagonal). In Section 7 we introduce the specific algebras  $D_q(n)$ ,  $J_q^n(n)$ ,  $J_q^0(n)$ , and  $J_q^z(n)$  and we find their canonical forms. The associated varieties (in the terminology of quadratic algebras) are determined in Section 8, and in Section 9 we discuss the symplectic leaves. The centers are determined in Section 10, the quantum algebra  $\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q^\vartheta(n)$  is analyzed in Section 11 and, finally, in Section 12 the rank  $r$  matrices are considered.

## 2. DEFINITIONS, ORE, BACKGROUND

Fix an  $n \times n$  Cartan matrix  $A = (a_{ij})$  of finite type. Then there exists a vector  $(d_1, d_2, \dots, d_n)$  with relatively prime positive integral entries  $d_i$  such that  $(d_i a_{ij})$  is symmetric and positive definite. Let  $\Pi = \{\alpha_1, \dots, \alpha_n\}$  denote a choice of simple roots and let the usual symmetric bilinear form on the root lattice  $Q$  be given as

$$(2.1) \quad (\alpha_i | \alpha_j) = d_i a_{ij}.$$

Let  $P$  denote the weight lattice generated by the fundamental dominant weights  $\lambda_1, \dots, \lambda_n$ , where

$$(2.2) \quad (\lambda_i | \alpha_j) = \delta_{ij} d_j.$$

Let  $q \in \mathbb{C}^*$  be the quantum parameter. As usual, for  $n \in \mathbb{Z}$  and  $d \in \mathbb{Z}_+$  we let

$$(2.3) \quad [n]_d = (q^{dn} - q^{-dn}) / (q^d - q^{-d}), [n]_d! = [1]_d [2]_d \cdots [n]_d,$$

$$(2.4) \quad \binom{n}{j}_d = [n]_d! / [n-j]_d! [j]_d! \text{ for } j \in \mathbb{Z}_+ \setminus \{0\}, \binom{n}{0}_d = 1.$$

We shall omit the subscript  $d$  when  $d = 1$ .

Following [2], let  $\mathfrak{g}$  be the finite dimensional simple Lie algebra with Cartan matrix  $(a_{ij})$ . The enveloping algebra  $\mathcal{U}_q(\mathfrak{g})$  is the  $\mathbb{C}$ -algebra on generators  $E_i, F_i$  ( $1 \leq i \leq n$ ),  $L_i = L_{\lambda_i}$ ,  $i = 1, \dots, n$ , and the following defining relations:

$$(2.5) \quad \begin{aligned} L_i L_j &= L_j L_i, \quad L_i L_i^{-1} = L_i^{-1} L_i = 1 \\ L_i E_j &= q_i^{\delta_{ij}} E_j L_i, \quad L_i F_i = q_i^{-\delta_{ij}} F_i L_i, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q^{d_i} - q^{-d_i}}, \quad i, j = 1, 2, \dots, n \\ \sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s}_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s &= 0, \text{ if } i \neq j \\ \sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s &= 0, \text{ if } i \neq j, \end{aligned}$$

where for  $\xi = \sum n_i \lambda_i \in P$ ,  $L_\xi := \prod_j L_j^{n_j}$ ,  $K_{\alpha_i} = \prod_j L_j^{a_{ji}}$ , and  $q_i = q^{d_i}$ .

Let  $\mathfrak{g}$  be a finite dimensional Lie algebra corresponding to a noncompact hermitian symmetric space. We have

$$(2.6) \quad \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{p}^-)\mathcal{U}(\mathfrak{k})\mathcal{U}(\mathfrak{p}^+),$$

where  $\mathfrak{p}^-$  and  $\mathfrak{p}^+$  are abelian subalgebras of  $\mathfrak{g}$ , which are furthermore invariant under the maximal compact subalgebra  $\mathfrak{k}$ , and where

$$(2.7) \quad \mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+.$$

In [13], a quantum version of the above decomposition was found:

$$(2.8) \quad \mathcal{U}_q(\mathfrak{g}) = A^+\mathcal{U}_q(\mathfrak{k})A^-,$$

where  $A^-$  and  $A^+$  are quadratic algebras. We will describe the quadratic algebras  $A^+$  explicitly in case of  $su(n, n)$ ; the construction of  $A^-$  is similar. For a simple compact root vector  $E_\mu$  and  $E_\alpha$  an arbitrary element of  $\mathcal{U}_q(\mathfrak{g})$  of weight  $\alpha$ , set

$$(2.9) \quad (adE_\mu)(E_\alpha) = E_\mu E_\alpha - q^{\langle \alpha, \mu \rangle} E_\alpha E_\mu,$$

where, as usual,  $\langle \alpha, \mu \rangle = \frac{2(\alpha, \mu)}{(\mu, \mu)}$ .

In case of  $A_{2n-1} \equiv su(n, n)$ , the set of simple compact roots breaks up into two orthogonal sets:

$$(2.10) \quad \Sigma_c = \{\nu_1, \nu_2, \dots, \nu_{n-1}\} \cup \{\mu_1, \mu_2, \dots, \mu_{n-1}\}.$$

Thus

$$(2.11) \quad E_{\mu_i} E_{\nu_j} = E_{\nu_j} E_{\mu_i},$$

for all  $i, j$ .

Assume moreover that these roots have been labeled in such a way that

$$(2.12) \quad \langle \beta, \mu_1 \rangle = \langle \beta, \nu_1 \rangle = \langle \mu_i, \mu_{i+1} \rangle = \langle \nu_i, \nu_{i+1} \rangle = -1, \text{ for all } i, j,$$

where  $\beta$  is the unique noncompact simple root.

We can then define

$$(2.13) \quad Z_{i,j} = (adE_{\mu_{i-1}}) \cdots (adE_{\mu_1})(adE_{\nu_{j-1}}) \cdots (adE_{\nu_1})(E_\beta) \text{ for } i, j = 1, 2, \dots, n.$$

In [14], it was proved that the quadratic algebra  $A_+$  is generated by  $Z_{i,j}$ ,  $i, j = 0, 1, \dots, n-1$ , and is isomorphic to the standard quantized matrix algebra  $M_q(n)$  whose defining relations are:

$$(2.14) \quad \begin{aligned} Z_{i,j}Z_{i,k} &= qZ_{i,k}Z_{i,j} \text{ if } j < k, \\ Z_{i,j}Z_{k,j} &= qZ_{k,j}Z_{i,j} \text{ if } i < k, \\ Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} \text{ if } i < s, t < j, \\ Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} + (q - q^{-1})Z_{i,t}Z_{s,j} \text{ if } i < s, j < t, \end{aligned}$$

where  $i, j, k, s, t = 1, 2, \dots, n$ , and  $q \in \mathbb{C}$  is the quantum parameter.

**Definition 2.1.** Let  $\wp = (\zeta_1, \dots, \zeta_n, \xi_1, \dots, \xi_n) \in P^{2n}$ . Let  $\tilde{Z}_{i,j} = Z_{i,j}L_{\zeta_i}L_{\xi_j}$ . Let  $M_q^\wp(n)$  be the subalgebra generated by  $\tilde{Z}_{i,j}$  for all  $i, j = 1, 2, \dots, n$ . The algebra  $M_q^\wp(n)$  is called a modification of  $M_q(n)$ , or a modified algebra.

Observe that according to this terminology,  $M_q(n)$  itself is also a modified algebra.

Let  $\mu^i = \mu_1 + \dots + \mu_{i-1}$  for  $i = 2, 3, \dots, n$ ,  $\mu^1 = 0$  and  $\nu^j = \nu_1 + \dots + \nu_{j-1}$  for  $j = 2, 3, \dots, n$ ,  $\nu^1 = 0$ . We denote by  $\alpha_{i,j} = \mu^i + \beta + \nu^j$  the root of  $Z_{i,j}$  in the enveloping algebra.

The generators of  $M_q^\wp(n)$  satisfy the following relations:

$$(2.15) \quad \begin{aligned} \tilde{Z}_{i,j}\tilde{Z}_{i,k} &= q^{(\alpha_{i,k}|\zeta_i+\xi_j)-(\alpha_{i,j}|\zeta_i+\xi_k)+1}\tilde{Z}_{i,k}\tilde{Z}_{i,j} \text{ if } j < k, \\ \tilde{Z}_{i,j}\tilde{Z}_{k,j} &= q^{(\alpha_{k,j}|\zeta_i+\xi_j)-(\alpha_{i,j}|\zeta_k+\xi_j)+1}\tilde{Z}_{k,j}\tilde{Z}_{i,j} \text{ if } i < k, \\ \tilde{Z}_{i,j}\tilde{Z}_{s,t} &= q^{(\alpha_{s,t}|\zeta_i+\xi_j)-(\alpha_{i,j}|\zeta_s+\xi_t)}\tilde{Z}_{s,t}\tilde{Z}_{i,j} \text{ if } i < s \text{ and } t < j, \\ \tilde{Z}_{i,j}\tilde{Z}_{s,t} &= q^{(\alpha_{s,t}|\zeta_i+\xi_j)-(\alpha_{i,j}|\zeta_s+\xi_t)}\tilde{Z}_{s,t}\tilde{Z}_{i,j} \\ &\quad + (q - q^{-1})q^{(\alpha_{s,t}|\zeta_i+\xi_j)-(\alpha_{s,j}|\zeta_i+\xi_t)}\tilde{Z}_{i,t}\tilde{Z}_{s,j} \text{ if } i < s \text{ and } j < t. \end{aligned}$$

For later use we consider the following relations

$$\begin{aligned}
(2.16) \quad x_{i,j}x_{i,k} &= q^{(\alpha_{i,k}|\zeta_i+\xi_j)-(\alpha_{i,j}|\zeta_i+\xi_k)+1}x_{i,k}x_{i,j} \text{ if } j < k, \\
x_{i,j}x_{k,j} &= q^{(\alpha_{k,j}|\zeta_i+\xi_j)-(\alpha_{i,j}|\zeta_k+\xi_j)+1}x_{k,j}x_{i,j} \text{ if } i < k, \\
x_{i,j}x_{s,t} &= q^{(\alpha_{s,t}|\zeta_i+\xi_j)-(\alpha_{i,j}|\zeta_s+\xi_t)}x_{s,t}x_{i,j} \text{ if } i < s \text{ and } t < j, \\
x_{i,j}x_{s,t} &= q^{(\alpha_{s,t}|\zeta_i+\xi_j)-(\alpha_{i,j}|\zeta_s+\xi_t)}x_{s,t}x_{i,j} \text{ if } i < s \text{ and } j < t.
\end{aligned}$$

**Definition 2.2.** The algebra  $\overline{M_q^\wp(n)}$  whose defining relations are those of (2.16) is called the associated quasipolynomial algebra.

**Definition 2.3.** Write the equations (2.16) in the form  $z_{i,j}z_{s,t} = q^{h_{[i,j],[s,t]}}z_{s,t}z_{i,j}$ . The  $n^2 \times n^2$  matrix

$$(2.17) \quad \mathcal{M}(M_q^\wp(n)) = \{h_{[i,j],[s,t]}\}$$

is called the defining matrix of  $M_q^\wp(n)$ .

**Theorem 2.4.** Let  $M_q^\wp(n)$  be any modified algebra. Then  $M_q^\wp(n)$  is in fact an iterated Ore extension and hence a domain. Its Hilbert series is the same as that of the commutative polynomial ring in  $n^2$  variables. Hence, (2.15) are the defining relations of the modified algebra  $M_q^\wp(n)$ .

*Proof.* To prove that  $M_q^\wp(n)$  is an iterated Ore extension, we start from the base field  $\mathbb{C}$  and add generators  $\tilde{Z}_{i,j}$  one by one according to lexicographic ordering. For each  $(s, t)$ , let  $M(s, t)$  be the subalgebra of  $M_q^\wp(n)$  generated by  $\tilde{Z}_{i,j}$  with  $(i, j) < (s, t)$ . Then by the relations of the algebra  $M_q^\wp(n)$ , the subalgebra  $M(s, t)$  is spanned by the ordered monomials in that set of generators. Let  $S = M(s, t)(\tilde{Z}_{s,t})$ . By the PBW theorem for quantum enveloping algebras ([25], [21]), we see that  $M(s, t) \subset S$  and  $S$  is a free  $M(s, t)$ -module with basis  $1, \tilde{Z}_{i,j}, \tilde{Z}_{i,j}^2, \dots$ . By (2.15), we see that for each  $a \in M(s, t)$  we have

$$(2.18) \quad \tilde{Z}_{i,j}a = \sigma_{s,t}(a)\tilde{Z}_{i,j} + D_{s,t}(a).$$

Again by the PBW theorem, we see that  $\sigma_{s,t}(a)$  and  $D_{s,t}(a)$  are uniquely determined and therefore  $\sigma_{s,t}$  is an automorphism of  $M(s,t)$  and  $D_{s,t}$  is a  $\sigma_{s,t}$ -derivation. Hence,

$$(2.19) \quad S = M(s,t)[\sigma_{s,t}, D_{s,t}, \tilde{Z}_{s,t}].$$

This completes the proof. □

### 3. THE DEGREE OF A PRIME ALGEBRA

The main tool used to compute the degree of  $M_q^\wp(n)$  is the theory developed in [5] by De Concini and Procesi. Indeed, our situation (c.f. Theorem 2.4) is such that we may specialize their result into the following

**Proposition 3.1.** *The degree of  $M_q^\wp(n)$  is equal to the degree of the associated quasipolynomial algebra  $\overline{M_q^\wp(n)}$ .*

It is well known that a skew-symmetric matrix over  $\mathbb{Z}$  such as our matrix  $\mathcal{M}(M_q^\wp(n))$  can be brought into a block diagonal form by an element  $W \in SL(\mathbb{Z})$ . Specifically, there is a  $W \in SL(\mathbb{Z})$  and a sequence of  $2 \times 2$  matrices  $S(m_i) = \begin{pmatrix} 0 & -m_i \\ m_i & 0 \end{pmatrix}$ ,  $i = 1, \dots, N$ , with  $m_i \in \mathbb{Z}$  for each  $i = 1, \dots, N$ , such that

$$(3.1) \quad W \cdot \mathcal{M}(M_q^\wp(n)) \cdot W^t = \begin{cases} \text{diag}(S(m_1), \dots, S(m_N), 0) & \text{with } N = \frac{n^2-1}{2}, \text{ if } n \text{ is odd} \\ \text{diag}(S_1(m_1), \dots, S(m_N)) & \text{with } N = \frac{n^2}{2}, \text{ if } n \text{ is even} \end{cases}.$$

**Definition 3.2.** *Any matrix of the form of the right-hand-side in (3.1) will be called a canonical form of  $\mathcal{M}(M_q^\wp(n))$ .*

Thus, a canonical form of  $\mathcal{M}(M_q^\wp(n))$  reduces the associated quasipolynomial algebra to the tensor product of twisted Laurent polynomial algebras in two variables with commutation relation  $xy = q^r yx$ . As a special case of [5, Proposition 7.1] it follows in particular that the degree of a twisted Laurent polynomial algebra in two variables is equal to  $m/(m,r)$ , where  $(m,r)$  is the greatest common divisor of  $m$  and  $r$ . The formula for the general case follows easily from this.

## 4. THE MODIFIED DETERMINANT AND THE MODIFIED LAPLACE EXPANSION

The quantum determinant  $\det_q$  of  $M_q(n)$  is defined as follows ([23]):

$$(4.1) \quad \det_q = \sum_{\sigma \in D_n} (-q)^{l(\sigma)} Z_{1,\sigma(1)} Z_{2,\sigma(2)} \cdots Z_{n,\sigma(n)}.$$

**Definition 4.1.** *An element  $x \in M_q(n)$  is called covariant if for any  $Z_{i,j}$  there exists an integer  $n_{i,j}$  such that*

$$(4.2) \quad x Z_{i,j} = q^{n_{i,j}} Z_{i,j} x.$$

Clearly,  $Z_{1,n}$  and  $Z_{n,1}$  are covariant.

Let  $p \leq n$  be a positive integer. Given any two subsets  $I = \{i_1, i_2, \dots, i_p\}$  and  $J = \{j_1, j_2, \dots, j_p\}$  of  $\{1, 2, \dots, n\}$ , each having cardinality  $p$ , it is clear that the subalgebra of  $M_q(n)$  generated by the elements  $Z_{i_r, j_s}$  with  $r, s = 1, 2, \dots, p$  is isomorphic to  $M_q(p)$ , so we can talk about its determinant. Such a determinant is called a subdeterminant of  $\det_q$ , and will be denoted by  $\det_q(I, J)$ . If  $I = \{1, 2, \dots, n\} \setminus \{i\}$ ,  $J = \{1, 2, \dots, n\} \setminus \{j\}$ ,  $\det_q(I, J)$  will be denoted by  $A(i, j)$ .

The following proposition was proved by Parshall and Wang ([23]) (their  $q$  is our  $q^{-1}$ ):

**Proposition 4.2.** *Let  $i, k \leq n$  be fixed integers. Then*

$$(4.3) \quad \delta_{i,k} \det_q = \sum_{j=1}^n (-q)^{j-k} Z_{i,j} A(k, j) = \sum_{j=1}^n (-q)^{i-j} A(i, j) Z_{k,j}$$

$$(4.4) \quad = \sum_{j=1}^n (-q)^{j-k} Z_{j,i} A(j, k) = \sum_{j=1}^n (-q)^{i-j} A(j, i) Z_{j,k}.$$

The above formulas are called the quantum Laplace expansions. In the following we will establish the modified versions of these expansions.

Clearly, there is an element  $\det_q^\wp \in M_q^\wp(n)$  such that

$$(4.5) \quad \det_q^\wp \Pi_{i,j} L_{-\zeta_i} L_{-\xi_j} = \det_q.$$

The element  $\det_q^\wp$  is called the modified determinant of  $M_q^\wp(n)$ . Similarly we define the modified subdeterminant  $\det_q^\wp(I, J)$  of  $M_q^\wp(n)$  and, if  $I = \{1, \dots, n\} \setminus \{i\}$  and  $J = \{1, \dots, n\} \setminus$

$\{j\}$ ,  $A^\wp(i, j) = \det_q^\wp(I, J)$ . Let  $w_{i,j}$  be the weight of  $A^\wp(i, j)$  in  $\mathcal{U}_q(\mathfrak{g})$ . It follows easily that we have

$$\begin{aligned}
 (4.6) \quad \delta_{i,k} \det_q^\wp &= \sum_{j=1}^n (-q)^{j-k} q^{-(w_{i,j}, \zeta_i + \xi)} \tilde{Z}_{i,j} A_q^\wp(k, j) \\
 &= \sum_{j=1}^n (-q)^{i-j} q^{-(\alpha_{k,j}, \sum_{r \neq k, t \neq j} (\zeta_r + \xi_t))} A_q^\wp(i, j) \tilde{Z}_{k,j} \\
 &= \sum_{j=1}^n (-q)^{j-k} q^{-(w_{j,k}, \zeta_j + \xi_i)} \tilde{Z}_{j,i} A_q^\wp(j, k) \\
 &= \sum_{j=1}^n (-q)^{i-j} q^{-(\alpha_{j,k}, \sum_{r \neq j, t \neq k} (\zeta_r + \xi_t))} A_q^\wp(j, i) \tilde{Z}_{j,k}.
 \end{aligned}$$

The above formulas are called the modified quantum Laplace expansions.

By using induction on  $s$  it is easy to prove that

**Lemma 4.3.** *If  $i < k$  and  $j < l$  then*

$$\tilde{Z}_{i,j}^s \tilde{Z}_{k,l} = q^{s(\alpha_{k,l}, \zeta_i + \xi_j) - s(\alpha_{i,j}, \zeta_k + \xi_l)} \tilde{Z}_{k,l} \tilde{Z}_{i,j}^s + q^{(\alpha_{k,l}, \zeta_i + \xi_j) - (\alpha_{i,j}, \zeta_k + \xi_l)} (q - q^{1-2s}) \tilde{Z}_{i,j}^{s-1} \tilde{Z}_{i,l} \tilde{Z}_{k,j}.$$

**Corollary 4.4.** *If  $q$  is an  $m$ th root of unity, then  $\tilde{Z}_{i,j}^m$  is central for all  $i, j = 1, 2, \dots, n$ .*

## 5. SYMPLECTIC STRUCTURES

We denote by  $\lambda_\beta, \lambda_{\nu_1}, \dots, \lambda_{\mu_{n-1}}$  the fundamental weights corresponding to the simple roots  $\beta, \nu_1, \dots, \mu_{n-1}$ . Let  $\lambda_i$  be any one of these, let  $c \in \mathbb{C}^*$ , and define a map  $\check{\lambda}_i(c) : M_q^\wp(n) \mapsto M_q^\wp(n)$  by

$$(5.1) \quad \check{\lambda}_i(c)(\tilde{Z}_{s,t}) = c^{(\alpha_{s,t}, \lambda_i)} \tilde{Z}_{s,t}.$$

Observe that  $\check{\lambda}_i(q)(\tilde{Z}_{s,t}) = L_i \tilde{Z}_{s,t} L_i^{-1}$ .

Let  $\mathcal{S}_{\text{mult}}$  denote the group generated by the maps  $\check{\lambda}_\beta(c_1), \check{\lambda}_{\nu_1}(c_2), \dots, \check{\lambda}_{\mu_{n-1}}(c_{2n-1})$  for  $c_1, \dots, c_{2n-1} \in \mathbb{C}^*$ . Obviously we have:

**Lemma 5.1.**  *$\mathcal{S}_{\text{mult}}$  is contained in the automorphism group of  $M_q^\wp(n)$ , is independent of  $q$  and  $\wp$ , and is isomorphic to  $(\mathbb{C}^*)^{2n-1}$ .*

Observe that  $\mathcal{S}_{\text{mult}}$  also acts on  $M(n, \mathbb{C})$  via (5.1).

**Lemma 5.2.** *For  $\chi = (\psi_1, \psi_2, \dots, \psi_n, \phi_1, \phi_2, \dots, \phi_n) \in (\mathbb{C}^*)^{2n}$ , let the automorphism  $\check{\chi}$  of  $M_q^\varphi(n)$  be given by*

$$(5.2) \quad \check{\chi}(Z_{i,j}) = \psi_i \phi_j Z_{i,j} \text{ for all } i, j = 1, 2, \dots, n.$$

Then  $\check{\chi}$  is implemented by an element of  $S_{\text{mult}}$ .

*Proof.* Write the fundamental dominant weights as  $\lambda_\beta, \lambda_{\mu_1}, \dots, \lambda_{\nu_{n-1}}$ . Clearly  $\check{\lambda}_\beta(c)$  is just multiplication by  $c$ ,  $\check{\lambda}_{\mu_i}(c)$  corresponds to  $\chi = (\underbrace{1, \dots, 1}_i, \underbrace{c, \dots, c}_{n-i}, \underbrace{1, \dots, 1}_n)$ , and  $\check{\lambda}_{\nu_j}(c)$  corresponds to  $\chi = (\underbrace{1, \dots, 1}_n, \underbrace{1, \dots, 1}_j, \underbrace{c, \dots, c}_{n-j})$ . Notice that the action of  $\chi^l = (\underbrace{c, \dots, c}_n, \underbrace{1, \dots, 1}_n)$  equals the action of  $\chi^r = (\underbrace{1, \dots, 1}_n, \underbrace{c, \dots, c}_n)$ . The claim follows easily from this.  $\square$

We consider  $M_\epsilon^\varphi(n)$  where  $\epsilon$  is a primitive  $m$ th root of unity for some positive integer  $m \neq 2$ . Let  $\mathcal{Z}_\epsilon^\varphi$  denote the part of its center generated by the elements  $\tilde{Z}_{i,j}^m$ . For an  $n \times n$  matrix  $a = \{a_{i,j}\}$  let  $R(a)$  denote the quotient

$$(5.3) \quad R(a) = M_\epsilon^\varphi(n) / I(\tilde{Z}_{i,j}^m - a_{i,j}),$$

and let  $\pi_a$  denote the canonical projection.

This gives us a bundle of algebras over  $M(n, \mathbb{C})$  and  $M_\epsilon^\varphi(n)$  may be considered as a space of global sections of this bundle by the prescription

$$(5.4) \quad M_\epsilon^\varphi(n) \ni \tilde{Z} : \quad \tilde{Z}(a) = \pi_a(\tilde{Z}) \in R(a).$$

For  $a \in M(n, \mathbb{C})$ , let the  $\mathbb{C}$  algebra homomorphism  $\Psi_a : \mathcal{Z}_\epsilon^\varphi \mapsto M(n, \mathbb{C})$  be defined as

$$(5.5) \quad \Psi_a(\tilde{Z}_{i,j}^m) = a_{i,j}.$$

Similar to [4] we obtain for each  $M_\epsilon^\varphi(n)$  a Poisson structure  $\{\cdot, \cdot\}_\varphi$  on  $M(n, \mathbb{C})$  defined by (identifying coordinates and coordinate functions)

$$(5.6) \quad \{a_{ij}, a_{st}\}_\varphi(a) = \Psi_a \left( \lim_{q \rightarrow \epsilon} \frac{1}{m(q^m - 1)} [\tilde{Z}_{i,j}^m, \tilde{Z}_{s,t}^m] \right),$$

where the right hand side commutator is computed in  $M_q^\varphi(n)$ .

We shall occasionally denote the Poisson structure from  $M_q(n)$  (corresponding to  $\wp = (0, \dots, 0)$ ) as  $\{\cdot, \cdot\}_0$ . Let

$$(5.7) \quad R = \frac{1}{2} \sum_{1 \leq i < j \leq n} e_{i,j} \wedge e_{j,i},$$

where  $\{e_{i,j}\}$  denotes the standard basis of  $M(n, \mathbb{C})$ . Then it is easy to see that the Poisson tensor  $\pi(g)$  at  $g \in M(n, \mathbb{C})$  is given as  $\pi(g) = -2(l_g^* R - r_g^* R)$ . The factor -2 is of no practical importance, but we wish to keep this difference between the structure defined by (5.6) and the one (on the regular points) considered in ([26]) and ([11, Appendix A]). Specifically, the present Poisson structure is given as follows:

$$(5.8) \quad \begin{aligned} \{Z_{i,j}, Z_{i,k}\}_0 &= Z_{i,k} Z_{i,j} \text{ if } j < k, \\ \{Z_{i,j}, Z_{k,j}\}_0 &= Z_{k,j} Z_{i,j} \text{ if } i < k, \\ \{Z_{i,j}, Z_{s,t}\}_0 &= 0 \text{ if } i < s, t < j, \\ \{Z_{i,j}, Z_{s,t}\}_0 &= 2Z_{s,t} Z_{i,j} \text{ if } i < s, j < t. \end{aligned}$$

The Hamiltonian vector field  $\theta_{ij}$  corresponding to  $a_{ij}$  is then given by

$$(5.9) \quad \begin{aligned} \theta_{ij}(a_{st}) &= \{a_{ij}, a_{st}\}_\wp, \text{ hence} \\ \theta_{ij} &= \sum_{st} \{a_{ij}, a_{st}\}_\wp \frac{\partial}{\partial a_{st}}. \end{aligned}$$

The Hamiltonian vector field  $\theta_f$  corresponding to an arbitrary  $C^\infty$ -function  $f$  may then be defined as

$$(5.10) \quad \theta_f = - \sum_{st} \theta_{st}(f) \frac{\partial}{\partial a_{st}},$$

or, equivalently,

$$(5.11) \quad \theta_f = \sum_{ij} \left( \frac{\partial f}{\partial a_{ij}} \right) \theta_{ij}.$$

It is clear that the assignment  $\mathcal{D}$  to each  $a$  of a subspace in  $T_a(M_q(n))$  given by

$$(5.12) \quad M(n, \mathbb{C}) \ni a \mapsto \mathcal{D}(a) = \{\xi_f(a) \mid f \in C^\infty\},$$

is an involutive distribution.

**Definition 5.3.** *By a symplectic leaf  $\mathcal{L}$  we mean a maximal integral manifold of  $\mathcal{D}$ . By a symplectic loaf we mean a set of the form  $\mathcal{S}_{\text{mult}}(\mathcal{L})$  where  $\mathcal{L}$  is a symplectic leaf.*

It is well known (see e.g. [11]) (and is also elementary to see directly here) that the action by  $\mathcal{S}_{\text{mult}}$  normalizes the Hamiltonian action.

Along with the Hamiltonian vector fields  $\theta_{ij}$  we may also consider derivations  $\delta_{ij}$  of  $M_\epsilon^\wp(n)$ , defined by

$$(5.13) \quad \delta_{ij}(\tilde{Z}) = \lim_{q \rightarrow \epsilon} \frac{1}{m(q^m - 1)} [\tilde{Z}_{i,j}^m, \tilde{Z}],$$

for an arbitrary element  $\tilde{Z} \in M_\epsilon^\wp(n)$ .

If we think of  $\tilde{Z}$  as a section of the above bundle, it is clear that  $\delta_{ij}$  is a lifting of  $\theta_{ij}$ . More generally, for any  $C^\infty$ -function  $f$  and any section  $\tilde{Z}$  we may define

$$(5.14) \quad \delta_f(\tilde{Z}) = \sum_{st} \left( \frac{\partial f}{\partial a_{st}} \right) \delta_{st}(\tilde{Z}),$$

and we write

$$(5.15) \quad \nabla_{\theta_f} \tilde{Z} = \delta_f(\tilde{Z}).$$

**Proposition 5.4.** *Parallel transport along an integral curve of a Hamiltonian vector field gives rise to an algebra isomorphism between fibers.*

*Proof.* The above  $\nabla$  is a connection along symplectic leaves, hence the following argument makes sense: Consider two parallel sections,  $s_1$  and  $s_2$  along an integral curve of a Hamiltonian vector field  $\theta_H$ . Then

$$(5.16) \quad \nabla_{\theta_H}(s_1 \cdot s_2) = \delta_H(s_1 \cdot s_2) = (\nabla_{\theta_H} s_1)s_2 + s_1(\nabla_{\theta_H} s_2) = 0.$$

Thus, parallel transport yields the isomorphism.  $\square$

We wish to show now that the loaves for the various quantizations of  $n \times n$  matrices are the same. In order to do that, we introduce an auxiliary Poisson manifold  $M(n, \mathbb{C}) \times (\mathbb{C}^*)^{2n-1}$ . Actually, this Poisson manifold seems to be of fundamental importance.

Consider the subalgebra of  $\mathcal{U}_q(\mathfrak{g})$  generated by the elements  $Z_{i,j}$ ,  $i, j = 1, \dots, n$  and  $L_i$ ,  $i = 1, \dots, 2n-1$ . This may be viewed, in an obvious way, as a semi-direct product

$$(5.17) \quad \mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q^\wp(n)$$

for any  $\wp \in P^{2n-1}$ . Using this, a construction analogous to (5.6) makes  $M(n, \mathbb{C}) \times (\mathbb{C}^*)^{2n-1}$  into a Poisson manifold where, if  $\ell_1, \dots, \ell_{2n-1}$  denote the standard coordinate functions on  $(\mathbb{C}^*)^{2n-1}$ , and  $\ell \equiv (\ell_1, \dots, \ell_{2n-1})$ ,

$$(5.18) \quad \begin{aligned} \{a_{ij}, a_{st}\}_{\wp}(a, \ell) &= \{a_{ij}, a_{st}\}_{\wp}(a) \quad (\text{as it were}) \\ \{\ell_k, a_{ij}\}_{\wp}(a, \ell) &= (\lambda_k | \alpha_{i,j}) a_{ij} \ell_k \\ \{\ell_r, \ell_s\}_{\wp}(a, \ell) &= 0. \end{aligned}$$

**Lemma 5.5.** *A symplectic leaf in  $M(n, \mathbb{C})$  defined by the Poisson structure  $\{\cdot, \cdot\}_{\wp}$  is equal to the projection onto the first factor in  $M(n, \mathbb{C}) \times (\mathbb{C}^*)^{2n-1}$  of a symplectic leaf in the full space.*

*Proof.* The flow of the Hamiltonian vector field  $\theta_{\ell_k}$  is as follows:

$$(5.19) \quad (\{a_{st}\}, \ell) \mapsto (\{e^{(\lambda_k | \alpha_{s,t})} a_{st}\}, \ell)$$

while the flow of  $\theta_{a_{st}}$  on  $(a, \ell_1, \dots, \ell_{2n-1})$  is equal to the old flow in the first factor  $a = (\{a_{ij}\})$  while  $\ell_k \mapsto e^{(\lambda_k | \alpha_{s,t})} \ell_k$  for  $k = 1, \dots, 2n-1$ .  $\square$

If  $\wp = (\zeta_1, \dots, \zeta_n, \xi_1, \dots, \xi_n)$  write

$$(5.20) \quad \begin{aligned} \zeta_i &= \sum_{j=1}^{2n-1} \zeta_i^j \lambda_j \text{ for } i = 1, \dots, n, \text{ and} \\ \xi_i &= \sum_{j=1}^{2n-1} \xi_i^j \lambda_j \text{ for } i = 1, \dots, n. \end{aligned}$$

Define, for  $i, j = 1, \dots, n$  the functions  $\psi_i^{\wp}$  and  $\phi_j^{\wp}$  on  $(\mathbb{C}^*)^{2n-1}$  by

$$(5.21) \quad \begin{aligned} \psi_i(l) &= \prod_{j=1}^{2n-1} \ell_j^{\zeta_i^j}, \text{ and} \\ \phi_i(l) &= \prod_{j=1}^{2n-1} \ell_j^{\xi_i^j}. \end{aligned}$$

Observe that if we define the functions  $\tilde{a}_{ij}(a, \ell) = a_{ij}(a) \psi_i(l) \phi_j(\ell)$  then we may write

$$(5.22) \quad \{\tilde{a}_{ij}, \tilde{a}_{st}\}_0 = \sum_{(a,b),(c,d)} c_{(a,b),(c,d)}^{(i,j),(s,t)} \tilde{a}_{ab} \tilde{a}_{cd}$$

where the coefficients  $c_{(a,b),(c,d)}^{(i,j),(s,t)}$  are the constants in

$$(5.23) \quad \{a_{ij}, a_{st}\}_\varphi = \sum_{(a,b),(c,d)} c_{(a,b),(c,d)}^{(i,j),(s,t)} a_{ab} a_{cd}.$$

The following is then obvious either from the above or from the way the different algebras are constructed

**Lemma 5.6.** *The Poisson structure on  $M(n, \mathbb{C}) \times (\mathbb{C}^*)^{2n-1}$  obtained from  $\varphi$  is equal to that corresponding to  $\varphi = 0$  expressed in the coordinate system  $(\{\tilde{a}_{ij}\}, \ell)(a, \ell) = (\{a_{ij}\psi_i(\ell)\phi_j(\ell)\}, \ell)$ .*

**Proposition 5.7.** *The symplectic leaves are the same in  $M(n, \mathbb{C})$  for all choices of  $\varphi$ .*

*Proof.* This follows directly from Lemma 5.5 and Lemma 5.6 since  $\{a_{ij}\psi_i(\ell)\phi_j(\ell)\}$  according to Lemma 5.2 can be obtained from  $\{a_{ij}\}$  through the action of  $\mathcal{S}_{\text{mult}}$ .  $\square$

We shall investigate further the various Poisson structures in Section 9.

## 6. VERMA AND CYCLIC MODULES

We now introduce and study some modules which turn out to be very useful.

**Definition 6.1.** *For an integer  $m$  set*

$$(6.1) \quad m' = \begin{cases} m & \text{if } m \text{ is odd} \\ \frac{m}{2} & \text{if } m \text{ is even} \end{cases}.$$

**Definition 6.2.** *Suppose our modified algebra  $M_q^\varphi(n)$  satisfies*

$$(6.2) \quad \forall i, j : \tilde{Z}_{i, n+1-i} \tilde{Z}_{j, n+1-j} = \tilde{Z}_{j, n+1-j} \tilde{Z}_{i, n+1-i}.$$

Let  $\Lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$  and let  $I(\Lambda)$  be the left ideal in  $M_q^\varphi(n)$  generated by the elements  $\tilde{Z}_{i,j}$  with  $i+j \geq n+2$  together with the elements  $\tilde{Z}_{k, n+1-k} - \lambda_k$  for  $k = 1, \dots, n$  and  $\tilde{Z}_{i,j}^{m'}$  with  $i+j \leq n$ . The restricted Verma module  $\overline{M_\varphi(\Lambda)}$  is defined as

$$(6.3) \quad \overline{M_\varphi(\Lambda)} = M_q^\varphi(n)/I(\Lambda).$$

We denote by  $v_\Lambda$  the image of 1 in the quotient.

**Remark 6.3.** *It seems natural to affix the name Verma to these modules since they do have much of the flavour of the usual ones. Notice, however, that what corresponds to the Cartan subalgebra is here the opposite diagonal.*

**Theorem 6.4.** *Let  $m > 2$  be an integer. Then the restricted Verma-module  $\overline{M_\varphi(\Lambda)}$  with the highest weight  $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  is irreducible if and only if  $\lambda_i \neq 0$  for all  $i$ .*

*Proof.* This was proved for the case of  $M_q(n)$  in [15]. We can extend this result to the present situation by considering the induced module

$$(6.4) \quad \overline{M_\varphi(\Lambda)} \uparrow = (\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}]_m \times_s M_q^\varphi(n)) \otimes_{M_q^\varphi(n)} \overline{M_\varphi(\Lambda)},$$

where  $\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}]_m$  denotes the quotient of  $\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}]$  generated by the elements  $L_i^m - 1$  for  $i = 1, \dots, 2n - 1$ . Consider the subspace  $S = \mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \otimes_{\mathbb{C}} \mathbb{C} \cdot v_\Lambda$ . We know that  $M_q(n)$  is a subalgebra of  $\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q^\varphi(n)$  and it follows that the commutative algebra  $\{Z_{i,n+1-i} \mid i = 1, \dots, n\} \subset M_q(n)$  leaves  $S$  invariant. Hence, there is a common eigenvector  $v_{\tilde{\Lambda}}$ , and it follows easily that  $S = \mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \otimes_{\mathbb{C}} \mathbb{C} \cdot v_{\tilde{\Lambda}}$  and that  $\tilde{\Lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_n)$  with all  $\tilde{\lambda}_i \neq 0$ . In fact,

$$(6.5) \quad \mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q^\varphi(n) \otimes_{M_q^\varphi(n)} \overline{M_\varphi(\Lambda)} = \mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q(n) \otimes_{M_q(n)} \overline{M_0(\tilde{\Lambda})}.$$

Finally observe that

$$(6.6) \quad \{x \in \overline{M_\varphi(\Lambda)} \uparrow \mid \tilde{Z}_{i,j} \cdot x = 0 \quad \forall \tilde{Z}_{i,j} \in M_q^\varphi(n) \text{ with } i + j \geq n + 2\}$$

$$(6.7) \quad = \{x \in \overline{M_\varphi(\Lambda)} \uparrow \mid Z_{i,j} \cdot x = 0 \quad \forall Z_{i,j} \in M_q(n) \text{ with } i + j \geq n + 2\},$$

and that this set of primitive vectors is invariant under the subalgebras  $\{\tilde{Z}_{i,n+1-i} \mid i = 1, \dots, n\}$ ,  $\{Z_{i,n+1-i} \mid i = 1, \dots, n\}$ , and  $\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}]$ .  $\square$

**Corollary 6.5.** *Rank  $\mathcal{M}(M_q^\varphi(n)) \geq n^2 - n$ , where  $\mathcal{M}(M_q^\varphi(n))$  is the defining matrix of the algebra  $M_q^\varphi(n)$ , provided  $M_q^\varphi(n)$  satisfies (6.2).*

To deal with the case of  $J_q^0(n)$ , especially with  $m$  is even, we now introduce the concept of a “restricted minimally generalized Verma module for  $J_q^0(n)$ ”

**Definition 6.6.** Let  $\Lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ , let  $\phi \in \mathbb{C}$ , and let  $I^0(\Lambda, \phi)$  be the left ideal in  $J_q^0(n)$  generated by the elements  $\tilde{Z}_{i,j}$  with  $i+j \geq n+2$  together with the elements  $\tilde{Z}_{k,n+1-k} - \lambda_k$  for  $k = 1, \dots, n$  and the element  $\tilde{Z}_{n-1,1}^m - \phi$ . Let  $\overline{I^0(\Lambda, \phi)}$  denote the left ideal in  $J_q^0(n)$  generated by  $I^0(\Lambda, \phi)$  together with the elements  $\tilde{Z}_{i,j}^{m'}$  for  $i+j = n$  (except  $(i,j) = (n-1,1)$ ). The restricted minimally generalized Verma module  $\overline{M^0(\Lambda, \Phi)}$  is given as

$$(6.8) \quad \overline{M^0(\Lambda, \phi)} = J_q^0(n) / \overline{I^0(\Lambda, \phi)}.$$

We denote by  $\overline{v_{\Lambda, \Phi}}$  the image of 1 in the quotient.

**Theorem 6.7.** The module  $\overline{M^0(\Lambda, \phi)}$  is irreducible for  $\phi = 1$  and  $\Lambda = (1, \dots, 1)$ . It has dimension  $m \cdot (m')^{(n^2-n-2)/2}$ .

*Proof.* This follows by a mixture of the proofs of Theorem 3.7 and Theorem 3.11 in [15]. By the irreducibility of the Baby Verma module for the case of  $J_q^0(n-1)$  (based on the generators  $\tilde{Z}_{i,j}$  with  $i = 1, \dots, n-1$  and  $j = 2, \dots, n$ ), it follows that any invariant subspace must contain a primitive vector  $v_p$  of the form

$$(6.9) \quad v_p = \tilde{Z}_{1,1}^{i_1} \tilde{Z}_{2,1}^{i_2} \cdots \tilde{Z}_{k,1}^{i_k} \tilde{Z}_{n-1,1}^\alpha \cdot \overline{v_{\Lambda, \phi}} + \dots,$$

where the power  $\alpha$  may be 0 or  $m'$ .

By looking at the action of  $\tilde{Z}_{n,2}$  it follows that

$$(6.10) \quad v_p = (\tilde{Z}_{1,1}^{i_1} \tilde{Z}_{2,1}^{i_2} \cdots \tilde{Z}_{k,1}^{i_k} + \beta \cdot \tilde{Z}_{1,1}^{i_1} \tilde{Z}_{2,1}^{i_2} \cdots \tilde{Z}_{k,1}^{i_k-1} \tilde{Z}_{n-1,1} \tilde{Z}_{k,2}) \cdot \tilde{Z}_{n-1,1}^\alpha \cdot \overline{v_{\Lambda, \phi}} + \dots$$

It is now easy to see that the assumption that  $v_p$  is primitive leads to the same contradictions as those in the proof of Theorem 3.7 in [15].

Observe that  $\tilde{Z}_{n-1,1}^{m'} \cdot \overline{v_{\Lambda, \phi}}$  is a primitive vector which is different from  $\overline{v_{\Lambda, \phi}}$  if  $m$  is even. However, it does not generate a non-trivial invariant subspace since we can multiply it with  $\tilde{Z}_{n-1,1}^{m'}$  and thus get back to the highest weight vector. Also observe that we can separate it from the highest weight vector since  $\tilde{Z}_{n,1} \tilde{Z}_{n-1,1}^{m'} = -\tilde{Z}_{n-1,1}^{m'} \tilde{Z}_{n,1}$ .  $\square$

**Remark 6.8.** The modules  $\overline{M^0(\Lambda, \phi)}$  are generically irreducible.

**Remark 6.9.** *One may wonder why the generalized Verma modules of [15] no longer are irreducible. (If they were, one would get a contradiction with the degree). The reason is that the vector*

$$(6.11) \quad \tilde{Z}_{n-1,1}^{m'} \tilde{Z}_{n-2,2}^{m'} \cdots \tilde{Z}_{1,n-1}^{m'} \cdot v_{\Lambda,\phi}$$

*is a non-trivial primitive vector.*

**Remark 6.10.** *The modules  $\overline{M^0(\Lambda, \phi)}$  may of course be defined for a wide class of algebras. The unitarity result will hold provided that there are non trivial relations  $Z_{i,j}Z_{i+a,j} = q^*Z_{i+a,j}Z_{i,j}$  with  $q^* \neq 1$  and likewise in the column variable;  $Z_{i,j}Z_{i,j+b} = q^*Z_{i,j+b}Z_{i,j}$ . This condition is not satisfied by  $J_q^z(n)$ .*

## 7. SOME QUADRATIC ALGEBRAS

We now introduce four quantized matrix algebras; each has its own justification. We shall see that they all are modifications of  $M_q(n)$ . We further compute their degrees as functions of  $n$  and  $m$ .

The so-called Dipper Donkin quantized matrix algebra  $D_q(n)$  is an associative algebra over the complex numbers  $\mathbb{C}$  generated by elements  $D_{i,j}$ ,  $i, j = 1, 2, \dots, n$  subject to the following relations:

$$(7.1) \quad \begin{aligned} D_{i,j}D_{s,t} &= qD_{s,t}D_{i,j} \text{ if } i > s \text{ and } j \leq t, \\ D_{i,j}D_{s,t} &= D_{s,t}D_{i,j} + (q-1)D_{s,j}D_{i,t}, \text{ if } i > s \text{ and } j > t, \\ D_{i,j}D_{i,k} &= D_{i,k}D_{i,j} \text{ for all } i, j, k. \end{aligned}$$

Secondly, let  $J_q^0(n)$  be the associative algebra generated by elements  $J_{i,j}$  for  $i, j = 1, \dots, n$  and defining relations:

$$(7.2) \quad J_{i,j}J_{s,t} = q^{s+t-i-j}J_{s,t}J_{i,j}, \text{ if } (s-i)(t-j) \leq 0,$$

$$(7.3) \quad q^{1-t+j}J_{i,j}J_{s,t} = q^{s-i-1}J_{s,t}J_{i,j} + (q - q^{-1})J_{i,t}J_{s,j} \text{ if } s > i \text{ and } t > j.$$

Thirdly, let  $J_q^z(n)$  be the associative algebra generated by elements  $M_{i,j}$ ,  $i, j = 1, 2, \dots, n$  subject to the following relations:

$$(7.4) \quad \begin{aligned} M_{i,j}M_{s,t} &= M_{s,t}M_{i,j} \text{ if } (s-i)(t-j) \leq 0, \\ qM_{i,j}M_{s,t} &= q^{-1}M_{s,t}M_{i,j} + (q - q^{-1})M_{i,t}M_{s,j} \text{ if } i < s \text{ and } j < t, \end{aligned}$$

where  $i, j, k, s, t = 1, 2, \dots, n$ .

Finally, let  $J_q^n(n)$  be the associative algebra generated by elements  $N_{i,j}$  subject to the following relations:

$$(7.5) \quad \begin{aligned} N_{i,j}N_{s,t} &= q^{s-t-i+j-2}N_{s,t}N_{i,j}, \text{ if } s \geq i, \text{ and } t < j, \\ N_{i,j}N_{s,t} &= q^{s-i}N_{s,t}N_{i,j}, \text{ if } s > i, \text{ and } t = j \\ q^{t-j-1}N_{i,j}N_{s,t} &= q^{s-i-1}N_{s,t}N_{i,j} + (q - q^{-1})N_{i,t}N_{s,j} \text{ if } s > i \text{ and } t > j. \end{aligned}$$

To make it easier to write up the following relations, we define the symbols  $L(\lambda_{\mu_n})$  and  $L(\lambda_{\nu_n})$  to be the real number 1.

**Proposition 7.1.** *Let*

$$(7.6) \quad \widetilde{D}_{i,j} = Z_{i,j}L(\lambda_{\mu_i})^{-1}L(\lambda_{\nu_j}), i, j = 1, 2, \dots, n.$$

Let  $\widetilde{D}_q(n)$  be the subalgebra generated by these elements. Then  $\widetilde{D}_q(n)$  is isomorphic to  $D_{q^{-2}}(n)$ .

*Proof.* By direct calculations we see that the  $\widetilde{D}_{i,j}$ 's satisfy the defining relations of  $D_q(n)$  with the quantum parameter  $q^{-2}$ . By the PBW theorem for the enveloping algebra, the Hilbert series of  $\widetilde{D}_q(n)$  is equal to that of the Dipper Donkin quantized matrix algebra. This completes the proof.  $\square$

Similarly we have

**Proposition 7.2.** *The algebra  $J_q^0(n)$  is isomorphic to the algebra generated by the elements*

$$(7.7) \quad J_{i,j} = Z_{i,j}L(\lambda_{\beta})^{-(i+j)}L(\lambda_{\mu_i})^{-1}L(\lambda_{\nu_j})^{-1},$$

the algebra  $J_q^z(n)$  is isomorphic to the algebra generated by the elements

$$(7.8) \quad M_{i,j} = Z_{i,j}L(\lambda_{\mu_i})^{-1}L(\lambda_{\nu_j})^{-1}, i, j = 1, 2, \dots, n,$$

and the algebra  $J_q^n(n)$  is isomorphic to the algebra generated by the elements

$$(7.9) \quad N_{i,j} = Z_{i,j}L(\lambda_\beta)^{-i+j}L(\lambda_{\mu_i})^{-1}L(\lambda_{\nu_j}) \text{ for } i, j = 1, \dots, n.$$

The degrees of the algebras  $M_q(n)$  and  $D_q(n)$  were computed in [15] and [16]. They are  $m^{n-1}(m')^{\frac{(n-2)(n-1)}{2}}$  and  $m^{\lfloor \frac{n^2}{2} \rfloor}$ , respectively. We now sketch a computation of the degrees of  $J_q^0(n)$ ,  $J_q^z(n)$ , and  $J_q^n(n)$ . We denote the defining matrices of these algebras by  $\mathcal{M}_q^0(n)$ ,  $\mathcal{M}_q^z(n)$ , and  $\mathcal{M}_q^n(n)$ , respectively.

**Lemma 7.3.** *Consider the quasipolynomial algebra  $\overline{J_q^0(n)}$ . Let*

$$(7.10) \quad \begin{aligned} X(1) &= x_{1,1}x_{2,2} \cdots x_{n,n} \text{ and} \\ X(j) &= x_{1,j}x_{2,j+1} \cdots x_{n-j+1,n}x_{n-j+1,1}x_{n-j+2,2} \cdots x_{n,j} \text{ for } j = 2, 3, \dots, n. \end{aligned}$$

Then we have

$$(7.11) \quad \begin{aligned} x_{s,t}X(1) &= q^{(n-2)(n+1-s-t)}X(1)x_{s,t} \text{ for all } s, t = 1, 2, \dots, n \text{ and} \\ x_{s,t}X(j) &= q^{(n-1)(n+1-s-t)}X(j)x_{s,t} \text{ for all } s, t = 1, \dots, n \text{ and } j = 2, 3, \dots, n. \end{aligned}$$

Hence  $X(j)x_{1,n}^{rm-n+1}$  and  $X(1)x_{1,n}^{rm-n+2}$  are central elements of the quasipolynomial algebra  $\overline{J_q^0(n)}$  for all  $j = 2, 3, \dots, n$ , where  $r$  is the smallest positive integer such that  $rm - n + 1 \geq 0$ .

*Proof.* This follows by checking directly the four cases  $s$  vs.  $n - j + 1$ , and  $t$  vs.  $j$ , where vs. either is  $\leq$  or  $>$ .  $\square$

**Theorem 7.4.** *Let  $D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and  $D_j = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$  for  $j = 2, n + 1, \dots, \frac{n^2-n}{2}$ . A canonical form of  $J_q^0(n)$  is  $\text{diag}(D_1, D_2, \dots, D_{\frac{n^2-n}{2}}, 0, \dots, 0)$ .*

*Proof.* By the central elements we already found we know that

$$(7.12) \quad \text{rank } \mathcal{M}_q^0(n) \leq n^2 - n.$$

Thus, by Corollary 6.5,  $\text{rank}(\mathcal{M}_q^0) = n^2 - n$ . Next, it is easy to see by direct inspection of the defining matrix that in case  $m = 2$ , the degree is 2. But it then follows by Theorem 6.7 that the entries of a canonical form of  $J_q^0(n)$  all are powers of the integer 2, except 1 which can only be  $D_1$ . Indeed, by Theorem 6.7 the form must be as stated.  $\square$

Now let us consider the algebra  $J_q^z(n)$ .

**Proposition 7.5.**  $\text{rank } \mathcal{M}_q^z(n) = n^2 - n$ .

*Proof.* Let  $I(n) = \{[i, j] \mid i, j = 1, 2, \dots, n\}$  with lexicographic order. The skew-symmetric matrix  $\mathcal{M}_q^0(n)$  can be written as follows:

$$(7.13) \quad \mathcal{M}_q^0(n) = H + 2\mathcal{M}_q^z(n),$$

where  $H = (h_{[i,j],[s,t]})_{i,j,s,t=1}^n$  and  $h_{[i,j],[s,t]} = s + t - i - j$ .

We have already proved that  $\text{rank } \mathcal{M}_q^0(n) = n^2 - n$ . Obviously, the rows of  $H$  can be generated by  $\mathbf{T} = (1, 1, \dots, 1)$  and  $\mathbf{W}$  which is the  $(1, n)$ -th row of  $H$ . If we sum up all the rows of  $\mathcal{M}_q^z(n)$  we get a vector  $X = (x_{i,j}) \in \mathbb{C}^{n^2}$ , where  $x_{i,j} = \#\{[s, t] \in I(n) \mid s > i, t > j\} - \#\{[s, t] \in I(n) \mid s < i, t < j\} = (n - i)(n - j) - (i - 1)(j - 1) = (n - 1)(n + 1 - i - j)$ . This means that  $X$  is  $(n - 1)$  times the  $(1, n)$ -th row of  $H$ . So

$$(7.14) \quad n^2 - n - 1 \leq \text{rank } \mathcal{M}_q^z(n) \leq n^2 - n + 1.$$

However,  $\text{rank } \mathcal{M}_q^z(n)$  must be an even integer, so we get the result.  $\square$

**Theorem 7.6.** Let  $D_i = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$  for  $i = 1, 2, \dots, \frac{n^2-n}{2}$ . Then a canonical form of  $\mathcal{M}_q^z(n)$  is  $\text{diag}(D_1, D_2, \dots, D_{\frac{n^2-n}{2}}, 0, \dots, 0)$ .

*Proof.* We now know that the rank of  $\mathcal{M}_q^z(n)$  is  $n^2 - n$  and the entries of a canonical form of  $\mathcal{M}_q^z(n)$  are clearly all even. Hence the assertion follows Theorem 6.4.  $\square$

The final case,  $J_q^n(n)$ , is more difficult to handle. We know from experiments that the canonical form contains matrices of the form  $\begin{pmatrix} 0 & 2^p \\ -2^p & 0 \end{pmatrix}$  with  $p > 1$ . Indeed, the maximal occurring  $p$ , as a function of  $n$ , appears to be increasing. We shall be content to compute the rank of the canonical form (which gives the degree when  $m$  is “good”):

Let  $B$  be the integral anti-symmetric matrix with entries  $b_{[i,j],[s,t]}$  defined by

$$(7.15) \quad b_{[i,j],[s,t]} = -2 \text{ if } s \geq i \text{ and } t < j,$$

$$(7.16) \quad b_{[i,j],[s,t]} = 2 \text{ if } i \geq s \text{ and } j < t, \text{ and}$$

$$(7.17) \quad b_{[i,j],[s,t]} = 0 \text{ otherwise .}$$

Let  $H$  be the integral anti-symmetric matrix with entries  $h_{[i,j],[s,t]} = s - i + j - t$ . Then it is obvious that  $H$  is of rank two and the rows of  $H$  is spanned by the  $1 \times n^2$  row  $P = (r_{[s,t]})$  with  $r_{[s,t]} = s - t$  and the  $1 \times n^2$  row  $\mathbf{T}$  in which all entries are 1. The defining matrix of  $J_q^n(n)$  is then equal to  $H + B$ .

Consider the sum of the  $(k, k)$ th rows of  $B$  for all  $k = 1, 2, \dots, n$ . This is  $-2P$ . Hence the rank of  $H + B$  is the same as the rank of  $B$  since it has got to be even. But  $B$  is the twice the transposed of the defining matrix of  $D_q(n)$ . Thus we obtain

**Proposition 7.7.** *The rank of  $\mathcal{M}_q^n(n)$  is  $n^2$  if  $n$  is even and  $n^2 - 1$  if  $n$  is odd.*

We end this section by illustrating how closely related e.g.  $J_q^z(n)$  and  $J_q^0(n)$  are: Let  $A_2$  be the quantum plane i.e. an associative algebra generated by  $x, y$  subject to the following relation:

$$(7.18) \quad yx = qxy.$$

**Lemma 7.8.** *Let  $\tilde{Z}_{i,j} = x^{i+j}y \otimes M_{i,j}$  for all  $i, j$ . Then the  $\tilde{Z}_{i,j}$  generate a subalgebra of  $A_2 \otimes J_q^z(n)$  which is isomorphic with  $J_q^0(n)$ .*

*Proof.* If  $(s - i)(t - j) \leq 0$ , then

$$(7.19) \quad \tilde{Z}_{i,j} \tilde{Z}_{s,t} = x^{i+j}y x^{s+t}y \otimes M_{i,j} M_{s,t}$$

$$(7.20) \quad = q^{s+t} x^{i+j+s+t} y^2 \otimes M_{i,j} M_{s,t} = q^{s+t-i-j} x^{s+t} y x^{i+j} y \otimes M_{s,t} M_{i,j}$$

$$(7.21) \quad = q^{s+t-i-j} \tilde{Z}_{s,t} \tilde{Z}_{i,j}.$$

If  $s > i, t > j$ , we have

$$(7.22) \quad q^{1+j-t} \tilde{Z}_{i,j} \tilde{Z}_{s,t} = q^{1+j+s} [x^{i+j+s+t} y^2 \otimes M_{i,j} M_{s,t}]$$

$$(7.23) \quad = q^{j+s} [x^{i+j+s+t} y^2 \otimes (q^{-1} M_{s,t} M_{i,j} + (q - q^{-1}) M_{i,t} M_{s,j})]$$

$$(7.24) \quad = q^{j+s-1-i-j} \tilde{Z}_{s,t} \tilde{Z}_{i,j} + (q - q^{-1}) \tilde{Z}_{i,t} \tilde{Z}_{s,j}.$$

This completes the proof. □

**Remark 7.9.** Similarly, one can embed  $J_q^z(n)$  into  $A_2 \otimes J_q^0(n)$  by the map  $\tau : J_q^z(n) \longrightarrow A_2 \otimes J_n$  defined by

$$(7.25) \quad M_{i,j} \mapsto xy^{i+j} \otimes \tilde{Z}_{i,j}.$$

## 8. THE VARIETIES OF THE ALGEBRAS $J_q^0(n)$ , $J_q^z(n)$ , $J_q^n(n)$ , AND $D_n$

In this section we consider the associated varieties of the modified algebras. Let  $V$  be a complex linear space and let  $T(V)$  be the tensor algebra on  $V$ . Let  $R \subset V \otimes V$  be a subspace and let  $(R)$  be the ideal of  $T(V)$  generated by  $R$ . Set  $A = T(V)/(R)$ . This is a quadratic algebra. The elements of  $V \otimes V$  may be viewed as bilinear forms on  $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ : If

$$(8.1) \quad f = \sum_{i,j} \alpha_{i,j} x_i \otimes x_j \in R$$

and  $(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ , then

$$(8.2) \quad f(p, q) = \sum_{i,j} \alpha_{i,j} x_i(p) x_j(q).$$

Hence we may associate to  $R$  the subvariety

$$(8.3) \quad \Gamma(R) := \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid f(p, q) = 0 \text{ for all } f \in R\}.$$

We call  $\Gamma(R)$  the associated variety.

In [28] the associated variety of the standard quantized matrix algebra was determined. Among other thing it turned out to be independent of the quantum parameter  $q$ . In this section we (again) assume that the  $q^2 \neq 1$  and we consider first the associated variety  $\Gamma_n^0$  of the algebra  $J_q^0(n)$ . In some sense, this is the nicest.

Let  $((a_{i,j}), (b_{i,j})) \in \Gamma_n^0$ , where  $(a_{i,j})$  and  $(b_{i,j})$  are two  $n \times n$  complex matrices. Then we have

**Lemma 8.1.** *Let the notations be as above. Then  $a_{i,j} = 0$  if and only if  $b_{i,j} = 0$ .*

*Proof.* We assume that  $a_{s,t} \neq 0$  for some  $(s, t)$  and  $a_{i,j} = 0$  but  $b_{i,j} \neq 0$ . By

$$(8.4) \quad a_{i,j} b_{i,k} = q^{k-j} a_{i,k} b_{i,j},$$

we have  $a_{i,k} = 0$  for all  $k = 1, 2, \dots, n$ .

If  $(s - i)(t - j) \leq 0$  we have

$$(8.5) \quad a_{i,j}b_{s,t} = q^{s+t-i-j}a_{s,t}b_{i,j}$$

which implies that  $b_{i,j} = 0$ . Contradiction.

If  $s > i$  and  $t > j$  we have

$$(8.6) \quad q^{1-t+j}a_{i,j}b_{s,t} = q^{s-i-1}a_{s,t}b_{i,j} + (q - q^{-1})a_{i,t}b_{s,j}$$

which together with  $a_{i,t} = 0$  imply  $b_{i,j} = 0$  which again is a contradiction.

Similarly one can prove that if  $s < i$  and  $t < j$  we also get  $b_{i,j} = 0$ . This completes the proof.  $\square$

**Lemma 8.2.** *If  $(a_{i,j})$  is a rank one  $n \times n$  complex matrix, then  $((a_{i,j}), (q^{i+j}a_{i,j})) \in \Gamma_n^0$ .*

*Proof.* By direct verification.  $\square$

**Lemma 8.3.** *Let  $((a_{i,j}), (b_{i,j})) \in \Gamma_n^0$  and suppose that  $(a_{i,j})$  is a rank one complex matrix. Then  $b_{i,j} = q^{i+j}a_{i,j}$  for all  $i, j = 1, 2, \dots, n$ .*

*Proof.* We assume that  $a_{i,j} \neq 0$  for some  $(i, j)$ , then  $b_{i,j} \neq 0$  and by multiplying through by some non-zero complex number we can assume that  $b_{i,j} = q^{i+j}a_{i,j}$ .

For any  $(s, t)$ , if  $(s - i)(t - j) \leq 0$  we have

$$(8.7) \quad a_{i,j}b_{s,t} = q^{s+t-i-j}a_{s,t}b_{i,j},$$

so  $b_{s,t} = q^{s+t}a_{s,t}$ .

If  $s > i$  and  $t > j$ , we have

$$(8.8) \quad q^{1-t+j}a_{i,j}b_{s,t} = q^{s-i-1}a_{s,t}b_{i,j} + (q - q^{-1})a_{i,t}b_{s,j}.$$

Since  $b_{s,j} = q^{s+j}a_{s,j}$  and since rank 1 of the matrix  $(a_{i,j})$  implies that  $a_{i,t}a_{s,j} = a_{i,j}a_{s,t}$ , we get  $b_{s,t} = q^{s+t}a_{s,t}$ .

Similarly, one can prove that if  $s < i$  and  $t < j$  then  $b_{s,t} = q^{s+t}a_{s,t}$ . This completes the proof.  $\square$

Now we assume that the matrix  $(a_{i,j})$  is indecomposable. Let  $a_{i,j}$  be the first non-zero entry in the matrix  $(a_{i,j})$  according to the lexicographic ordering and assume  $b_{i,j} = q^{i+j}a_{i,j}$ .

Let

$$(8.9) \quad I_1 = \{(i, j) \mid b_{i,j} = q^{i+j}a_{i,j}\},$$

$$(8.10) \quad I_2 = \{(i, j) \mid b_{i,j} \neq q^{i+j}a_{i,j}\}.$$

Then it is easy to see that if  $a_{i,k} \neq 0$  and  $(i, k) \in I_1$  for some  $k$ , then  $(i, j) \in I_1$  for all  $j$ . Similarly, if  $a_{k,j} \neq 0$  and  $(k, j) \in I_1$  for some  $k$ , then  $(i, j) \in I_1$  for all  $i$ . Indeed, if  $(k, l) \in I_1$  and if  $(s - k)(t - l) \leq 0$  then  $(s, t) \in I_1$ . Since  $I_1 \neq \emptyset$  and  $(a_{i,j})$  is indecomposable, it follows easily that  $I_1 = I(n)$ , i.e.  $b_{i,j} = q^{i+j}a_{i,j}$  for all  $i, j$ . For any  $i < s$  and  $j < t$  we have

$$(8.11) \quad q^{1-t+j}a_{i,j}b_{s,t} = q^{s-i-1}a_{s,t}b_{i,j} + (q - q^{-1})a_{i,t}b_{s,j}.$$

Hence  $a_{i,j}a_{s,t} = a_{i,t}a_{s,j}$ . This proves that  $\text{rank}(a_{i,j}) = 1$ .

Now we assume that the matrix  $(a_{i,j})$  is decomposable. Then  $\text{rank}(a_{i,j}) \geq 2$ . Let  $a_{i,j}a_{s,t} \neq 0$ ,  $(i, j) \in I_1$ ,  $(s, t) \in I_2$ . As above,  $(s - i)(t - j) \leq 0$  is impossible. So without losing generality we assume that  $s > i$  and  $t > j$ . We then must have  $a_{i,t} = a_{s,j} = 0$ . By

$$(8.12) \quad q^{1-t+j}a_{i,j}b_{s,t} = q^{s-i-1}a_{s,t}b_{i,j} + (q - q^{-1})a_{i,t}b_{s,j}$$

we get that  $b_{s,t} = q^{s+t-2}b_{s,t}$  for  $(s, t) \in I_2$ . More generally this proves that the matrix  $(a_{i,j})$  is in fact a direct sum of indecomposable matrices,

$$(8.13) \quad (a_{i,j}) = \text{diag}(D_1, D_2, \dots, D_r),$$

where each  $D_i$ 's is either zero or of rank one. Furthermore, the above analysis of how the relation between  $a_{s,t}$  and  $b_{s,t}$  follows from  $I_1$  clearly implies (since  $q^2 \neq 1$ ), that at most two of them are non-zero. Summarizing, we have proved

**Theorem 8.4.** *Let  $q$  be generic or  $q$  is an  $m$ th root of unity ( $m \neq 2$ ). Let  $((a_{i,j}), (b_{i,j})) \in \Gamma_n^0$ . Then the matrix  $(a_{i,j})$  is either of rank one and  $b_{i,j} = q^{i+j}a_{i,j}$  for all  $i, j$  or  $(a_{i,j})$  of the following form:*

$$(8.14) \quad (a_{i,j}) = \text{diag}(0, 0, \dots, 0, D_1, 0, \dots, 0, D_2, 0, \dots, 0)$$

where  $D_i$  are rank one matrices. In this case,

$$(8.15) \quad (b_{i,j}) = \text{diag}(0, 0, \dots, T_1, 0, \dots, 0, T_2, 0, \dots, 0)$$

where  $T_1 = (q^{i+j}a_{i,j}), T_2 = (q^{i+j-2}a_{i,j})$ .

Let us now consider the variety  $\Gamma_n^z$  of  $J_q^z(n)$ .

**Theorem 8.5.** *Let  $((a_{i,j}), (b_{i,j})) \in \Gamma_n^z$ . Then the matrix  $(a_{i,j})$  is either of rank one and  $b_{i,j} = a_{i,j}$  for all  $i, j$  or  $(a_{i,j})$  of the following form:*

$$(8.16) \quad (a_{i,j}) = \text{diag}(0, 0, \dots, 0, D_1, 0, \dots, 0, D_2, 0, \dots, 0)$$

where  $D_i$  are rank one matrices (of arbitrary shape). Then

$$(8.17) \quad (b_{i,j}) = \text{diag}(0, 0, \dots, T_1, 0, \dots, 0, T_2, 0, \dots, 0)$$

where  $T_1 = D_1, T_2 = q^{-2}D_2$ .

*Proof.* The proof is almost the same as that of  $J_q^0(n)$ . □

**Theorem 8.6.** *Let  $\Gamma_n^D$  denote the variety of  $D_q(n)$  and let  $((a_{ij}), (b_{ij}))$  be a point in  $\Gamma_n^D$ . Then either there exists a non-zero  $1 \times n$  row  $R$  and a  $c \in \mathbb{C}^*$  such that*

$$(8.18) \quad (a_{ij}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ R \\ 0 \\ \dots \\ 0 \\ cR \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad (b_{ij}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ qR \\ 0 \\ \vdots \\ 0 \\ qcR \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

or  $(a_{ij}) = \text{diag}(A_1, A_2, \dots, A_n)$  where  $A_i$  is a  $1 \times s_i$  complex row vector for some positive integer  $s_i$  and  $(b_{ij}) = (a_{ij})$ .

*Proof.* Consider the relations (7.1) and let  $((a_{ij}), (b_{ij}))$  be a point in the variety  $\Gamma(D_n)$ . First of all an elementary computation shows that  $a_{ij} = 0 \implies b_{ij} = 0$ . Moreover,

$$(8.19) \quad a_{ij}b_{ik} = a_{ik}b_{ij} \text{ for all } i, j, k.$$

Hence there exist  $c_1, c_2, \dots, c_n \in \mathbb{C}^*$  such that

$$(8.20) \quad b_{ij} = c_i a_{ij} \text{ for all } i, j = 1, 2, \dots, n.$$

If there exist  $i > s$  and  $j \leq t$  such that  $a_{ij}a_{st} \neq 0$ , then

$$(8.21) \quad a_{ij}b_{st} = qa_{st}b_{ij},$$

and so  $c_s = qc_i$ . Thus it is impossible to have three non-zero entries  $a_{ij}, a_{st}, a_{lk}$  such that  $i > s > l$  and  $j \leq t \leq k$ .

For  $i > s$  and  $j > t$ , if  $a_{ij}a_{st}a_{sj}a_{it} \neq 0$ , then

$$(8.22) \quad a_{ij}b_{st} = a_{st}b_{ij} + (q-1)a_{sj}b_{it}.$$

Therefore  $a_{ij}a_{st} = a_{sj}a_{it}$ .

The above argument proves that for any  $2 \times 2$  submatrix

$$(8.23) \quad \begin{pmatrix} a_{st} & a_{sj} \\ a_{it} & a_{ij} \end{pmatrix},$$

if all entries are non-zero, the rank is 1. But we can furthermore see that the number of zero entries cannot be 1. In fact, if  $a_{ij}$  or  $a_{st}$  is zero, by

$$(8.24) \quad a_{ij}b_{st} = a_{st}b_{ij} + (q-1)a_{sj}b_{it}$$

we get  $a_{sj}a_{it} = 0$ . If  $a_{sj} = 0$  but the other entries are non-zero then, since  $a_{it}a_{st} \neq 0$ , we get  $c_s = qc_i$ . But

$$(8.25) \quad a_{ij}b_{st} = a_{st}b_{ij}$$

implies that  $c_i = c_s$  which is a contradiction. Similarly one can dismiss the case  $a_{it} = 0$  but the other entries are non-zero.

If  $\text{rank}(a_{ij}) = 1$ , then there exists a non-zero  $1 \times n$  row  $R$  such that

$$(8.26) \quad (a_{ij}) = \begin{pmatrix} d_1 R \\ d_2 R \\ \dots \\ d_n R \end{pmatrix}$$

for certain constants  $d_1, \dots, d_n$ . By the above observations, at most two  $d_i$  are non-zero and the first assertion follows.

If  $\text{rank}(a_{ij}) \geq 2$  there exists a non-degenerate submatrix

$$(8.27) \quad \begin{pmatrix} a_{st} & a_{sj} \\ a_{it} & a_{ij} \end{pmatrix}.$$

By the above argument we must have  $a_{sj} = a_{it} = 0$ . If the matrix  $(a_{ij})$  does not have a  $2 \times 2$  submatrix with all entries non-zero then  $(a_{ij}) = \text{diag}(A_1, A_2, \dots, A_n)$  where  $A_i$  is a  $1 \times s_i$  complex row vector for some positive integer  $s_i$  and by the above discussion we must have  $(b_{ij}) = c(a_{ij})$  for some  $c \in \mathbb{C}^*$ . If there is a rank one  $2 \times r$  submatrix  $S$  with all entries are non-zero and we assume that  $r \geq 2$  is maximal among the possible choices, then there are some non-zero entries in  $(a_{ij})$  outside  $S$  because  $\text{rank}(a_{ij}) \geq 2$ . Clearly, those non-zero entries cannot sit in the middle of, on top of, or below the matrix  $S$ , since there are no triples  $a_{ij}, a_{st}, a_{lk}$  of non-zero entries with  $i > s > l$  and  $j \leq t \leq k$ . Now assume that there is a non-zero  $a_{lk}$  located to the lower right the submatrix  $S$ . Let the numbers of the two rows of the submatrix  $S$  be  $r$  and  $t$  with  $r < t$ . Hence we have three non-zero entries  $a_{lk}, a_{rj}, a_{tj}$  where  $a_{rj}, a_{tj}$  are entries in  $S$  and  $r < t < l$  and  $j < k$ . Obviously  $a_{lj} = a_{rk} = a_{tk} = 0$  and so  $c_l = c_r = c_t$ , but  $c_r = qc_t$  since both  $a_{rj}$  and  $a_{tj}$  are non-zero and this is a contradiction. Similarly one can dismiss any other location of a non-zero entry outside of  $S$ . But this means that the rank of  $(a_{ij})$  is 1 which is contrary to our assumption. Hence the matrix  $(a_{ij})$  does not have a rank 1  $2 \times r$  submatrix with all entries non-zero. Therefore

$$(8.28) \quad (a_{ij}) = \text{diag}(A_1, A_2, \dots, A_n)$$

where  $A_i$  is a  $1 \times s_i$  complex row vector for some positive integer  $s_i$ . It is then clear that the matrix  $(b_{ij})$  must be a multiple of the matrix  $(a_{ij})$ . This completes the proof.  $\square$

**Theorem 8.7.** *Let  $\Gamma_n^n$  be the variety of  $J_q^n(n)$  and let  $((a_{ij}), (b_{ij}))$  be a point in  $\Gamma_n^n$ . Then either there exists a non-zero  $n \times 1$  column  $R$  and a  $c \in \mathbb{C}^*$  such that*

$$(8.29) \quad (a_{ij}) = (0, \dots, 0, R, 0, \dots, 0, cR, 0, \dots, 0)$$

and

$$(8.30) \quad (b_{ij}) = (0, \dots, 0, R', 0, \dots, 0, cq^2R', 0, \dots, 0)$$

where  $R = \begin{pmatrix} r_{1i} \\ r_{2i} \\ \dots \\ r_{ni} \end{pmatrix}$  and  $R' = \begin{pmatrix} q^{1-i}r_{1i} \\ q^{2-i}r_{2i} \\ \dots \\ q^{n-i}r_{ni} \end{pmatrix}$  or  $(a_{ij}) = \text{diag}(A_1, A_2, \dots, A_n)$  where  $A_i$  is an  $s_i \times 1$  complex column for some positive integer  $s_i$  and  $(b_{ij}) = (q^{i-j}a_{ij})$ .

*Proof.* This follows by arguments analogous to those in the proof for  $D_q(n)$ . □

## 9. STRUCTURE AND DIMENSIONS OF SYMPLECTIC LEAVES

The dimensions of the symplectic leaves in the case of the *regular points* of  $M(n, \mathbb{C})$  can be computed by the method of the Manin double [26], [19] as explained in e.g. [11]. Specifically, let  $n^\pm$  denote the set of strictly upper and lower triangular matrices in  $M(n, \mathbb{C})$ , and let  $N^\pm = \exp(n^\pm)$ . Let  $h$  denote the diagonal subalgebra of  $M(n, \mathbb{C})$ , let  $h_0$  denote the subalgebra of  $h$  consisting of trace 0 elements, and let  $H_0$  denote the diagonal elements of determinant 1. By  $B_0^\pm$  we denote the upper and lower triangular matrices, respectively, in  $SL(n, \mathbb{C})$  and we denote the analogous subgroups of  $GL(n, \mathbb{C})$  by  $B^\pm$ . Identify  $SL(n, \mathbb{C})$  with the diagonal in  $D_0 = SL(n, \mathbb{C}) \times SL(n, \mathbb{C})$ . Let  $SL_r(n, \mathbb{C})$  denote the subgroup of  $D_0$  generated by  $N^+ \times 1, 1 \times N^-$ , and  $A_0 = \{x, x^{-1} \mid x \in H_0\}$ , and denote by  $sl_r(n, \mathbb{C})$  the Lie algebra of this subgroup. Analogously, define  $GL_r(n, \mathbb{C})$  and  $gl_r(n, \mathbb{C})$  by removing the determinant 1 and trace 0 condition from  $A_0$  and  $h_0$ , respectively.

We denote by  $\langle \cdot, \cdot \rangle$  the standard bilinear form  $\langle x, y \rangle = \text{tr } xy$  both on  $M(n, \mathbb{C})$  and on  $sl(n, \mathbb{C})$ , and we define the bilinear form  $B$  on  $M(n, \mathbb{C}) \times M(n, \mathbb{C})$  by

$$(9.1) \quad B((x_1, y_1), (x_2, y_2)) = \frac{1}{2}(\langle x_1, x_2 \rangle - \langle y_1, y_2 \rangle).$$

Through the bilinear form  $B$ ,  $sl_r(n, \mathbb{C})$  is identified with  $sl(n, \mathbb{C})^*$  and  $gl_r(n, \mathbb{C})$  with  $M(n, \mathbb{C})^*$ .

The traditional setting is to view  $SL_r(n, \mathbb{C})/\Gamma$ , where  $\Gamma = \{x \in H_0 \mid x^2 = 1\}$ , as sitting inside  $D_0/SL_r(n, \mathbb{C})$ . The latter is a Poisson manifold, and  $SL_r(n, \mathbb{C})/\Gamma$  is an open Poisson submanifold.

Let  $\alpha \in sl_r(n, \mathbb{C})$ . Through the bilinear form  $B$  above,  $\alpha$  induces a right invariant 1-form  $\alpha_r(x)$  on  $SL(n, \mathbb{C})$ . The right dressing vector field  $\rho(\alpha)$  is defined by

$$(9.2) \quad \forall \xi \in \Omega^1(M(n, \mathbb{C})) : \langle \rho_x(\alpha), \xi \rangle = \pi_x(\alpha_r(x), \xi).$$

Secondly,  $\alpha \in sl_r(n, \mathbb{C})$  gives rise to a vector field on  $SL_r(n, \mathbb{C})/\Gamma$  through the left action on  $D_0/SL_r(n, \mathbb{C})$ , and this can be lifted to a vector field  $\sigma(\alpha)$  on  $SL(n, \mathbb{C})$ . The key result is then

**Theorem 9.1** ([26],[19]). *For all  $x \in SL(n, \mathbb{C})$ ,*

$$(9.3) \quad \rho_x(\alpha) = -\sigma_x(\alpha).$$

It follows from the above ([11]) that  $SL(n, \mathbb{C})$  is a disjoint union of the sets  $\mathcal{L}_{\omega_1, \omega_2} = B_0^+ \omega_1 B_0^+ \cap B_0^- \omega_2 B_0^-$  where  $(\omega_1, \omega_2) \in W \times W$ . Each set  $\mathcal{L}_{\omega_1, \omega_2}$  is a union of symplectic leaves of the same dimension. This dimension may be computed by placing one self at a good point in  $D_0/SL_r(n, \mathbb{C})$ , e.g.  $[\omega_1, \omega_2]$  even though this, when  $\omega_1 \neq \omega_2$ , is not in  $SL_r(n, \mathbb{C})/\Gamma$ .

This picture extends in an obvious way to  $GL(n, \mathbb{C})$ . In particular, we have the following

**Corollary 9.2.** *The symplectic loaves in  $GL(n, \mathbb{C})$  are precisely the sets*

$$(9.4) \quad B^+ \omega_1 B^+ \cap B^- \omega_2 B^-.$$

Let us now take a closer look at the Poisson brackets (5.18)

$$(9.5) \quad \{l_k, a_{i,j}\} = (\lambda_k, \alpha_{i,j}) a_{i,j} l_k.$$

In this expression,  $(\lambda_k, \alpha_{i,j})$  is exactly the exponent of the multiplication operator  $\check{\lambda}_k$  (5.1). Thus, it follows that if  $\theta_k$  denotes the vector field defined by  $\check{\lambda}_k$ , then

$$(9.6) \quad \{l_k, a_{i,j}\} = l_k da_{i,j}(\theta_k).$$

Let  $s_i^r$  denote the  $r$ th scalar row operator and  $s_j^c$  the  $j$ th scalar column operator. Then  $s_i^r$  acts from the left and  $s_j^c$  from the right. Specifically, let  $d_i$  denote the diagonal matrix in  $gl(n, \mathbb{C})$  with 1 at the  $i$ th place and zeros elsewhere. Then

$$(9.7) \quad s_i^r f(Z) = \frac{d}{dt} \Big|_{t=0} f(e^{td_i} Z) \text{ and } s_j^c f(Z) = \frac{d}{dt} \Big|_{t=0} f(Ze^{td_j}).$$

We now wish to determine the Poisson structure on  $M(n, \mathbb{C})$  obtained through a modification  $\varphi$ . The functions  $z_{i,j}$ 's are transformed into  $z_{i,j}\phi_i\psi_j$ . Recalling that the Poisson bracket  $\{f, g\}$  only depends on  $df$  and  $dg$ , it follows easily, letting  $l_k \rightarrow 1$ , that the modified Poisson bracket  $\{\omega, \xi\}^*$  between two 1-forms  $\omega, \xi$  on  $M(n, \mathbb{C})$  is given as

$$(9.8) \quad \{\omega, \xi\}^* = \{\omega^*, \xi^*\} = \{\omega, \xi\} + \sum_k \{\omega(s_k^r)\phi_k, \xi\} + \sum_k \{\omega(s_k^c)\psi_k, \xi\} + \{\omega, \xi(s_k^r)\phi_k\} + \{\omega, \xi(s_k^c)\psi_k\} \quad .$$

Let us for the rest of this section assume that the modifications are of the form  $\tilde{Z}_{i,j} = Z_{i,j}\phi_i\psi_j$  where  $\phi_i$  only involves the fundamental roots corresponding to  $\beta, \mu_1, \dots, \mu_{n-1}$ , and where  $\psi_j$  only involves the fundamental roots corresponding to  $\beta, \nu_1, \dots, \nu_{n-1}$ . We let (c.f. Lemma 5.2)  $x_i$  and  $y_j$  denote the right and left invariant vector fields, respectively, corresponding to  $\phi_i$  and  $\psi_j$ ,  $i, j = 1, \dots, n$ . Specifically,

$$(9.9) \quad \{\phi_k, \xi\} = \xi(x_k), \text{ and } \{\psi_k, \xi\} = \xi(y_k).$$

Then

$$(9.10) \quad \{\omega, \xi\}^* = \xi(\rho_\omega) + \xi(\sum_k \omega(d_k^r)x_k) + \xi(\sum_k \omega(d_k^c)y_k) + \omega(\sum_k \xi(d_k^r)x_k) + \omega(\sum_k \xi(d_k^c)y_k).$$

Summarizing,

**Proposition 9.3.** *Let  $\rho_\omega$  denote the dressing vector field corresponding to the 1-form  $\omega$  in the unmodified Poisson structure and let  $\rho_\omega^*$  denote the dressing vector field defined by  $\omega$  with respect to the modified Poisson structure. Then,*

$$(9.11) \quad \rho_\omega^* = \rho_\omega + \sum_k \omega(d_k^r)x_k + \sum_k \omega(d_k^c)y_k - \sum_k \omega(x_k^r)d_k - \sum_k \omega(y_k^c)d_k.$$

Let

$$(9.12) \quad r_1 = \sum_k d_k \wedge x_k \text{ and } r_2 = \sum_k d_k \wedge y_k$$

be elements in  $h \wedge h$ , where we indentify  $x_k$  and  $y_k$  with their values at 0 and where  $h$  denotes the diagonal subalgebra of  $\mathfrak{g} = M(n, \mathbb{C})$ .

**Corollary 9.4.** *The modifications considered have the form (c.f. (5.7))*

$$(9.13) \quad \tilde{\pi}(g) = \pi(g) + (l_g)_*(r_2) + (r_g)_*(r_1).$$

This class of modifications is of the form considered by Semenov-Tian-Shansky in [26]. Indeed, viewed under appropriate identifications as a skew-symmetric map  $\mathfrak{g} \rightarrow \mathfrak{g}$ , any  $\tilde{r} = \sum_{1 \leq i < j \leq n} e_{i,j} \wedge e_{j,i} + r$ , with  $r \in h \wedge h$ , satisfies the Yang-Baxter identity

$$(9.14) \quad [\tilde{r}X, \tilde{r}Y] = \tilde{r}([\tilde{r}X, Y] + [X, \tilde{r}Y]) - [X, Y], \quad X, Y \in \mathfrak{g}.$$

We introduce the following elements of  $h$  for  $k = 1, \dots, n$ :

$$(9.15) \quad h_k = d_{k+1} + \dots + h_n \text{ and } a_k = k(d_1 + \dots + d_n),$$

where, naturally,  $h_n$  is defined to be 0. We then have

Algebra	$x_k$	$y_k$
$D_q(n)$	$-h_k$	$h_k$
$J_q^0(n)$	$-h_k - a_k$	$-h_k - a_k$
$J_q^z(n)$	$-h_k$	$-h_k$
$J_q^n(n)$	$-h_k - a_k$	$h_k + a_k$

The following then follows from Propositions 7.1 and 7.2.

**Proposition 9.5.** *Consider the following elements in  $M(n, \mathbb{C}) \wedge M(n, \mathbb{C})$ ,*

$$(9.16) \quad r = \sum_{\alpha \in \Delta^+} e_\alpha \wedge e_{-\alpha}, \quad r_0 = \sum_{k=1}^n d_k \wedge h_k, \quad \text{and } r_s = \sum_{k=1}^n d_k \wedge a_k.$$

The Poisson structures  $\pi_s(g)$ ,  $\pi_D(g)$ ,  $\pi_{J^0}(g)$ , and  $\pi_{J^z}(g)$  on  $M(n, \mathbb{C})$  corresponding to the algebras  $M_q(n)$ ,  $D_q(n)$ ,  $J_q^0(n)$ , and  $J_q^z(n)$  are then given as follows:

$$(9.17) \quad \begin{aligned} \pi_s(g) &= -(l_g)_*r + (r_g)^*r, & \pi_D(g) &= -(l_g)_*(r - r_0) + (r_g)^*(r - r_0) \\ \pi_{J^0}(g) &= -(l_g)_*(r + r_0 + r_s) + (r_g)^*(r - r_0 - r_s), & \pi_{J^z}(g) &= -(l_g)_*(r + r_0) + (r_g)^*(r - r_0) \\ \pi_{J^n}(g) &= -(l_g)_*(r - r_0 - r_s) + (r_g)^*(r - r_0 - r_s). \end{aligned}$$

The following is an easy consequence of [26, Theorem 2, p. 1242]

**Proposition 9.6.** *Multiplication  $M(n, \mathbb{C}) \times M(n, \mathbb{C}) \rightarrow M(n, \mathbb{C})$  induces Poisson mappings*

$$(9.18) \quad \begin{aligned} M(n, \mathbb{C})_D \times M(n, \mathbb{C})_{J^z} &\rightarrow M(n, \mathbb{C})_{J^z} \\ M(n, \mathbb{C})_{J^n} \times M(n, \mathbb{C})_{J^0} &\rightarrow M(n, \mathbb{C})_{J^0}. \end{aligned}$$

In case  $r_1 = -r_2$ , the dimensions may be computed by the method devised by Semenov-Tian-Shansky. Indeed, these dimensions have already been computed in [6] and [12]. In case  $r_1 \neq -r_2$  it seems to be difficult to obtain the answer in full generality. However, in case  $\omega_1 = \omega_2 = \omega$  one may obtain satisfactory results:

When computing at the point  $[(\omega, \omega)]$  it is easy to see that the only Hamiltonian (dressing) vector fields that are being modified are those corresponding to elements of the form  $(a, -a) \in gl_r(n, \mathbb{C})$ , where  $a \in h$ . (Observe that we have to move into  $gl(n, \mathbb{C})$ ). Set

$$(9.19) \quad \forall a \in h : T_R(a) = \langle a, d_k \rangle \cdot y_k - \langle a, y_k \rangle \cdot d_k \text{ and } T_L(a) = \langle a, d_k \rangle \cdot x_k - \langle a, x_k \rangle \cdot d_k.$$

When we compute at the point  $[(\omega, \omega)]$  we make all vector fields into right actions. Observe that  $\sigma(a) \star [(\omega, \omega)] = [(a \cdot \omega, a^{-1} \cdot \omega)] = [(\omega(\omega^{-1}a\omega), \omega(\omega^{-1}a^{-1}\omega))] = [(\omega, \omega)]$ .

**Proposition 9.7.** *The modified dressing vector field corresponding to  $a \in h$  is given by*

$$(9.20) \quad \tilde{\rho}(a)T_R(\omega^{-1}a\omega) + \omega^{-1}(T_L(a))\omega.$$

The right hand side of (9.20) may be identified with an element of  $h$ . Let  $L_\omega$  denote the linear map  $h \xrightarrow{\omega} h$  given by

$$(9.21) \quad h \ni a \mapsto L_\omega(a) = T_R(\omega^{-1}a\omega) + \omega^{-1}(T_L(a))\omega.$$

We can now give formulas for the dimensions of some symplectic leaves for the modifications we have considered, where we use the known formula ([11]) from the standard case.

**Proposition 9.8.** *The dimension of the symplectic leaf through the point  $(\omega, \omega)$  is given as  $2 \cdot \ell(\omega) + \text{rank } L_\omega$ .*

In general it appears to be difficult to compute  $\text{rank } L$  in terms of  $\omega$ . However, we have the following partial result.

**Proposition 9.9.** *Let  $\omega_\ell$  denote the longest element of the Weyl group. Then  $L_{\omega_\ell}$  is zero for  $J_q^0(n)$  and  $J_q^z(n)$  whereas in the cases of  $D_q(n)$  and  $J_q^n(n)$ , the rank of  $L_{\omega_\ell}$  is  $n$  for  $n$  even and  $n - 1$  for  $n$  odd.*

*Proof.* It is easy to see that there are many cancellations and simplifications in this special case. Thus, the claim about  $J_q^0(n)$  and  $J_q^z(n)$  follows by easy inspection. For the remaining cases, one is quickly reduced to finding the rank of (e.g.)  $T_L$ . For  $D_q(n)$  the matrix of  $T_L$  is skew-symmetric with 1's below the diagonal; a matrix with the stated rank. For  $J_q^n(n)$  it is slightly more complicated, but after a few simple manipulations, one may decompose the matrix into an invertible  $4 \times 4$  matrix and a skewsymmetric matrix  $M$  whose  $i, j$ th entry below the diagonal is  $i - j + 1$ . The last is the sum of a rank 2 matrix  $A$  (with entries  $a_{i,j} = i - j$ ) and a matrix as for the case of the Dipper Donkin algebra. But a combination of the columns of  $A$ , namely the column vector with 1's at all places, is in the span of  $M$ . The claim follows from this, since the rank must be even.  $\square$

## 10. THE CENTERS OF THE ALGEBRAS $J_q^0(n)$ , $J_q^z(n)$ , AND $J_q^n(n)$

In [15] and [16] the center of the standard quantized matrix algebra and the center of the Dipper-Donkin quantized matrix algebra were determined explicitly. A strategy one may try when computing the center of any modified algebra in our family is the following: Our modification is based on the standard quantized matrix algebra  $M_q(n)$ . In [15] it was

proved that the subdeterminants in the left upper or right lower corner are covariant. Since our modifications are by multiplication by some monomials in the  $L_i$ 's, the corresponding modified subdeterminants are still covariant. Although for different modified algebra one may need to use different method to compute its degree, it seems that we can get the whole center of the modified algebra by combining the modified subdeterminants in some proper ways (c.f. [15] and [16]). As already seen in Section 7, there is a close relationship between the size of the center and the degree. We now first look at the center of  $J_q^0(n)$  since by Lemma 7.3, the center of its associated quasipolynomial algebra  $\overline{J_q^0(n)}$  is within reach.

For any  $B = (b_{i,j})_{i,j=1}^n \in J_q^0(n)(\mathbb{Z}_+)$  we define

$$(10.1) \quad J^B = \prod J_{i,j}^{b_{i,j}},$$

where the factors are arranged according to lexicographic ordering. We denote the generators of  $\overline{J_q^0(n)}$  by  $x_{i,j}$ ,  $i, j = 1, 2, \dots, n$ , and define the symbol  $x^B$  analogously in terms of the same ordering.

Let  $C$  be the center of  $J_q^0(n)$  and  $\overline{C}$  the center of  $\overline{J_q^0(n)}$ . For any  $P \in C$ , the leading term of  $P$  must be of the form  $cJ^B$ ,  $c \in \mathbb{C}$ , for some  $B = (a_{i,j})_{i,j=1}^n \in M_n(\mathbb{Z}_+)$ . For any  $J_{k,l}$  the leading term of  $PJ_{k,l}$  is  $cq^{r_{k,l}}J^{B+E_{k,l}}$  where  $r_{k,l} = \sum_{(i,j)>(k,l)}(k+l-i-j)b_{i,j} + \sum_{i>k,j>l}2b_{i,j}$ . The leading term of  $J_{k,l}P$  is  $cq^{l_{k,l}}J^{B+E_{k,l}}$  where  $l_{k,l} = \sum_{(i,j)<(k,l)}(i+j-k-l)b_{i,j} - 2\sum_{i<k,j<l}b_{i,j}$ . Since  $PJ_{k,l} = J_{k,l}P$  we get  $q^{l_{k,l}} = q^{r_{k,l}}$  and this implies that  $cX^B$  is a central element of the twisted polynomial algebra  $\overline{J_q^0(n)}$ . Hence we can define a map  $\Lambda : C \rightarrow \overline{C}$  by

$$(10.2) \quad P \mapsto cX^B$$

if the leading term of  $P$  is  $cJ^B$ ,  $c \in \mathbb{C}$ .

Clearly,  $\Lambda(P) = 0$  implies  $P = 0$ .

**Theorem 10.1.** *Let  $q$  be a primitive  $m$ th root of unity and let  $s$  be the minimal positive integer such that  $sm - n + 1 \geq 0$ . Then*

(a) *If  $m$  is odd, then the center of  $\overline{J_q^0(n)}$  is generated by  $x_{i,j}^m, x_{1,n}^{m-r}x_{n,1}^r$  for  $r = 1, \dots, m-1$ ,  $x_{1,n}^{sm-n+1}X(j)$  for  $j = 2, 3, \dots, n$ , and  $x_{1,n}^{sm-n+2}X(1)$ .*

(b) *If  $m$  is even,  $m = 2m'$  say, then the center of  $\overline{J_q^0(n)}$  is generated by  $x_{i,j}^{m'}x_{j,i}^{m'}, x_{1,n}^{m-r}x_{n,1}^r$  for  $r = 1, \dots, m-1$ ,  $x_{1,n}^{n-1}X(j)$  for  $j = 2, 3, \dots, n$ , and  $x_{1,n}^{n-2}X(1)$ .*

*Proof.* By Theorem 7.4 and Proposition 3.1 we know that the degree of the quasipolynomial algebra  $\overline{J_q^0(n)}$  is:

$$(10.3) \quad \deg \overline{J_q^0(n)} = m \cdot (m')^{\frac{n^2-n-2}{2}}.$$

The result now follows from [5, Proposition 7.1]. □

In the following, the quantum determinants are those corresponding to  $J_q^0(n)$ . Let

$$(10.4) \quad J(k) = \det_q(\{1, 2, \dots, k\}, \{n - k + 1, \dots, n\}) \det_q(\{k, k + 1, \dots, n\}, \{1, 2, \dots, n - k + 1\}),$$

for  $k = 2, 3, \dots, n$  and

$$(10.5) \quad J(1) = \det_q.$$

Then we have

**Theorem 10.2.** *Let  $q$  be a primitive  $m$ th root of unity for some odd positive integer  $m$ . Then the center of  $J_q^0(n)$  is generated by the elements  $J_{i,j}^m, J(k)J_{1,n}^{m-n+1}$  for all  $k = 2, 3, \dots, n$ ,  $J_{1,n}^{m-n+2}J(1)$ , and  $J_{1,n}^{m-r}J_{n,1}^r$  for  $r = 1, 2, \dots, m$ .*

*Proof.* Let  $C'$  be the central subalgebra generated by the central elements stated in the theorem. For any  $Y \in C$  we use induction on the leading term of  $Y$  to prove that  $Y$  belongs to  $C'$ . By Theorem 10.1, we know that there is a central element  $Y' \in C'$  which has the same leading term as that of  $Y$ . Hence,  $Y - Y' \in C'$ . This completes the proof. □

Similarly, we get

**Theorem 10.3.** *Let  $q$  be a primitive  $m$ th root of unity for some even positive integer  $m = 2m'$ . The center of  $J_q^0(n)$  is generated by the elements  $J_{i,j}^{m'}J_{j,i}^{m'}$  for  $i, j = 1, 2, \dots, n$ ,  $J(k)J_{1,n}^{m-n+1}$  for  $k = 2, 3, \dots, n$ ,  $J_{1,n}^{m-n+2}J(1)$ , and  $J_{1,n}^{m-r}J_{n,1}^r$  for  $r = 1, 2, \dots, m$ .*

We next consider the center of the algebra  $J_q^z(n)$ . Let  $M(k)$  be the minor  $\det_q(\{n - k + 1, n - k + 2, \dots, n\}, \{1, 2, \dots, k\})$  and let  $\tau$  be the anti-automorphism sending  $M_{i,j}$  to  $M_{j,i}$ . Then in a similar way we get

**Theorem 10.4.** *Let  $q$  be a primitive  $m$ th root of unity (odd or even) and let  $C$  be the center of the algebra  $J_q^z(n)$ . Then  $C$  is generated by the elements  $M_{i,j}^m, M_{1,n}, M_{n,1}$ , and  $M(k)^r \tau(M(n-k+1)^r) \det_q^{m-r}$  for  $k = 2, 3, \dots, n-1$  and  $r = 1, 2, \dots, m-1$ .*

Finally, we consider  $J_q^n(n)$ . Let  $q$  be a primitive  $m$ th root of unity and let  $A = (a_{st}) \in M_n(\mathbb{Z}_+)$  where

$$(10.6) \quad a_{st} = 1 \text{ if } s+t \text{ is even,}$$

$$(10.7) \quad a_{st} = m-1 \text{ if } s+t \text{ is odd.}$$

Let  $\overline{J_q^n(n)}$  be the associated quasipolynomial algebra of  $J_q^n(n)$ , and denote the generators of  $\overline{J_q^n(n)}$  by  $\overline{N_{ij}}$ .

**Proposition 10.5.** *The element  $\overline{N}^{(n-2)A-I}$  is a central element of  $\overline{J_q^n(n)}$  provided that  $n$  is odd.*

*Proof.*

$$(10.8) \quad \begin{aligned} \overline{N_{ij}} \overline{N}^A &= q^{\sum_{s,t} (s-i-t+j)(-1)^{s+t} + 2\sum_{s=1}^i \sum_{t=j+1}^n (-1)^{s+t} - 2\sum_{s=i}^n \sum_{t=1}^{j-1} (-1)^{s+t}} \overline{N}^A \overline{N_{ij}} \\ &= q^{j-i} \overline{N}^A \overline{N_{ij}} \text{ for all } i, j. \end{aligned}$$

Since  $n$  is odd we have

$$(10.9) \quad \sum_{s,t} (s-i-t+j)(-1)^{s+t} = \sum_{s,t} (-i+j)(-1)^{s+t} = j-i,$$

$$(10.10) \quad 2 \sum_{s=1}^i \sum_{t=j+1}^n (-1)^{s+t} - 2 \sum_{s=i}^n \sum_{t=1}^{j-1} (-1)^{s+t} = 0,$$

and

$$(10.11) \quad \overline{N_{ij}} \overline{N}^I = q^{(n-2)(j-i)} \overline{N}^I \overline{N_{ij}} \text{ for all } i, j.$$

This completes the proof. □

Let  $I = \{t+1, t+2, \dots, n\}, J = \{1, 2, \dots, n-t\}$ , and let  $\phi_t = \det_q(I, J)$ . Let  $I^* = \{1, 2, \dots, t\}, J^* = \{n-t+1, n-t+2, \dots, n\}$  and let  $\phi_t^* = \det_q(I^*, J^*)$ , where the determinant

is the modified determinant. Let  $a_1 = n - 3$ ,  $a_i = (n - 2)$  if  $i$  is odd and  $i \neq 1$ , and  $a_i = (n - 2)(m - 1)$  if  $i$  is even. Set

$$(10.12) \quad \Omega(n) = \prod_{i=1}^n \phi_i^{a_i} \prod_{j=2}^n \psi_j^{a_{n-j+1}}.$$

Then the element  $\Omega(n)$  is a central element of  $J_q^n(n)$ . Due to our weaker result concerning the canonical form in this case, we also need an extra assumption on  $m$  for our result concerning the center of  $J_q^n(n)$ .

**Theorem 10.6.** *Let  $q$  be an  $m$ th root of unity for some “good” integer  $m$ . Then the center of  $J_q^n(n)$  is generated by  $N_{ij}^m$  for all  $i, j = 1, 2, \dots, n$  if  $n$  is even and is generated by  $N_{ij}^m$  and  $\Omega(n)$  for all  $i, j = 1, 2, \dots, n$  if  $n$  is odd.*

$$11. \quad \mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q^\wp(n)$$

As should hopefully be clear from the preceding sections,  $\mathcal{A}_n = \mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q^\wp(n)$  is, in some sense, the most fundamental algebra. We here briefly study some of its properties.

**Proposition 11.1.**  $\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q^\wp(n)$  is an iterated Ore extension.

*Proof.* Since the elements  $L_i$  are covariant, this is obvious. □

Obviously,  $\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q^\wp(n)$  is a quadratic algebra, and the associated quasipolynomial algebra may be taken to be  $\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s \overline{M_q^\wp(n)}$ .

**Theorem 11.2.** Let  $S_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  for  $i = 1, 2, \dots, 3n - 3$  and  $S_j = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$  for  $j = 3n - 2, \dots, \frac{n^2+n}{2}$ . Then  $\text{diag}(S_1, S_2, \dots, S_{\frac{n^2+n}{2}}, 0, \dots, 0)$  is a canonical form of the algebra  $\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q^\wp(n)$ . In particular, the degree is given by

$$(11.1) \quad \deg \mathcal{A}_n = m^{3n-3} (m')^{(n-2)(n-3)/2}.$$

*Proof.* This relies heavily on the result (and method) for  $M_q(n)$  ([15]). Write down the defining matrix for the associated quasipolynomial algebra. This may be taken in the form

$$(11.2) \quad \begin{pmatrix} M & C \\ -C^t & 0 \end{pmatrix},$$

where  $M$  is the defining matrix of  $M_q(n)$ . But it is easy to see that  $C$  can be used to remove the first  $n$  rows and columns of this matrix together with rows and columns  $i \cdot n + 1$  for  $i = 1, \dots, n-1$ . This is done at the expense of  $2n-1$  blocks  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . What remains is exactly the defining matrix  $M_q(n-1)$ . The result follows immediately from this.  $\square$

Recall that the usual coproduct on  $M_q(n)$  is given as

$$(11.3) \quad \Delta(Z_{i,j}) = \sum_{\alpha} Z_{i,\alpha} \otimes Z_{\alpha,j}.$$

Though we know from experiments that it is not possible to define coproducts on all modified algebras  $M_q^{\varphi}(n)$ , it is interesting that it is possible to define a structure of bialgebra (in fact, several, due to a certain ambiguity) on  $\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q(n)$ :

**Lemma 11.3.** *Define  $\Delta(Z_{i,j})$  as in (11.3) and set*

$$(11.4) \quad \begin{aligned} \Delta(L_{\mu_i}) &= L_{\mu_i} \otimes 1, \\ \Delta(L_{\nu_j}) &= 1 \otimes L_{\nu_j}, \\ \Delta(L_{\beta}) &= L_{\beta} \otimes 1, \\ \varepsilon(Z_{i,j}) &= \delta_{i,j} \quad \text{and} \quad \forall i = 1, \dots, 2n-1 : \varepsilon(L_i) = 1. \end{aligned}$$

*Then this is a bialgebra structure on  $\mathbb{C}[L_1^{\pm 1}, \dots, L_{2n-1}^{\pm 1}] \times_s M_q(n)$ .*

*Proof.* This follows easily from the way  $M_q(n)$  is constructed, c.f. (2.13).  $\square$

**Remark 11.4.** *More generally, one may set  $\Delta(L_{\beta}) = L_{\beta}^a \otimes L_{\beta}^b$  for any pair  $a, b$  of integers with  $a + b = 1$ .*

## 12. RANK R

In this section we shall consider the subsets of lower rank matrices. To begin with we consider the standard quantum matrix algebra  $M_q(n)$  and  $M(n, \mathbb{C})$  with the standard Poisson structure. As usual,  $q$  is a primitive  $m$ th root of unity.

**Proposition 12.1.** *In  $M_q(n)$ ,*

$$(12.1) \quad (\det_q(\{Z_{i,j}\}))^m = \det(\{Z_{i,j}^m\}).$$

*Proof.* By the quantum Laplace expansion, the quantum determinant is a sum of  $q$ -commuting terms (c.f. [15]). The claim then follows easily by the quantum binomial formula.  $\square$

**Corollary 12.2.**

$$(12.2) \quad \forall s, t = 1, \dots, n \quad \{Z_{s,t}, \det(\{Z_{i,j}\})\} = 0.$$

*Proof.* This follows easily since both  $Z_{i,j}^m$  and  $\det_q$  are central elements (c.f. [15]).  $\square$

**Lemma 12.3.**

$$(12.3) \quad \{Z_{i,j}, A_j^i\} = 2\left(\sum_{s<i} (-1)^{i-s} Z_{s,j} A_j^s - \sum_{j<t} (-1)^{t-j} Z_{i,t} A_t^i\right).$$

*Proof.* We use induction on  $n$ . The formula is true for  $n = 2$ . By Laplace expansion we have

$$(12.4) \quad A_j^i = \sum_{s=1}^{i-1} (-1)^{s-1} Z_{s,1} A_{j,1}^{i,s} - \sum_{s=i+1}^n (-1)^{s-1} Z_{s,1} A_{j,1}^{i,t}.$$

Hence

$$(12.5) \quad \begin{aligned} \{Z_{i,j}, A_j^i\} = & \\ & \sum_{s=1}^{i-1} (-1)^{s-1} Z_{s,1} \{Z_{i,j}, A_{j,1}^{i,s}\} - 2 \sum_{s=1}^{i-1} (-1)^{s-1} Z_{s,j} Z_{i,1} A_{j,1}^{i,s} - \sum_{s=i+1}^n (-1)^{s-1} Z_{s,1} \{Z_{i,j}, A_{j,1}^{i,s}\}. \end{aligned}$$

Applying the inductive hypothesis to  $A_{j,1}^s$ , and changing the enumeration appropriately, we get

$$(12.6) \quad \{Z_{i,j}, A_{j,1}^{i,s}\} = -2 \sum_{r=1}^{s-1} (-1)^{i-r} Z_{r,j} A_{j,1}^{r,s} + 2 \sum_{r=s+1}^{i-1} (-1)^{i-r} Z_{r,j} A_{j,1}^{r,s} - 2 \sum_{j<l} (-1)^{l-j} Z_{i,l} A_{l,1}^{i,s}$$

for  $s = 1, 2, \dots, i-1$ .

Similarly, we have

$$(12.7) \quad \{Z_{i,j}, A_{j,1}^{i,s}\} = 2\left(\sum_{r<i} (-1)^{i-r} Z_{r,j} A_{j,1}^{r,s} - \sum_{j<l} (-1)^{l-j} Z_{i,l} A_{l,1}^{i,s}\right)$$

for  $s = i+1, \dots, n$ .

Now we get

$$(12.8) \quad \{Z_{i,j}, A_j^i\} =$$

$$-2 \sum_{s=1}^{i-1} Z_{s,1} \sum_{r=1}^{s-1} (-1)^{i-r} Z_{r,j} A_{j,1}^{r,s} + 2 \sum_{s=1}^{i-1} Z_{s,1} \sum_{r=s+1}^{i-1} (-1)^{i-r} Z_{r,j} A_{j,1}^{r,s}$$

$$-2 \sum_{s=1}^{i-1} Z_{s,1} \sum_{j < l} (-1)^{l-j} Z_{i,l} A_{l,1}^{i,s} - 2 \sum_{r=1}^{i-1} (-1)^{r-1} Z_{r,j} Z_{i,1} A_{j,1}^{i,r}$$

$$-2 \sum_{s=i+1}^n (-1)^{s-1} Z_{s,1} \sum_{r < i} (-1)^{i-r} Z_{r,j} A_{j,1}^{r,s} + 2 \sum_{s=i+1}^n (-1)^{s-1} Z_{s,1} \sum_{j < l} (-1)^{l-j} Z_{i,l} A_{l,1}^{i,s}.$$

The assertion now follows by considering the Laplace expansion of  $A_j^r$  along the first column for  $r < i$  and of  $A_i^j$  along the first column for  $j < l$ .  $\square$

By the same method it follows that

**Lemma 12.4.**

$$(12.9) \quad \{Z_{i,j}, A_j^n\} = \sum_{k < j} (-1)^{j-k} Z_{i,k} A_k^n - \sum_{s > j} (-1)^{j-s} Z_{i,s} A_s^n.$$

**Proposition 12.5.** *In the space of all polynomials on  $M(n, \mathbb{C})$ , the ideal generated by all  $r \times r$  minors is invariant under Hamiltonian flow.*

*Proof.* By, if necessary, deleting and/or renaming columns and rows, this follows from Corollary 12.2, Lemma 12.3, and Lemma 12.4.  $\square$

**Corollary 12.6.** *The space of matrices of rank  $r$  is preserved by Hamiltonian flow in the standard Poisson structure.*

*Proof.* We know by Proposition 12.5 that the space of matrices of rank  $\leq r$  is invariant. But clearly, the rank cannot decrease along a Hamiltonian flow since by reversing time it would then be possible to increase rank.  $\square$

**Proposition 12.7.**

$$(12.10) \quad \Delta(Z_{i,j}^m) = \sum_{\alpha=1}^n Z_{i,\alpha}^m \otimes Z_{\alpha,j}^m.$$

*Proof.* Similar to the proof of Proposition 12.1.  $\square$

The following is important because all tensor categories are important.

**Corollary 12.8.** *The space of matrices of rank less than or equal to  $r$  form a tensor category. Indeed, if in two representations  $\pi_1, \pi_2$ ,  $\{Z_{i,j}^m\}$  is represented by matrices  $A$  and  $B$ , respectively, then it is represented by  $A \cdot B$  in the tensor product.*

**Remark 12.9.** *Special cases of the above is when  $A^2 = a \cdot A$  for some  $r$ th root  $a$  of 1 (e.g.  $a = 1$ ).*

Turning, finally, to the other Poisson structures on  $M(n, \mathbb{C})$  defined by our modifications, we recall that according to Proposition 9.3, the modified vector fields differ from the original ones by left and/or right multiplication operators. Hence

**Corollary 12.10.** *The space of matrices of rank  $\leq r$  is preserved by Hamiltonian flow for a modified Poisson structure.*

As for tensor categories, we do not have as precise results for the modified algebras, but observe that it is possible to start with two irreducible modules  $I_1, I_2$  of a modified algebra  $M_q^\varphi(n)$ . These may then be induced to the semi-direct product, and the tensor product may be formed of the induced representations according to Lemma 11.3. Finally, the result may be decomposed into irreducible  $M_q^\varphi(n)$  modules.

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