DOUBLE QUANTUM SCHUBERT CELLS AND QUANTUM MUTATIONS

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ABSTRACT. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra of s simple finite dimensional Lie algebra over $\mathbb{C}$. To each pair $w^{\mathfrak{a}} \leq w^{\mathfrak{c}}$ of minimal left coset representatives in the quotient space $W_p\backslash W$ we construct explicitly a quantum seed $\mathcal{Q}_q(\mathfrak{a},\mathfrak{c})$. We define Schubert creation and annihilation mutations and show that our seeds are related by such mutations. We also introduce more elaborate seeds to accommodate our mutations. The quantized Schubert Cell decomposition of the quantized generalized flag manifold can be viewed as the result of such mutations having their origins in the pair $(\mathfrak{a},\mathfrak{c}) = (\mathfrak{e},\mathfrak{p})$, where the empty string $\mathfrak{e}$ corresponds to the neutral element. This makes it possible to give simple proofs by induction. We exemplify this in three directions: Prime ideals, upper cluster algebras, and the diagonal of a quantized minor.

1. Introduction

We study a class of quadratic algebras connected to quantum parabolics and double quantum Schubert cells. We begin by considering a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and a parabolic sub-algebra $\mathfrak{p} \subset \mathfrak{g}$. Then we consider a fixed Levi decomposition

$$\mathfrak{p} = \mathfrak{l} + \mathfrak{u}, \tag{1}$$

with $\mathfrak{u} \neq 0$ and $\mathfrak{l}$ the Levi subalgebra.

The main references for this study are the articles by A. Berenstein and A. Zelevinski [3] and by C. Geiss, B. Leclerc, J. Schröer [15]. We also refer to [22] for further background.

Let, as usual, W denote the Weyl group. Let $W_p = \{w \in W \mid w(\triangle^-)\cap\triangle^+ \subseteq \triangle^+(\mathfrak{l})\}$ and W^p, by some called the Hasse Diagram of $G\backslash P$, denote the usual set of minimal length coset representatives of $W_p\backslash W$. Our primary input is a pair of Weyl group elements $w^{\mathfrak{a}}, w^{\mathfrak{c}} \in W^p$ such that $w^{\mathfrak{a}} \leq w^{\mathfrak{c}}$. We will often, as here, label our elements w by "words" $\mathfrak{a}$; $w = w^{\mathfrak{a}}$, in a fashion similar, though not completely identical, to that of [3]. Details follow in later sections, but we do mention here that the element e in W is labeled by $\mathfrak{e}$ corresponding to the empty string; $e = \omega^{\mathfrak{e}}$ while the longest elements in W^p is labeled by $\mathfrak{p}$.

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To each pair $w^{\mathfrak{a}}, w^{\mathfrak{c}}$ as above we construct explicitly a quantum seed
$$\mathcal{Q}_q(\mathfrak{a},\mathfrak{c}) := (\mathcal{C}_q(\mathfrak{a},\mathfrak{c}), \mathcal{L}_q(\mathfrak{a},\mathfrak{c}), \mathcal{B}_q(\mathfrak{a},\mathfrak{c})). \tag{2}$$
The cluster $\mathcal{C}_q(\mathfrak{a},\mathfrak{c})$ generates a quadratic algebra $\mathcal{A}_q(\mathfrak{a},\mathfrak{c})$ in the space of functions on $\mathcal{U}_q(\mathfrak{n})$.

After that we define transitions
$$\mathcal{Q}_q(\mathfrak{a},\mathfrak{c}) \to \mathcal{Q}_q(\mathfrak{a}_1,\mathfrak{c}_1). \tag{3}$$
We call our transitions quantum Schur (creation/annihilation) mutations and prove that they are indeed just (composites of) quantum mutations in the sense of Berenstein and Zelevinski. These actually have to be augmented by what we call creation/annihilation mutations which are necessary since we have to work inside a larger ambient space. To keep the full generality, we may also have to restrict our seeds to sub-seeds.

The natural scene turns out to be
$$\mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c}) := (\mathcal{C}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c}), \mathcal{L}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c}), \mathcal{B}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})), \tag{4}$$
which analogously is determined by a triple $w^{\mathfrak{a}}, w^{\mathfrak{b}}, w^{\mathfrak{c}} \in W^p$ such that $w^{\mathfrak{a}} \leq w^{\mathfrak{b}} \leq w^{\mathfrak{c}}$.

Later we extend our construction to even
$$\mathcal{Q}_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n) \text{ and } \mathcal{A}_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n), \tag{5}$$
though we do not use it here for anything specific.

It is a major point of this study to establish how our seeds and algebras can be constructed, inside an ambient space, starting from a single variable (indeed: none). In this sense the quantized generalized flag manifold of $(G/P)_q$ as built from quantized Schubert Cells can be built from a single cell. Furthermore, we prove that we can pass between our seeds by Schubert creation and annihilation mutations inside a larger ambient space.

This sets the stage for (simple) inductive arguments which is a major point of this article, and is what we will pursue here.

We first prove by induction that the two-sided sided ideal $I(\det_s^{\mathfrak{a},\mathfrak{c}})$ in $\mathcal{A}_q(\mathfrak{a},\mathfrak{c})$ generated by the quantized minor $\det_s^{\mathfrak{a},\mathfrak{c}}$ is prime.

Then we prove that each upper cluster algebra $\mathbb{U}(\mathfrak{a},\mathfrak{c})$ equals its quadratic algebra $\mathcal{A}_q(\mathfrak{a},\mathfrak{c})$.

There is a sizable overlap between these result and results previously obtained by K. Goodearl M. Yakimov ([16],[17]).

We further use our method to study the diagonal of a quantum minor.

The idea of induction in this context was introduced in [20] and applications were studied in the case of a specific type of parabolic related to type A_n.

Further ideas relating to explicit constructions of compatible pairs in special cases were studied in [21].

2. A little about quantum groups and cluster algebras

2.1. **2.1 Quantum Groups.** We consider quantized enveloping algebras $U = \mathcal{U}_q(\mathfrak{g})$ in the standard notation given either eg. by Jantzen ([23]) or by Berenstein and Zelevinsky ([3]), though their assumptions do not coincide completely. To be completely on the safe side, we state our assumptions and notation, where it may differ: Our algebra is a Hopf algebra defined in the usual fashion from a semi-simple finite-dimensional complex Lie algebra $\mathfrak{g}$. They are algebras over $\mathbb{Q}(q)$. Φ denotes a given set of roots and throughout, $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_R\}$ a fixed choice of simple roots. Our generators are then given as

$$\{E_\alpha, F_\alpha, K^\alpha\}_{\alpha\in\Pi},$$

but we will allow elements of the form K^η for any integer weight. W denotes the Weyl group defined by Φ.

Finally we let $\{\Lambda_\alpha \mid \alpha \in \Pi\}$ denote the set of fundamental weights. We assume throughout that the diagonalizing elements d_α are determined by

$$\forall\alpha \in \Pi : (\Lambda_\alpha, \alpha) = d_\alpha. \tag{6}$$

Lemma 2.1 ((2.27) in [14])**.** *Let $\alpha_i \in \Phi$. Then*

$$(\sigma_i + 1)(\Lambda_i) + \sum_{j\neq i} a_{ji}(\Lambda_j) = 0.$$

2.2. **Quantum Cluster Algebras.** We take over without further ado the terminology and constructions of ([3]). Results from [15] are also put to good use.

Definition 2.2. *We say that two elements A, B in some algebra over $\mathbb{C}$ q-commute if, for some $r \in \mathbb{R}$:*

$$AB = q^r BA. \tag{7}$$

To distinguish between the original mutations and the more elaborate ones we need here, and to honor the founding fathers A. Berenstein, S. Fomin, and A. Zelevinski, we use the following terminology:

Definition 2.3. *A quantum mutation as in* [3] *is called a BFZ-mutation.*

2.3. **A simple observation.** If $\underline{a} = (a_1, a_2, \dots, a_{\mathfrak{m}})$ and $\underline{f} = (f_1, f_2, \dots, f_{\mathfrak{m}})$ are vectors then

Lemma 2.4. *([20])*

$$\mathcal{L}_q(\underline{a})^T = (\underline{f})^T \Leftrightarrow \forall i : X_i X^{\underline{a}} = q^{f_i} X^{\underline{a}} X_i. \tag{8}$$

In particular, if there exists a j such that $\forall i : f_i = -\delta_{i,j}$ then the column vector $\underline{a}$ can be the jth column in the matrix $\mathcal{B}$ of a compatible pair.

However simple this actually is, it will have a great importance later on.

3. On Parabolics

The origin of the following lies in A. Borel [4], and B. Kostant [30]. Other main contributors are [2] and [40]. See also [6]. We have also found ([39]) useful.

Definition 3.1. *Let $w \in W$. Set*

$$\Phi_\omega = \{\alpha \in \Delta^+ \mid w^{-1}\alpha \in \Delta^-\} = w(\Delta^-) \cap \Delta^+.$$

We have that $\ell(w) = \ell(w^{-1}) = |\Phi_\omega|$.

We set $\Phi_\omega = \Delta^+(w)$.

From now on, we work with a fixed parabolic $\mathfrak{p}$ with a Levi decomposition

$$\mathfrak{p} = \mathfrak{l} + \mathfrak{u}, \tag{9}$$

where $\mathfrak{l}$ is the Levi subalgebra, and where we assume $\mathfrak{u} \neq 0$,

Let

Definition 3.2.

$$\begin{aligned} W_p &= \{w \in W \mid \Phi_\omega \subseteq \Delta^+(\mathfrak{l})\}, \\ W^p &= \{w \in W \mid \Phi_\omega \subseteq \Delta^+(\mathfrak{u})\}. \end{aligned}$$

W^p is a set of distinguished representatives of the right coset space $W_p \backslash W$.

It is well known (see eg ([39])) that any $w \in W$ can be written uniquely as $w = w_p w^p$ with $w_p \in W_p$ and $w^p \in W^p$.

One defines, for each w in the Weyl Group W, the Schubert cell X_w. This is a cell in $\mathbb{P}(V)$, the projective space over a specific finite-dimensional representation of $\mathfrak{g}$. The closure, X_w, is called a Schubert variety. The main classical theorems are

Theorem 3.3 (Kostant,[30])**.**

$$G/P = \sqcup_{w\in W^p} X_w.$$

Theorem 3.4 ([40])**.** *Let $w, w' \in W^p$. Then*

$$X_{w'} \subseteq X_w$$

if and only $w' \leq w$ in the usual Bruhat ordering.

If $\omega^{\mathfrak{r}} = \omega_m\tilde{\omega}$ and $\omega_m = \omega_n\hat{\omega}$ with $\omega_n, \omega_m \in W^P$ and all Weyl group elements reduced, we say that $\omega_n <_L \omega_m$ if $\hat{\omega} \neq e$. This is the weak left Bruhat order.

4. The quadratic algebras

Let $\omega = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_t}$ be an element of the Weyl group written in reduced form. Following Lusztig ([35]), we construct roots $\gamma_i = \omega_{i-1}(\alpha_i)$ and elements $Z_{\gamma_i} \in \mathcal{U}_q(\mathfrak{n}_\omega)$.

The following result is well known, but notice a change $q \to q^{-1}$ in relation to ([22]).

Theorem 4.1 ([32],[31])**.** *Suppose that $1 \leq i < j \leq t$. Then*

$$Z_iZ_j = q^{-(\gamma_i,\gamma_j)}Z_jZ_i + \mathcal{R}_{ij},$$

where $\mathcal{R}_{ij}$ is of lower order in the sense that it involves only elements Z_k with $i < k < j$. Furthermore, the elements

$$Z_t^{a_t}\dots Z_2^{a_2}Z_1^{a_1}$$

with $a_1, a_2, \dots, a_t \in \mathbb{N}_0$ form a basis of $\mathcal{U}_q(\mathfrak{n}_\omega)$.

Our statement follows [23],[24]. Other authors, eg. [32], [15] have used the other Lusztig braid operators. The result is just a difference between q and q^{-1}. Proofs of this theorem which are more accessible are available ([8],[24]).

It is known that this algebra is isomorphic to the algebra of functions on $\mathcal{U}_q(\mathfrak{n}_\omega)$ satisfying the usual finiteness condition. It is analogously equivalent to the algebra of functions on $\mathcal{U}_q^-(\mathfrak{n}_\omega)$ satisfying a similar finiteness condition. See eg ([15]) and ([23]).

5. BASIC STRUCTURE

Let $\omega^{\mathfrak{p}}$ be the maximal element in W^p. It is the one which maps all roots in $\Delta^+(\mathfrak{u})$ to Δ^-. (Indeed: To $\Delta^-(\mathfrak{u})$.) Let w_0 be the longest element in W and w_L the longest in the Weyl group of $\mathfrak{l}$, Then

$$w^{\mathfrak{p}}w_L = w_0. \tag{10}$$

Let $\omega^{\mathfrak{r}} = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r} \in W^p$ be written in a fixed reduced form. Then $\ell(\omega^{\mathfrak{r}}) = r$. We assume here that $r \geq 1$. We set $e = \omega^{\mathfrak{e}}$ and $\ell(\omega^{\mathfrak{e}}) = 0$ where $\mathfrak{e}$ denotes the empty set, construed as the empty sequence. We also let $\mathfrak{r}$ denote the sequence $i_1, i_2, \dots, i_r$ if $\mathfrak{r} \neq \mathfrak{e}$. If a sequence $\mathfrak{s}$ corresponds to an analogous element $\omega^{\mathfrak{s}} \in W^p$ we define

$$\mathfrak{s} \leq \mathfrak{r} \Leftrightarrow \omega^{\mathfrak{s}} \leq_L \omega^{\mathfrak{r}}. \tag{11}$$

Set

$$\Delta^+(\omega^{\mathfrak{r}}) = \{\beta_{i_1}, \dots, \beta_{i_r}\}. \tag{12}$$

Definition 5.1. *Let* $\mathbf{b}$ *denote the map* $\Pi \to \{1, 2, \dots, R\}$ *defined by* $\mathbf{b}(\alpha_i) = i$. *Let* $\overline{\pi}_{\mathfrak{r}} : \{1, 2, \dots, r\} \to \Pi$ *be given by*

$$\overline{\pi}_{\mathfrak{r}}(j) = \alpha_{i_j}. \tag{13}$$

If $\overline{\pi}_{\mathfrak{r}}(j) = \alpha$ *we say that* α *(or* σ_α*) occurs at position* j *in* $w^{\mathfrak{r}}$*, and we say that* $\overline{\pi}_{\mathfrak{r}}^{-1}(\alpha)$ *are the positions at which* α *occurs in* w*. Set*

$$\pi_{\mathfrak{r}} = \mathbf{b} \circ \overline{\pi}_{\mathfrak{r}}. \tag{14}$$

$\pi_{\mathfrak{e}}$ is construed as a map whose image is the empty set.

Recall from ([22]):

Definition 5.2. *Let* $\omega^{\mathfrak{r}} \in W^p$ *be given and suppose* $s \in Im(\pi_{\mathfrak{r}})$*. Then* $s = \pi_{\mathfrak{r}}(n)$ *for some* n *and we set* $\omega_n := \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_n}$*. Suppose* $\omega_n = \omega_1\sigma_{i_n}\omega_2\dots\omega_t\sigma_{i_n}$ *and* $\omega_i \in W \setminus \{e\}$ *for* $i > 1$*. Further assume that each* ω_i *is reduced and does not contain any* σ_{i_n}*. We denote this simply as* $n \leftrightarrow (s, t)$*. We further write* $\beta_n \leftrightarrow \beta_{s,t}$ *and*

$$\omega_n \leftrightarrow \omega_{s,t} \tag{15}$$

if n, s, t *are connected as above. It is convenient to set* $\omega_{s,0} = e$ *for all* $s \in \{1, 2, \dots, R\}$*.*

For a fixed $s \in \{1, 2, \dots, R\}$ *we let* $s_{\mathfrak{r}}$ *denote the maximal such* t*. If there is no such decomposition we set* $t = 0$*. So, in particular,* $s_{\mathfrak{e}} = 0$*, and* $s_{\mathfrak{r}}$ *is the number of times* σ_s *occurs in* $\omega^{\mathfrak{r}}$*. Finally we set (cf. ([22]))*

$$\mathbb{U}(\mathfrak{r}) = \{(s, t) \in \mathbb{N} \times \mathbb{N}_0 \mid 1 \leq s \leq R \text{ and } 0 \leq t \leq s_{\mathfrak{r}}\}. \tag{16}$$

Notice that if $(s,t) \in \mathbb{U}(\mathfrak{r})$ then we may construct a subset $\mathbb{U}(\mathbf{s},\mathbf{t})$ of $\mathbb{U}$ by the above recipe, replacing $\omega^{\mathfrak{r}}$ by $\omega_{s,t}$. In this subset t is maximal. Likewise, if $\mathfrak{s} \leq \mathfrak{r}$ we have of course $\mathbb{U}(\mathfrak{s}) \subseteq \mathbb{U}(\mathfrak{r})$ and may set $\mathbb{U}(\mathfrak{r} \setminus \mathfrak{s}) = \mathbb{U}(\mathfrak{r}) \setminus \mathbb{U}(\mathfrak{s})$.

6. Key structures and background results

6.1. **Quantized minors.** Following a construction of classical minors by S. Fomin and A. Zelevinsky [14], the last mentioned and A. Berenstein have introduced a family of quantized minors $\Delta_{u\cdot\lambda,v\cdot\lambda}$ in [3]. These are elements of the quantized coordinate ring $\mathcal{O}_q(G)$. The results by K. Brown and K. Goodearl ([5]) were important in this process.

The element $\Delta_{u\cdot\lambda,v\cdot\lambda}$ is determined by $u,v \in W$ and a positive weight λ. We will always assume that $u \leq_L v$.

6.2. **Identifications.** There is a well-known pairing between $\mathcal{U}^{\leq}$ and $\mathcal{U}^{\geq}$ ([23]) and there is a unique bilinear form on $\mathcal{U}_q(\mathfrak{n})$. With this we can identify $(\mathcal{U}^{\geq})^*$ with $\mathcal{U}^{\geq}$. One can even define a product in $(\mathcal{U}_q(\mathfrak{n}))^*$ that makes it isomorphic to $\mathcal{U}_q(\mathfrak{n})$ [15]. We can in this way identify the elements $\Delta_{u\cdot\lambda,v\cdot\lambda}$ with elements of $\mathcal{U}^{\geq}$.

6.3. **Key results from** [3] **and** [15]**.** The quantized minors are by definitions functions on $\mathcal{U}_q(\mathfrak{g})$ satisfying certain finiteness conditions. What is needed first are certain commutation relations that they satisfy. Besides this, they can be restricted to being functions on $\mathcal{U}_q(\mathfrak{b})$ and even on $\mathcal{U}_q(\mathfrak{n})$. Our main references here are ([3]) and ([15]); the details of the following can be found in the latter.

Lemma 6.1 ([3])**.** *The element $\triangle_{u\lambda,v\lambda}$ indeed depends only on the weights $u\lambda, v\lambda$, not on the choices of u,v and their reduced words.*

Theorem 6.2 (A version of Theorem 10.2 in [3])**.** *For any $\lambda,\mu \in P^+$, and $s,s',t,t' \in W$ such that*

$$\ell(s's) = \ell(s') + \ell(s), \ell(t't) = \ell(t') + \ell(t),$$

the following holds:

$$\triangle_{s's\lambda,t'\lambda}\triangle_{s'\mu,t't\mu} = q^{(s\lambda|\mu)-(\lambda|t\mu)}\triangle_{s'\mu,t't\mu}\triangle_{s's\lambda,t'\lambda}.$$

It is very important for the following that the conditions essentially are on the Weyl group elements. The requirement on λ,μ is furthermore independent of those.

An equally important fact we need is the following q-analogue of [14, Theorem 1.17]:

Theorem 6.3 ([15], Proposition 3.2). *Suppose that for $u, v \in W$ and $i \in I$ we have $l(us_i) = l(u) + 1$ and $l(vs_i) = l(v) + 1$. Then*

$$\Delta_{us_i(\Lambda_i),vs_i(\Lambda_i)}\,\Delta_{u(\Lambda_i),v(\Lambda_i)} = (q^{-d_i})\Delta_{us_i(\Lambda_i),v(\Lambda_i)}\,\Delta_{u(\Lambda_i),vs_i(\Lambda_i)} + \prod_{j\neq i}\Delta^{-a_{ji}}_{u(\Lambda_j),v(\Lambda_j)} \tag{17}$$

holds in $\mathcal{O}_q(\mathfrak{g})$.

(That a factor q^{-d_i} must be inserted for the general case is clear.)

One considers in [15], and transformed to our terminology, modified elements

$$D_{\xi,\eta} = \triangle_{\xi,\eta}K^{-\eta}. \tag{18}$$

We suppress here the restriction map ρ, and our $K^{-\eta}$ is denoted as $\triangle^{\star}_{\eta,\eta}$ in [15]. The crucial property is that

$$K^{-\eta}\triangle_{\xi_1,\eta_1} = q^{-(\eta,\xi_1-\eta_1)}\triangle_{\xi_1,\eta_1}K^{-\eta}. \tag{19}$$

The family $D_{\xi,\eta}$ satisfies equations analogous to those in Theorem 6.2 subject to the same restrictions on the relations between the weights.

The following result is important:

Proposition 6.4 ([15]). *Up to a power of q, the following holds:*

$$Z_{c,d} = D_{\omega^{\mathfrak{r}}_{c,d-1}(\Lambda_c),\omega^{\mathfrak{r}}_{c,d}(\Lambda_c)}. \tag{20}$$

We need a small modification of the elements $D_{\xi,\eta}$ of [15]:

Definition 6.5.

$$E_{\xi,\eta} := q^{\frac{1}{4}(\xi-\eta,\xi-\eta)+\frac{1}{2}(\rho,\xi-\eta)}D_{\xi,\eta}. \tag{21}$$

It is proved in ([29]), ([38])) that $E_{\xi,\eta}$ is invariant under the dual bar anti-homomorphism augmented by $q \to q^{-1}$.

Notice that this change does not affect commutators:

$$D_1D_2 = q^{\alpha}D_2D_1 \Leftrightarrow E_1E_2 = q^{\alpha}E_2E_1 \tag{22}$$

if $E_i = q^{x_i}D_i$ for $i = 1, 2$.

Definition 6.6. *We say that*

$$E_{\xi,\eta} < E_{\xi_1,\eta_1} \tag{23}$$

if $\xi = s's\lambda$, $\eta = t'\lambda$, $\xi_1 = s'\mu$ and $\eta_1 = t't\mu$ and the conditions of Theorem 6.2 are satisfied.

The crucial equation is

Corollary 6.7.

$$E_{\xi,\eta} < E_{\xi_1,\eta_1} \Rightarrow E_{\xi,\eta}E_{\xi_1,\eta_1} = q^{(\xi-\eta,\xi_1+\eta_1)}E_{\xi_1,\eta_1}E_{\xi,\eta}. \tag{24}$$

6.4. **Connecting with the toric frames.**

Definition 6.8. *Suppose that $\triangle_i$, $i = 1, \dots, r$ is a family of mutually q-commuting elements. Let $n_1, \dots, n_r \in \mathbb{Z}$. We then set*

$$N(\prod_{i=1}^r \triangle_i^{n_i}) = q^m \prod_{i=1}^r \triangle_i^{n_i}, \tag{25}$$

where q^m is determined by the requirement that

$$q^{-m}\triangle_r^{n_r}\dots\triangle_2^{n_2}\triangle_1^{n_1} = q^m\triangle_1^{n_1}\triangle_2^{n_2}\dots\triangle_r^{n_r}. \tag{26}$$

It is easy to see that

$$\forall \mu \in S_r : N(\prod_{i=1}^r \triangle_{\mu(i)}^{n_{\mu(i)}}) = N(\prod_{i=1}^r .\triangle_i^{n_i}) \tag{27}$$

It is known through [3] that eg. the quantum minors are independent of the choices of the reduced form of $\omega_{\mathfrak{p}}^{\mathfrak{r}}$. Naturally, this carries over to $\omega^{\mathfrak{r}}$. The quadratic algebras we have encountered are independent of actual choices. In the coming definition we wish to maintain precisely the right amount of independence.

Let us now formulate Theorem 6.3 in our language while using the language and notation of toric frames from [3]. In the following Theorem we first state a formula which uses our terminology, and then we reformulate it in the last two lines in terms of toric frames M. These frames are defined by a cluster made up by certain elements of the form $E_{\xi,\eta}$ to be made more precise later.

Theorem 6.9.

$$\begin{aligned} E_{us_i\Lambda_i,vs_i\Lambda_i} &= N\left(E_{us_i\Lambda_i,v\Lambda_i}E_{u\Lambda_i,vs_i\Lambda_i}E^{-1}_{u\Lambda_i,v\Lambda_i}\right) && (28)\\ &+ N\left((\prod_{j\neq i} E^{-a_{ji}}_{u(\Lambda_j),v(\Lambda_j)})E^{-1}_{u(\Lambda_i),v(\Lambda_i)}\right) \\ &= M(E_{us_i(\Lambda_i),v(\Lambda_i)} + E_{u(\Lambda_i),vs_i(\Lambda_i)} - E_{u(\Lambda_i),v(\Lambda_i)}) && (29)\\ &+ M(\sum_{j\neq i} -a_{ji}E_{u(\Lambda_j),v(\Lambda_j)} - E_{u(\Lambda_i),v(\Lambda_i)}). && (30) \end{aligned}$$

Proof of Theorem 6.9: We first state a lemma whose proof is omitted as it is straightforward.

Lemma 6.10. *Let Δ_{ξ_k} be a family of q-commuting elements of weights ξ_k, $k = 1, \dots, r$ in the sense that for any weight b:*

$$\forall k = 1, \dots, r : K^b \Delta_{\xi_k} = q^{(b,\xi_k)} \Delta_{\xi_k} K^b. \tag{31}$$

Let α be defined by

$$\Delta_{\xi_r} \cdots \Delta_{\xi_1} = q^{-2\alpha} \Delta_{\xi_1} \cdots \Delta_{\xi_r} \tag{32}$$

Furthermore, let $b_1, \dots, b_r$ be integer weights. Then

$$\begin{gathered}(\Delta_{\xi_1}\Delta_{\xi_2} \cdots \Delta_{\xi_r}) K^{b_1} K^{b_2} \cdots K^{b_r} = \\ q^{\sum_{k<\ell}(b_k,\xi_\ell)} (\Delta_{\xi_1} K^{b_1})(\Delta_{\xi_2} K^{b_2}) \cdots (\Delta_{\xi_r} K^{b_r}), \text{ and,} \\ (\Delta_{\xi_r} K^{b_r}) \cdots (\Delta_{\xi_1} K^{b_1}) = \\ q^{-2\alpha} q^{(\sum_{k<\ell} - \sum_{\ell<k})(b_\ell,\xi_k)} (\Delta_{\xi_1} K^{b_1}) \cdots (\Delta_{\xi_r} K^{b_r}), \text{ so that} \\ (\Delta_{\xi_1} K^{b_1}) \cdots (\Delta_{\xi_r} K^{b_r}) = \\ q^{\alpha} q^{-\frac{1}{2}(\sum_{k<\ell} - \sum_{\ell<k})(b_\ell,\xi_k)} N\left((\Delta_{\xi_1} K^{b_1}) \cdots (\Delta_{\xi_r} K^{b_r})\right).\end{gathered} \tag{33}$$

Finally,

$$\begin{gathered}q^{-\alpha}(\Delta_{\xi_1}\Delta_{\xi_2} \cdots \Delta_{\xi_r}) K^{b_1} K^{b_2} \cdots K^{b_r} = \\ q^{-\frac{1}{2}(\sum_{\ell\neq k})(b_\ell,\xi_k)} N\left((\Delta_{\xi_1} K^{b_1}) \cdots (\Delta_{\xi_r} K^{b_r})\right).\end{gathered} \tag{34}$$

We apply this lemma first to the case where the elements ξ_k are taken from the set $\{-\mathrm{sign}(a_{ki})(u\Lambda_k - v\Lambda_k) \mid a_{ki} \neq 0\}$ and where each element corresponding to an $a_{ki} < 0$ is taken $-a_{ki}$ times. Then $r = \sum_{k\neq i} |a_{ji}| + 1$. The terms considered actually commute so that here, $\alpha = 0$. The weights b_k are chosen in the same fashion, but here $b_k = \mathrm{sign}(a_{ki})(v\Lambda_k)$. We have that

$$\sum_{\ell\neq k} (b_\ell, \xi_k) = \left(\sum_\ell b_\ell, \sum_k \xi_k\right) - \sum_k (b_k, \xi_k). \tag{35}$$

It follows from (2.1) that $\sum_\ell b_\ell = -vs_i\lambda_i$ and $\sum_k \xi_k = (us_i\Lambda_i - vs_i\lambda_i)$. Now observe that for all k: $-(v\Lambda_k, (u-v)\Lambda_k) = \frac{1}{2}(\xi_k, \xi_k)$. Let $\xi_0 = (us_i - vs_i)\Lambda_i$. The individual summands in $\sum_k (b_k, \xi_k)$ can be treated analogously. Keeping track of the multiplicities and signs, it follows that

$$\begin{gathered}q^{-\alpha}(\Delta_{\xi_1}\Delta_{\xi_2} \dots \Delta_{\xi_r}) K^{b_1} K^{b_2} \dots K^{b_r} = \\ q^{-\frac{1}{4}(\xi_0,\xi_0) + \frac{1}{4}\sum_k \varepsilon_k (\xi_k,\xi_k)} N\left((\Delta_{\xi_1} K^{b_1}) \dots (\Delta_{\xi_r} K^{b_r})\right).\end{gathered} \tag{36}$$

Let us turn to the term

$$q^{-d_i}\Delta_{us_i\Lambda_i,v\Lambda_i}\Delta_{u\Lambda_i,vs_i\Lambda_i}\Delta^{-1}_{u\Lambda_i,v\Lambda_i}K^{-vs_i\Lambda_i}. \tag{37}$$

We can of course set $K^{-vs_i\Lambda_i} = K^{-v\Lambda_i}K^{-vs_i\Lambda_i}K^{v\Lambda_i}$. Furthermore, it is known (and easy to see) that

$$\begin{gathered}\Delta^{-1}_{u\Lambda_i,v\Lambda_i}\Delta_{u\Lambda_i,vs_i\Lambda_i}\Delta_{us_i\Lambda_i,v\Lambda_i} = \\ q^{-2d_i}\Delta_{us_i\Lambda_i,v\Lambda_i}\Delta_{u\Lambda_i,vs_i\Lambda_i}\Delta^{-1}_{u\Lambda_i,v\Lambda_i},\end{gathered} \tag{38}$$

so that $\alpha = d_i$ here. We easily get again that $\sum_\ell b_\ell = -vs_i\lambda_i$ and $\sum_k \xi_k = (us_i\Lambda_i - vs_i\lambda_i)$.

Let us introduce elements $\tilde{E}_{\xi,\eta} = q^{\frac{1}{4}(\xi-\eta,\xi-\eta)}\Delta_{\xi,\eta}K^{-\eta}$. It then follows that (c.f. Theorem 6.3)

$$\begin{aligned}\tilde{E}_{us_i\Lambda_i,vs_i\lambda_i} &= N\left(\tilde{E}_{us_i\Lambda_i,v\Lambda_i}\tilde{E}_{u\Lambda_i,vs_i\Lambda_i}\tilde{E}^{-1}_{u\Lambda_i,v\Lambda_i}\right) \\ &+ N\left((\prod_{j\neq i}\tilde{E}^{-a_{ji}}_{u(\Lambda_j),v(\Lambda_j)})\tilde{E}^{-1}_{u(\Lambda_i),v(\Lambda_i)}\right).\end{aligned} \tag{39}$$

The elements $E_{\xi,\eta}$ differ from the elements $\tilde{E}_{\xi,\eta}$ by a factor which is q to an exponent which is linear in the weight $(\xi - \eta)$. Hence an equation identical to the above holds for these elements. □

7. Compatible pairs

We now construct some general families of quantum clusters and quantum seeds. The first, simplest, and most important, correspond to double Schubert Cells:

Let $\mathfrak{e} \leq \mathfrak{s} < \mathfrak{t} < \mathfrak{v} \leq \mathfrak{p}$.

Set

$$\begin{aligned}\mathbb{U}^{d,\mathfrak{t},\mathfrak{v}} &:= \{(a,j) \in \mathbb{U}(\mathfrak{p}) \mid a_\mathfrak{t} < j \leq a_\mathfrak{v}\}, \\ \mathbb{U}^{d,\mathfrak{t},\mathfrak{v}}_{R<} &:= \{(a,j) \in \mathbb{U}(\mathfrak{p}) \mid a_\mathfrak{t} < j < a_\mathfrak{v}\}, \\ \mathbb{U}^{u,\mathfrak{s},\mathfrak{t}} &:= \{(a,j) \in \mathbb{U}(\mathfrak{p}) \mid a_\mathfrak{s} \leq j < a_\mathfrak{t}\}, \\ \mathbb{U}^{u,\mathfrak{s},\mathfrak{t}}_{L<} &:= \{(a,j) \in \mathbb{U}(\mathfrak{p}) \mid a_\mathfrak{s} < j < a_\mathfrak{t}\}.\end{aligned}$$

Further, set

$$\mathbb{U}^{d,\mathfrak{t}} = \mathbb{U}^{d,\mathfrak{t},\mathfrak{p}}, \quad (40)$$

$$\mathbb{U}^{u,\mathfrak{t}} = \mathbb{U}^{d,\mathfrak{e},\mathfrak{t}}. \quad (41)$$

It is also convenient to define

Definition 7.1.

$$E_s(i,j) := E_{\omega^{\mathfrak{p}}_{(s,i)}\Lambda_s,\omega^{\mathfrak{p}}_{(s,j)}\Lambda_s} \quad (0\leq i<j\leq s_{\mathfrak{p}}). \quad (42)$$

For $j'\geq s_{\mathfrak{t}}$ we set

$$E^d_{\mathfrak{t}}(s,j') := E_s(s_{\mathfrak{t}},j'). \quad (43)$$

For $j'\leq s_{\mathfrak{t}}$ we set

$$E^u_{\mathfrak{t}}(s,j') := E_s(j',s_{\mathfrak{t}}). \quad (44)$$

Finally, we set

$$\mathcal{C}^d_q(\mathfrak{t},\mathfrak{v}) = \{E^d_{\mathfrak{t}}(s,j') \mid (s,j')\in\mathbb{U}^{d,\mathfrak{t},\mathfrak{v}}\}, \quad (45)$$

$$\mathcal{C}^u_q(\mathfrak{s},\mathfrak{t}) = \{E^u_{\mathfrak{t}}(s,j');(s,j')\in\mathbb{U}^{u,\mathfrak{s},\mathfrak{t}}\}, \quad (46)$$

$$\mathcal{C}^d_q(\mathfrak{t}) = \mathcal{C}^d_q(\mathfrak{t},\mathfrak{p}), \text{ and} \quad (47)$$

$$\mathcal{C}^u_q(\mathfrak{t}) = \mathcal{C}^u_q(\mathfrak{s},\mathfrak{t}). \quad (48)$$

It is clear that $\mathcal{C}^d_q(\mathfrak{t},\mathfrak{v})\subseteq\mathcal{C}^d_q(\mathfrak{t})$ for any $\mathfrak{v}>\mathfrak{t}$ and $\mathcal{C}^u_q(\mathfrak{s},\mathfrak{t})\subseteq\mathcal{C}^u_q(\mathfrak{t})$ for any $\mathfrak{s}<\mathfrak{t}$.

Lemma 7.2. *The elements in the set $\mathcal{C}^d_q(\mathfrak{t})$ are q-commuting and the elements in the set $\mathcal{C}^u_q(\mathfrak{t})$ are q-commuting.*

The proof is omitted as it is very similar to the proof of Proposition 7.19 which comes later.

Definition 7.3. *$\mathcal{A}^d_q(\mathfrak{t},\mathfrak{v})$ denotes the $\mathbb{C}$-algebra generated by $\mathcal{C}^d_q(\mathfrak{t},\mathfrak{v})$ and $\mathcal{A}^u_q(\mathfrak{s},\mathfrak{t})$ denotes the $\mathbb{C}$-algebra generated by $\mathcal{C}^u_q(\mathfrak{s},\mathfrak{t})$. Further, $\mathcal{F}^d_q(\mathfrak{t},\mathfrak{v})$ and $\mathcal{F}^u_q(\mathfrak{s},\mathfrak{t})$ denote the corresponding skew-fields of fractions. Likewise, $\mathbf{L}^d_q(\mathfrak{t},\mathfrak{v})$ and $\mathbf{L}^u_q(\mathfrak{s},\mathfrak{t})$ denote the respective Laurent quasi-polynomial algebras. Finally, $\mathcal{L}^d_q(\mathfrak{t},\mathfrak{v})$ and $\mathcal{L}^u_q(\mathfrak{s},\mathfrak{t})$ denote the symplectic forms associated with the clusters $\mathcal{C}^d_q(\mathfrak{t},\mathfrak{v})$, and $\mathcal{C}^u_q(\mathfrak{s},\mathfrak{t})$, respectively.*

Definition 7.4. *Whenever* $\mathfrak{a} < \mathfrak{b}$*, we set*

$$\forall s \in Im(\pi_{\mathfrak{b}}) : \det_s^{\mathfrak{a},\mathfrak{b}} := E_{\omega^{\mathfrak{a}}\Lambda_s, \omega^{\mathfrak{b}}\Lambda_s}. \tag{49}$$

We conclude in particular that

Proposition 7.5. *The elements* $\det_s^{\mathfrak{t},\mathfrak{p}}$ *q-commute with all elements in the algebra* $\mathcal{A}_q^d(\mathfrak{t})$ *and the elements* $\det_s^{\mathfrak{c},\mathfrak{t}}$ *q-commute with all elements in the algebra* $\mathcal{A}_q^d(\mathfrak{t})$*.*

Definition 7.6. *An element* C *in a quadratic algebra* $\mathcal{A}$ *that q-commutes with all the generating elements is said to be covariant.*

As a small aside, we mention the following easy generalization of the result in ([22]):

Proposition 7.7. *It* $\mathfrak{a} < \mathfrak{b}$*, then the spaces* $\mathcal{A}_q^u(\mathfrak{a},\mathfrak{b})$ *and* $\mathcal{A}_q^d(\mathfrak{a},\mathfrak{b})$ *are quadratic algebras. In both cases, the center is given by* $Ker(\omega^{\mathfrak{a}} + \omega^{\mathfrak{b}})$*. The semi-group of covariant elements in generated by* $\{\det_s^{\mathfrak{a},\mathfrak{b}} \mid s \in Im(\pi_{\mathfrak{b}})\}$*.*

We now construct some elements in $\mathbf{L}_q^d(\mathfrak{t})$ and $\mathbf{L}_q^u(\mathfrak{t})$ of fundamental importance. They are indeed monomials in the elements of $\left[\mathcal{C}_q^d(\mathfrak{t})\right]^{\pm 1}$ and $\left[\mathcal{C}_q^u(\mathfrak{t})\right]^{\pm 1}$, respectively.

First a technical definition:

Definition 7.8. $p(a,j,k)$ *denotes the largest non-negative integer for which*

$$\omega^{\mathfrak{p}}_{(k,p(a,j,k))}\Lambda_k = \omega^{\mathfrak{p}}_{(a,j)}\Lambda_k.$$

We also allow $E_a(j,j)$ which is defined to be 1.

Here are then the first building blocks:

Definition 7.9.

$$\forall (a,j) \in \mathbb{U}^{d,\mathfrak{t}} :$$
$$H^d_{\mathfrak{t}}(a,j) := E_a(a_{\mathfrak{t}},j)E_a(a_{\mathfrak{t}},j-1)\prod_{a_{ka}<0} E_k(k_{\mathfrak{t}},p(a,j,k))^{a_{ka}} \tag{50}$$
$$\forall (a,j) \in \mathbb{U}^{d,\mathfrak{t}} \text{ with } j < a_{\mathfrak{p}} :$$
$$B^d_{\mathfrak{t}}(a,j) := H^d_{\mathfrak{t}}(a,j)(H^d_{\mathfrak{t}}(a,j+1))^{-1}. \tag{51}$$

The terms $E(k_{\mathfrak{t}},p(a,j,k))$ and $E_a(a_{\mathfrak{t}},j-1)$ are well-defined but may become equal to 1. Also notice that, where defined, $H^d_{\mathfrak{t}}(a,j), B^d_{\mathfrak{t}}(a,j) \in \mathbf{L}_q^d(\mathfrak{t})$.

Lemma 7.10. *If $E_{\xi,\eta} < H^d_{\mathfrak{t}}(a,j)$ in the sense that it is less than or equal to each factor E_{ξ_1,η_1} of $H^d_{\mathfrak{t}}(a,j)$ (and $<$ is defined in (23)), then*

$$E_{\xi,\eta}H^d_{\mathfrak{t}}(a,j) = q^{(\xi-\eta,\omega^{\mathfrak{t}}(\alpha_a))}H^d_{\mathfrak{t}}(a,j)E_{\xi,\eta}. \tag{52}$$

If $E_{\xi,\eta} \geq H^d_{\mathfrak{t}}(a,j)$, then

$$E_{\xi,\eta}H^d_{\mathfrak{t}}(a,j) = q^{(-\xi-\eta,\omega^{\mathfrak{t}}(\alpha_a))}H^d_{\mathfrak{t}}(a,j)E_{\xi,\eta}. \tag{53}$$

Proof. This follows from (23) by observing that we have the following pairs (ξ_1,η_1) occurring in $H^d_{\mathfrak{t}}(a,j)$:

$$(\omega^{\mathfrak{t}}\Lambda_a, \omega(a,j)\Lambda_a), (\omega^{\mathfrak{t}}\Lambda_a, \omega(a,j)\sigma_a\Lambda_a),$$

and

$$(-\omega^{\mathfrak{t}}\Lambda_k, -\omega(a,j)\Lambda_k) \text{ with multiplicity } (-a_{ka}).$$

Furthermore, as in (2.1), $\Lambda_a + \sigma_a\Lambda_a + \sum_k a_{ka}\Lambda_k = 0$ and, equivalently, $2\Lambda_a + \sum_k a_{ka}\Lambda_k = \alpha_a$. □

Proposition 7.11. *$\forall (a,j),(b,j') \in \mathbb{U}^{d,\mathfrak{t}}, j < a_{\mathfrak{p}}$ the following holds:*

$$E^d_{\mathfrak{t}}(b,j')B^d_{\mathfrak{t}}(a,j) = q^{-2(\Lambda_a,\alpha_a)\delta_{j,j'}\delta_{a,b}}B^d_{\mathfrak{t}}(a,j)E^d_{\mathfrak{t}}(b,j'). \tag{54}$$

Proof. It is clear from the formulas (52-53) that if an element $E_{\xi,\eta}$ either is bigger than all factors in $B^d_{\mathfrak{t}}(s,j)$ or smaller than all factors, then it commutes with this element. The important fact now is that the ordering is independent of the fundamental weights Λ_i - it depends only on the Weyl group elements. The factors in any $H^d_{\mathfrak{t}}$ are, with a fixed $\mathfrak{t}$, of the form $E_{\omega^{\mathfrak{t}}\Lambda_i,\omega\Lambda_i}$ or $E_{\omega^{\mathfrak{t}}\Lambda_a,\omega\circ\sigma_a\Lambda_a}$ for some $\omega \geq \omega^{\mathfrak{t}}$. The elements $E_{\xi,\eta} = E^d_{\mathfrak{t}}(b,j')$ we consider thus satisfy the first or the second case in Lemma 7.11 for either terms $H^d_{\mathfrak{t}}(a,j)$ and $H^d_{\mathfrak{t}}(a,j+1)$. Clearly, we then need only consider the in-between case $H^d_{\mathfrak{t}}(a,j) \leq E_{\xi,\eta} \leq H^d_{\mathfrak{t}}(a,j+1)$, and here there appears a factor $q^{-2(\xi,\omega^{\mathfrak{t}}(\alpha_a))}$ in the commutator with $\xi = \omega^{\mathfrak{t}}\Lambda_b$. This accounts for the term $-2(\Lambda_a,\alpha_a)\delta_{a,b}$. Finally, if $a = b$ the previous assumption forces $j = j'$. □

Let us choose an enumeration

$$\mathcal{C}^d_q(\mathfrak{t}) = \{c_1, c_2, \ldots, c_N\} \tag{55}$$

so that each $(a,j) \leftrightarrow k$ and let us use the same enumeration of the elements $B^d_{\mathfrak{t}}(a,j)$. Set, for now $B^d_{\mathfrak{t}}(a,j) = b_k$ if $(a,j) \leftrightarrow k$. Let us also agree that the, say n, non-mutable elements $\det_s^{\mathfrak{t},\mathfrak{p}}$ of $\mathcal{C}_q^d(\mathfrak{t})$ are written last, say numbers $N-n+1, N-n+2, \dots, N-n$. Then, as defined,

$$\forall j = 1, \dots, N-n : b_j = q^{\alpha_j} \prod_k c_k^{b_{kj}} \tag{56}$$

for some integers b_{kj} and some, here inconsequential, factor q^{α_j}. The symplectic form yields a matrix which we, abusing notation slightly, also denote $\mathcal{L}_q^d(\mathfrak{t})$ such that

$$\forall i,j = 1, \dots, N : \left(\mathcal{L}_q^d(\mathfrak{t})\right)_{ij} = \lambda_{ij} \tag{57}$$

and

$$\forall i,j = 1, \dots, N : c_j c_j = q^{\lambda_{ij}} c_j c_i. \tag{58}$$

Similarly, we let $\mathcal{B}_q^d(\mathfrak{t})$ denote the matrix

$$\forall i = 1, \dots, N \ \forall j = 1, \dots, N-n : \left(\mathcal{B}_q^d(\mathfrak{t})\right)_{ij} = b_{ij}. \tag{59}$$

Then, where defined,

$$c_i b_j = \prod_k q^{\lambda_{ik} b_{kj}} b_j c_i, \tag{60}$$

and Proposition 7.11 may then be restated as

$$\forall i = 1, \dots, N \ \forall j = 1, \dots, N-n : \sum_k \lambda_{ik} b_{kj} = -2(\Lambda_s, \alpha_s)\delta_{ij}, \tag{61}$$

where we assume that $i \leftrightarrow (s, \ell)$.

We have then established

Theorem 7.12. *The pair $(\mathcal{L}_q^d(\mathfrak{t}), \mathcal{B}_q^d(\mathfrak{t}))$ is a compatible pair and hence,*

$$\mathcal{Q}_q^d(\mathfrak{t}) := (\mathcal{C}_q^d(\mathfrak{t}), \mathcal{L}_q^d(\mathfrak{t}), \mathcal{B}_q^d(\mathfrak{t})) \tag{62}$$

is a quantum seed with the n non-mutable elements $\det_s^{\mathfrak{t},\mathfrak{p}}$, $(s, s_{\mathfrak{p}}) \in \mathbb{U}^d(\mathfrak{t})$. The entries of the diagonal of the matrix $\tilde{D} = (\mathcal{B}_q^d(\mathfrak{t}))^T \mathcal{L}_q^d(\mathfrak{t})$ are in the set $\{2(\Lambda_s, \alpha_s) \mid s = 1, \dots, R\}$.

It $\mathfrak{v} > \mathfrak{t}$, we let $(\mathcal{L}_q^d(\mathfrak{t}, \mathfrak{v}), \mathcal{B}_q^d(\mathfrak{t}, \mathfrak{v}))$ denote the part of the compatible pair $(\mathcal{L}_q^d(\mathfrak{t}), \mathcal{B}_q^d(\mathfrak{t}))$ that corresponds to the cluster $\mathcal{C}_q^d(\mathfrak{t}, \mathfrak{v})$ and we let $\mathcal{Q}_q^d(\mathfrak{t}, \mathfrak{v})$ be the corresponding triple. It is then obvious by simple restriction, that we in fact have obtained

Theorem 7.13. *The pair* $(\mathcal{L}_q^d(\mathfrak{t},\mathfrak{v}),\mathcal{B}_q^d(\mathfrak{t},\mathfrak{v}))$ *is a compatible pair and hence,*

$$\mathcal{Q}_q^d(\mathfrak{t},\mathfrak{v}) := (\mathcal{C}_q^d(\mathfrak{t},\mathfrak{v}),\mathcal{L}_q^d(\mathfrak{t},\mathfrak{v}),\mathcal{B}_q^d(\mathfrak{t},\mathfrak{v})) \tag{63}$$

is a quantum seed with the n *non-mutable elements* $\det_s^{\mathfrak{t},\mathfrak{v}}$, $(s,s_{\mathfrak{v}})\in\mathbb{U}^d(\mathfrak{t},\mathfrak{v})$.

The case of $\mathcal{C}_q^u(\mathfrak{t})$ is completely analogous: Define

Definition 7.14.

$$H_{\mathfrak{t}}^u(a,j) := E_a(j,a_{\mathfrak{t}})E_a(j-1,a_{\mathfrak{t}})\prod_{a_{ka}<0}E_k(p(a,j,k),k_{\mathfrak{t}})^{a_{ka}} \;(1\leq j<a_{\mathfrak{t}}),$$

$$B_{\mathfrak{t}}^u(a,j) := H_{\mathfrak{t}}^u(a,j+1)(H_{\mathfrak{t}}^u(a,j))^{-1} \;(1\leq j<a_{\mathfrak{t}}). \tag{64}$$

The terms $E(p(a,j,k),k_{\mathfrak{t}})$ *are well-defined but may become equal to* 1*. Notice also the exponents on the terms* $H_{\mathfrak{t}}^u$*.*

The terms $E(p(a,j,k),k_{\mathfrak{t}})$ are well-defined but may become equal to 1. As defined, $H_{\mathfrak{t}}^u(a,j)$, and $B_{\mathfrak{t}}^u(a,j)$ are in $\mathbf{L}_q^u(\mathfrak{t})$.

Proposition 7.15. $\forall (a,j),(b,j')\in\mathbb{U}^{u,\mathfrak{t}}, 1\leq j$ *the following holds:*

$$E_{\mathfrak{t}}^u(b,j')B_{\mathfrak{t}}^u(a,j) = q^{2(\Lambda_a,\alpha_a)\delta_{j,j'}\delta_{a,b}}B_{\mathfrak{t}}^u(a,j)E_{\mathfrak{t}}^u(b,j'). \tag{65}$$

We then get in a similar way

Theorem 7.16. *The pair* $(\mathcal{L}_q^u(\mathfrak{t}),\mathcal{B}_q^u(\mathfrak{t}))$ *is a compatible pair and hence,*

$$\mathcal{Q}_q^u(\mathfrak{t}) := (\mathcal{C}_q^u(\mathfrak{t}),\mathcal{L}_q^u(\mathfrak{t}),\mathcal{B}_q^u(\mathfrak{t})) \tag{66}$$

is a quantum seed with the n *non-mutable elements* $\det_s^{\mathfrak{e},\mathfrak{t}}$, $(s,s_{\mathfrak{t}})\in\mathbb{U}^u(\mathfrak{t})$.

Naturally, we even have

Theorem 7.17. *The pair* $(\mathcal{L}_q^u(\mathfrak{s},\mathfrak{t}),\mathcal{B}_q^u(\mathfrak{s},\mathfrak{t}))$ *is a compatible pair and hence,*

$$\mathcal{Q}_q^u(\mathfrak{s},\mathfrak{t}) := (\mathcal{C}_q^u(\mathfrak{s},\mathfrak{v}),\mathcal{L}_q^u(\mathfrak{s},\mathfrak{t}),\mathcal{B}_q^u(\mathfrak{s},\mathfrak{t})) \tag{67}$$

is a quantum seed with the n *non-mutable elements* $\det_s^{\mathfrak{s},\mathfrak{t}}$, $(s,s_{\mathfrak{s}})\in\mathbb{U}^u(\mathfrak{s},\mathfrak{t})$.

We now wish to consider more elaborate seeds. The first generalization is the most important:

Let

$$\mathfrak{e}\leq\mathfrak{a}\leq\mathfrak{b}\leq\mathfrak{c}\leq\mathfrak{p},\;\text{ but }\mathfrak{a}\neq\mathfrak{c}. \tag{68}$$

$$\mathcal{C}_q^d(\mathfrak{a},\mathfrak{b},\mathfrak{c}) := \{E_{\mathfrak{a}}^d(s,j) \mid (a,j)\in(\mathbb{U}^{d,\mathfrak{b}}\setminus\mathbb{U}^{d,\mathfrak{c}})=\mathbb{U}^{d,\mathfrak{b},\mathfrak{c}}\}, \tag{69}$$

$$\mathcal{C}_q^u(\mathfrak{a},\mathfrak{b},\mathfrak{c}) := \{E_{\mathfrak{c}}^u(s,j) \mid (s,j)\in(\mathbb{U}^{u,\mathfrak{b}}\setminus\mathbb{U}^{u,\mathfrak{a}})=\mathbb{U}^{u,\mathfrak{a},\mathfrak{b}}\}. \tag{70}$$

In (69), $\mathfrak{a}=\mathfrak{b}$ is allowed, and in (70), $\mathfrak{b}=\mathfrak{c}$ is allowed.

Definition 7.18.

$$\begin{aligned}\mathcal{C}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c}) &:= \mathcal{C}_q^d(\mathfrak{a},\mathfrak{b},\mathfrak{c})\cup\mathcal{C}_q^u(\mathfrak{a},\mathfrak{b},\mathfrak{b}),\\ \mathcal{C}_q^o(\mathfrak{a},\mathfrak{b},\mathfrak{c}) &:= \mathcal{C}_q^u(\mathfrak{a},\mathfrak{b},\mathfrak{c})\cup\mathcal{C}_q^d(\mathfrak{b},\mathfrak{b},\mathfrak{c}).\end{aligned}$$

Proposition 7.19. *The elements of* $\mathcal{C}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ *and* $\mathcal{C}_q^o(\mathfrak{a},\mathfrak{b},\mathfrak{c})$*, respectively, q-commute.*

Proof. The two cases are very similar, so we only prove it for the first case. We examine 3 cases, while using the following mild version of Theorem 6.2: $\triangle_{s's\lambda,t'\lambda}$ and $\triangle_{s'\mu,t't\mu}$ q commute for any $\lambda,\mu\in P^+$, and $s,s',t,t'\in W$ for which $\ell(s's)=\ell(s')+\ell(s),\ell(t't)=\ell(t')+\ell(t)$.

Case 1: $E_{\mathfrak{a}}^d(s,t)$ and $E_{\mathfrak{a}}^d(s_1,t_1)$ for $(s,t)\in\mathbb{U}^{d,\mathfrak{b},\mathfrak{c}}$ and $(s,t)<(s_1,t_1)$: Set $\lambda=\Lambda_s,\mu=\Lambda_{s_1}$, $s=1, s'=\omega^{\mathfrak{a}}$, and $t'=\omega^{\mathfrak{c}}(s,t), t't=\omega^{\mathfrak{c}}(s_1,t_1)$.

Case 2: $E_{\mathfrak{b}}^u(s,t)$ and $E_{\mathfrak{b}}^u(s_1,t_1)$ for $(s,t)\in\mathbb{U}^{u,\mathfrak{a},\mathfrak{b}}$ and $(s,t)>(s_1,t_1)$: Set $\lambda=\Lambda_s,\mu=\Lambda_{s_1}$, $t=1$, $t'=\omega^{\mathfrak{b}}$ and $s'=\omega^{\mathfrak{p}}(s_1,t_1), s's=\omega^{\mathfrak{r}}(s,t)$.

Case 3: $E_{\mathfrak{b}}^u(s,t)$ and $E_{\mathfrak{a}}^d(s_1,t_1)$ for $(s,t)\in\mathbb{U}^{u,\mathfrak{a},\mathfrak{b}}$ and $(s_1,t_1)\in\mathbb{U}^{d,\mathfrak{b},\mathfrak{c}}$: Set $\lambda=\Lambda_s,\mu=\Lambda_{s_1}$, $s'=\omega^{\mathfrak{a}}$, $s=\omega^{\mathfrak{p}}(s,t)$, $t'=\omega^{\mathfrak{b}}$ and $t't=\omega^{\mathfrak{p}}(s_1,t_1)$. □

Notice that the ordering in $\mathbb{U}^{u,\mathfrak{a},\mathfrak{b}}$ (Case 2) is the opposite of that of the two other cases.

We also define, for $\mathfrak{a}<\mathfrak{b}$,

$$\begin{aligned}\mathcal{C}_q^u(\mathfrak{a},\mathfrak{b}) &= \mathcal{C}_q^u(\mathfrak{a},\mathfrak{b},\mathfrak{b}), \text{ and}\\ \mathcal{C}_q^d(\mathfrak{a},\mathfrak{b}) &= \mathcal{C}_q^d(\mathfrak{a},\mathfrak{a},\mathfrak{b}).\end{aligned} \tag{71}$$

We let $\mathcal{L}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ and $\mathcal{L}_q^o(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ denote the corresponding symplectic matrices. We proceed to construct compatible pairs and give the details for just $\mathcal{C}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})$. We will be completely explicit except in the special cases $E^u_{\omega^{\mathfrak{a}}\Lambda_s,\omega^{\mathfrak{b}}\Lambda_s}=\det_s^{\mathfrak{a},\mathfrak{b}}$ where we only give a recipe for $B_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}})$. Notice, however, the remark following (77).

$$B_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,j) := \begin{cases} B^d_{\mathfrak{a}}(s,j) & \text{if } (s,j)\in \mathbb{U}^{d,\mathfrak{b},\mathfrak{c}}_{R<} \\ & \\ B^u_{\mathfrak{b}}(s,j) & \text{if } (s,j)\in \mathbb{U}^{u,\mathfrak{a},\mathfrak{b}}_{L<} \end{cases}. \tag{72}$$

We easily get from the preceding propositions:

Proposition 7.20. *Let $E(b,j')\in\mathcal{C}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ and let $B_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,j)$ be as in the previous equation. Then*

$$E(b,j')B_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,j) = q^{-2(\Lambda_s,\alpha_s)\delta_{j,j'}\delta_{s,b}}B_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,j)E(b,j'), \tag{73}$$

and $B_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,j)$ is in the algebra $\mathcal{A}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ generated by the elements of $\mathcal{C}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})$.

This then leaves the positions $(s,s_{\mathfrak{c}})\in\mathbb{U}^{d,\mathfrak{b},\mathfrak{c}}$ and $(s,s_{\mathfrak{a}})\in\mathbb{U}^{u,\mathfrak{a},\mathfrak{b}}$ to be considered. Here, the first ones are considered as the non-mutable elements. In the ambient space $\mathcal{A}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})$, the positions in remaining cases define elements that are, in general, mutable.

The elements in these cases are of the form $E_{\omega^{\mathfrak{a}}\Lambda_s,\omega^{\mathfrak{b}}\Lambda_s}$ for some s. To give a recipe we define the following elements in $\mathcal{A}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})$:

$$\tilde{B}_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}}) := \tag{74}$$
$$\left(H^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}+1)H^d_{\mathfrak{a}}(s,s_{\mathfrak{b}}+1)\right)^{-1}E_s(s_{\mathfrak{a}},s_{\mathfrak{b}})^2\prod\nolimits_{a_{ka}<0}E_k(k_{\mathfrak{a}},k_{\mathfrak{b}})^{a_{ks}}.$$

If $\omega(s,s_{\mathfrak{a}}+1)=\omega^{\mathfrak{a}}\omega_x\sigma_s$ and $\omega(s,s_{\mathfrak{b}}+1)=\omega^{\mathfrak{b}}\omega_y\sigma_s$, and if we set $u=\omega^{\mathfrak{a}}\omega_x$, $v=\omega^{\mathfrak{b}}\omega_y$ this takes the simpler form

$$\tilde{B}_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}}) = E^{-1}_{u\sigma_s\Lambda_s,v\Lambda_s}E^{-1}_{u\Lambda_s,v\sigma_s\Lambda_s}\prod_{a_{ks<0}}E^{-a_{ks}}_{\omega^{\mathfrak{a}}\Lambda_k,v\Lambda_k}\prod_{a_{ks<0}}E^{-a_{ks}}_{u\Lambda_k,\omega^{\mathfrak{b}}\Lambda_k}\prod_{a_{ks<0}}E^{a_{ks}}_{\omega^{\mathfrak{a}}\Lambda_k,\omega^{\mathfrak{b}}\Lambda_k}. \tag{75}$$

Proposition 7.21.

$$\forall\ell: E_{\omega^{\mathfrak{a}}\Lambda_\ell,\omega^{\mathfrak{b}}\Lambda_\ell}\tilde{B}_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}}) = q^{-2\delta_{\ell.s}(\lambda_s,\alpha_s)}\tilde{B}_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}})E_{\omega^{\mathfrak{a}}\Lambda_\ell,\omega^{\mathfrak{b}}\Lambda_\ell}. \tag{76}$$

Besides this, $\tilde{B}_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}})$ commutes with everything in the cluster except possibly elements of the form

$$E_{\omega^{\mathfrak{a}}\Lambda_\ell,\omega^{\mathfrak{b}}\tilde{\omega}_y\Lambda_\ell},\ \text{and } E_{\omega^{\mathfrak{a}}\tilde{\omega}_x\Lambda_\ell,\omega^{\mathfrak{b}}\Lambda_\ell},$$

with $1<\tilde{\omega}_x<\omega_x$ and $1<\tilde{\omega}_y<\omega_y$.

The exceptional terms above are covered by Proposition 7.20 which means that we can in principle make a modification $\tilde{B}_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}}) \to B_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}})$ where the latter expression commutes with everything except $E_{\omega^{\mathfrak{a}}\Lambda_s,\omega^{\mathfrak{b}}\Lambda_s}$ where we get a factor $q^{-2(\Lambda_s,\alpha_s)}$.

If $\omega_y = 1$ we get a further simplification where now $u = \omega^{\mathfrak{a}}\omega_x$ and $v = \omega^{\mathfrak{b}}$:

$$\tilde{B}_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}}) = E^{-1}_{u\sigma_s\Lambda_s,v\Lambda_s}E^{-1}_{u\Lambda_s,v\sigma_s\Lambda_s}\prod_{a_{ks}<0}E^{-a_{ks}}_{u\Lambda_k,v\Lambda_k}. \tag{77}$$

Here we actually have $\tilde{B}_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}}) = B_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}})$, and this expression has the exact form needed for the purposes of the next section.

We let $\mathcal{B}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ and $\mathcal{B}_q^o(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ denote the corresponding symplectic matrices and can now finally define our quantum seeds:

$$\mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c}) := (\mathcal{C}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c}),\mathcal{L}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c}),\mathcal{B}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})). \tag{78}$$

Definition 7.22.

$$\mathcal{Q}_q^o(\mathfrak{a},\mathfrak{b},\mathfrak{c}) := (\mathcal{C}_q^o(\mathfrak{a},\mathfrak{b},\mathfrak{c}),\mathcal{L}_q^o(\mathfrak{a},\mathfrak{b},\mathfrak{c}),\mathcal{B}_q^o(\mathfrak{a},\mathfrak{b},\mathfrak{c})). \tag{79}$$

According to our analysis above we have established

Theorem 7.23. *They are indeed seeds. The non-mutable elements are in both cases the elements* $\det_s^{\mathfrak{a},\mathfrak{c}}; s \in Im(\pi_{\omega^{\mathfrak{c}}})$.

Let us finally consider a general situation where we are given a finite sequence of elements $\{\omega^{\mathfrak{r}_i}\}_{i=1}^n \in W^p$ such that

$$\mathfrak{e} \le \mathfrak{r}_1 < \cdots < \mathfrak{r}_n \le \mathfrak{p}. \tag{80}$$

Observe that

$$\forall (s,t) \in \mathbb{U}(\mathfrak{r}_k) : \omega^{\mathfrak{r}_k}_{(s,t)} = \omega^{\mathfrak{p}}_{(s,t)}. \tag{81}$$

It may of course well happen that for some a, and some $\mathfrak{r}_i < \mathfrak{r}_j$,

$$\omega^{\mathfrak{r}_i}\Lambda_a = \omega^{\mathfrak{r}_j}\Lambda_a. \tag{82}$$

Definition 7.24. *Given (80) we define*

$$\begin{aligned}\mathcal{C}_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n) &= \mathcal{C}_q^d(\mathfrak{r}_1,\mathfrak{r}_{n-1},\mathfrak{r}_n)\cup\mathcal{C}_q^u(\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_{n-1})\cup\dots \\ &= \bigcup_{0<2i\le n}\mathcal{C}_q^d(\mathfrak{r}_i,\mathfrak{r}_{n-i},\mathfrak{r}_{n-i+1})\cup\bigcup_{0<2j\le n-1}\mathcal{C}_q^u(\mathfrak{r}_j,\mathfrak{r}_{j+1},\mathfrak{r}_{n-j}).\end{aligned} \tag{83}$$

It is also convenient to consider

$$\begin{aligned}\mathcal{C}_q^o(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n) &= \mathcal{C}_q^u(\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_n)\cup\mathcal{C}_q^d(\mathfrak{r}_2,\mathfrak{r}_{n-1},\mathfrak{r}_n)\cup\dots \\ &= \bigcup_{0<2i\leq n}\mathcal{C}_q^u(\mathfrak{r}_i,\mathfrak{r}_{i+1},\mathfrak{r}_{n-i+1})\cup\bigcup_{0<2j\leq n-1}\mathcal{C}_q^d(\mathfrak{r}_{j+1},\mathfrak{r}_{n-j},\mathfrak{r}_{n-j+1}).\end{aligned}\tag{84}$$

Notice that

$$\begin{aligned}\mathcal{C}_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n) &= \mathcal{C}_q^d(\mathfrak{r}_1,\mathfrak{r}_{n-1},\mathfrak{r}_n)\cup\mathcal{C}_q^o(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-2},\mathfrak{r}_{n-1}) \\ \mathcal{C}_q^o(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n) &= \mathcal{C}_q^u(\mathfrak{r}_1,\mathfrak{r}_2,\mathfrak{r}_n)\cup\mathcal{C}_q(\mathfrak{r}_2,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n)\end{aligned}\tag{85}$$

For the last equations, notice that $\mathcal{C}_q^d(\mathfrak{e},\mathfrak{r},\mathfrak{r})=\emptyset=\mathcal{C}_q^d(\mathfrak{r},\mathfrak{r},\mathfrak{r})$.

Proposition 7.25. *The spaces*

$$\mathcal{C}_q^o(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n) \text{ and } \mathcal{C}_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n)\tag{86}$$

each consists of q-commuting elements.

Proof. This is proved in the same way as Proposition 7.19. □

Our goal is to construct seeds out of these clusters using (and then generalizing) Proposition 7.23.

With Proposition 7.25 at hand, we are immediately given the corresponding symplectic matrices

$$\mathcal{L}_q^o(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n) \text{ and } \mathcal{L}_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n).\tag{87}$$

The construction of the accompanying B-matrices

$$\mathcal{B}_q^o(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n) \text{ and } \mathcal{B}_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n)\tag{88}$$

takes a little more work, though in principle it is straightforward. The idea is in both cases to consider an element in the cluster as lying in a space

$$\mathcal{C}_q^d(\mathfrak{r}_i,\mathfrak{r}_{n-i},\mathfrak{r}_{n-i+1})\cup\mathcal{C}_q^u(\mathfrak{r}_i,\mathfrak{r}_{i+1},\mathfrak{r}_{n-i}) \subseteq \mathcal{C}_q(\mathfrak{r}_i,\mathfrak{r}_{n-i},\mathfrak{r}_{n-i+1}) \text{ or}\tag{89}$$

$$\mathcal{C}_q^u(\mathfrak{r}_i,\mathfrak{r}_{i+1},\mathfrak{r}_{n-i+1})\cup\mathcal{C}_q^d(\mathfrak{r}_{i+1},\mathfrak{r}_{n-i},\mathfrak{r}_{n-i+1}) \subseteq \mathcal{C}_q^o(\mathfrak{r}_i,\mathfrak{r}_{i+1},\mathfrak{r}_{n-i+1})\tag{90}$$

as appropriate. Then we can use the corresponding matrices $\mathcal{B}_q(\mathfrak{r}_i,\mathfrak{r}_{n-i},\mathfrak{r}_{n-i+1})$ or $\mathcal{B}_q^o(\mathfrak{r}_i,\mathfrak{r}_{i+1},\mathfrak{r}_{n-i+1})$ in the sense that one can extend these matrices to the full rank by inserting rows of zeros. In this way, we can construct columns even for the troublesome elements of the form $E(a_{\mathfrak{r}_i},a_{\mathfrak{r}_j})$ that may belong to such spaces. Indeed, we may start by including $E(a_{\mathfrak{r}_{\frac{n+0}{2}}},a_{\mathfrak{r}_{\frac{n+2}{2}}})$ (n even) or $E(a_{\mathfrak{r}_{\frac{n-1}{2}}},a_{\mathfrak{r}_{\frac{n+1}{2}}})$ (n odd) in a such space in which they may be seen as mutable. Then these spaces have new non-mutable elements which can be handled by viewing them in appropriate spaces. The only ones which we cannot capture are the elements $\det_s^{\mathfrak{r}_1,\mathfrak{r}_n} = E(s_{\mathfrak{r}_1}, s_{\mathfrak{r}_n})$.

Definition 7.26. *In both cases, the elements* $\det_s^{\mathfrak{r}_1,\mathfrak{r}_n}$, $s \in Im(\pi_{\mathfrak{r}_1})$ *are the non-mutable elements. We let* $\mathcal{N}_q(\mathfrak{r}_1, \mathfrak{r}_n)$ *denote the set of these.*

Proposition 7.27.

$$\mathcal{Q}_q(\mathfrak{r}_1, \dots, \mathfrak{r}_{n-1}, \mathfrak{r}_n) \text{ and } \mathcal{Q}^o_q(\mathfrak{r}_1, \dots, \mathfrak{r}_{n-1}, \mathfrak{r}_n) \tag{91}$$

are quantum seeds.

8. Mutations

Here is the fundamental setup: Let $\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}, \omega^{\mathfrak{c}} \in W^p$ satisfy

$$\mathfrak{a} < \mathfrak{c} \text{ and } \mathfrak{a} \leq \mathfrak{b} \leq \mathfrak{c}. \tag{92}$$

Definition 8.1. *A root* $\gamma \in \triangle^+(\mathfrak{c})$ *is an* **increasing-mutation site** *of* $\omega^{\mathfrak{b}} \in W^p$ *(in reference to* $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$*) if there exists a reduced form of* $\omega^{\mathfrak{c}}$ *as*

$$\omega^{\mathfrak{c}} = \hat{\omega}\sigma_\gamma\omega^{\mathfrak{b}}. \tag{93}$$

Let $W^p \ni \omega^{\mathfrak{b}'} = \sigma_\gamma\omega^{\mathfrak{b}}$. *It follows that*

$$\omega^{\mathfrak{b}'} = \omega^{\mathfrak{b}}\sigma_{\alpha_s} \tag{94}$$

for a unique $s \in Im(\pi_{\mathfrak{b}'})$. *Such a site will henceforth be called an* $\mathfrak{m}^+$ *site.*

We will further say that γ *is a* **decreasing-mutation site**, *or* $\mathfrak{m}^-$ *site (in reference to* $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$*) of* $\omega^{\mathfrak{b}} \in W^p$ *in case there exists a rewriting of* $\omega^{\mathfrak{b}}$ *as* $\omega^{\mathfrak{b}} = \sigma_\gamma\omega^{\mathfrak{b}''}$ *with* $\mathfrak{a} \leq \omega^{\mathfrak{b}''} \in W^p$. *Here,*

$$\omega^{\mathfrak{b}} = \omega^{\mathfrak{b}''}\sigma_{\alpha_s} \tag{95}$$

for a unique $s \in Im(\pi_{\mathfrak{b}})$. *We view such sites as places where replacements are possible and will use the notation*

$$\mathfrak{m}^+_{\mathfrak{a},\mathfrak{c}} : (\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \to (\mathfrak{a}, \mathfrak{b}', \mathfrak{c}), \tag{96}$$

and

$$\mathfrak{m}^-_{\mathfrak{a},\mathfrak{c}} : (\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \to (\mathfrak{a}, \mathfrak{b}'', \mathfrak{c}), \tag{97}$$

respectively, for the replacements while at the same time defining what we mean by replacements.

Notice that $\mathfrak{a} = \mathfrak{b}$ *and* $\mathfrak{b}' = \mathfrak{c}$ *are allowed in the first while* $\mathfrak{b} = \mathfrak{c}$ *and* $\mathfrak{b}'' = \mathfrak{a}$ *are allowed in the second.*

Furthermore,

$$\mathfrak{m}_{\mathfrak{a},\mathfrak{c}} : (\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \to (\mathfrak{a}, \mathfrak{b}_1, \mathfrak{c})$$

denotes the composition of any finite number of such maps $\mathfrak{m}^{\pm}_{\mathfrak{a},\mathfrak{c}}$ *(in any order, subject to the limitations at any step stipulated above)*

We will further extend the meaning of $\mathfrak{m}_{\mathfrak{a},\mathfrak{c}}$ *also to include the replacements*

$$\mathcal{C}_q(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \to \mathcal{C}_q(\mathfrak{a}, \mathfrak{b}_1, \mathfrak{c}),$$

and even

$$\mathcal{Q}_q(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \to \mathcal{Q}_q(\mathfrak{a}, \mathfrak{b}_1, \mathfrak{c}).$$

At the seed level, we will refer to the replacements as **Schubert mutations***.*

Similarly, we can define maps $\mathfrak{m}^{o,\pm}_{\mathfrak{a},\mathfrak{c}}$*, and after that mutations as composites*

$$\mathfrak{m}^{o}_{\mathfrak{a},\mathfrak{c}} : \mathcal{Q}^o_q(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \to \mathcal{Q}^o_q(\mathfrak{a}, \mathfrak{b}_1, \mathfrak{c}).$$

We need to define another kind of replacement: Consider

$$\mathfrak{a} < \mathfrak{b}_1 < \mathfrak{b} < \mathfrak{c}. \tag{98}$$

Definition 8.2. *We say that* $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ *is a* **d-splitting** *of* $(\mathfrak{a}, \mathfrak{c})$ *if*

$$\mathcal{C}_q(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = \mathcal{C}_q(\mathfrak{a}, \mathfrak{c}).$$

In this case we will also say that $(\mathfrak{a}, \mathfrak{c})$ *is a* **d-merger** *of* $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$*.*

To make this more definitive, one might further assume that $\mathfrak{b}$ is maximal amongst those satisfying (98), but we will not need to do this here.

Similarly,

Definition 8.3. *We say that* $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ *is a* **u-splitting** *of* $(\mathfrak{a}, \mathfrak{c})$ *if*

$$\mathcal{C}^o_q(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = \mathcal{C}^o_q(\mathfrak{a}, \mathfrak{c}).$$

Similarly, we will in this case also say that $(\mathfrak{a}, \mathfrak{c})$ *is a* **u-merger** *of* $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$*.*

Our next definition combines the two preceding:

Definition 8.4. *A Schubert creation replacement*

$$a^{+}_{\mathfrak{a},\mathfrak{c}} : (\mathfrak{a}, \mathfrak{c}) \to (\mathfrak{a}, \mathfrak{b}_1, \mathfrak{c})$$

consists in a d-splitting

$$(\mathfrak{a}, \mathfrak{c}) \to (\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$$

followed by a replacement $m_{\mathfrak{a},\mathfrak{c}}$ *applied to* $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$*. A Schubert annihilation replacement*

$$a^{-}_{\mathfrak{a},\mathfrak{c}} : (\mathfrak{a}, \mathfrak{b}_1, \mathfrak{c}) \to (\mathfrak{a}, \mathfrak{c})$$

is defined as the reverse process.

Schubert creation/annihilation mutations $a^{o,\pm}_{\mathfrak{a},\mathfrak{c}}$ *are defined analogously;*

$$a^{o,+}_{\mathfrak{a},\mathfrak{c}} : \mathcal{Q}^o_q(\mathfrak{a},\mathfrak{c}) \to \mathcal{Q}^o_q(\mathfrak{a},\mathfrak{b}_1,\mathfrak{c}),$$

and

$$a^{o,-}_{\mathfrak{a},\mathfrak{c}} : \mathcal{Q}^o_q(\mathfrak{a},\mathfrak{b}_1,\mathfrak{c}) \to \mathcal{Q}^o_q(\mathfrak{a},\mathfrak{c}).$$

We finally extend these Schubert creation/annihilation mutations into (we could do it more generally, but do not need to do so here)

$$\mathcal{Q}_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n) \to \mathcal{Q}_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-2},\dots,\mathfrak{r}_{n\pm 1})$$

by inserting/removing an $\mathfrak{r}_x$ *between* $\mathfrak{r}_{\frac{n}{2}}$ *and* $\mathfrak{r}_{\frac{n}{2}+1}$ *(n even) or between* $\mathfrak{r}_{\frac{n-1}{2}}$ *and* $\mathfrak{r}_{\frac{n+1}{2}}$ *(n odd). Similar maps are defined for the spaces* $\mathcal{Q}^o_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n)$*.*

In the sequel, we will encounter expressions of the form $\check{B}(u,v,s)$;

$$\check{B}(u,v,s) = E^{-1}_{u\sigma_s\Lambda_s,v\Lambda_s} E^{-1}_{u\Lambda_s,v\sigma_s\Lambda_s} \prod_{a_{ks}<0} E^{a_{ks}}_{u\Lambda_k,v\Lambda_k} \tag{99}$$

where

$$E(u\Lambda_s,v\Lambda_s)\check{B}(u,v,s) = q^{-2(\Lambda_s,\alpha_s)}\check{B}(u,v,s)E(u\Lambda_s,v\Lambda_s), \tag{100}$$

and where $\check{B}(u,v,s)$ commutes with all other elements in a given cluster.

Definition 8.5. *We say that* $\check{B}(u,v,s)$ *implies the change*

$$E_{u\Lambda_s,v\Lambda_s} \to E_{u\sigma_s\Lambda_s,v\sigma_s\Lambda_s}.$$

We will only encounter such changes where the set with $E_{u\Lambda_s,v\Lambda_s}$ removed from the initial cluster, and $E_{u\sigma_s\Lambda_s,v\sigma_s\Lambda_s}$ added, again is a cluster.

We further observe that a (column) vector with -1 at positions corresponding to $E_{u\sigma_s\Lambda_s,v\Lambda_s}$ and $E_{u\Lambda_s,v\sigma_s\Lambda_s}$ and a_{ks} at each position corresponding to a $E_{u\Lambda_k,v\Lambda_k}$ with $a_{ks}<0$ has the property that the symplectic form of the original cluster, when applied to it, returns a vector whose only non-zero entry is $-2(\Lambda_s,\alpha_s)$ at the position corresponding to $E_{u\Lambda_s,v\Lambda_s}$. Hence, this can be a column vector of the B of a potential compatible pair.

Even more can be ascertained: It can be seen that the last two lines of Theorem 6.9 precisely states that with a B matrix like that, the following holds:

Proposition 8.6. *The change* $E_{u\Lambda_s,v\Lambda_s} \to E_{u\sigma_s\Lambda_s,v\sigma_s\Lambda_s}$ *implied by* $\check{B}(u,v,s)$ *is the result of a BFZ mutation.*

Theorem 8.7. *The Schubert mutation*

$$\mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})\to\mathcal{Q}_q(\mathfrak{a},\mathfrak{b}',\mathfrak{c})$$

implied by a replacement $\mathfrak{m}^+_{\mathfrak{a},\mathfrak{c}}$ *as in (96) is the result of series of BFZ mutations.*

Proof. The number s is given by (94) and remains fixed throughout. We do the replacement in a number of steps. We set $\mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})=\mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})(0)$ and perform changes

$$\begin{gathered}\mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})=\mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})(0)\to\\ \mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})(1)\to\cdots\to\mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})(t_0)=\mathcal{Q}_q(\mathfrak{a},\mathfrak{b}',\mathfrak{c}).\end{gathered}\tag{101}$$

We will below see that $t_0=s_{\mathfrak{b}}-s_{\mathfrak{a}}-1$. We set

$$\text{If } 0\leq t\leq t_o: \mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})(t)=(\mathcal{C}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})(t),\mathcal{L}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})(t),\mathcal{B}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})(t)).\tag{102}$$

The intermediate seeds $\mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})(t)$ with $0<t<t_0$ are not defined by strings $\tilde{\mathfrak{a}}\leq\tilde{\mathfrak{b}}\leq\tilde{\mathfrak{c}}$. At each t-level, only one column is replaced when passing from $\mathcal{B}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})(t)$ to $\mathcal{B}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})(t+1)$, and here (77) is applied. Of course, the whole $\mathcal{B}$ matrix is given by (72) and (75) for a suitable seed.

Specifically, using (77) we introduce a family of expressions $\check{B}$ as in (99)

$$\begin{aligned}B^{\mathfrak{a},\mathfrak{b}(t),\mathfrak{c}}_{m^+}(s,t)&=E^{-1}_{\omega(s_{\mathfrak{a}}+t+1)\Lambda_s,\omega^{\mathfrak{b}}\Lambda_s}E^{-1}_{\omega(s_{\mathfrak{a}}+t)\Lambda_s,\omega^{\mathfrak{b}'}\Lambda_s}\prod E^{-a_{js}}_{\omega(s,s_{\mathfrak{a}}+t+1)\Lambda_j,\omega^{\mathfrak{b}'}\Lambda_j}\\ &=(E^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}+t+1)E^u_{\mathfrak{b}'}(s,s_{\mathfrak{a}}+t))^{-1}\prod E^u_{\mathfrak{b}}(j,\overline{p}(j,s,s_{\mathfrak{a}}+t+1))^{-a_{js}},\end{aligned}\tag{103}$$

implying the changes

$$E^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}+t)\to E^u_{\mathfrak{b}'}(s,s_{\mathfrak{a}}+t+1).\tag{104}$$

If $\omega(s,s_{\mathfrak{a}}+t+1)=u_t\sigma_s$ and $v=\omega^{\mathfrak{b}}$ then this corresponds to

$$((u_t\sigma_s\Lambda_s,v\Lambda_s)(u_t\Lambda_s,v\sigma_s\Lambda_s)(u\Lambda_j,v\Lambda_j)^{a_{js}})^{-1}\tag{105}$$

Here are then in details how the changes are performed:

$$\begin{aligned}
&\text{S}tep(0): \\
\mathcal{C}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c}) \ni E^d_{\mathfrak{a}}(s,s_{\mathfrak{b}}+1) &\to E^u_{\mathfrak{b}'}(s,s_{\mathfrak{a}}) \in \mathcal{C}_q(\mathfrak{a},\mathfrak{b}(0),\mathfrak{c})\ (renaming),\\
B_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}}) &\to B_{m^+}^{\mathfrak{a},\mathfrak{b}(0),\mathfrak{c}}(s,0)\ (renaming),\\
\mathcal{L}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c}) &\to \mathcal{L}_q(\mathfrak{a},\mathfrak{b}(0),\mathfrak{c})\ (renaming),\\
\text{S}tep(1): &\quad (implied\ by\ B_{m^+}^{\mathfrak{a},\mathfrak{b}(0),\mathfrak{c}}(s,0)),\\
\mathcal{C}_q(\mathfrak{a},\mathfrak{b}(0),\mathfrak{c}) \ni E^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}) &\to E^u_{\mathfrak{b}'}(s,s_{\mathfrak{a}}+1) \in \mathcal{C}^d_q(\mathfrak{a},\mathfrak{b}(1),\mathfrak{c}),\\
B_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}}+1) &\to B_{m^+}^{\mathfrak{a},\mathfrak{b}(1),\mathfrak{c}}(s,1)(by\ (77)),\\
\mathcal{L}_q(\mathfrak{a},\mathfrak{b}(0),\mathfrak{c}) &\to \mathcal{L}_q(\mathfrak{a},\mathfrak{b}(1),\mathfrak{c})\ (implied),\\
\text{S}tep(2): &\quad (implied\ by\ B_{m^+}^{\mathfrak{a},\mathfrak{b}(1),\mathfrak{c}}(s,1)),\\
\mathcal{C}^d_q(\mathfrak{a},\mathfrak{b}(1),\mathfrak{c}) \ni E^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}+1) &\to E^u_{\mathfrak{b}'}(s,s_{\mathfrak{a}}+2) \in \mathcal{C}^d_q(\mathfrak{a},\mathfrak{b}(2),\mathfrak{c}),\\
&\vdots\\
\text{S}tep(t+1): &\quad (implied\ by\ B_{m^+}^{\mathfrak{a},\mathfrak{b}(t),\mathfrak{c}}(s,t)),\\
\mathcal{C}^d_q(\mathfrak{a},\mathfrak{b}(t),\mathfrak{c}) \ni E^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}+t) &\to E^u_{\mathfrak{b}'}(s,s_{\mathfrak{a}}+t+1) \in \mathcal{C}^d_q(\mathfrak{a},\mathfrak{b}(t+1),\mathfrak{c}),\\
B_q^{\mathfrak{a},\mathfrak{b},\mathfrak{c}}(s,s_{\mathfrak{a}}+t) &\to B_{m^+}^{\mathfrak{a},\mathfrak{b}(t),\mathfrak{c}}(s,t)(by\ (77)),\\
\mathcal{L}_q(\mathfrak{a},\mathfrak{b}(t),\mathfrak{c}) &\to \mathcal{L}_q(\mathfrak{a},\mathfrak{b}(t+1),\mathfrak{c})\ (implied).
\end{aligned}$$

The last step is $t = s_{\mathfrak{b}} - s_{\mathfrak{a}} - 1$. $\mathfrak{b}(0) = \mathfrak{b}$, $\mathfrak{b}(s_{\mathfrak{b}} - s_{\mathfrak{a}} - 1) = \mathfrak{b}'$.

It is easy to see that all intermediate sets indeed are seeds.

What is missing now is to connect, via a change of basis transformation of the compatible pair, with the "E, F" matrices of [3]. Here we notice that both terms

$$(E^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}+t+1)E^u_{\mathfrak{b}'}(s,s_{\mathfrak{a}}+t))^{-1}(E^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}+t))^{-1} \tag{106}$$

and

$$\prod E^u_{\mathfrak{b}}(j,\overline{p}(j,s,s_{\mathfrak{a}}+t+1))^{-a_{js}}(E^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}+t))^{-1} \tag{107}$$

have the same q-commutators as $E^u_{\mathfrak{b}'}(s,s_{\mathfrak{a}}+t+1)$. The two possibilities correspond to the two signs in formulas (3.2) and (3.3) in [3].

Indeed, the linear transformation

$$E(t): E^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}+t) \to -E^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}+t+1) - E^u_{\mathfrak{b}'}(s,s_{\mathfrak{a}}+t) - E^u_{\mathfrak{b}}(s,s_{\mathfrak{a}}+t) \tag{108}$$

results in a change-of-basis on the level of forms:

$$\begin{aligned}\mathcal{L}_q(\mathfrak{a},\mathfrak{b}(t),\mathfrak{c})\to\mathcal{L}_q(\mathfrak{a},\mathfrak{b}(t+1),\mathfrak{c}) &= E^T(t)\mathcal{L}_q(\mathfrak{a},\mathfrak{b}(t),\mathfrak{c})E(t), \\ \mathcal{B}^{\mathfrak{a},\mathfrak{b}(t),\mathfrak{c}}_{m^+}(s,t)\to\mathcal{B}^{\mathfrak{a},\mathfrak{b}(t+1),\mathfrak{c}}_{m^+}(s,t+1) &= E(t)\mathcal{B}^{\mathfrak{a},\mathfrak{b}(t),\mathfrak{c}}_{m^+}(s,t)F(t),\end{aligned} \quad (109)$$

where $F(t)$ is a truncated part of $E(t)^T$ (the restriction to the mutable elements).

With this, the proof is complete. □

Theorem 8.8. *Any $\mathcal{Q}_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n)$ can be obtained from $\mathcal{Q}_q(\mathfrak{e},\mathfrak{p})$ as a sub-seed and any $\mathcal{Q}^o_q(\mathfrak{r}_1,\dots,\mathfrak{r}_{n-1},\mathfrak{r}_n)$ can be obtained from $\mathcal{Q}^o_q(\mathfrak{e},\mathfrak{p})$ as a sub-seed through a series of Schubert creation and annihilation mutations. These mutations are, apart from the trivial actions of renaming, splitting, merging, or simple restrictions, composites of BFZ-mutations.*

Proof. Apart from mergers and splittings (Definition 8.4), the mutations are composites of mutations of the form $\mathcal{Q}_q(\mathfrak{a},\mathfrak{b},\mathfrak{c})\to\mathcal{Q}_q(\mathfrak{a},\mathfrak{b}',\mathfrak{c})$. □

Corollary 8.9. *The algebras $\mathcal{A}^{d,\mathfrak{a},\mathfrak{c}}_q$ and $\mathcal{A}^{u,\mathfrak{a},\mathfrak{c}}_q$ are mutation equivalent and indeed are equal. We denote henceforth this algebra by $\mathcal{A}^{\mathfrak{a},\mathfrak{c}}$. This is the quadratic algebra generated by the elements $\beta_{c,d}$ with $c_{\mathfrak{a}}<d\le c_{\mathfrak{c}}$.*

We similarly denote the corresponding skew-field of fractions by $\mathcal{F}^{\mathfrak{a},\mathfrak{c}}_q$.

9. Prime

Definition 9.1.

$$\det{}^{\mathfrak{a},\mathfrak{c}}_s := E_{\omega^{\mathfrak{a}}\Lambda_s,\omega^{\mathfrak{c}}\Lambda_s}. \quad (110)$$

Theorem 9.2. *The 2 sided ideal $I(\det^{\mathfrak{a},\mathfrak{c}}_s)$ in $\mathcal{A}_q(\mathfrak{a},\mathfrak{c})$ generated by the covariant and non-mutable element $\det^{\mathfrak{a},\mathfrak{c}}_s$ is prime for each s.*

Proof. Induction. The induction start is trivially satisfied. Let us then divide the induction step into two cases. First, let Z_γ be an annihilation-mutation site of $\omega^{\mathfrak{c}}$ such that $\omega^{\mathfrak{c}}=\sigma_\gamma\omega^{\mathfrak{c}_1}=\omega^{\mathfrak{c}_1}\sigma_{\alpha_s}$ with $\omega^{\mathfrak{c}_1}\in W^p$. We have clearly $\mathcal{A}_q(\mathfrak{a},\mathfrak{c})=\mathcal{A}_q(\mathfrak{a},\mathfrak{c}_1)\cup I(\det^{\mathfrak{a},\mathfrak{c}}_s)$. Furthermore, $\mathcal{A}_q(\mathfrak{a},\mathfrak{c})\setminus\mathcal{A}_q(\mathfrak{a},\mathfrak{c}_1)=I_\ell(Z_\gamma)$, where $I_\ell(Z_\gamma)$ denotes the left ideal generated by Z_γ. We might as well consider the right ideal, but not the 2-sided ideal since in general there will be terms $\mathcal{R}$ of lower order, c.f. Theorem 4.1.

It follows that

$$\det{}^{\mathfrak{a},\mathfrak{c}}_s = M_1Z_\gamma+M_2 \quad (111)$$

where $M_1, M_2 \in \mathcal{A}_q(\mathfrak{a}, \mathfrak{c}_1)$ and $M_1 \neq 0$. Indeed, M_1 is a non-zero multiple of $\det_s^{\mathfrak{a},\mathfrak{c}_1}$. (If $s_{\mathfrak{c}} = 1$ then $M_1 = 1$ and $M_2 = 0$.) We also record, partly for later use, that Z_γ q-commutes with everything up to correction terms from $\mathcal{A}_q(\mathfrak{a}, \mathfrak{c}_1)$.

Notice that we use Corollary 8.9.

Now consider an equation

$$\det_s^{\mathfrak{a},\mathfrak{c}} p_1 = p_2 p_3 \tag{112}$$

with $p_1, p_2, p_3 \in \mathcal{A}_q(\mathfrak{a}, \mathfrak{c})$. Use (111) to write for each $i = 1, 2, 3$

$$p_i = \sum_{k=0}^{n_i} (\det_s^{\mathfrak{a},\mathfrak{c}})^k N_{i,k} \tag{113}$$

where each $N_{i,k} \in \mathbf{L}_q(\mathfrak{a}, \mathfrak{c}_1)$ and assume that $N_{i,0} \neq 0$ for $i = 2, 3$ Then $0 \neq N_{0,2} N_{0,3} \in \mathbf{L}_q(\mathfrak{a}, \mathfrak{c}_1)$. At the same time,

$$N_{0,2} N_{0,3} = \sum_{k=1}^{n_i} (\det_s^{\mathfrak{a},\mathfrak{c}})^k \tilde{N}_{i,k} \tag{114}$$

for certain elements $\tilde{N}_{i,k} \in \mathbf{L}_q(\mathfrak{a}, \mathfrak{c}_1)$.

Using the linear independence ([3, Proposition 10.8]) we easily get a contradiction by looking at the leading term in $\det_s^{\mathfrak{a},\mathfrak{c}}$).

Now in the general case, the s in $\det_s^{\mathfrak{a},\mathfrak{c}}$ is given and we may write $\omega^{\mathfrak{c}} = \omega^{\mathfrak{c}_2} \sigma_s \tilde{\omega}$ where σ_s does not occur in $\tilde{\omega}$. Let $\omega^{\mathfrak{c}_1} = \omega^{\mathfrak{c}_2} \sigma_s$. It is clear that $\det_s^{\mathfrak{a},\mathfrak{c}} = \det_s^{\mathfrak{a},\mathfrak{c}_1}$ and by the previous, $\det_s^{\mathfrak{a},\mathfrak{c}_1}$ is prime in $\mathcal{A}_q(\mathfrak{a}, \mathfrak{c}_1)$. We have that $\mathcal{A}_q(\mathfrak{a}, \mathfrak{c}_1)$ is an algebra in its own right. Furthermore,

$$\mathcal{A}_q(\mathfrak{a}, \mathfrak{c}) = \mathcal{A}_q(\mathfrak{a}, \mathfrak{c}_1)[Z_{\gamma_1}, \dots, Z_{\gamma_n}], \tag{115}$$

where the Lusztig elements $Z_{\gamma_1}, \dots, Z_{\gamma_n}$ are bigger than the generators of $\mathcal{A}_q(\mathfrak{a}, \mathfrak{c}_1)$. In a PBW basis we can put them to the right. They even generate a quadratic algebra $\tilde{\mathcal{A}}_q$ in their own right! The equation we need to consider are of the form

$$p_1 p_2 = \det_s^{\mathfrak{a},\mathfrak{c}_1} p_3 \tag{116}$$

with $p_1, p_2, p_3 \in \mathcal{A}_q(\mathfrak{a}, \mathfrak{c})$. The claim that at least one of p_1, p_2 contains a factor of $\det_{q,s}^{\mathfrak{r}_1}$ follows by easy induction on the $\tilde{\mathcal{A}}_q$ degree of $p_1 p_2$, i.e. the sum of the $\tilde{\mathcal{A}}_q$ degrees of p_1 and p_2. □

10. Upper

Let $\omega^{\mathfrak{a}}, \omega^{\mathfrak{c}} \in W^p$ and $\mathfrak{a} < \mathfrak{c}$.

Definition 10.1. *The cluster algebra $\mathbf{A}_q(\mathfrak{a},\mathfrak{c})$ is the $\mathbb{Z}[q]$-algebra generated in the space $\mathcal{F}_q(\mathfrak{a},\mathfrak{c})$ by the inverses of the non-mutable elements $\mathcal{N}_q(\mathfrak{a},\mathfrak{c})$ together with the union of the sets of all variables obtainable from the initial seed $\mathcal{Q}_q(\mathfrak{a},\mathfrak{c})$ by composites of quantum Schubert mutations. (Appropriately applied)*

Observe that we include $\mathcal{N}_q(\mathfrak{a},\mathfrak{c})$ in the set of variables.

Definition 10.2. *The upper cluster algebra $\mathbf{U}_q(\mathfrak{a},\mathfrak{c})$ connected with the same pair $\omega^{\mathfrak{a}}, \omega^{\mathfrak{c}} \in W^p$ is the $\mathbb{Z}[q]$-algebra in $\mathcal{F}_q(\mathfrak{a},\mathfrak{c})$ given as the intersection of all the Laurent algebras of the sets of variables obtainable from the initial seed $\mathcal{Q}_q(\mathfrak{a},\mathfrak{c})$ by composites of quantum Schubert mutations. (Appropriately applied)*

Proposition 10.3.

$$\mathcal{A}_q(\mathfrak{a},\mathfrak{c}) \subseteq \mathbf{A}_q(\mathfrak{a},\mathfrak{c}) \subset \mathbf{U}_q(\mathfrak{a},\mathfrak{c}).$$

Proof. The first inclusion follows from [15], the other is the quantum Laurent phenomenon. □

Remark 10.4. *Our terminology may seem a bit unfortunate since the notions of a cluster algebra and an upper cluster algebra already have been introduced by Berenstein and Zelevinsky in terms of all mutations. We only use quantum line mutations which form a proper subset of the set of all quantum mutations. However, it will be a corollary to what follows that the two notions in fact coincide, and for this reason we do not introduce some auxiliary notation.*

Theorem 10.5.

$$\mathbf{U}_q(\mathfrak{a},\mathfrak{c}) = \mathcal{A}_q(\mathfrak{a},\mathfrak{c})[(\det_s^{\mathfrak{a},\mathfrak{c}})^{-1}; s \in Im(\pi_{\mathfrak{c}})].$$

Proof. This follows by induction on $\ell(\omega^{\mathfrak{c}})$ (with start at $\ell(\omega^{\mathfrak{a}})+1$) in the same way as in the proof of [20, Theorem 8.5], but for clarity we give the details: Let the notation and assumptions be as in the proof of Theorem 9.2. First of all, the induction start is trivial since we there are looking at the generator of a Laurent quasi-polynomial algebra. Let then $u \in \mathbf{U}_q(\mathfrak{a},\mathfrak{c})$. We will argue by contradiction, and just as in the proof of [20, Theorem 8.5], one readily sees that one may assume that $u \in \mathcal{A}_q(\mathfrak{a},\mathfrak{c}_1)[(\det_s^{\mathfrak{a},\mathfrak{c}_1})^{-1},\det_s^{\mathfrak{a},\mathfrak{c}}]$. Using (111) we may now write

$$u = \left(\sum_{i=0}^K Z_\gamma^i p_i(\det_s^{\mathfrak{a},\mathfrak{c}_1})^{k_i}\right)(\det_s^{\mathfrak{a},\mathfrak{c}_1})^{-\rho}, \tag{117}$$

with $p_i \in \mathcal{A}_q(\mathfrak{a},\mathfrak{c}_1)$, $p_i \notin I(\det_s^{\mathfrak{a},\mathfrak{c}_1})$, and $k_i \geq 0$. Our assumption is that $\rho > 0$. recall that the elements $\det_s^{\mathfrak{a},\mathfrak{c}_1}$ and $\det_s^{\mathfrak{a},\mathfrak{c}}$ are covariant and define prime ideals in the appropriate algebras. Using the fact that $\mathbf{U}_q(\mathfrak{a},\mathfrak{c})$ is an algebra containing $\mathcal{A}_q(\mathfrak{a},\mathfrak{c})$, we can assume that the expression in the left bracket in (117) is not in $I(\det_s^{\mathfrak{a},\mathfrak{c}})$ and we may further assume that $p_i \neq 0 \Rightarrow k_i < \rho$. To wit, one can remove the factors of $\det_s^{\mathfrak{a},\mathfrak{c}}$, then remove the terms with $k_i \geq \rho$, then possibly repeat this process a number of times.

Consider now the cluster $\mathcal{C}_q^u(\mathfrak{a},\mathfrak{c})$. We know that u can be written as a Laurent quasi-polynomial in the elements of $\mathcal{C}_q^u(\mathfrak{a},\mathfrak{c})$. By factoring out, we can then write

$$u = p\prod_{(c,d)\in\mathbb{U}^{u,\mathfrak{a},\mathfrak{c}}}(E_{\mathfrak{c}}^u(c,d))^{-\alpha_{c,d}}, \tag{118}$$

where $p \in \mathcal{A}_q(\mathfrak{a},\mathfrak{c})$, and $\alpha_{c,d} \geq 0$. We will compare this to (117). For the sake of this argument set $\tilde{\mathbb{U}}^{u,\mathfrak{a},\mathfrak{c}}) = \{(c,d) \in \mathbb{U}^{u,\mathfrak{a},\mathfrak{c}}) \mid \alpha_{c,d} > 0\}$.

Of course, $\det_s^{\mathfrak{a},\mathfrak{c}} \in \mathcal{C}^u(\mathfrak{e},\mathfrak{r})$.

"Multiplying across", we get from (117) and (118), absorbing possibly some terms into p:

$$(\sum_{i=0}^K Z^i p_i(\det_s^{\mathfrak{a},\mathfrak{c}_1})^{k_i})\prod_{(c,d)\in\tilde{\mathbb{U}}^{u,\mathfrak{a},\mathfrak{c}}}(E_{\mathfrak{c}}^u(c,d))^{\alpha_{c,d}} = p(\det_s^{\mathfrak{a},\mathfrak{c}_1})^{\rho}. \tag{119}$$

Any factor of $\det_s^{\mathfrak{a},\mathfrak{c}}$ in p will have to be canceled by a similar factor of $E_{\mathfrak{c}}^u(s,0)$ in the left-hand side, so we can assume that p does not contain no factor of $\det_s^{\mathfrak{a},\mathfrak{c}}$. After that we can assume that $(s,0) \notin \tilde{\mathbb{U}}^{u,\mathfrak{a},\mathfrak{c}}$ since clearly $\det_s^{\mathfrak{a},\mathfrak{c}_1} \notin I(\det_s^{\mathfrak{a},\mathfrak{c}})$. Using that $k_i < \rho$ it follows that there must be a factor of $(\det_s^{\mathfrak{a},\mathfrak{c}_1})$ in $\prod_{(c,d)\in\tilde{\mathbb{U}}^{u,\mathfrak{a},\mathfrak{c}}}(E_{\mathfrak{c}}^u(c,d))^{\alpha_{c,d}}$. Here, as but noticed, $d = 0$ is excluded. The other terms do not contain $Z_{s,1}$ but $(\det_s^{\mathfrak{a},\mathfrak{c}_1})$ does. This is an obvious contradiction. □

11. The diagonal of a quantized minor

Definition 11.1. *Let $\mathfrak{a} < \mathfrak{b}$. The diagonal, $\mathbb{D}_{\omega^{\mathfrak{a}}(\Lambda_s),\omega^{\mathfrak{b}}(\Lambda_s)}$, of $E_{\omega^{\mathfrak{a}}(\Lambda_s),\omega^{\mathfrak{b}}(\Lambda_s)}$ is set to*

$$\mathbb{D}_{\omega^{\mathfrak{a}}(\Lambda_s),\omega^{\mathfrak{b}}(\Lambda_s)} = q^{\alpha} Z_{s,s_{\mathfrak{a}}+1} \cdots Z_{s,s_{\mathfrak{b}}}, \tag{120}$$

where

$$Z_{s,s_{\mathfrak{b}}} \cdots Z_{s,s_{\mathfrak{a}}+1} = q^{2\alpha} Z_{s,s_{\mathfrak{a}}+1} \cdots Z_{s,s_{\mathfrak{b}}} + \mathcal{R} \tag{121}$$

where the terms $\mathcal{R}$ are of lower order

Proposition 11.2.

$$E_{\omega^{\mathfrak{a}}(\Lambda_s),\omega^{\mathfrak{b}}(\Lambda_s)} = \mathbb{D}_{\omega^{\mathfrak{a}}(\Lambda_s),\omega^{\mathfrak{b}}(\Lambda_s)} + \mathcal{R}$$

The terms in $\mathcal{R}$ are of lower order in our ordering induced by $\leq_L$. They can in theory be determined from the fact that the full polynomial belongs to the dual canonical basis. ([3],[15]*).*

Proof. We prove this by induction on the length $s_{\mathfrak{b}} - s_{\mathfrak{a}}$ of any s-diagonal. When this length is 1 we have at most a quasi-polynomial algebra and here the case is clear. Consider then a creation-mutation site where we go from length r to $r+1$: Obviously, it is only the very last determinant we need to consider. Here we use the equation in Theorem 6.3 but reformulate it in terms of the elements $E_{\xi,\eta}$, cf. Theorem 6.9.

Set $\omega^{\mathfrak{b}_1} = \omega^{\mathfrak{b}}\sigma_s$ and consider $E_{\omega^{\mathfrak{a}}(\Lambda_s),\omega^{\mathfrak{b}_1}(\Lambda_s)}$. Its weight is given as

$$\omega^{\mathfrak{b}_1}(\Lambda_s) - \omega^{\mathfrak{a}}(\Lambda_s) = \beta_{s,s_{\mathfrak{a}}+1} \cdots + \beta_{s,s_{\mathfrak{b}}+1}.$$

In the recast version of Theorem 6.3, the terms on the left hand side are covered by the induction hypothesis. The second term on the right hand side contains no element of the form $Z_{s,s_{\mathfrak{b}_1}}$ and it follows that we have an equation

$$(Z_{s,s_{\mathfrak{a}}+2} \cdots Z_{s,s_{\mathfrak{b}}}) E_{\omega^{\mathfrak{a}}(\Lambda_s),\omega^{\mathfrak{b}_1}(\Lambda_s)} = (Z_{s,s_{\mathfrak{a}}+2} \cdots Z_{s,s_{\mathfrak{b}}+1})(Z_{s,s_{\mathfrak{a}}+1} \cdots Z_{s,s_{\mathfrak{b}}}) + \mathcal{R}. \tag{122}$$

The claim follows easily from that. □

Recall that in the associated quasi-polynomial algebra is the algebra with relations corresponding to the top terms, i.e., colloquially speaking, setting the lower order terms $\mathcal{R}$ equal to 0. Let

$$d_{\omega^{\mathfrak{r}}_{s,t_1}(\Lambda_s),\omega^{\mathfrak{r}}_{s,t}(\Lambda_s)} = z_{s,t_1+1} \cdots z_{s,t}. \tag{123}$$

The following shows the importance of the diagonals:

Theorem 11.3.

$$d_{u\cdot\Lambda_{i_0},v\cdot\Lambda_{i_0}} d_{u_1\cdot\Lambda_{i_1},v_1\cdot\Lambda_{i_1}} = q^G d_{u_1\cdot\Lambda_{i_1},v_1\cdot\Lambda_{i_1}} d_{u\cdot\Lambda_{i_0},v\cdot\Lambda_{i_0}} \Leftrightarrow \tag{124}$$

$$\mathbb{D}_{u\cdot\Lambda_{i_0},v\cdot\Lambda_{i_0}} \mathbb{D}_{u_1\cdot\Lambda_{i_1},v_1\cdot\Lambda_{i_1}} = q^G \mathbb{D}_{u_1\cdot\Lambda_{i_1},v_1\cdot\Lambda_{i_1}} \mathbb{D}_{u\cdot\Lambda_{i_0},v\cdot\Lambda_{i_0}} + \mathcal{R} \tag{125}$$

In particular, if the two elements $E_{u\cdot\Lambda_{i_0},v\cdot\Lambda_{i_0}} E_{u_1\cdot\Lambda_{i_1},v_1\cdot\Lambda_{i_1}}$ q-commute:

$$E_{u\cdot\Lambda_{i_0},v\cdot\Lambda_{i_0}} E_{u_1\cdot\Lambda_{i_1},v_1\cdot\Lambda_{i_1}} = q^G E_{u_1\cdot\Lambda_{i_1},v_1\cdot\Lambda_{i_1}} E_{u\cdot\Lambda_{i_0},v\cdot\Lambda_{i_0}} \tag{126}$$

then G can be computed in the associated quasi-polynomial algebra:

$$d_{u\cdot\Lambda_{i_0},v\cdot\Lambda_{i_0}} d_{u_1\cdot\Lambda_{i_1},v_1\cdot\Lambda_{i_1}} = q^G D_{u_1\cdot\Lambda_{i_1},v_1\cdot\Lambda_{i_1}} d_{u\cdot\Lambda_{i_0},v\cdot\Lambda_{i_0}}. \tag{127}$$

Remark 11.4. *One can also compute G directly using the formulas in* [3].

Remark 11.5. *The elements $E_{\xi,\eta}$ that we consider belong to the dual canonical basis. As such, they can in principle be determined from the highest order terms $\mathbb{D}_{\xi,\eta}$.*

12. Litterature

References

[1] R. J. Baston and M.R. Eastwood, The Penrose Transform: its Interaction with Representation Theory, Oxford University Press, Oxford (1989)

[2] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Schubert Cells and cohomology of the spaces G/P, Russian Math. Surveys 28 (1973), 1-26.

[3] A. Berenstein and A. Zelevinsky, Quantum cluster algebras, Adv. Math. 195 (2005), 405455

[4] A. Borel, Kahlerian coset spaces of semisimple Lie groups, Proc. Nat. Acad, Sci. U.S.A. 40 (1954), 1147-1151.

[5] K. Brown and K. Goodearl, Lectures on Algebraic Quantum Groups, Birkhuser, 2002.

[6] A. Căp and Jan Slovák, *Parabolic Geometries I.* Mathematical Surveys and Monographs Vol. 154, AS.M.S. (2009)

[7] C. De Concini and V. Lyubashenko, Quantum function algebra at roots of 1, Adv. in Math. 108 (1994), 205-262.

[8] De Concini, C.; Procesi, C. Quantum groups. D-modules, representation theory, and quantum groups (Venice, 1992), 31-140, Lecture Notes in Math., 1565, Springer, Berlin, 1993.

[9] R. Dipper and S. Donkin, Quantum GL_n, Proc. London Math. Soc. 63 (1991), 165–211.

[10] V.G. Drinfeld, Hopf algebras and the quantum Yang-Baxter equations. Soviet Math. Dokl., 32 (1985), 254-258.

[11] V.G. Drinfeld, Quantum groups, in **Proceedings of the ICM 1986**, Amer. Math. Soc., Providence, RI, 1987.

[12] L.D. Faddeev, N.Yu. Reshetikhin, and L.A. Takhtajan, Quantization of Lie groups and Lie algebras, in **Algebraic Analysis**, Academic Press 1988.

[13] S. Fomin, S. Zelevinsky, Andrei (2002), Cluster algebras. I. Foundations, J. Amer.Math.Soc. 15 (2002), 497529.

[14] S. Fomin and A. Zelevinsky, Double Bruhat cells and total positivity, J. Amer. Math. Soc. 12 (1999), 335380.
[15] C. Geiss, B. Leclerc, J. Schröer Cluster structures on quantum coordinate rings, Sel. Math. new Series May 2013, Volume 19, Issue 2, pp 337-397
[16] K. Goodearl and M, Yakimov, Quantum cluster algebras and quantum nilpotent algebras, Proc. Nat. Acad. Sci. USA 111, no. 27 (2014) 9696-9703.
[17] K. Goodearl and M, Yakimov, Quantum cluster algebra structures on quantum nilpotent algebras, arXiv:1309.7869.
[18] H.P. Jakobsen, Quantized hermitian symmetric spaces. In: "Lie Theory and its Applications in Physics", 105-116, World Scientific, Singapore (1996)
[19] (with S. Jøndrup) Quantized Rank R Matrices. J. Algebra 246 (2001), 70-96.
[20] (with H. Zhang) Double-partition Quantum Cluster Algebras J. Algebra 372 (2012), 172-203.
[21] (with C. Pagani) Quantized Matrix Algebras and Quantum Seeds, J. of Linear and Multilinear Algebra 63, 713-753 (2015)
[22] The center of $\mathcal{U}_q(\mathfrak{n}_\omega)$. Submitted 2015.
[23] J.C. Jantzen, *Lectures on Quantum Groups*, A.M.S. Graduate Studies in Mathematics Vol. 6 (1996)
[24] J.C. Jantzen, Note to Remark to 8.24 3) in *Lectures on Quantum Groups*, Unpublished (2001).
[25] M. Jimbo, A q-difference analogue of $\mathcal{U}(\mathfrak{g})$ and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985), 6369.
[26] M. Jimbo, A q-analogue of $\mathcal{U}(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation, Lett. Math. Phys. 11 (1986), 247-252.
[27] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. 63 (1991), 465–516.
[28] M. Kashiwara, Global crystal bases of quantum groups, Duke Math. J. 69 (1993), 455–485.
[29] Y. Kimura, Quantum unipotent subgroup and dual canonical basis, Kyoto J. Math. **52** (2012), 277–331.
[30] B. Kostant, Lie Algebra Cohomology and the Generalized Borel-Weil Theorem, Ann. Math. Vol. 74 (1961), pp. 329-387.
[31] S.Z. Levendorskii, A short proof of the P.B.W.-theorem for quantized enveloping algebras, preprint 1991
[32] S.Z. Levendorskii, Ya. S. Soibelman, Some applications of the quantum Weyl Groups, J. Geom. Phys. 7 (1990), 241-254.
[33] S. Levendorskii and Y. Soibelman, Algebras of functions on compact quantum groups, Schubert Cells and quantum tori, Comm. Math. Phys 139 (1991), 141-170.
[34] G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, Adv. in Math. 70 (1988), 237-249.
[35] G. Lusztig, Canonical bases arising from quantized enveloping algebras, J Amer. Math. Soc. 3 (1990), 447-498.
[36] G. Lusztig **Introduction to Quantum Groups.** Progress In Mathematics 110, Birkhuser 1993.
[37] J.C. McConnell and J.C. Robson, **Noncommutative Noetherian Rings**, Wiley Interscience, 1987.
[38] M. Reineke, Multiplicative properties of dual canonical bases of quantum groups, J. Algebra 211 (1999), 134-149.
[39] K. Sagerschnig, Schubert Cell Decomposition and Homology of Generalized Flag Manifolds, Dissertation Universität Wien 2003.
[40] R. Steinberg, Lectures on Chevalley groups, Yale University Press, New Haven, Conn. 1967.

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