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Tensor Products, Reproducing Kernels, and Power Series

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Tensor products of holomorphic discrete series representations in reproducing kernel Hilbert spaces are decomposed by considering power series expansions of functions in the direction perpendicular to the diagonal in $\mathscr{D} \times \mathscr{D}$.

INTRODUCTION

Let $\mathscr{D}$ be a connected complex domain and let G be a group of holomorphic transformations of $\mathscr{D}$. Let H_i $(i = 1, 2)$ be reproducing kernel Hilbert spaces of holomorphic functions from $\mathscr{D}$ to finite dimensional complex vector spaces V_i such that each H_i carries a unitary representation U_i of G. Then $U_1 \otimes U_2$ is unitary in a reproducing kernel Hilbert space of holomorphic functions from $\mathscr{D} \times \mathscr{D}$ to $V_1 \otimes V_2$. Since $\mathscr{D}$ sits naturally in $\mathscr{D} \times \mathscr{D}$, one can attempt to decompose $U_1 \otimes U_2$ by restricting it to $\mathscr{D}$. Clearly this will not give the complete decomposition, since a function can vanish on $\mathscr{D}$ without being zero on $\mathscr{D} \times \mathscr{D}$. One is then lead to consider "derivatives perpendicular to the diagonal." We make this notion precise for $G = SU(n, n)$ and $G = Sp(n, \mathbf{R})$, and show that one in fact can get the decomposition this way.

In Section 1 we give some key relations obtainable from quite general properties of reproducing kernels. In Section 2 we use these to find composition series for modules of holomorphic functions. Finally, in Section 3, we introduce the Hilbert space structures and treat the case of tensor products of holomorphic discrete series. This is illustrated by some examples with $G = SU(2, 2)$.

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1. THE KERNEL AS INTERTWINING OPERATOR

Recall from [1, 4] that if $\mathscr{D}_j$ denotes the generalized upper half-plane $(j = 1, 2)$;

$$\mathscr{D}_1 = \{z = x + iy \mid x \text{ and } y \text{ are real } n \times n \text{ matrices, } x = x^t, y = y^t, \text{ and } y > 0\},$$

and

$$\mathscr{D}_2 = \{z = x + iy \mid x \text{ and } y \text{ are complex } n \times n \text{ matrices, } x = x^*, y = y^*, \text{ and } y > 0\},$$

if $G_1 = Sp(n, \mathbf{R})$ and $G_2 = SU(n, n)$, then for $j = 1, 2$,

$$\begin{aligned} K_{1,j}(gz, gw) &= \left(\frac{gz - gw^*}{2i}\right)^{-1} = (cz + d)\, K_{1,1}(z, w)(cw + d)^* \\ K_{2,j}(gz, gw) &= \left(\frac{gz - gw^*}{2i}\right) = (zc^* + d^*)^{-1}\, K_{2,j}(z, w)(wc^* + d^*)^{*-1}, \end{aligned} \tag{1.1}$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_j$, $z, w \in \mathscr{D}$, and $gz = (az + b)(cz + d)^{-1}$. For this action, $g(z^*) = (gz)^*$. In both cases, $j = 1$ and $j = 2$, $*$ denotes the complex adjoint operator.

Before continuing, we shall, due to the large similarity between the two cases, drop the subscript j. In some of the cases to come, we may have to pass to a covering group of G. When it is obvious, we do this without comments.

The relations (1.1) have been used as follows: Suppose that π^+ and π^- are finite dimensional holomorphic representations of $GL(n, \mathbf{C})$ in complex vector spaces V_{π^+} and V_{π^-} satisfying that $\pi^+(g)^* = \pi^+(g^*)$, and $\pi^-(g)^* = \pi^-(g^*)$. From the relations (1.1) it follows that

$$K_{\pi^+,\pi^-,\alpha,\beta}(z, w) = \frac{\pi^+(K_1(z, w)) \otimes \pi^-(K_2(z, w))}{\det\left(\dfrac{z - w^*}{2i}\right)^{\alpha+\beta}} \tag{1.2}$$

has the potential for being a reproducing kernel for a Hilbert space $\mathscr{H}_{\pi^+,\pi^-,\alpha,\beta}$ of holomorphic functions from $\mathscr{D}$ to $V_{\pi^+} \otimes V_{\pi^-}$, on which G acts unitarily by

$$(U_{\pi^+,\pi^-,\alpha,\beta}(g)f)(z) = \frac{\pi^+((cz + d)^{-1})}{\det(cz + d)^\alpha} \otimes \frac{\pi^-(zc^* + d^*)}{\det(zc^* + d^*)^\beta} f(g^{-1}z). \tag{1.3}$$

Here, $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and (cf. [5]) $\det(zc^* + d^*) = \det(cz + d)$. We define the operator valued function $J_{\pi^+,\pi^-,\alpha,\beta}(g, z)$ from $G \times \mathscr{D}$ to $\mathrm{Hom}(V_{\pi^+} \otimes V_{\pi^-}, V_{\pi^+} \otimes V_{\pi^-})$ by

$$(U_{\pi^+,\pi^-,\alpha,\beta}(g)f)(z) = J_{\pi^+,\pi^-,\alpha,\beta}(g^{-1}, z)^{-1} f(g^{-1}z). \tag{1.4}$$

That $U_{\pi^+,\pi^-,\alpha,\beta}$ does define an action follows from the defining properties of G (cf. [4]), but in order to have $K_{\pi^+,\pi^-,\alpha,\beta}$ define a Hilbert space, we must have that

$$\sum_{i,j=1}^{N} \langle K_{\pi^+,\pi^-,\alpha,\beta}(w_j, w_i)\, v_i, v_j\rangle \geq 0$$

for all $w_i \in \mathscr{D}$ and all v_i in $V_{\pi^+} \otimes V_{\pi^-}$ ($i = 1, 2,..., N$; N arbitrary). Special cases of this have been studied in detail in [1, 4, 8] and some integral formulas concerning a more general case, which appears naturally here, have recently been announced [2]. The Eqs. (1.1) however contain more information:

In analogy with the above, we can consider actions of G on spaces of anti-holomorphic functions on $\mathscr{D}$. If f is a such, with values in $V_{\pi^+} \otimes V_{\pi^-}$, we define

$$(U_{\overline{\pi^+},\overline{\pi^-},\alpha,\beta}(g)f)(z) = \frac{J_{\pi^+,\pi^-,0,0}(g^{-1}, z)^*}{\det(cz + d)^{*\alpha+\beta}} f(g^{-1}z). \tag{1.5}$$

Finally, we consider the module $(\mathscr{H} \otimes \mathscr{A})_{\pi^+\otimes\pi^-}$ of functions from $\mathscr{D} \times \mathscr{D}$ to $(V_{\pi^+} \otimes V_{\pi^-}) \otimes (V_{\pi^+} \otimes V_{\pi^-})$ that are holomorphic in the first variable (z_1) and are anti-holomorphic in the second (z_2). We interpret $\det(z_1 - z_2^*)^{-\gamma}$ as an invertible multiplication operator acting on functions in $(\mathscr{H} \otimes \mathscr{A})_{\pi^+\otimes\pi^-}$, and it follows readily from (1.1) that

$$\begin{aligned}(U_{\pi^+,\pi^-,\alpha,\beta} &\otimes U_{\overline{\pi^+},\overline{\pi^-},\alpha,\beta}) \det(z_1 - z_2^*)^{-(\alpha+\beta)} \\ &= \det(z_1 - z_2^*)^{-(\alpha+\beta)}\, U_{\pi^+,\pi^-,0,0} \otimes U_{\overline{\pi^+},\overline{\pi^-},0,0}. \end{aligned}\tag{1.6}$$

More generally one can use the kernels $K_{\pi_1,\pi_2,\alpha,\beta}(z_1, z_2)$ to build up invertible multiplication operators which, similarly to the above, will intertwine certain representations of G.

The situation gets different if we consider modules $(\mathscr{H} \otimes \mathscr{H})$ of holomorphic functions on $\mathscr{D} \times \mathscr{D}$. We need an analogue of (1.1). However such an analogue does exist, since (1.1) implies:

For all z_1, z_2 in $\mathscr{D}$ and g in G

$$(gz_1 - gz_2) = (z_1c^* + d^*)^{-1}(z_1 - z_2)(cz_2 + d)^{-1}. \tag{1.7}$$

The difference between this, and the preceding case is, that the analogues of the multiplication operators in (1.6) etc. either are not everywhere defined, or are non-invertible. Also, the requirement that the operators should be holomorphic gives constraints, e.g. in $\det(z_1 - z_2)^{-\gamma}$, γ must be integer. Still, one can get some information about tensor products from this. As an example, let

$$(U_{n,n}(g)f)(z_1, z_2) = \det(cz_1 + d)^{-n} \det(cz_2 + d)^{-n} f(g^{-1}z_1, g^{-1}z_2), \tag{1.8}$$

let H_0 denote the set of holomorphic functions from $\mathscr{D} \times \mathscr{D}$ to $\mathbf{C}$, and $H_k = \{f \in H_0 \mid f = \det(z_1 - z_2)^k g, g \in H_0\}$. Then each H_k is invariant under $U_{n,n}$,

$$H_0 \supset H_1 \supset \cdots \supset H_k \supset \cdots,$$

and $U_{n,n}$'s restriction to H_k is equivalent to $U_{n+k,n+k}$ acting on H_0. Moreover, since if $(Rf)(z) = f(z, z)$ denotes the restriction map, sending holomorphic functions on $\mathscr{D} \times \mathscr{D}$ to holomorphic functions on $\mathscr{D}$,

$$RU_{n,n} = W_{2n}R, \tag{1.9}$$

where $(W_{2n}(g)h)(z) = \det(cz + d)^{-2n}\, h(g^{-1}z)$, it follows that for the projected action of $U_{n,n}$ on H_k/H_{k+1}, W_{2n+2k} appears as composition factor.

The reason that the above is incomplete is, basically, that the function $(z_1 - z_2) \to \det(z_1 - z_2)$ is of too high homogeneity. In the next section we shall use (1.7) to complete the picture.

2. Module Decomposition of Holomorphic Tensor Products

We let K denote the maximal compact subgroup of G and assume given two unitary (not necessarily irreducible) representations π_1 and π_2 in finite dimensional complex vector spaces V_{π_1} and V_{π_2}. Let $J_{\pi_j}: G \times \mathscr{D} \to GL(V_{\pi_j})$ be functions satisfying $(j = 1, 2)$

$$J_{\pi_j}(g_1 g_2, z) = J_{\pi_j}(g_1, g_2 z) \cdot J_{\pi_j}(g_2, z)$$

$$J_{\pi_j}(k, i) = \pi_j(k) \qquad \text{for all } k \in K$$

$$J_{\pi_j}(1, z) = 1,$$

and $z \to J_{\pi_j}(g, z)$ is holomorphic for all g in G.

Let us consider $U_{\pi_1} \otimes U_{\pi_2}$ acting on the space $(\mathscr{H} \otimes \mathscr{H})_{\pi_1 \otimes \pi_2}$ of holomorphic functions from $\mathscr{D} \times \mathscr{D}$ to $V_{\pi_1} \otimes V_{\pi_2}$ by

$$(U_{\pi_1} \otimes U_{\pi_2}(g)f)(z_1, z_2)) = (J_{\pi_1}(g^{-1}, z_1)^{-1} \otimes J_{\pi_2}(g^{-1}, z_2)^{-1}) f(g^{-1}z_1, g^{-1}z_2).$$

For each point (z, z) on the diagonal of $\mathscr{D} \times \mathscr{D}$ (which we identify with $\mathscr{D}$) there exists an open neighborhood N_z such that the restriction of any holomorphic function to N_z is a holomorphic function of y and z, where, for z_1 and z_2 in $\mathscr{D}$

$$y = \frac{z_1 - z_2}{2}; \qquad z = \frac{z_1 + z_2}{2}. \tag{2.2}$$

We let D_r denote the subspace of functions which in some N_z as functions of y are homogeneous of degree r, and define $S_r = \bigcup_{s=0}^{\infty} D_{r+s}$. Since $\mathscr{D}$ is connected, D_r is independent of z, and hence so is S_r. Thus,

$$(\mathscr{H} \otimes \mathscr{H})_{\pi_1 \otimes \pi_2} = S_0 \supset S_1 \supset \cdots \supset S_r \supset \cdots \tag{2.3}$$

and

$$S_r / S_{r+1} \cong D_r .$$

LEMMA 2.1. *Any element of S_r can be written as a finite linear combination of functions of the form*

$$F_{M^z, f} \colon (z_1, z_2) \to (\operatorname{tr} y M^z)^r f(z_1, z_2), \tag{2.4}$$

where $f \in (\mathscr{H} \otimes \mathscr{H})_{\pi_1 \otimes \pi_2}$, and $z \to M^z$ is a holomorphic function from $\mathscr{D}$ to $M(n, \mathbf{C})$.

Proof. The space of polynomials in n^2 variables of homogeneous degree r is isomorphic to $\bigotimes_s^r \mathbf{C}^{n^2}$; the rth fold symmetrized tensor product of $\mathbf{C}^{n^2}$. This space is spanned by vectors of the form $x \otimes x \otimes \cdots \otimes x$ (r times), and $\mathbf{C}^{n^2} = M(n, \mathbf{C})$. The remaining part then follows by Taylor expansion.

LEMMA 2.2. *S_r is invariant under the action $U_{\pi_1} \otimes U_{\pi_2}$, for any $r = 0, 1, 2, \ldots$.*

Proof. We need only check what happens to a function of the form $F_{M^z, f}$, and for this case, we can use (1.7) to see that

$$(U_{\pi_1} \otimes U_{\pi_2}(g)) F_{M^z, f} = F_{(V(g)M)^z, (U_{\pi_1} \otimes U_{\pi_2}(g)) f} + (\text{higher order terms}) \tag{2.5}$$

where $(V(g)M)^z = (cz + d)^{-1} M^{g^{-1}z} (zc^* + d^*)^{-1}$. ∎

We define the representation $U_{(\tau_1 \otimes \tau_2)_r, \pi_1, \pi_2}$ on the space of holomorphic functions from $\mathscr{D}$ to $(\bigotimes_s^r \mathbf{C}^{n^2}) \otimes V_{\pi_1} \otimes V_{\pi_2}$ by

$$\begin{aligned} &(U_{(\tau_1 \otimes \tau_2)_r, \pi_1, \pi_2}(g) f)(z) \\ &\quad = \bigotimes_s^r ((cz + d)^{-1} \otimes {}^t(zc^* + d^*)^{-1}) \otimes J_{\pi_1}(g^{-1}, z)^{-1} \otimes J_{\pi_2}(g^{-1}, z)^{-1} f(g^{-1} z). \end{aligned} \tag{2.6}$$

Here, $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, ${}^t A = \bar{A}^* =$ transposed matrix of A, $(\tau_1(g) f)(z) = (cz + d)^{-1} f(g^{-1} z)$, $(\tau_2(g) f) = {}^t(zc^* + d^*)^{-1} f(g^{-1} z)$, and by convention, $(\tau_1 \otimes \tau_2)_0$ denotes the trivial representation.

Then, since we can identify the complex dual of $\mathbf{C}^n$ with $\mathbf{C}^n$, we see that the representation $M^z \to (cz + d)^{-1} M^{g^{-1}z} (zc^* + d^*)^{-1}$ is equivalent to the representation $U_{(\tau_1 \otimes \tau_2)_1, 1, 1}$ acting on functions from $\mathscr{D}$ to $\mathbf{C}^n \otimes \mathbf{C}^n$. Moreover, it is straightforward that the map

$$(T(F_{M^z, f}))(z) = \left(\bigotimes_s^r M^z \right) \otimes f(z, z) \tag{2.7}$$

extends to be a well-defined map from S_r to the module of functions on $\mathscr{D}$ with values in $(\otimes_s^r(\mathbf{C}^n \otimes \mathbf{C}^n)) \otimes V_{\pi_1} \otimes V_{\pi_2}$ with kernel S_{r+1}, and thus that the action of $U_{\pi_1} \otimes U_{\pi_2}$ on S_r/S_{r+1} is equivalent to $U_{(\tau_1\otimes\tau_2)_r,\pi_1,\pi_2}$.

Another way of phrasing this is: Let, as before, R denote the restriction map $(Rf)(z) = f(z, z)$. Let us use the n^2 entries of y; $y_1, y_2, \ldots, y_{n^2}$, as the polynomial variables, and pick a basis e_ρ of $\otimes_s^r \mathbf{C}^{n^2}$, where ρ runs through the set of multi-indices $\rho = (\rho_1, \rho_2, \ldots, \rho_{n^2})$ with $\rho_1 + \rho_2 + \cdots + \rho_{n^2} = r$. Then we can define a map RT_r from S_r to the space of holomorphic functions from $\mathscr{D}$ to $(\otimes_s^r \mathbf{C}^{n^2}) \otimes V_{\pi_1} \otimes V_{\pi_2}$ by defining the component $(RT_r f)_\rho$ of $RT_r f$ in the subspace $\mathbf{C}e_\rho \otimes V_{\pi_1} \otimes V_{\pi_2}$ to be

$$(RT_r f)_\rho(z) = e_\rho \otimes \left(R\left(\frac{\partial^{\rho_1}}{\partial y_1^{\rho_1}} \cdots \frac{\partial^{\rho_{n^2}}}{\partial y_{n^2}^{\rho_{n^2}}} f\right)\right)(z).$$

Then the kernel of RT_r is S_{r+1}, and the above is equivalent to stating that

$$(RT_r)(U_{\pi_1} \otimes U_{\pi_2}) = (U_{(\tau_1\otimes\tau_2)_r,\pi_1,\pi_2})(RT_r) \tag{2.8}$$

(up to a linear automorphism of $(\otimes_s^r \mathbf{C}^{n^2}) \otimes V_{\pi_1} \otimes V_{\pi_2}$).

We collect the above to

THEOREM 2.1. *Under the action of* $U_{\pi_1} \otimes U_{\pi_2}$, $(\mathscr{H} \otimes \mathscr{H})_{\pi_1\otimes\pi_2}$ *has a chain of invariant subspaces,*

$$(\mathscr{H} \otimes \mathscr{H})_{\pi_1\otimes\pi_2} = S_0 \supset S_1 \supset \cdots \supset S_r \supset S_{r+1} \supset \cdots,$$

and on S_r/S_{r+1}, $U_{\pi_1} \otimes U_{\pi_2}$ *is equivalent to* $U_{(\tau_1\otimes\tau_2)_r,\pi_1,\pi_2}$.

We stress that the representations $U_{(\tau_1\otimes\tau_2)_r,\pi_1,\pi_2}$ in general will be reducible. However, they are obtained by holomorphic induction from K, and can therefore be broken down by decomposing finite-dimensional representations of K. The resulting representations will then most often be irreducible. We refer to [4] for the motivation for this.

3. HILBERT SPACE STRUCTURES

Assume that the representations U_{π_i} are unitary in reproducing kernel Hilbert spaces H_i of holomorphic functions from $\mathscr{D}$ to V_{π_i} $(i = 1, 2)$. That is, assume that there are functions $(z, w) \to K_i(z, w)$ from $\mathscr{D} \times \mathscr{D}$ to $\mathrm{Aut}(V_{\pi_i}, V_{\pi_i})$ such that

1. For any $v \in V_{\pi_i}$ and any $w \in \mathscr{D}$ the map $z \to K_i(z, w)v$ belongs to H_i,
2. For any $v \in V_{\pi_i}$ and any $f \in H_i$, $\langle v, f(w)\rangle_{V_{\pi_i}} = \langle K(\cdot, w)v, f(\cdot)\rangle_{H_i}$.

Note in this connection that if π_i is an irreducible unitary representation of K

and if U_{π_i} is unitary in a Hilbert space $H_0 \subset \mathscr{H}$, then H_0 is unique, and U_{π_1} is irreducible [3].

By Morera's theorem, the Hilbert space completion of $H_1 \otimes H_2$, H, is a space of holomorphic functions from $\mathscr{D} \times \mathscr{D}$ to $V_{\pi_1} \otimes V_{\pi_2}$ for which $(z_1, z_2, w_1, w_2) \to K_1(z_1, w_1) \otimes K_2(z_2, w_2)$ is a reproducing kernel. Moreover, by Weierstrass's theorem, the modules $H_r = S_r \cap H$ are closed subspaces of H. Let us for simplicity assume that $H_r \neq H_{r+1}$ for all $r = 0, 1, 2,\ldots$. The decomposition of H then consists of a series of steps, out of which the $(r+1)$st is the following: $U_{\pi_1} \otimes U_{\pi_2}$ is unitary on H_r, and H_{r+1} is a closed invariant subspace. Thus, we can give H_r/H_{r+1} a Hilbert space structure such that H_r is unitarily equivalent to $H_r/H_{r+1} \oplus H_{r+1}$. Moreover, under this isomorphism the restriction of $U_{\pi_1} \otimes U_{\pi_2}$ to H_r, $(U_{\pi_1} \otimes U_{\pi_2})_r$, decomposes as

$$(U_{\pi_1} \otimes U_{\pi_2})_r = U^s_{(\tau_1 \otimes \tau_2)_r, \pi_1, \pi_2} \oplus (U_{\pi_1} \otimes U_{\pi_2})_{r+1}. \tag{3.1}$$

Here, the s on $U^s_{(\tau_1 \otimes \tau_2)_r, \pi_1, \pi_2}$ signifies that it may happen that only a subspace of $U_{(\tau_1 \otimes \tau_2)_r, \pi_1, \pi_2}$ is being picked up. On this subspace the representation is clearly unitary.

We shall here treat a case, where we in fact pick up the entire $U_{(\tau_1 \otimes \tau_2)_i, \pi_1, \pi_2}$. We shall only treat the $SU(n, n)$ case. Simple modifications will then yield the $Sp(n, \mathbf{R})$ case. Specifically, let

$$C^+ = \{k \in M(n, \mathbf{C}) \mid k = k^* \text{ and } k > 0\},$$

and assume that there are continuous functions $k \to M_i(k) \in GL(V_{\pi_i})$, $M_i(k)^* = M_i(k) > 0$, such that

$$K_i(z, w) = \int_{C^+} e^{i \operatorname{tr}(z - w^*)k} M_i(k)\, dm(k)$$

where $dm(k)$ is Lebesgue measure. This means that H_i is equivalent to the space of functions $\varphi: C^+ \to V_{\pi_i}$ with $\langle \varphi, \varphi \rangle = \int_{C^+} \langle M^{-1}(k)\, \varphi(k), \varphi(k) \rangle\, dm(k)$, the isomorphism being given by

$$(F\varphi)(z) = \int_{C^+} e^{i \operatorname{tr} zk} \varphi(k)\, dm(k)$$

on a dense set of vectors. The representations we discuss here include the holomorphic discrete series.

LEMMA 3.1. *Let v_i be a vector in V_{π_i}. Then there exists a function $f_{v_i}: \mathscr{D} \to V_{\pi_i}$ such that $f_{v_i}(z)$ belongs to the subspace spanned by v_i for all z, and such that if p is any polynomial in the n^2 entries of $z \in \mathscr{D}$, then $p(z) f_{v_i}(z)$ belongs to H_i.*

Proof. Let $\varphi_i: C^+ \to V_{\pi_i}$ be any C^∞-function whose support is compact and contained in the interior of C^+, such that $\varphi(k)$ belongs to the subspace spanned by v_i for all k. Then there exist differential operators $D_p(k)$ and $\widetilde{D_p(k)}$ such that

$$p(z) \int_{C^+} e^{i\,\mathrm{tr}\, zk} \varphi_i(k)\, dm(k) = \int_{C^+} (D_p(k)\, e^{i\,\mathrm{tr}\, zk})\, \varphi_i(k)\, dm(k)$$

$$= \int_{C^+} e^{i\,\mathrm{tr}\, zk} (\widetilde{D_p(k)}\, \varphi_i(k))\, dm(k).$$

This completes the proof since the last function clearly belongs to H_i.

This lemma now implies, by expanding homogeneous polynomials of degree r in $(z_1 - z_2)$ into products of polynomials in z_1 and z_2, that in this case, for any one-dimensional subspace of $(\bigotimes_s^r \mathbf{C}^{n^2}) \otimes V_{\pi_1} \otimes V_{\pi_2}$, there are functions in $RT_r(H_r)$ that take their values solely in this subspace. We can therefore state

THEOREM 3.2. *For the above U_{π_1} and U_{π_2},*

$$U_{\pi_1} \otimes U_{\pi_2} \cong \bigoplus_{r=0}^{\infty} U_{(\tau_1 \otimes \tau_2)_r, \pi_1, \pi_2} \tag{3.2}$$

as unitary representations.

As for the limits of holomorphic discrete series, our method is less applicable and only for a few cases with $G = SU(2, 2)$ has it been extended. However, the decomposition of those can be obtained in a straightforward manner from [6], whereas it is much harder to extract the tensor products of holomorphic discrete series from that. This was pointed out to us by Michele Vergne. Thus the two papers are almost orthogonal.

We illustrate the above with a few examples with $G = SU(2, 2)$. This group is complicated enough to give an impression of the analysis involved, and yet small enough that the combinatorics does not get too messy. Also, the results here are of potential relevance to theoretical physics, in particular to I. E. Segal's unified theory [10].

EXAMPLE 1. Let $(U_n(g) f)(z) = \det(cz + d)^{-n-2} f(g^{-1}z)$, and consider $U_n \otimes U_m$ for $n, m \geqslant 0$ (cf. [4]). We can extend these representations to $U(2, 2)$ and then decompose under this group. This will give the same decomposition as for $SU(2, 2)$, but $U(2, 2)$ is more convenient. We must now decompose $U_{(\tau_1 \otimes \tau_2)_r, n, m}$, which is the representation equivalent to $U_n \otimes U_m$ on S_r/S_{r+1} (Theorem 2.1), and for this purpose, we can let $n, m = -2$. We are then looking at a representation of the form $(U(g) f)(z) = J^r(g^{-1}, z)^{-1} f(g^{-1}z)$, where

$$J^r(k, i)(M \otimes \cdots \otimes M)$$
$$= [(a + ib)\, M(a - ib)^*] \otimes \cdots \otimes [(a + ib)\, M(a - ib)^*] \tag{3.3}$$

for $k = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in K$. K, the maximal subgroup of $U(2, 2)$, is isomorphic to $U(2) \times U(2)$ by $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \to (a + ib, a - ib) = (u_1, u_2) \in U(2) \times U(2)$. The equivalent representation of K on the space P^r of homogeneous polynomials in four complex variables y of degree r is then

$$(J^r(u_1, u_2, i)p)(y) = p(u_2^{-1} y u_1). \tag{3.4}$$

The decomposition of this representation for the general case is known [9]. However, for $U(2) \times U(2)$ it can also be found by straightforward arguments:

Let, for each $r \in \mathbf{N}$, $P_0{}^r$ denote the set of homogeneous polynomials of degree r that are orthogonal (in the standard inner product) to the ideal generated by the second order polynomial $\det(y)$ in P^r. Then

$$\begin{aligned} P^{2n} &= P_0^{2n} \oplus \det y P_0^{2n-2} \oplus \det y^2 P_0^{2n-4} \oplus \cdots \oplus \mathbf{C} \det y^n \\ P^{2n+1} &= P_0^{2n+1} \oplus \det y P_0^{2n-1} \oplus \det y^2 P_0^{2n-3} \oplus \cdots \oplus \det y^n P_0{}^1. \end{aligned} \tag{3.5}$$

This decomposition is invariant under the action of K, since $\det u_2^{-1} y u_1 = \det u_2^{-1} \det y \det u_1$. We also observe that the representation $u_2 \to (u_2^{-1})^t$ of $U(2)$ is unitarily equivalent to the representation $u_2 \to (\det u_2^{-1})\, u_2$.

Let $g \to \tau_n(g)$ denote the nth fold symmetrized tensor product of the defining representation $\tau = \tau_1$ of $GL(2, \mathbf{C})$. Then it follows from the theory of highest weight (see e.g. [12]) that there must be a subspace of P_0^{2n} that transforms according to $(\tau_{2n}(u_1) \otimes \tau_{2n}(u_2))/(\det u_2^{2n})$, and hence that

$$\frac{\tau_{2n}(u_1) \otimes \tau_{2n}(u_2)}{\det u_2^{2n}} \oplus \frac{(\tau_{2n-2}(u_1) \otimes \tau_{2n-2}(u_2)) \det u_1}{\det u_2^{2n-1}} \oplus \cdots$$

$$\oplus \frac{\det u_1{}^n}{\det u_2{}^n} \subseteq J^{2n}(u_1, u_2, i) \tag{3.6}$$

(Observe that $(u_1, u_2) \in SU(2, 2) \Leftrightarrow \det u_1 = \det u_2^{-1}$).

Now the dimension of P^{2n} is $\binom{3+2n}{3}$, and $\tau_i(u_1) \otimes \tau_i(u_2)$ acts in a space of dimension $(i + 1)^2$. Therefore, by counting dimensions, we conclude that the above sum of subspaces exhausts P^{2n}. An entirely similar argument works for P^{2n+1}. Finally, $(u_1, u_2) \to \tau_i(u_1) \otimes \tau_i(u_2)$ is in fact an outer tensor product, and hence irreducible, since τ_i is irreducible.

We introduce the following notation:

$$(D(n, m, \beta)f)(z) = \frac{\tau_n((cz + d)^{-1}) \otimes \tau_m(zc^* + d^*)}{\det(cz + d)^\beta} f(g^{-1}z) \tag{3.7}$$

for $f\colon \mathscr{D} \to (\otimes_s^n \mathbf{C}^2) \otimes (\otimes_s^m \mathbf{C}^2)$. Then, by collecting the various terms from the decomposition, we get (letting $\tau_0 = 1$)

PROPOSITION 3.3. *For $n, m \geqslant 0$*

$$U_n \otimes U_m \cong \bigoplus_{l=0}^{\infty} \bigoplus_{j=0}^{\infty} D(l, l, l + 2j + n + m + 4).$$

If we let $U_n \otimes_s U_n$ denote the restriction of $U_n \otimes U_n$ to the space of symmetric functions, we must exclude all terms arising from functions whose leading terms in y are homogeneous of an odd degree. Doing this, we get

COROLLARY 3.4. *As representation of $SU(2, 2)$, for $n \geqslant 0$*

$$U_n \otimes_s U_n \cong \bigoplus_{l=0}^{\infty} \bigoplus_{j=0}^{\infty} D(2l, 2l, 2l + 2j + 2n + 4).$$

Before continuing with a few more examples, we wish to make a remark of a general nature, which is easily expressed for $U_n \otimes U_n$. Suppose we extend our module of holomorphic functions on $\mathscr{D} \times \mathscr{D}$ to include functions that are meromorphic, but holomorphic on the subset where $\det(z_1 - z_2) \neq 0$. Then, as we saw in Section 1, $\det(z_1 - z_2)^p$, $p \in \mathbf{Z}$ becomes an invertible intertwining operator, and all $U_n \otimes U_n$'s are equivalent. Moreover, if we let

$$y_0 \frac{\partial}{\partial y_0} + y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_3} = K_y$$

denote the scale operator in the y-direction (cf. (2.2)) and if $D_r = \{f(z_1, z_2) = f(y, z) \mid K_y f = rf\}$, then $(U_n \otimes U_n) D_r \subseteq \bigcup_{s=0}^{\infty} D_{r+2s}$; also for r negative. Finally, we observe that the restriction of $U_n \otimes U_n$ to the maximal parabolic subgroup $P = \{\left(\begin{smallmatrix} a & b \\ 0 & a^{*-1} \end{smallmatrix}\right) \mid ab^* = ba^*, a \in SL(2, \mathbf{C})\}$ leaves D_r invariant.

EXAMPLE 2. Let f be a holomorphic function from $\mathscr{D}$ to $\mathbf{C}^2$, and let

$$\begin{aligned} (T(1, n)f)(g) &= \frac{(cz + d)^{-1}}{\det(cz + d)^{n+2}} f(g^{-1}z) \\ (T(2, m)f)(g) &= \frac{zc^* + d^*}{\det(cz + d)^{m+3}} f(g^{-1}z). \end{aligned} \tag{3.8}$$

The decomposition of $T(1, n) \otimes T(1, m)$, $T(1, n) \otimes T(2, m)$, and $T(2, n) \otimes T(2, m)$ follows readily from that of $U_n \otimes U_m$, by noting that as representations, for $n \geqslant 2$,

$$\begin{aligned} &\tau_n(u_1) \otimes \tau_n(u_2) \otimes u_1 \otimes u_1 \\ &\quad = \tau_{n+2}(u_1) \otimes \tau_n(u_2) \oplus 2 \det u_1 \tau_n(u_1) \otimes \tau_n(u_2) \\ &\qquad \oplus (\det u_1)^2\, \tau_{n-2}(u_1) \otimes \tau_n(u_2), \end{aligned}$$

and (3.9)

$$\tau_n(u_1) \otimes \tau_n(u_2) \otimes u_1 \otimes u_2$$
$$= \tau_{n+1}(u_1) \otimes \tau_{n+1}(u_2) \bigoplus \det u_2 \tau_{n+1}(u_1) \otimes \tau_{n-1}(u_2)$$
$$\bigoplus \det u_1 \tau_{n-1}(u_1) \otimes \tau_{n+1}(u_2) \bigoplus \det(u_1 u_2)\, \tau_{n-1}(u_1) \otimes \tau_{n-1}(u_2),$$

with similar formulas for $n = 1$.

PROPOSITION 3.5. *For $n, m \geqslant 0$*

$$T(1, n) \otimes T(2, m) = 2 \bigoplus_{l=1}^{\infty} \bigoplus_{j=0}^{\infty} D(l, l, l + 2j + n + m + 6)$$
$$\oplus \bigoplus_{j=0}^{\infty} D(0, 0, 2j + n + m + 6)$$
$$\oplus \bigoplus_{l=2}^{\infty} \bigoplus_{j=0}^{\infty} D(l, l - 2, l + 2j + n + m + 3)$$
$$\oplus \bigoplus_{l=2}^{\infty} \bigoplus_{j=0}^{\infty} D(l - 2, l, l + 2j + n + m + 5)$$
$$\oplus \bigoplus_{l=0}^{\infty} D(l, l, l + n + m + 4).$$

PROPOSITION 3.6. *For $n, m \geqslant 0$*

$$T(1, n) \otimes T(1, m) = 2 \bigoplus_{l=1}^{\infty} \bigoplus_{j=0}^{\infty} D(l, l, l + 2j + n + m + 5)$$
$$\oplus \bigoplus_{j=0}^{\infty} D(2, 0, 2j + n + m + 4)$$
$$\oplus \bigoplus_{l=3}^{\infty} \bigoplus_{j=0}^{\infty} D(l, l - 2, l + 2j + n + m + 2)$$
$$\oplus \bigoplus_{l=2}^{\infty} \bigoplus_{j=0}^{\infty} D(l - 2, l, l + 2j + n + m + 6)$$
$$\oplus \bigoplus_{j=0}^{\infty} D(0, 0, 2j + n + m + 5).$$

PROPOSITION 3.7. *For $n, m \geqslant 0$*

$$T(2, n) \otimes T(2, m) = 2 \bigoplus_{l=1}^{\infty} \bigoplus_{j=0}^{\infty} D(l, l, l + 2j + n + m + 5)$$

$$\oplus \bigoplus_{j=0}^{\infty} D(0, 2, 2j + n + m + 6)$$

$$\oplus \bigoplus_{l=3}^{\infty} \bigoplus_{j=0}^{\infty} D(l - 2, l, l + 2j + n + m + 4)$$

$$\oplus \bigoplus_{l=2}^{\infty} \bigoplus_{j=0}^{\infty} D(l, l - 2, l + 2j + n + m + 4)$$

$$\oplus \bigoplus_{j=0}^{\infty} D(0, 0, 2j + n + m + 5).$$

We observe that if $s: \mathbf{C}^2 \otimes \mathbf{C}^2 \to \mathbf{C}^2 \otimes \mathbf{C}^2$ denotes the symmetrization map; $s(v_1 \otimes v_2) = \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$, and if $a = 1 - s$, then, if f and g are holomorphic maps from $\mathscr{D}$ to $\mathbf{C}^2$, $s(f(z_1) \otimes g(z_2) + g(z_1) \otimes f(z_2))$ is a symmetric function of z_1 and z_2, whereas $(z_1, z_2) \to a(f(z_1) \otimes g(z_2) + g(z_1) \otimes f(z_2))$ changes sign when z_1 and z_2 change place. With this in mind, it is easy to see that

COROLLARY 3.8. *For $n \geqslant 0$*

$$T(1, n) \otimes_s T(1, n) = \bigoplus_{l=1}^{\infty} \bigoplus_{j=0}^{\infty} D(2l, 2l - 2, 2l + 2j + 2n + 2)$$

$$\oplus \bigoplus_{l=1}^{\infty} \bigoplus_{j=0}^{\infty} D(2l - 2, 2l, 2l + 2j + 2n + 6)$$

$$\oplus \bigoplus_{l=1}^{\infty} \bigoplus_{j=0}^{\infty} D(l, l, l + 2j + 2n + 5).$$

COROLLARY 3.9. *For $m \geqslant 0$*

$$T(2, m) \otimes_s T(2, m) = \bigoplus_{l=1}^{\infty} \bigoplus_{j=0}^{\infty} D(2l - 2, 2l, 2l + 2j + 2m + 4)$$

$$\oplus \bigoplus_{l=1}^{\infty} \bigoplus_{j=0}^{\infty} D(2l, 2l - 2, 2l + 2j + 2m + 4)$$

$$\oplus \bigoplus_{l=1}^{\infty} \bigoplus_{j=0}^{\infty} D(l, l, l + 2j + 2m + 5).$$

Note added in proof. W. Schmid has kindly informed us that S. Martens has used the same technique of differentiation in the direction normal to a complex submanifold in a study of the characters of the holomorphic discrete series. (See: S. Martens, *Proc. Nat. Acad. Sci. USA* **72** (1975), 3275–3276, for a summary.)

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