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A CONSTRUCTION OF TOPOLOGICAL QUANTUM FIELD THEORIES FROM 6J-SYMBOLS*

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We outline a construction of topological quantum field theories based on generalised $6j$-symbols associated with the representation theory of algebras satisfying certain properties and argue that deformations of the enveloping algebras of classical simple Lie algebras satisfy these properties when the value of the deformation parameter equals a primitive root of -1.

1. Introduction

We shall in this contribution describe a construction of a class of three-dimensional topological quantum field theories starting from data given by the representation theory of certain algebraic structures. This work is inspired by the discrete approach to three-dimensional quantum gravity followed by Ponzano and Regge [1] (see also [2]) using angular momentum theory, and by the recent work by Turaev and Viro [3], who carried out a construction of topological quantum field theories based on the representation theory of $sl(2)_q$, with q a simple root of unity. Subsequently, Ocneanu [4] carried out a similar construction based on data from the theory of embeddings of subfactors of von Neumann algebras.

Our purpose will be to provide a rather general framework for constructing euclidean three-dimensional topological quantum field theories from so-called generalized $6j$-symbols associated with the representation theory of a certain class of associative algebras. In particular we argue that quantum groups corresponding to the classical simple Lie algebras belong to this class. The construction is implemented by exact discretizations of the appropriate functional integrals, in the sense that the manifolds are being triangulated but the resulting quantities are shown to be independent of the triangulations. Since a triangulation of a manifold can be viewed as equipping it with a discrete Riemannian structure ([5-7]), this may be rephrased by saying that the functional integrals do not depend on the Riemannian metric, which is a characteristic of the topological nature of a quantum field theory [8]. Thus the additional complication of integrating over metrics, which is inherent in theories of quantum gravity, is avoided in these models, making them more tractable for exact treatment.

2. Topological quantum field theories

Before embarking on the construction let us briefly recall the key properties of a three-dimensional topological quantum field theory [8,9].

In the following three-manifolds as well as surfaces will be assumed to be smooth oriented and compact. With any three-manifold M with

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boundary components $\Sigma_1, \dots, \Sigma_n$ is associated a finite dimensional complex Hilbert space

$$V_{\partial M} = V_{\Sigma_1} \otimes \cdots \otimes V_{\Sigma_n}, \qquad (1)$$

where ∂M denotes the oriented boundary of M, and V_Σ denotes a finite dimensional complex Hilbert space with inner product $\langle \cdot, \cdot \rangle_\Sigma$ associated with any connected surface Σ, such that

$$V_{\Sigma^*} = V_\Sigma^*, \qquad (2)$$

where Σ^* denotes the surface Σ with opposite orientation and V_Σ^* denotes the dual space to V_Σ.

Moreover, if Σ and Σ' are surfaces, an isomorphism

$$U(f) : V_\Sigma \to V_{\Sigma'} \qquad (3)$$

is associated to any orientation preserving diffeomorphism $f : \Sigma \to \Sigma'$, in such a way that

$$U(f_1 \circ f_2) = U(f_1)U(f_2) \qquad (4)$$

for any pair of such diffeomorphisms $f_2 : \Sigma \to \Sigma'$ and $f_1 : \Sigma' \to \Sigma''$.

Finally, M determines a vector

$$Z(M) \in V_{\partial M}, \qquad (5)$$

such that the following conditions hold:

i) If $F : M \to M'$ is an orientation preserving diffeomorphism between manifolds M and M' and we set $f = F|_{\partial M}$, then

$$Z(M') = U(f)Z(M). \qquad (6)$$

ii) For any manifold M we have

$$Z(M^*) = Z(M)^*, \qquad (7)$$

where M^* denotes M with opposite orientation and where $v \to v^*$ denotes the canonical conjugate linear isomorphism between $V_{\partial M}$ and $V_{\partial M}^*$.

iii) If M_1 and M_2 denote two manifolds such that there is a (not necessarily connected) surface Σ such that $\partial M_1 = \Sigma_1 \cup \Sigma$ and $\partial M_2 = \Sigma_2 \cup \Sigma^*$ and if $M_1 \amalg_\Sigma M_2$ denotes the manifold obtained by gluing M_1 to M_2 along Σ, then

$$Z(M_1 \amalg_\Sigma M_2) = (Z(M_1), Z(M_2))_\Sigma, \qquad (8)$$

where the right hand side denotes the vector in $V_{\Sigma_1} \otimes V_{\Sigma_2}$ obtained by contracting $Z(M_1)$ and $Z(M_2)$ with respect to V_Σ, i.e. if

$$Z(M_1) = \sum_i f_i^1 \otimes g_i \qquad (9)$$

$$, \quad Z(M_2) = \sum_j f_j^2 \otimes g_j^*, \qquad (10)$$

where $f_i^1 \in V_{\Sigma_1}$, $f_j^2 \in V_{\Sigma_2}$, $g_i \in V_\Sigma$ and $g_j^* \in V_{\Sigma^*}$ then

$$(Z(M_1), Z(M_2))_\Sigma = \sum_{i,j} (g_i, g_j^*)_\Sigma f_i^1 \otimes f_j^2, \qquad (11)$$

where $(\cdot, \cdot)_\Sigma$ denotes the bilinear form yielding the duality between V_Σ and V_{Σ^*}.

iv) If Σ is a surface, then

$$Z(\Sigma \times [0,1]) = 1_{V_\Sigma}, \qquad (12)$$

as an element in $V_\Sigma \otimes V_\Sigma^* = \mathrm{End}(V_\Sigma)$, and, furthermore,

$$V_\phi = \mathbb{C}. \qquad (13)$$

For further discussion of these axioms and some variations thereof as well as some consequences we refer to [9,10]. In the next section, we shall describe a rather general framework in which they may be implemented.

3. Intertwiner spaces and 6j-symbols

The building blocks of our construction will be the so-called 6j-symbols associated with the representation theory of certain associative $*$-algebras, which we now introduce.

Let $\mathfrak{A}$ be an associative algebra over the complex numbers with an antilinear involution $a \to$

a^* and assume we are given a finite set I of irreducible $*$-representations of $\mathfrak{A}$ on Hilbert spaces H_i, $i \in I$, as well as a notion of tensor product of representations, which we denote by $i \underline{\otimes} j$, and which is a $*$-representation of $\mathfrak{A}$ on a Hilbert space $H_{i\underline{\otimes} j}$ satisfying the following properties:

1) Associativity: For $i,j,k \in I : (i\underline{\otimes}j)\underline{\otimes}k = i\underline{\otimes}(j\underline{\otimes}k)$;

2) Reducibility: For $i,j \in I$ there exists a finite decomposition

$$i\underline{\otimes}j = \bigoplus_{s=1}^{n_{ij}} i_s \tag{14}$$

and correspondingly

$$H_{i\underline{\otimes}j} = \bigoplus_{k\in I} V_{ij}^k \otimes H_k\,, \tag{15}$$

where V_{ij}^k can be identified with the space of intertwiners between $i\underline{\otimes}j$ and k, i.e. mappings $\alpha : H_{i\underline{\otimes}j} \to H_k$ such that $\alpha \circ (i\underline{\otimes}j)(a) = k(a) \circ \alpha$ for all $a \in \mathfrak{A}$. In particular the multiplicity of k in $i\underline{\otimes}j$ is

$$N_{ij}^k = \dim V_{ij}^k\,. \tag{16}$$

Note that V_{ij}^k is a Hilbert space with inner product $\langle\alpha,\beta\rangle$ given by

$$\langle\alpha,\beta\rangle 1_k = \alpha \circ \beta^* \tag{17}$$

which makes sense, since k is irreducible, and $\alpha\circ\beta^*$ is an operator in H_k commuting with $k(\mathfrak{A})$ and hence proportional to the identity 1_k on H_k.

3) Existence of "trivial" and "dual" representation: There exists a distinguished representation $0 \in I$ such that
(i) $0\underline{\otimes}i = i\underline{\otimes}0 = i$ for $i \in I$;
(ii) For each $i \in I$ there exists a unique $i^\vee \in I$, called the dual of i, such that $N_{ii^\vee}^0 = 1$, and $N_{ij}^0 = 0$ for $j \neq i^\vee$;
(iii) $(i^\vee)^\vee = i$, i.e. $N_{i^\vee i}^0 = 1$ and $N_{i^\vee j}^0 = 0$ for $j \neq i$.

Remark 3.1 *The standard tensor product $\pi\otimes\rho$ of two representations π and ρ of an associative algebra $\mathfrak{A}$ yields a representation of $\mathfrak{A}\otimes\mathfrak{A}$ and not of $\mathfrak{A}$. If $\mathfrak{A}$ is a bialgebra with co-multiplication $\Delta : \mathfrak{A} \to \mathfrak{A}\otimes\mathfrak{A}$ one may define a tensor product representation of $\mathfrak{A}$ by*

$$\pi\underline{\otimes}\rho = (\pi\otimes\rho)\circ\Delta\,, \tag{18}$$

and it is this type of tensor product one should have in mind in the following. Moreover, in interesting cases, as eg. for quantum groups at roots of unity discussed in section 5, the space $\mathcal{H}_{i\underline{\otimes}j}$ does not equal $\mathcal{H}_i\otimes\mathcal{H}_j$, but rather a quotient of it.

For the following we also have to assume that tensor products of intertwiners u and v between representations of $\mathfrak{A}$ make sense as intertwiners:

4): Given representations π, π', ρ, ρ' of $\mathfrak{A}$, which are decomposable into representations in I and intertwiners $u : H_\pi \to H_{\pi'}$, $v : H_\rho \to H_{\rho'}$, there is a canonical intertwiner $u\underline{\otimes}v : \mathcal{H}_{\pi\underline{\otimes}\rho} \to \mathcal{H}_{\pi'\underline{\otimes}\rho'}$ between $\pi\underline{\otimes}\rho$ and $\pi'\underline{\otimes}\rho'$ depending bilinearly on u and v and fulfilling:

$$(u'\underline{\otimes}v')(u\underline{\otimes}v) = (u'u)\underline{\otimes}(v'v) \tag{19}$$

$$1_\pi\underline{\otimes}1_\rho = 1_{\pi\underline{\otimes}\rho} \tag{20}$$

$$(u\underline{\otimes}v)\underline{\otimes}w = u\underline{\otimes}(v\underline{\otimes}w) \tag{21}$$

$$(u\underline{\otimes}v)^* = u^*\underline{\otimes}v^* \tag{22}$$

for any intertwiners u, v, u', v', w between appropriate representations.

Remark 3.2 *In the following we shall fix, once and for all, intertwiners*

$$\psi_k \in V_{kk^\vee}^0\,, \qquad \varphi_i \in V_{0i}^i \quad \text{and} \quad {}_i\varphi \in V_{i0}^i \tag{23}$$

which are partial isometries and unique up to phases. In fact φ_i and ${}_i\varphi$ are unitaries according to 3 i). And we assume this can be done such that *i)* $1_i \underline{\otimes} \varphi_i =_i \varphi \underline{\otimes} 1_j$;

ii) $\varphi_k \circ (1_0 \underline{\otimes} \alpha)\alpha \circ (\varphi_i \underline{\otimes} 1_j)$;

iii) ${}_k\varphi \otimes (\alpha \underline{\otimes} 1_0)\alpha \circ (1_i \underline{\otimes}_j \varphi)$

for all $\alpha \in V_{ij}^k$.

The assumptions made so far seem quite natural once it is assumed that the representations in I form a closed system under tensor products. We shall need two more purely technical assumptions, which will be explained below.

Using the intertwiner spaces V_{ij}^k we may now associate a vector space $V_{(\Sigma,\mathcal{S})}$ with any surface Σ equipped with a triangulation $\mathcal{S}$ as follows.

First, consider an oriented triangle whose links are decorated by arrows and by indices $i, j, k \in I$ in cyclic order. If all three arrows point in positive direction we associate with this labeled triangle the vector space

$$V_t = V_{ij}^{k^\vee}, \tag{24}$$

whereas if some of the arrows point in negative direction we replace in this definition the corresponding labels by their dual ones. Of course this definition only makes sense if $V_{ij}^{k^\vee}$, $V_{ki}^{j^\vee}$ and $V_{jk}^{i^\vee}$ can be canonically identified. There is indeed an obvious linear mapping $\alpha \to \tilde{\alpha}$ from $V_{ij}^{k^\vee}$ to $V_{ki}^{j^\vee}$ given by

$$\psi_k \circ (1_k \underline{\otimes} \alpha) = \psi_{j^\vee} \circ (\tilde{\alpha} \underline{\otimes} 1_j) \tag{25}$$

as intertwiners from $H_{k \underline{\otimes} i \underline{\otimes} j}$ to H_0, which is easily shown to be an isomorphism. However, in order to use this isomorphism to identify the three spaces it is necessary (and sufficient) that

$$\tilde{\tilde{\tilde{\alpha}}} = \alpha \tag{26}$$

for any $\alpha \in V_{ij}^{k^\vee}$. This identity does not follow from the assumptions above alone, so we shall in the following assume that it holds. It is possible to rephrase this assumption in terms of the equality of certain compositions of intertwiners (see [11]) but we shall not do so here.

Having defined the spaces V_t associated with labeled triangles t, we now decorate each link in $\mathcal{S}$ by an arrow in an arbitrary way. By labeling each link ℓ in $\mathcal{S}$ by a label $i_\ell \in I$, the triangles in $\mathcal{S}$ become labeled and we may define

$$V_{(\Sigma,\mathcal{S})} = \bigoplus_{(i_\ell)_{\ell \in \mathcal{S}}} \bigotimes_{t \in \mathcal{S}} V_t \tag{27}$$

where the direct sum is over all labelings $(i_\ell)_{\ell \in \mathcal{S}}$ of the links in $\mathcal{S}$ (but the configuration of arrows is fixed) and the tensor product is over all triangles in $\mathcal{S}$. Note that according to the definition of V_t the space $V_{(\Sigma,\mathcal{S})}$ is independent of the chosen fixed configuration of arrows.

Next, consider the triangulated surface $(\Sigma^*, \mathcal{S}^*)$, i.e. $(\Sigma, \mathcal{S})$ with opposite orientation, and with the same decoration of arrows. For each oriented triangle t in $\mathcal{S}$ the oppositely oriented triangle t^* occurs in $\mathcal{S}^*$, and for each labeling $(i_\ell)_{\ell \in \mathcal{S}}$ it is easily seen that if

$$V_t = V_{ij}^{k^\vee} \tag{28}$$

then

$$V_{t^*} = V_{i^\vee k^\vee}^{j}. \tag{29}$$

The bilinear form $(\ ,\ )_{ij}^{k^\vee} : V_{ij}^{k^\vee} \times V_{i^\vee k^\vee}^{j} \to \mathbb{C}$ given by

$$(\alpha, \beta)_{ij}^{k^\vee} 1_j = \beta \circ (1_{i^\vee} \underline{\otimes} \alpha) \circ (\psi_{i^\vee}^* \underline{\otimes} 1_j) \circ \varphi_j^* \tag{30}$$

can be shown to be non-degenerate and symmetric, in the sense that $(\alpha, \beta)_{ij}^{k^\vee} = (\beta, \alpha)_{i^\vee k^\vee}^{j}$, as a consequence of our assumptions, and this yields a duality between $V_{ij}^{k^\vee}$ and $V_{i^\vee k^\vee}^{j}$. Moreover, it can be shown that this duality is consistent with the isomorphisms $\alpha \to \tilde{\alpha}$. This yields a duality between V_t and V_{t^*}. The corresponding bilinear form will be denoted by $(\cdot, \cdot)_t$.

Thus, as a consequence of the definition of $V_{(\Sigma,S)}$ the identification

$$V^*_{(\Sigma,S)} = V_{(\Sigma^*,S)} \tag{31}$$

follows from the duality implied by the non-degenerate, symmetric bilinear form $(\cdot,\cdot)_{(\Sigma,S)}$: $V_{(\Sigma,S)} \times V_{(\Sigma^*,S^*)} \to \mathbb{C}$ given by

$$\left(\bigotimes_{\ell\in S}\alpha_\ell, \bigotimes_{\ell\in S}\beta_{\ell^*}\right)_{(\Sigma,S)} = \left(\prod_{\ell\in S} F_{i_\ell}^{-1}\right)\prod_{\ell\in S}(\alpha_\ell,\beta_{\ell^*})_\ell, \tag{32}$$

where $\alpha_\ell \in V_\ell$, $\beta_{\ell^*} \in V_{\ell^*}$ for any labeling $(i_\ell)_{\ell\in S}$ of the links in S, and where the non-zero factor $\prod_{\ell\in S} F_{i_\ell}$ has been inserted for later convenience. The numbers F_i , $i \in I$, will be defined below.

Next we introduce the $6j$-symbols. For each set of six indices $i,j,k,\ell,p,q \in I$ the corresponding $6j$-symbol is the linear map

$$F_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} : V^i_{jp}\otimes V^p_{k\ell} \to V^i_{q\ell}\otimes V^q_{jk} \tag{33}$$

defined by

$$\left\langle F_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}\alpha\otimes\beta\,,\ \gamma\otimes\delta\right\rangle 1_i = \alpha\circ(1_j\underline{\otimes}\beta)(\delta^*\underline{\otimes}1_\ell)\circ\gamma^*, \tag{34}$$

which makes sense since the right-hand side is an operator in $\mathcal{H}_i$ commuting with $i(\mathfrak{A})$, and is linear in $\alpha \in V^i_{jp}$, $\beta \in V^p_{k\ell}$ and conjugate linear in $\gamma \in V^i_{q\ell}$, $\delta \in V^q_{jk}$.

We thus have

$$\begin{aligned} F_{pq}\begin{bmatrix} jk \\ i\ell \end{bmatrix} &\in \mathrm{Hom}\left(V^i_{jp}\otimes V^p_{k\ell}\,,\ V^i_{q\ell}\otimes V^q_{jk}\right) \\ &= V^i_{q\ell}\otimes V^q_{jk}\otimes\left(V^i_{jp}\right)^*\otimes\left(V^p_{k\ell}\right)^* \\ &= V^i_{q\ell}\otimes V^q_{jk}\otimes V^p_{j^\vee i}\otimes V^\ell_{k^\vee p} \subseteq V_{(S^2,\partial T_0)} \end{aligned} \tag{35}$$

where $V_{(S^2,\partial T_0)}$ is the vector space associated with the triangulated boundary $(S_2,\partial T_0)$ of the

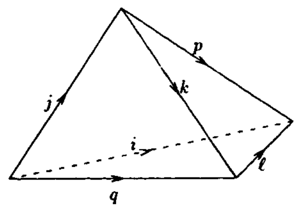

Fig. 1.

labeled tetrahedron T_0 indicated in Fig. 1, and S^2 denotes the two-sphere.

Since the symmetry group of the oriented tetrahedron contains twelve elements we may in this way construct twelve vectors in $V_{(S^2,\partial T_0)}$. However, it turns out that after suitable normalization they are in fact equal, as formulated in the following theorem.

Theorem 3.1 *If, for $i \in I$, we set*

$$F_i = \left\langle F_{00}\begin{bmatrix} i & i^\vee \\ i & i \end{bmatrix}{}_i\phi\otimes\psi_{i^\vee}\,,\ \phi_i\otimes\psi_i\right\rangle \tag{36}$$

and assume that F_i is non-zero and real, then the vector

$$W(T_0) \equiv F_pF_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} \in V_{(S^2,\partial T_0)}\,, \tag{37}$$

is invariant under the action of the tetrahedral symmetry group and hence defines a unique vector in $V_{(S^2,\partial T_0)}$.

In order to prove this theorem one has to show that any vector that can be obtained from $F_pF_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}$ by a replacement of indices corresponding to applying a tetrahedral symmetry to T_0 (with the configuration of arrows fixed and still using the convention that reversing an arrow is equivalent to replacing the corresponding label i by $i^\vee$) is related to $F_pF_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} \in$

$V^i_{q\ell}\otimes V^q_{jk}\otimes V^p_{j^\vee i}\otimes V^\ell_{k^\vee p}$ by a suitable tensor product of the isomorphisms $\alpha \to \tilde{\alpha}$ and their inverses.

Since the tetrahedral symmetry group is generated by two of its elements it is seen to be sufficient to prove that

$$F_p F_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} = F_{p^\vee} F_{p^\vee q^\vee}\begin{bmatrix} \ell & i^\vee \\ k^\vee & j \end{bmatrix} \quad (38)$$

and

$$F_p F_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} = F_i F_{ik^\vee}\begin{bmatrix} q^\vee & j \\ \ell & p \end{bmatrix} \quad (39)$$

up to compositions with isomorphisms $\alpha \to \tilde{\alpha}$, which can be read off from the formulae by considering the inital and final spaces of the left hand sides as compared to the right hand sides.

The first of these identities can be obtained by applying twice the identity

$$\begin{gathered}\left\langle F_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}(\alpha\otimes\beta),\, \gamma\otimes\delta\right\rangle = \\ \left\langle \tilde{\beta}\otimes\tilde{\alpha},\, F_{qp^\vee}\begin{bmatrix} i^\vee & j \\ \ell^\vee & k \end{bmatrix}\tilde{\gamma}\otimes\delta\right\rangle,\end{gathered} \quad (40)$$

which on the other hand is obtained by a rather straightforward calculation using the definitions of $F_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}$ and the isomorphism $\alpha \to \tilde{\alpha}$ (see ref. [11]).

The second identity is obtained (see [11]) as a special case of the so-called pentagon identity, which reads

$$\begin{aligned}\sum_{q\in I} F^{(23)}_{qm}\begin{bmatrix} n & j \\ r & k \end{bmatrix} F^{(12)}_{ir}\begin{bmatrix} n & q \\ u & \ell \end{bmatrix} F^{(23)}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} \\ = P_{23} F^{(13)}_{pr}\begin{bmatrix} m & k \\ u & \ell \end{bmatrix} F^{(12)}_{im}\begin{bmatrix} n & j \\ u & p \end{bmatrix},\end{aligned}$$

as operators from $V^u_{ni}\otimes V^i_{jp}\otimes V^p_{k\ell}$ to $V^u_{r\ell}\otimes V^r_{mk}\otimes V^m_{nj}$, where the upper pair of indices in paranthesis indicate on which factors in the tensor product the $6j$-symbol acts and P_{23} denotes permutation of the second and third factor. This identity is a well-known consequence of associativity of the tensor product $\otimes$ (see eg. [11,12]).

Finally, we remark that the condition

$$F_i \in \mathbb{R}\setminus\{0\} \quad (41)$$

stated in the theorem is the second and last technical assumption mentioned above. It is equivalent to assuming that

$$F_i \neq 0 \quad \text{and} \quad F_{i^\vee} = F_i \quad \text{for} \quad i \in I. \quad (42)$$

4. The construction

We have now defined the basic building blocks, and may now describe the actual construction of topological QFT's based on these.

Let M be a smooth orientable compact manifold with boundary components $\Sigma_1,\ldots,\Sigma_n$ and let $\mathcal{T}$ be a triangulation of M. The induced triangulations of $\Sigma_1,\ldots,\Sigma_n$ are denoted $\mathcal{S}_1,\ldots,\mathcal{S}_n$. With the triangulated manifold $(M,\mathcal{T})$ we associate the Hilbert space

$$V_{(\partial M,\partial\mathcal{T})} = \bigotimes_{i=1}^{n} V_{(\Sigma_i,\mathcal{S}_i)}\,. \quad (43)$$

In order to define an appropriate vector $Z(M,\mathcal{T}) \in V_{(\partial M,\partial\mathcal{T})}$ we distribute in an arbitrary way arrows on the links in $\mathcal{T}$ and attach to each link $\ell \in \mathcal{T}$ a lable $i_\ell \in I$. Thus each tetrahedron $T \in \mathcal{T}$ becomes labeled. Setting

$$F \equiv \sum_{i\in I}\frac{1}{F_i^2} > 0 \quad (44)$$

we may thus define

$$\begin{aligned}Z(M,\mathcal{T}) = F^{-(|\mathcal{T}|-\frac{1}{2}|\partial\mathcal{T}|)}\,. \\ \sum_{(i_\ell)_{\ell\in\mathcal{T}}}\prod_{\ell\in\mathcal{T}\setminus\partial\mathcal{T}} F_{i_\ell}^{-1}\left(\bigotimes_{T\in\mathcal{T}} W(T)\right)_{\mathrm{int}\,\mathcal{T}} \\ \in V_{(\partial M,\partial\mathcal{T})}\end{aligned} \quad (45)$$

where $|\mathcal{T}|$ and $|\partial\mathcal{T}|$ denote the number of vertices in $\mathcal{T}$ and $\partial\mathcal{T}$ respectively, the summation is over all possible labelings of links in $\mathcal{T}$, with the configuration of arrows fixed, and the tensor product is over all tetrahedra T in $\mathcal{T}$ and $(\cdot)_{\text{int}\,\mathcal{T}}$ indicates that the tensor $(\otimes_{T\in\mathcal{T}} W(T))$ is contracted with respect to all interior triangles in $\mathcal{T}$ using the bilinear form $(\cdot,\cdot)_t$ for each labeled triangle t.

Let us note that if $(M_1,\mathcal{T}_1)$ and $(M_2,\mathcal{T}_2)$ are two triangulated three-manifolds such that

$$(\partial M_1,\partial\mathcal{T}_1) = (\Sigma_1,\mathcal{S}_1) \cup (\Sigma^*,\mathcal{S}^*) \tag{46}$$

and

$$(\partial M_2,\partial\mathcal{T}_2) = (\Sigma_2,\mathcal{S}_2) \cup (\Sigma,\mathcal{S}) \tag{47}$$

and if $(M_1 \amalg_\Sigma M_2, \mathcal{T}_1 \amalg_{\mathcal{S}} \mathcal{T}_2)$ denotes the triangulated manifold obtained by gluing $(M_1,\mathcal{T}_1)$ and $(M_2,\mathcal{T}_2)$ along $(\Sigma,\mathcal{S})$, then

$$Z(M_1 \amalg_\Sigma M_2, \mathcal{T}_1 \amalg_{\mathcal{S}} \mathcal{T}_2) = (Z(M_1,\mathcal{T}_1), Z(M_2,\mathcal{T}_2))_{(\Sigma,\mathcal{S})} \tag{48}$$

where $(\cdot,\cdot)_{(\Sigma,\mathcal{S})}$ as usual denotes contraction with respect to the bilinear form $(\cdot,\cdot)_{(\Sigma,\mathcal{S})}$ defined by (3.5). Note that the factors of F and of F_{i_ℓ} in the definition of $Z(M,\mathcal{T})$ and of the bilinear form $(\cdot,\cdot)_{(\Sigma,\mathcal{S})}$ have been arranged such that (48) holds.

Our purpose now will be to show that $Z(M,\mathcal{T})$ is independent of $\mathcal{T}$ (clearly it is independent of the chosen configuration of arrows). This can be done in two steps. First we show that $Z(M,\mathcal{T})$ does not depend on the interior of $\mathcal{T}$ and second we show that it is possible to replace the spaces $V_{(\partial M,\partial\mathcal{T})}$ by certain subspaces which can be canonically identified for different choices of $\partial\mathcal{T}$ defining a space $V_{\partial M}$ and such that the vectors $Z(M,\partial\mathcal{T})$ are identified, thus defining a vector $Z(M) \in V_{\partial M}$.

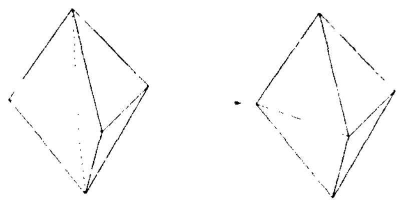

Fig. 2. Replacement of three tetrahedra glued along three triangles by two tetrahedra glued along one triangle.

Fig. 3. Collaps of two tetrahedra glued along two triangles to two triangles.

In order to accomplish the first step we note that any triangulation of M can be obtained from any other triangulation which is identical to the first one on ∂M by application of a sequence of deformations of the three types indicated on Fig. 2—4 and their inverses (see [3]).

Thus it is sufficient to prove that $Z(M,\mathcal{T})$ is invariant under these deformations.

The invariance under the first type of deformation is an easy consequence of the pentagon identity (41), whereas the two others follow from eq(40) combined with the identity

$$\sum_{m\in I} F_{qm}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}^* F_{pm}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} = \delta_{pq}\, 1_{V_{jp}^i \otimes V_{k\ell}^p}, \tag{49}$$

which in turn is an easy consequence of the complete reducibility assumption 2) in section

Fig. 4. Collaps of two tetrahedra glued along three triangles to one triangle.

3 (see [11]). In fact, invariance under the second type of deformation follows easily by noting that the composition of the two operators in (49) is equivalent to contracting the corresponding 6j-symbols with respect to two intertwiner spaces corresponding to two triangles, whereas invariance under the third type is obtained by setting $p = q$ and contracting with respect to an additional triangle, and using that

$$F = \sum_{k,\ell \in I} \frac{F_p N^p_{k\ell}}{F_k F_\ell} \tag{50}$$

for any $p \in I$ as shown in [11] (see also [3]). For detailed arguments we refer the reader to [11].

The independence of $Z(M, \mathcal{T})$ on the interior of $\mathcal{T}$ may now be used to accomplish the second step as follows. Let Σ be an arbitrary triangulated surface; let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two triangulations of Σ, and let

$$M_\Sigma = \Sigma \times [0, 1], \tag{51}$$

such that

$$\partial M_\Sigma = \Sigma^* \cup \Sigma, \tag{52}$$

where Σ^* is identified with $\Sigma \times \{0\}$ and Σ is identified with $\Sigma \times \{1\}$. It is then easy to see, using the two-dimensional analogues of the deformations described above, that it is possible to devise a triangulation $\mathcal{T}$ of M_Σ such that it agrees with $\mathcal{S}_1^*$ on Σ^* and with $\mathcal{S}_2$ on Σ. Then

$$Z(M_\Sigma, \mathcal{T}) \in V^*_{(\Sigma,\mathcal{S}_1)} \otimes V_{(\Sigma,\mathcal{S}_2)} \tag{53}$$

is independent of the choice of $\mathcal{T}$, as already shown, and may be considered as an operator

$$h_{\mathcal{S}_2,\mathcal{S}_1}(\Sigma) : V_{(\Sigma,\mathcal{S}_1)} \to V_{(\Sigma,\mathcal{S}_2)}, \tag{54}$$

since

$$V^*_{(\Sigma,\mathcal{S}_1)} \otimes V_{(\Sigma,\mathcal{S}_2)} \simeq \mathrm{Hom}\left(V_{(\Sigma,\mathcal{S}_1)}, V_{(\Sigma,\mathcal{S}_2)}\right). \tag{55}$$

Moreover, if $\mathcal{S}_3$ is a third triangulation of Σ we have

$$h_{\mathcal{S}_3,\mathcal{S}_2}(\Sigma) \circ h_{\mathcal{S}_2,\mathcal{S}_1}(\Sigma) = h_{\mathcal{S}_3,\mathcal{S}_1}(\Sigma). \tag{56}$$

This equation follows easily from (48) and the independence of $Z(M, \mathcal{T})$ on the interior of $\mathcal{T}$.

In particular, it follows from (56) that

$$(h_{\mathcal{S},\mathcal{S}}(\Sigma))^2 = h_{\mathcal{S},\mathcal{S}}(\Sigma), \tag{57}$$

i.e.

$$P_{\mathcal{S}}(\Sigma) = h_{\mathcal{S},\mathcal{S}}(\Sigma) \tag{58}$$

is a projection. Setting

$$V'_{(\Sigma,\mathcal{S})} = P_{\mathcal{S}}\, V_{(\Sigma,\mathcal{S})} \tag{59}$$

it is now easy to see that the restriction of $h_{\mathcal{S}_2,\mathcal{S}_1}(\Sigma)$ to $V'_{(\Sigma,\mathcal{S}_1)}$ maps $V'_{(\Sigma,\mathcal{S}_1)}$ isomorphically onto $V'_{(\Sigma,\mathcal{S}_2)}$ and that these maps allow a consistent identification of the spaces $V'_{(\Sigma,\mathcal{S})}$ for different triangulations $\mathcal{S}$, thus defining the desired space V_Σ. Moreover, it folows that

$$Z(M, \mathcal{T}) \in V'_{\partial M,\partial\mathcal{T}} \tag{60}$$

and that the vectors $Z(M, \mathcal{T})$ are mapped into each other by the maps $h_{\partial\mathcal{T}_2,\partial\mathcal{T}_1}(\partial M)$, thus defining the desired vector

$$Z(M) \in V_{\partial M}. \tag{61}$$

It is also easy to see that the restriction of the bilinear forms $(\cdot,\cdot)_{(\Sigma,\mathcal{S})}$ to $V'_{(\Sigma,\mathcal{S})} \times V'_{(\Sigma^*,\mathcal{S}^*)}$ define a unique non-degenerate, symmetric bilinear form on $V_\Sigma \times V_{\Sigma^*}$, by which we may identify V_{Σ^*} with V_Σ^*.

In addition, the inner products on $V_{(\Sigma,\mathcal{S})}$ define a unique inner product $\langle\cdot,\cdot\rangle_\Sigma$ on V_Σ. This follows from the fact that

$$Z(M^*,\mathcal{T}^*) = Z(M,\mathcal{T})^* \tag{62}$$

which is a consequence of (40). In fact (40) is easily seen to be equivalent to (62) when $(M,\mathcal{T})$ is a single tetrahedron. The general case can then be derived by induction on the size of $\mathcal{T}$ (see ref. [11]). Applying (62) to $(M,\mathcal{T}) = (M_\Sigma,\mathcal{T})$ implies that the restriction of $h_{\mathcal{S}_2,\mathcal{S}_1}(\Sigma)$ to $V'_{(\Sigma,\mathcal{S}_1)}$ is an isometry.

Thus V_Σ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_\Sigma$, and (62) implies that

$$Z(M^*) = Z(M)^* . \tag{63}$$

Finally, given an orientation preserving diffeomorphism $f : \Sigma_1 \to \Sigma_2$ between surfaces and a triangulation $\mathcal{S}$ of Σ, we define an isomorphism $U(f) : V_{\Sigma_1} \to V_{\Sigma_2}$ as follows. Given a triangulation $\mathcal{S}$ of Σ_1, and a configuration of arrows as well as a labeling of the links in $\mathcal{S}$ we obtain in an obvious way a triangulation $f(\mathcal{S})$ of Σ_2 and a corresponding configuration of arrows and labeling of the links in $f(\mathcal{S})$. We set

$$U_{\mathcal{S}}(f)\left(\bigotimes_{t\in\mathcal{S}} \alpha_t\right) := \bigotimes_{t\in f(\mathcal{S})} \alpha_t \in V_{(\Sigma_2,f(\mathcal{S}))} \tag{64}$$

for arbitrary $\alpha_t \in V_t$ for $t \in \mathcal{S}$. Obviously the operators $U_{\mathcal{S}}(f)$ are unitary for any triangulation $\mathcal{S}$ and it is easy to see that they define a unique unitary operator

$$U(f) : V_{\Sigma_1} \to V_{\Sigma_2} \tag{65}$$

fulfilling property i) in section 2.

We have thus indicated the proof of the following main result (see ref. [11]).

Theorem 4.1 *The Hilbert spaces V_Σ, the vectors $Z(M)$ and isomorphisms $U(f)$ constructed above fulfill all the properties i)–iv) in section 2.*

5. The quantum group case

In this section we argue that deformations of the enveloping algebras of the classical finite dimensional Lie algebras satisfy our conditions for suitable values of the deformation parameter q.

For completeness let us write down the defining relations for these algebras in terms of generators, which we shall denote by E_i, F_i, K_i, K_i^{-1}, where $1 \le i \le m$. Denoting by $(a_{ij})_{1\le i,j\le m}$ the Cartan matrix, which we for simplicity assume to be symmetric, for a classical simple Lie algebra $\mathcal{G}$ the relations for the deformed algebra $\mathfrak{A} \equiv U_q\mathcal{G}$ are

$$\begin{aligned} K_iK_j &= K_jK_i \\ K_iK_i^{-1} &= K_i^{-1}K_i = 1 \\ K_iE_jK_i^{-1} &= q^{a_{ij}/2}E_j \\ K_iF_jK_i^{-1} &= q^{-a_{ij}/2}F_j \\ [E_i,F_j] &= \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q-q^{-1}} \end{aligned} \tag{66}$$

$$\begin{aligned} \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_q E_i^{(1-a_{ij}-n)}E_jE_i^n &= 0, \quad i \neq j \\ \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_q F_i^{(1-a_{ij}-n)}F_jF_i^n &= 0, \quad i \neq j \end{aligned} \tag{67}$$

where

$$\begin{bmatrix} r \\ s \end{bmatrix}_q = \frac{[r]_q!}{[s]_q![r-s]_q!} \tag{68}$$

and

$$[r]_q! = \prod_{j=1}^{r} \frac{q^j - q^{-j}}{q - q^{-1}} \tag{69}$$

for q complex and $\neq \pm 1$. In the limit $q \to 1$ one recovers the relations defining $\mathcal{G}$ with $K_i = q^{H_i/2}$.

$\mathfrak{A}$ has a Hopf algebra structure with comultiplication $\Delta : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ given by

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i \\ \Delta(E_i) &= K_i \otimes E_i + E_i \otimes K_i^{-1} \\ \Delta(F_i) &= K_i \otimes F_i + F_i \otimes K_i^{-1} \end{aligned} \tag{70}$$

and with an antipode $S :\to \mathfrak{A} \to \mathfrak{A}$ given by

$$\begin{aligned} S(K_i) &= K_i^{-1} \\ S(E_i) &= -qE_i \\ S(F_i) &= -q^{-1}F_i. \end{aligned} \tag{71}$$

The counit $\varepsilon : \mathfrak{A} \to \mathbb{C}$ is given by

$$\begin{aligned} \varepsilon(K_i) &= 1 \\ \varepsilon(E_i) &= \varepsilon(F_i) = 0. \end{aligned} \tag{72}$$

We also note that for $|q| = 1$ $\mathfrak{A}$ is an associative $*$-algebra with the conjugate linear involution $a \to a^*$ given by

$$\begin{aligned} K_i^* &= K_i^{-1} \\ E_i^* &= F_i \end{aligned} \tag{73}$$

and we have

$$\Delta(a)^* = \Delta'(a^*) \tag{74}$$

for $a \in \mathfrak{A}$, where Δ' denotes the opposite comultiplication obtained from Δ by composing it with the flip P on $\mathfrak{A} \otimes \mathfrak{A}$ given by

$$P(a \otimes b) = b \otimes a\,. \tag{75}$$

There exists an invertible element $R \in \mathfrak{A} \otimes \mathfrak{A}$ such that

$$R\Delta(a) = \Delta'(a)R \quad \text{for} \quad a \in \mathfrak{A}\,. \tag{76}$$

Explicit formulae for R have been obtained in [13]. We note the following properties of R:

$$\begin{aligned} (\Delta \otimes 1)R &= R_{13}R_{12} \\ (1 \otimes \Delta)R &= R_{13}R_{23} \\ R_{12}R_{13}R_{23} &= R_{23}R_{13}R_{12} \end{aligned} \tag{77}$$

as elements in $\mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A}$ with standard notation. Moreover,

$$R^* = PR^{-1}, \tag{78}$$

where the $*$ operator on $\mathfrak{A} \otimes \mathfrak{A}$ acts as $(a \otimes b)^* = a^* \otimes b^*$.

We shall also need the fact that there exists (see [14,15]) a unitary central element $c \in \mathfrak{A}$ such that

$$R_{21}R_{12} \cdot \Delta(c) = c \otimes c \tag{79}$$

in $\mathfrak{A} \otimes \mathfrak{A}$, where $R_{12} \equiv R$ and $R_{21} = P(R)$.

If q is not a root of unity the representation theory for $\mathfrak{A}$ is essentially the same as for $\mathcal{G}$ in the sense that the finite dimensional highest weight representations of $\mathfrak{A}$ may be obtained as deformations of those of $\mathcal{G}$ (see [16,17]). If q is a root of unity the situation is different and it is convenient to distinguish representations according to whether their q-dimension vanishes or not. The q-dimension of a finite dimensional representation π of $\mathfrak{A}$ is defined as

$$\dim_q \pi = \operatorname{tr} \pi(K_{2\rho})\,, \tag{80}$$

where ρ is half the sum of positive roots and

$$K_\beta = \prod_{i=1}^{m} K_{\alpha_i}^{m_i}\,, \tag{81}$$

if $\beta = \sum_{i=1}^{m} m_i \alpha_i$ is an element in the root lattice.

Now let $q = e^{i\pi/\ell}$, where the integer ℓ is bigger than the Coxeter number for $\mathcal{G}$ and let J_0 be the set of dominant weights λ for $\mathcal{G}$ fulfilling

$$\langle \lambda + \rho, \alpha^\vee \rangle < \ell \quad \text{for all positive roots } \alpha\,. \tag{82}$$

The corresponding irreducible highest weight representations of $\mathcal{G}$ may then be deformed into corresponding irreducible inequivalent highest weight representations π_λ of $\mathfrak{A}$ with non-zero q-dimension. In fact, they constitute a maximal set of representations with these properties.

It follows from Shapovalov's determinant formula that these representations are $*$-representations with respect to a positive definite inner product, which is unique up to a positive factor (see [18]). For this to hold the restriction on the values of q is important. For $sl(2)_q$ it is easy to verify directly.

Defining the dual $\pi^\vee$ to a representation π of $\mathfrak{A}$ on H by

$$\pi^\vee(a) = \pi(S(a))^t , \qquad (83)$$

where the index t indicates the ordinary dual operator on the dual space H^*, it is possible to verify that $\pi_\lambda^\vee$ is equivalent to a representation $\pi_{\lambda^\vee}$ where also $\lambda^\vee$ satisfies (82), and that π_λ and $(\pi_\lambda^\vee)^\vee$ are equivalent. The latter follows from the existence of an element $u \in \mathfrak{A}$ such that

$$S^2(a) = uau^{-1} \quad \text{for all} \quad a \in \mathfrak{A} . \qquad (84)$$

We now let I denote a set of representatives for the equivalence classes of the representations π_λ, one for each class, such that if $i \in I$ then there is a representation $i^\vee \in I$ defined either by (83) or by the inverse of that relation, provided i and $i^\vee$ are not equivalent. This also defines the involution $i \to i^\vee$ on I. Moreover, we choose the antipode ε to represent π_0 and denote it by $0 \in I$.

The tensor product $\pi \underline{\otimes} \rho$ of two finite dimensional representations π and ρ on H_π and H_ρ respectively, is defined by

$$\pi \underline{\otimes} \rho(a) = \pi \otimes \rho(\Delta(a)) , \qquad a \in \mathfrak{A} , \quad (85)$$

as operators on $H_\pi \otimes H_\rho$. If we attemt to use this tensor product together with I, we encounter two difficulties in establishing properties i)–iv) and the additional technical assumptions in section 3:

i) $H_\pi \otimes H_\rho$ is not decomposable into modules in I.

ii) Due to (5.1), $i \underline{\otimes} j$ is generally not a $*$-representation with respect to the standard inner product on $H_\pi \otimes H_\rho$.

As to the first point, it has for some time been known for $sl(2)_q$ (see eg. ref. [19]) that $H_i \otimes H_j$ has a decomposition

$$H_i \otimes H_j = Z_{ij} \oplus H_{i \underline{\otimes} j} , \qquad (86)$$

where Z_{ij} is a direct sum of indecomposable modules with vanishing q-dimension and $H_{i \underline{\otimes} j}$ is a direct sum of modules in I, and, furthermore, such that associativity of the tensor product $H_{i \underline{\otimes} j}$ is maintained. It is a non-trivial problem to generalize these results to an arbitrary classical simple Lie algebra $\mathcal{G}$, but results to the effect that they are valid have recently been obtained, see refs. [15,20,21].

As concerns the second point we first note that, as a consequence of eqs(74) and (76), it follows that $i \underline{\otimes} j$, for $i, j \in I$, is a $*$-representation on $H_i \otimes H_j$ with respect to the non-degenerate sesquilinear form

$$\langle x, y \rangle_R = \langle x, i \otimes j(R) y \rangle , \quad x, y \in H_i \otimes H_j , \qquad (87)$$

where $\langle \cdot , \cdot \rangle$ on the right-hand side denotes the standard inner product on $H_i \otimes H_j$ inherited from the inner products on H_i and H_j. Hence, on each irreducible component H_k in $H_i \otimes H_j$ the restriction of $\langle \cdot , \cdot \rangle_R$ is proportional to the inner product on H_k. However, $\langle \cdot , \cdot \rangle_R$ is not symmetric because of eq(78), so the constant of proportionality c_{ij}^k is generally not real. In order to fix it we may exploit eq. (79) as follows. Restricting $\langle \cdot , \cdot \rangle_R$ to

$H_k \subseteq H_i \otimes H_j$ we have

$$c^k_{ij}\langle x, i\otimes j(R)y\rangle = \bar{c}^k_{ij}\langle x, i\otimes j(R^*)(y))\rangle . \tag{88}$$

Assuming, that $\langle\cdot,\cdot\rangle_R$ is non-degenerate (i.e. non-vanishing) on H_k we conclude that

$$c^k_{ij}\, i\otimes j(R) = \bar{c}^k_{ij}\; i\otimes j(R^*) \tag{89}$$

or, using (78),

$$i\otimes j(R_{21}R_{12}) = \frac{\bar{c}^k_{ij}}{c^k_{ij}} . \tag{90}$$

Observing that the unitary central element c acts as multiplication by a phase c_i on H_i, $i \in I$, it now follows from (79) and (90) that

$$\frac{\bar{c}^k_{ij}}{c^k_{ij}} = \frac{c_i c_j}{c_k} \tag{91}$$

and hence we can choose

$$c^k_{ij} = \frac{\sqrt{c_k}}{\sqrt{c_i}\sqrt{c_j}} \tag{92}$$

where the square-roots have to be chosen such that $c^k_{ij}\langle\, ,\,\rangle$ is positive definite on each H_k. Using that $\langle\, ,\,\rangle_R$ equals $\langle\, ,\,\rangle$ in the limit $q \to 1$ the square-roots can be selected by continuity.

Clearly the inner products

$$(\cdot\, ,\cdot) = c^k_{ij}\langle\cdot,\cdot\rangle_R \tag{93}$$

so defined on each irreducible component yield a well defined inner product on $H_{i\underline{\otimes} j}$, provided $\langle\cdot,\cdot\rangle_R$ is non-vanishing on each irreducible component in $H_{i\underline{\otimes} j}$. We shall not discuss this latter assumption further here. Suffice to say that for a given k corresponding to a fixed dominant weight of $\mathcal{G}$, it holds for sufficiently large ℓ by continuity.

It is important to notice that the so defined inner product on $H_{i\underline{\otimes} j}$ is compatible with associativity, i.e. it yields a unique inner product on $H_{i\underline{\otimes} j\underline{\otimes} k}$, $i,j,k \in I$, as is easily seen from eqs. (77—77) and the form (92) of the factors of proportionality.

Now conditions iii) and iv) may be verified if the tensor product $\alpha\underline{\otimes}\beta$ of two intertwiners $\alpha : H_\pi \to H_{\pi'}$, $\beta : H_\rho \to H_{\rho'}$ denotes the restriction of the ordinary tensor product $\alpha\otimes\beta$ to $H_{\pi\underline{\otimes}\rho} \subseteq H_\pi \otimes H_\rho$. Furthermore, the additional technical assumptions in section 3 may be verified if ${}_i\phi$, ϕ_i and ψ_i are chosen to be the canonical intertwiners (see ref. [11] for details).

This concludes our discussion of the quantum group case.

Let us finally mention that another interesting class of algebras that are likely to satisfy our assumptions is furnished by the chiral algebras of unitary rational conformal field theories in two dimensions (see eg. [12,22]), but it requires additional work to carry out the detailed arguments.

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