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On Singular Holomorphic Representations *

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Introduction

In this article the full set of irreducible unitary holomorphic representations of $U(p,q)$ is determined.

For a general Hermitian symmetric space G/K of the noncompact type the set of irreducible unitary representations of G on scalar valued holomorphic functions has been found by Wallach [14] and by Rossi and Vergne [11]. For the groups $Mp(n,\mathbb{R})$ and $SU(n,n)$ unitary irreducible representations on vector valued holomorphic functions have been obtained by Gross and Kunze [1] from the decomposition of tensor products of the harmonic (Segal-Shale-Weil) representation L. Later the complete description of these tensor products for the groups $Mp(n,\mathbb{R})$ and $U(p,q)$ was given by Kashiwara and Vergne [8] (see also [4]), and it was conjectured that any irreducible unitary representation with highest weight appears in the k-th fold tensor product of L for some k. As we shall see, for the groups $U(p,q)$ this is indeed so.

The case of $G=SU(2,2)$ has been treated in [9] and [2]. In both cases some rather technical computations of Γ-functions were successfully completed by means of the detailed knowledge of Clebsch-Gordan and Racah coefficients for $U(2)$, available through the physics literature. Based on results in [10] a different proof for $SU(2,2)$ was recently given [15].

For $G=SU(n,n)$ the representations are of the form $(U(g)f)(z) = J(g^{-1},z)^{-1} f((az+b)/(cz+d))$ where the automorphic factor $J(g,z)$ is a product of an automorphic factor $J_0(g,z)$, which does not contain $\det(cz+d)$ to any power, and $\det(cz+d)$ to an integer power. Let $U=U_{J_0,k}$ if the power is k. The key observation we make is: If $U_{J_0,k}$ and $U_{J_0,k-1}$ are unitary and the Hilbert space $H_{J_0,k}$ of $U_{J_0,k}$ is annihilated by constant coefficient differential operators, then $U_{J_0,k-1}$ is annihilated by constant coefficient differential operators of one order less. To establish this fact and to make full use of it we need the theory of tensor products of holomorphic representations as developed in [5] and [6].

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§1. On the Ring $\mathscr{P} = \mathbb{C}[z]$ of Polynomials in the Entries of z

Let S denote the set of irreducible polynomial representations τ of $Gl(n, \mathbb{C})$ for which $\tau(a^*) = \tau(a)^*$ for all a in $Gl(n, \mathbb{C})$. It is a classical result, due to Schur [13], that the representation P of $Gl(n, \mathbb{C}) \times Gl(n, \mathbb{C})$ on the space $\mathscr{P} = \mathbb{C}[z]$ of polynomials in the entries of a (generic) $n \times n$ complex matrix z given by, for $p \in \mathscr{P}$,

$$(P(a, b)p)(z) = p(b^{-1} z a) \tag{1.1}$$

decomposes into a multiplicity free sum of irreducible representations ρ of the form $\rho(a, b) = \tau(a) \otimes \tau'(b)$, where $\tau \in S$ and τ' is the contragredient representation. This result will be needed later. It may be proved quite simply:

Lemma 1.1.
$$P = \bigoplus_{\tau \in S} \tau \otimes \tau'.$$

Proof. Let $\tau = \tau_1 \otimes \tau_2$ be an irreducible representation of $Gl(n, \mathbb{C}) \times Gl(n, \mathbb{C})$ on a finite dimensional vector space V_τ and assume that τ's restriction to $U(n, \mathbb{C}) \times U(n, \mathbb{C})$ is unitary. Assume moreover that there exists a non-zero linear map T: $V_\tau \to \mathscr{P}$ such that

$$\forall v \in V_\tau, \forall z \in M(n, \mathbb{C}), \forall (a, b) \in Gl(n, \mathbb{C}) \times Gl(n, \mathbb{C}):$$
$$(Tv)(b^{-1} z a) = (T(\tau_1(a) \otimes \tau_2(b)) v)(z). \tag{1.2}$$

Let $\tilde{T} v = (Tv)$ (1), where 1 denotes the identity matrix. $\tilde{T}$ is clearly a non-zero linear functional on V_τ. It follows from (1.2) that

$$\tilde{T} v = \tilde{T}(\tau_1(a) \otimes \tau_2(a)) v \tag{1.3}$$

for all $a \in Gl(n, \mathbb{C})$. Hence the trivial representation is contained in the representation $a \to \tau_1(a) \otimes \tau_2(a)$ of $Gl(n, \mathbb{C})$. This happens exactly when $\tau_2 = \tau_1'$ and in this case the trivial representation is contained exactly once. Finally it follows from (1.2) that

$$(Tv)(z) = \tilde{T}(\tau_1(z) \otimes 1) v \tag{1.4}$$

for all z, where we have extended τ_1 to $M(n, \mathbb{C})$ by continuity. This implies that τ_1 must be polynomial. The converse is easily obtained. It is sufficient to construct $\tilde{T}$ which clearly may be taken to be

$$\tilde{T} v = \langle v, w \rangle \tag{1.5}$$

where $w \neq 0$ is the unique (up to scalar multiples) vector for which $(\tau(a) \otimes \tau'(a)) w = w$ for all a in $Gl(n, \mathbb{C})$.

The above result has been generalized to arbitrary Hermitian symmetric space of the non-compact type by Schmid [12].

Definition 1.2. I_r, $r = 1, \ldots, n$, denotes the ideal in $\mathscr{P}$ generated by all r-order minors (= subdeterminants).

It is a well-known result that these ideals are *prime*. Actually this may be proved quite simply from the above, but we omit the proof. The variety $V(I_r)$ of the ideal I_r is $V(I_r)=\{z\in M(n,\mathbb{C})|\operatorname{rank} z\leqq r-1\}$. It follows that P leaves the ideal I_r invariant. We denote by S_i the subset of S consisting of all $\tau=\tau(n_1,n_2,\dots,n_i,\dots,n_n)$ with $n_1\geqq n_2\geqq\dots\geqq n_i>0$. Obviously then

Lemma 1.3.
$$P|_{I_r}=\bigoplus_{\tau\in S_r}\tau\otimes\tau'.$$

§2. Construction of Irreducible Holomorphic Representations of $U(n, n)$

Let
$$G=U(n,n)=\left\{g\in Gl(2n,\mathbb{C})|g^*\begin{bmatrix}0 & i\cdot I_n\\ -i\cdot I_n & 0\end{bmatrix}g=\begin{bmatrix}0 & i\cdot I_n\\ -i\cdot I_n & 0\end{bmatrix}\right\}.$$

As usual we write the elements of G as $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ where a,b,c and d are elements of $M(n,\mathbb{C})$. The maximal compact subgroup K is taken to be
$$K=\left\{\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\middle|(a+ib,a-ib)\in U(n)\times U(n)\right\}.$$

We let $v=a+ib$ and $u=a-ib$. In what follows the notation etc. will be that of [8]. It is realy $SU(n,n)$ we are interested in but it is more convenient to work with K as above and the difference is insignificant.

Let
$$\begin{aligned}&\mathcal{D}=\{z\in M(n,\mathbb{C})|(z-z^*)/2i>0\},\ \text{and}\\ &\mathcal{O}_l=\{k\in M(n,\mathbb{C})|k=k^*,\ k\geqq 0,\ \text{and rank } k\leqq l\},\end{aligned}\tag{2.1}$$

for $l=0,1,\dots,n-1$. $\mathcal{D}$ is the unbounded realization of the Hermitian symmetric space G/K.

The starting point is the following theorem due to Rossi and Vergne [11, §§4.5-4.6]:

Theorem 2.1. *There is a semi-invariant measure μ_l on $\mathcal{O}_l\,(l=0,1,\dots,n-1)$ such that*
$$\det(z/i)^{-l}=\int_{\mathcal{O}_l}e^{itr(zk)}\,d\mu_l(k).\tag{2.2}$$

An analogous formula was derived for an arbitrary Hermitian symmetric space of the non-compact type.

We shall be looking at representations of G on vector valued holomorphic functions on $\mathcal{D}$ of the form
$$(U_{\tau_1,\tau_2,k}(g)f)(z)=\det(cz+d)^{-k}\,\tau_1(zc^*+d^*)\otimes\tau_2(cz+d)^{-1}f(az+b/cz+d),\tag{2.3}$$

where $g^{-1}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, τ_1 and τ_2 are irreducible representations of $U(n)(Gl(n,\mathbb{C}))$ on V_{τ_1} and V_{τ_2}, respectively, τ_2 belongs to the set S, and τ_1 is the contragredient of an element of S. The representation of K from which this representation is holomorphically induced is

$$(u, v) \to (\det v)^k \tau_1(u) \otimes \tau_2(v). \tag{2.4}$$

We let $\delta_k(u, v) = (\det v)^k$. The trivial representation of $Gl(n, \mathbb{C})$ is denoted by 1. If the representation (2.3) is unitary in a space of holomorphic functions on $\mathscr{D}$ the space is a reproducing kernel Hilbert space [3], and the kernel is given (uniquely up to multiplication by a positive real constant) by

$$K_{\tau_1, \tau_2, k}(z, w) = \det((z - w^*)/2i)^{-k} \tau_1((z - w^*)/2i) \otimes \tau_2((z - w^*)/2i)^{-1}. \tag{2.5}$$

If one can express the function $z \to K_{\tau_1, \tau_2, k}(z, 0)$ as the Fourier-Laplace transform of a function with values in the set of positive semi-definite matrices, as in (2.2), the kernel defines a Hilbert space on which the representation is unitary.

We use this observation on representations $V_{\tau_1, \tau_2, k}$ where either τ_1 or τ_2 is trivial. Representations where both τ_1 and τ_2 are non-trivial may then be obtained by forming tensor products of these simple representations.

Proposition 2.2. *Let $\tau = \tau(n_1, n_2, \ldots, n_i, 0, 0, \ldots, 0) \in S_i$ and let $i \leqq l$. Up to multiplication by a strictly positive real number*

$$\begin{aligned} \det(z/2i)^{-l} \tau(z/2i)^{-1} &= \int_{\mathcal{O}_l} \tau(k) e^{i tr(zk)} d\mu_l(k) \\ \det(z/2i)^{-l} \tau({}^t z/2i)^{-1} &= \int_{\mathcal{O}_l} \tau({}^t k) e^{i tr(zk)} d\mu_l(k). \end{aligned} \tag{2.5}$$

Proof. We need only prove the first formula. Let $F(z) = \int_{\mathcal{O}_l} \tau(k) e^{i tr(zk)} d\mu_l(k)$. Then, for $a \in Gl(n, \mathbb{C})$

$$F(a z a^*) = \det(a^* a)^{-l} \tau(a^{*-1}) F(z) \tau(a^{-1}).$$

Thus the claim follows if we can prove that $F(i) \neq 0$. (By the irreducibility of $\tau, F(i)$ is a constant multiple of the identity matrix). Let e_1 denote the highest weight vector for τ. $\langle \tau(k) e_1, e_1 \rangle$ is then a polynomial in $I_i \setminus I_{l+1}$. It is in fact a product of minors symmetric around the diagonal in k and thus, by the semi-invariance of the measure μ_l, $\langle \tau(k) e_1, e_1 \rangle > 0$ a.e. $[\mu_l]$. Hence $\langle F(i) e_1, e_1 \rangle \neq 0$. Q.E.D.

We let τ_2 be the representation τ in (2.5) and let $\tau_1(a) = \tau({}^t a^{-1})$.

Tensor products of holomorphic representations may be decomposed along the lines of [6]. We shall only need the subrepresentation corresponding to the restriction to the diagonal in $\mathscr{D} \times \mathscr{D}$. This representation always occurs in the decomposition. Thus we may state

Proposition 2.3 [8]. *The representations $U_{\tau_1, \tau_2, k}$ holomorphically induced from*

$$\tau_1(0, 0, \ldots, -m_1, \ldots, -m_j) \otimes \tau_2(n_1, \ldots, n_i, 0, \ldots, 0) \otimes \delta_k$$

with $i + j \leqq k$ are unitary.

If $U_{\tau_1, \tau_2, k}$ is unitary, then so is $U_{\tau_1, \tau_2, k+m}$ for any $m \in \mathbb{N}$, since we can tensor $U_{\tau_1, \tau_2, k}$ with the known ([14], [11]) unitary representation $U_{1,1,1}$. Thus

Lemma 2.4. *If $U_{\tau_1, \tau_2, k}$ is not unitary then neither is $U_{\tau_1, \tau_2, k-m}$ for any $m \in \mathbb{N}$.*

§3. On Missing K-types

The space $\mathscr{D}$ is an open subset of $M(n,\mathbb{C})$. The set $\mathscr{E}(V)$ of polynomials on the dual to $M(n,\mathbb{C})$ with values in a finite dimensional complex vector space V may be identified with the space of V-valued constant coefficient differential operators on $\mathscr{D}$. We write elements of $\mathscr{E}(V)$ as $p=p(\partial/\partial z)$. If $V=V_1'$ is the dual to the vector space V_1 we define for every holomorphic function f from $\mathscr{D}$ to V_1 and every element p in $\mathscr{E}(V_1')$ the $\mathbb{C}$-valued holomorphic function (p,f) by

$$(p,f)(z)=(p(\partial/\partial z),f(\cdot))(z). \tag{3.1}$$

In the following we shall make use of the bounded realization $\mathscr{B}$ of the Hermitian symmetric space G/K

$$\mathscr{B}=\{z\in M(n,\mathbb{C})\,|\,z\,z^*<1\}. \tag{3.2}$$

By the above remark a constant coefficient differential operator on $\mathscr{B}$ is also a constant coefficient differential operator on $\mathscr{D}$, and vice-versa. The Cayley transform C_0,

$$C_0(z)=(1+i\,z)/(1-i\,z) \tag{3.3}$$

is a biholomorphic map from $\mathscr{D}$ onto $\mathscr{B}$. More generally the map

$$\begin{aligned}(C_{\tau_1,\tau_2,k}f)(z)=\det(i(z+1)/\sqrt{2})^{-k}\,\tau_1((z+1)/\sqrt{2})\\ \otimes\tau_2(i(z+1)/\sqrt{2})^{-1}f((z-1)/i(z+1))\end{aligned} \tag{3.4}$$

sends holomorphic vector valued functions on $\mathscr{D}$ into holomorphic vector valued functions on $\mathscr{B}$. We let

$$U^b_{\tau_1,\tau_2,k}=C_{\tau_1,\tau_2,k}\,U_{\tau_1,\tau_2,k}\,C^{-1}_{\tau_1,\tau_2,k} \tag{3.5}$$

and if $U_{\tau_1,\tau_2,k}$ is unitary in a Hilbert space H of holomorphic functions on $\mathscr{D}$ we let H^b denote the corresponding Hilbert space of holomorphic functions on $\mathscr{B}$. The restriction of $U^b_{\tau_1,\tau_2,k}$ to K is given by

$$(U^b_{\tau_1,\tau_2,k}(u,v)f)(z)=(\det v)^k\,\tau_1(u)\otimes\tau_2(v)f(u^{-1}\,z\,v). \tag{3.6}$$

In particular, the K-types are polynomials on $\mathscr{B}$. The action of the full Lie algebra through $x\to dU^b_{\tau_1,\tau_2,k}(x)$ is described in [6, p. 35].

Lemma 3.1. *If $U_{\tau_1,\tau_2,k}$ is unitary on a Hilbert space H of holomorphic functions on $\mathscr{D}$, and p is a constant coefficient differential operator then*

$$\forall f\in H\colon\ (p,f)=0 \Leftrightarrow \forall g\in H^b\colon\ (p,g)=0.$$

Proof. The Cayley transform is in $G^{\mathbb{C}}$. In fact it is essentially of the form e^x for some $x\in\mathfrak{g}^{\mathbb{C}}$. The map (3.4) may be viewed as the extension of $U_{\tau_1,\tau_2,k}$ to that element. The claim then follows by power series expansion and the invariance of under $U_{\tau_1,\tau_2,k}$. The converse is analogous. Then term (zc^*+d^*) may be a cause for some concern. However, $J(g,z)=(z\,c^*+d^*)^{-1}$ for $g=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ satisfies

$J(g_1 g_2, z) = J(g_1, g_2 z) J(g_2, z)$. If we put $g_1 = g_2^{-1}$ we get

$$z c^* + d^* = (a - ((a z + b)/(c z + d)) c)^{-1}. \tag{3.7}$$

The right hand side is the expression that was used to obtain (3.4). Q.E.D.

Let G/K be a hermitian symmetric space of the non-compact type. Corresponding to a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$. If λ is a finite dimensional unitary representation of K on V_λ let

$$\mathcal{O}(\lambda) = \{\text{Analytic functions } \varphi\colon G \to V_\lambda | \\ \forall k \in K\colon \varphi(g k) = \lambda(k)^{-1} \varphi(g) \text{ and } \forall x \in \mathfrak{p}^-\colon r(x) \varphi = 0\},$$

where r denotes differentiation from the right. G acts on $\mathcal{O}(\lambda)$ by left translation. The space G/K is isomorphic to a bounded homogeneous domain $\mathcal{B} \subset \mathfrak{p}^+$, the Harish-Chandra realization of G/K. Through this identification the action of G on $\mathcal{O}(\lambda)$ becomes an action of G on the space of V_λ-valued holomorphic functions on $\mathcal{B}$. We denote this representation by U_λ.

Proposition 3.3. *Suppose λ_i is irreducible and that U_{λ_i} is unitary in a reproducing kernel Hilbert space H_{λ_i} of holomorphic functions $\mathcal{B} \to V_{\lambda_i}$, $i = 1, 2$. Then $U_{\lambda_1 \otimes \lambda_2}$ is unitary in a reproducing kernel Hilbert space $H_{\lambda_1 \otimes \lambda_2}$ of holomorphic functions $\mathcal{B} \to V_{\lambda_1} \otimes V_{\lambda_2}$ and*

$$H_{\lambda_1 \otimes \lambda_2} \supset H_{\lambda_1} \otimes V_{\lambda_2} + V_{\lambda_1} \otimes H_{\lambda_2}. \tag{3.8}$$

Proof. Consider $U_{\lambda_1} \otimes U_{\lambda_2}$ on the Hilbert space $H_{\lambda_1} \otimes H_{\lambda_2}$. Following the proof of Proposition 2.5 in [6] we observe that $H_{\lambda_1} \otimes H_{\lambda_2}$ is a reproducing kernel Hilbert space of holomorphic functions on $\mathcal{B} \times \mathcal{B}$, that the subspace H_0 of functions that vanish on the diagonal of $\mathcal{B} \times \mathcal{B}$ is a G-invariant Hilbert space, and that the restriction map R,

$$(R(f_1 \otimes f_2))(z) = f_1(z) \otimes f_2(z) \tag{3.9}$$

intertwines $U_{\lambda_1} \otimes U_{\lambda_2}$ with a subrepresentation $U^s_{\lambda_1 \otimes \lambda_2}$ of $U_{\lambda_1 \otimes \lambda_2}$. $R(H_{\lambda_1} \otimes H_{\lambda_2})$ is canonically a Hilbert space and $U^s_{\lambda_1 \otimes \lambda_2}$ is unitary on this space. But $R(H_{\lambda_1} \otimes H_{\lambda_2})$ contains all the constant functions $v_1 \otimes v_2$, $v_i \in V_{\lambda_i}$, hence $U^s_{\lambda_1 \otimes \lambda_2} = U_{\lambda_1 \otimes \lambda_2}$ and $R(H_{\lambda_1} \otimes H_{\lambda_2}) = H_{\lambda_1 \otimes \lambda_2}$. The inclusion (3.8) is then obvious. Q.E.D.

Corresponding to U_λ we let dU_λ denote the associated representation of $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$, and we define

$$W(\lambda) = \{dU_\lambda(u) \cdot v | u \in \mathcal{U}(\mathfrak{g}^{\mathbb{C}}), v \in V_\lambda\}. \tag{3.10}$$

$W(\lambda)$ is a $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$-module of polynomials on $\mathcal{B}$. If λ is irreducible, any subspace of polynomials on $\mathcal{B}$ invariant under all $dU_\lambda(u)$, $u \in \mathcal{U}(\mathfrak{g}^{\mathbb{C}})$, contains $W(\lambda)$. ($W(\lambda)$ is *the* irreducible quotient of the module $\mathcal{U}(\mathfrak{g}^{\mathbb{C}}) \underset{\mathcal{U}(\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+)}{\otimes} V_\lambda$.) In this case, if U_λ is unitary on a Hilbert space H_λ of holomorphic functions on $\mathcal{B}$, then $W(\lambda) \subset H_\lambda$ and $\overline{W(\lambda)} = H_\lambda$. By results in [3] H_λ is a reproducing kernel Hilbert space and the kernel is unique up to multiplication by positive reals.

Corollary 3.4. *If $W(\lambda_1)$ is unitarizable and if $W(\lambda_1 \otimes \lambda_2) \not\supset W(\lambda_1) \otimes V_{\lambda_2}$ then $W(\lambda_2)$ is not unitarizable.*

Let us return to the case at hand, $G=SU(n,n)$. As in the general case, Proposition 3.3 and Corollary 3.4 can be dualized into statements about annihilators of modules, and it is those that will be used in what follows:

We consider representations $U^b_{\tau_1,\tau_2,k}$ with τ_1 and τ_2 fixed (and arbitrary) and drop temporarily these subscripts. Let W_k denote the corresponding $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$-module (3.10).

We define a bilinear pairing B between the space $\mathscr{E}_k$ of $(V_{\tau_1}\otimes V_{\tau_2})'$-valued constant coefficient differential operators on $\mathscr{B}$ and the space $\mathscr{P}_k$ of $V_{\tau_1}\otimes V_{\tau_2}$-valued polynomials on $\mathscr{B}$ by, for $e\in\mathscr{E}_k$, and $p\in\mathscr{P}_k$,

$$B(e,p)=(e(\partial/\partial z),p(\cdot))(0), \tag{3.11}$$

and we define

$$A_k=\{e\in\mathscr{E}_k \mid \forall\, p\in W_k, \forall\, z\in\mathscr{B}\colon (e,p)(z)=0\},$$

and

$$W_k^0=\{e\in\mathscr{E}_k \mid \forall\, p\in W_k\colon B(e,p)=0\}. \tag{3.12}$$

Since W_k is invariant under differentiation we have

Lemma 3.5. $A_k=W_k^0$.

Thus $A_k\neq 0$ exactly when W_k is not equal to the set of all $V_{\tau_1}\otimes V_{\tau_2}$-valued polynomials on $\mathscr{B}$, i.e. when certain K-types are missing from W_k. K acts on $\mathscr{E}_k$ through the pairing B as

$$(R_k'(u,v)\,e)(\partial/\partial z)=(\det v)^{-k}\,\overline{\tau_1(u)}\otimes\overline{\tau_2(v)}\,e({}^tu(\partial/\partial z){}^tv^{-1}). \tag{3.13}$$

The elements of A_k may be taken to be homogeneous polynomials. Let $d(A_k)$ denote the lowest degree occuring in A_k. If $A_k\neq 0$ clearly $d(A_k)\geqq 1$.

Corollary 3.6. *Assume that W_k and W_{k-1} are unitarizable and that $A_k\neq 0$. Then $A_{k-1}\neq 0$ and $d(A_{k-1})=d(A_k)-1$.*

Proof. Let $\tilde U$ denote the representation $U^b_{1,1,1}$ corresponding to $(U_{1,1,1}(g)f)(z)=\det(cz+d)^{-1}f((az+b)/(cz+d))$, and let $\tilde W$ denote the corresponding $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$-module. ((3.10)). It is easy to see that $\tilde W$ contains all first order polynomials on $\mathscr{B}$. Let $e\in A_k$, $p\in W_{k-1}$, and $q\in\tilde W$. It follows from Proposition 3.3 that $p\in W_k$ and $q\cdot p\in W_k$. Hence $\langle[e,q],p\rangle=0$. Q.E.D.

Corollary 3.7. *If W_k is unitarizable and $d(A_k)=1$ then W_{k-1} is not unitarizable.*

§4. The Main Result

Using Corollary 3.7 we shall now prove that the representations in Proposition 2.3 exhaust the set of unitary holomorphic representations of $SU(n,n)$.

Let $\tau_2=\tau(n_1,\dots,n_i,0,\dots,0)$ and consider the reproducing kernel

$$K_{1,\tau_2,i}(z,w)=\int_{\mathcal{O}_i}\tau_2(k)e^{i\,tr(z-w^*)k}\,d\mu_i(k).$$

It is easy to see that a constant coefficient (homogeneous) V_{τ_2}'-valued differential operator $e(\partial/\partial z)$ annihilates $H_{1,\tau_2,i}$ if and only if

$$\forall v \in V_{\tau_2}, \forall k \in \mathcal{O}_i: (e({}^t k), \tau_2(k) v) = 0. \tag{4.1}$$

$\mathcal{O}_i$ is a sufficiently big part of the variety $V(I_{i+1})$ that we conclude

Lemma 4.1. *$e \in \mathcal{E}(V'_{\tau_2})$ annihilates $H_{1,\tau_2,i}$ if and only if*

$$\forall v \in V_{\tau_2}: (\tau_2(z)\, e(z), v) \in I_{i+1}.$$

As before we identify τ'_2 with the representation $a \to \tau_2({}^t a^{-1})$. For each first order polynomial e in $\mathcal{E}(V'_{\tau_2})$ and each vector v in V_{τ_2} let

$$p_{e,v}(z) = (\tau_2(z)\, e(z), v).$$

Then

$$p_{e,v}(b^{-1} z a) = (\tau_2(z)\, \tau_2(a)\, e(b^{-1} z a), \tau_2({}^t b^{-1}) v).$$

Let $(R(a,b)e)(z) = \tau_2(a)\, e(b^{-1} z a)$. It follows that the functions $p_{e,v}$ transform according to a subrepresentation of $(a,b) \to R(a,b) \otimes \tau_2({}^t b^{-1})$. A (very) special case of the Littlewood-Richardson rule (see e.g. [7]) asserts that one of the irreducible subrepresentations of R is

$$\rho_2 = \tau(0, \dots, 0, -1) \otimes \tau(n_1, \dots, n_i, 1, 0, \dots, 0)$$

on, say, $V_{\rho_2} \subset \mathcal{E}(V'_{\tau_2})$. Clearly for any polynomial e in V_{ρ_2}, $p_{e,v} \neq 0$ for some v's in V_{τ_2}. On the other hand the representations contained in $(a,b) \to \rho_2(a,b) \otimes \tau_2({}^t b^{-1})$ that occur in the space of polynomials $p_{e,v}$; $e \in V_{\rho_2}$, $v \in V_{\tau_2}$, must be of the form $\tau \otimes \tau'$ (Lemma 1.1) and hence, by Lemma 1.3,

$$\forall e \in V_{\rho_2}, \forall v \in V_{\tau_2}: (\tau_2(z)\, e(z), v) \in I_{i+1}.$$

Of course V_{ρ_2} also carries the representation $(a,b) \to \rho_2({}^t a^{-1}, {}^t b^{-1})$ which, by duality, is the one we should consider. This means:

Proposition 4.2. *The space $H_{1,\tau_2,i}$, $i = 1, \dots, n-1$, is annihilated by first order constant coefficient differential operators. The corresponding K-type that does not appear in $H_{1,\tau_2,i}$ is $(u,v) \to (\det v)^i \rho_2(u,v)$.*

Let $\tau_1 = \tau(0, \dots, 0, -m_1, \dots, -m_j)$ and let

$$\rho_1 = \tau(0, \dots, 0, -1, -m_1, \dots, -m_j) \otimes \tau(1, 0, \dots, 0).$$

By similar reasoning we have

Proposition 4.3. *The space $H_{\tau_1,1,j}$, $j = 1, \dots, n-1$, is annihilated by first order constant coefficient differential operators. The corresponding K-type that does not appear in $H_{\tau_1,1,j}$ is $(u,v) \to (\det v)^j \rho_1(u,v)$.*

Let τ_1 and τ_2 be as before and turn to the bounded domain $\mathcal{B}$. If all first order polynomials with values in $V_{\tau_1} \otimes V_{\tau_2}$ were contained in $H^b_{\tau_1,\tau_2,i+j}$, this would imply that the K-type $(u,v) \to (\det v)^{i+j} \rho_3(u,v)$, where

$$\rho_3 = \tau(0, \dots, 0, -1, -m_1, \dots, -m_j) \otimes \tau(n_1, \dots, n_i, 1, 0, \dots, 0),$$

should appear. However, using that $H^b_{\tau_1,\tau_2,i+j} \subset H^b_{\tau_1,1,j} \otimes H^b_{1,\tau_2,i}$, the first order polynomials that occur in $H^b_{\tau_1,\tau_2,i+j}$ are seen to be of the form $v_1 \otimes q + p \otimes v_2$, where $q \in H^b_{1,\tau_2,i}$, $p \in H^b_{\tau_1,1,j}$, and $v_i \in V_{\tau_i}$ $(i=1,2)$. Hence it follows from Proposition 4.2 and Proposition 4.3 that the above K-type is missing. Thus

Proposition 4.4. *The space $H_{\tau_1,\tau_2,i+j}$, $i,j \in \{1,\ldots,n-1\}$, is annihilated by first order constant coefficient differential operators. The corresponding K-type that does not appear in $H_{\tau_1,\tau_2,i+j}$ is $(u,v) \to (\det v)^{i+j} \rho_3(u,v)$.*

Combining the last three propositions with Lemma 2.4 and Corollary 3.7 we obtain

Theorem 4.5. *The representations $U_{\tau_1,\tau_2,k}$ of $SU(n,n)$, holomorphically induced from*

$$\tau_1(0,\ldots,0,-m_1,\ldots,-m_j) \otimes \tau_2(n_1,\ldots,n_i,0,\ldots,0) \otimes \delta_k$$

are unitary if and only if $i+j \leqq k$.

§ 5. $U(p,q)$

Only a few extra observations are needed to handle these groups along the lines of the preceding chapters. We shall be content to give these and then state the theorem. The notation essentially follows [8].

We assume $p>q$ and let $r=p-q$. Throughout τ_2 denotes an irreducible representation of $Gl(q,\mathbb{C})$ belonging to the set S, and τ_1 an irreducible representation of $Gl(p,\mathbb{C})$ whose contragredient belongs to S. Elements of $M(q,\mathbb{C})$ are denoted z_1, z_2, etc., and elements of $M(r,q,\mathbb{C})$, the space of linear maps from $\mathbb{C}^q$ to $\mathbb{C}^r$, are denoted u_1, u_2, etc. The maximal compact subgroup K is isomorphic to $U(p) \times U(q)$. The domain corresponding to $\mathcal{D}$ for $U(n,n)$ is again denoted $\mathcal{D}$, and is defined as

$$\mathcal{D} = \{(z,u) \mid z \in M(q,\mathbb{C}),\ u \in M(r,q,\mathbb{C}), \text{ and } (z-z^*)/i > u^*u\}. \tag{5.1}$$

Representations $U_{\tau_1,\tau_2,k}$, holomorphically induced from irreducible unitary representations of K, are again considered. If $U_{\tau_1,\tau_2,k}$ is unitary in a Hilbert space $H_{\tau_1,\tau_2,k}$, this space is a reproducing kernel Hilbert space ([3]) and the kernel is given by

$$K_{\tau_1,\tau_2,k}(z_1,u_1,z_2,u_2) = \det((z_1 - z_2^* - iu_2^*u_1)/2i)^{-k} \tau_1 \begin{bmatrix} (z_1-z_2^*)/2i & iu_2^*/\sqrt{2} \\ -iu_1/\sqrt{2} & 1 \end{bmatrix}$$
$$\otimes \tau_2((z_1 - z_2^* - iu_2^*u_1)/2i)^{-1}. \tag{5.2}$$

The τ_1-part of this kernel differs slightly from that of [8]. If we write elements g of $U(p,q)$ as

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha \in M(p,\mathbb{C}), \quad \text{etc.,} \quad \text{and let } p = \begin{pmatrix} z \\ u \end{pmatrix} \quad \text{for } (z,u) \in \mathcal{D},$$

the action of g on $\mathcal{D}$ is given as $g \cdot p = (\alpha p + \beta)(\gamma p + \delta)^{-1}$. A slight problem occurs when the automorphic factor $J_1(g,p) = \alpha - (g \cdot p)\gamma$ (cf. (3.7) and (2.3)) is considered:

For $p_0=\begin{pmatrix} i \\ 0 \end{pmatrix}$ the representation $k \to J_1(k, p_0)$ of K is not unitary. An equivalent representation which is unitary is

$$k \to \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} J_1(k, p_0) \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$$

($\sqrt{2}=\sqrt{2}\cdot I_q$ where I_q is the identity on $\mathbb{C}^q$). Thus, for the representations $U_{\tau_1,\tau_2,k}$ it is understood that the τ_1-part of the multiplier has been changed by $\mathrm{Ad}\left(\tau_1\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}\right)$. With this change the Eq. (5.2) follows from the covariance properties of the kernel. Later, when the bounded realization is considered the automorphic factor must be changed back to its original form. As this change does not affect the number of K-types it is insignificant and we shall make no further comments on this issue.

Let $\tau_2=\tau(n_1, \ldots, n_i, 0, \ldots, 0)$. Since, up to a constant multiple,

$$\begin{aligned}\det((z_1-z_2^*-iu_2^*u_1)/2i)^{-i}\tau_2((z_1-z_2^*-iu_2^*u_1)/2i)^{-1}\\ =\int_{\mathcal{O}_i}\tau_2(k)e^{itr(z_1-z_2^*-iu_2^*u_1)k}\,d\mu_i(k),\end{aligned} \tag{5.3}$$

where $\mathcal{O}_i=\{k\in M(q,\mathbb{C})\,|\,k=k^*, k\geqq 0, \text{ and rank } k\leqq i\}$, we get from Proposition 4.2

Proposition 5.1. *The Hilbert space* $H_{1,\tau_2,i}$ *is annihilated by first order constant coefficient differential operators.*

Let $\tau_1=\tau(0, \ldots, 0, -m_1, \ldots, -m_j)$ and let λ denote the contragredient representation, i.e. $\forall b\in Gl(p, \mathbb{C})$: $\lambda(b)=\tau({}^tb^{-1})$. Then, up to a constant multiple,

$$\begin{aligned}\det((z_1-z_2^*-iu_2^*u_1)/2i)^{-j}\tau_1\begin{bmatrix}(z_1-z_2^*)/2i & iu_2^*/\sqrt{2}\\ -iu_1/\sqrt{2} & 1\end{bmatrix}\\ =\int_{\mathcal{O}_j}\lambda({}^tk)\,e^{-tr\begin{bmatrix}(z_1-z_2^*)/2i & iu_2^*/\sqrt{2}\\ -iu_1/\sqrt{2} & 1\end{bmatrix}\cdot k}\,d\mu_j(k),\end{aligned} \tag{5.4}$$

where $\mathcal{O}_j=\{k\in M(p,\mathbb{C})\,|\,k=k^*,\ k\geqq 0, \text{ and rank } k\leqq j\}$.

Since in a reproducing kernel Hilbert space of holomorphic functions norm convergence implies uniform convergence on compacta, a constant coefficient differential operator that annihilates a dense subspace, annihilates the whole space. Thus, if we can find a differential operator $\tilde{p}(\partial/\partial z, \partial/\partial u)$ such that

$$\left(\tilde{p}(\partial/\partial z, \partial/\partial u), \int_{\mathcal{O}_j}\lambda({}^tk)\,e^{-tr\begin{bmatrix}(z_1-z_2^*)/2i & iu_2^*/\sqrt{2}\\ -iu_1/\sqrt{2} & 1\end{bmatrix}\cdot k}v\cdot d\mu_j\right)=0 \tag{5.5}$$

for all (z_1, u_1), (z_2, u_2) in $\mathcal{D}$, and all $v\in V_{\tau_1}$, then p annihilates $H_{\tau_1,1,j}$. With $k=\begin{bmatrix}k_1 & \beta\\ \beta^* & k_2\end{bmatrix}$ where $k_1\in M(q, \mathbb{C})$, etc., the condition (5.5) is equivalent to

$$(\tilde{p}(-{}^tk_1/2i, -i\,{}^t\beta/\sqrt{2}), \lambda({}^tk)v)=0 \tag{5.6}$$

on $\mathcal{O}_j$, for all $v \in V_{\tau_1}$. We shall be looking for first order polynomials $\tilde{p}$ and may therefore just as well look for first order polynomials p such that

$$(p({}^tk_1, {}^t\beta), \lambda({}^tk)v) = 0 \tag{5.7}$$

on $\mathcal{O}_j$, for all $v \in V_{\tau_1}$. Write elements z of $M(p, \mathbb{C})$ as $z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$, where $z_1 \in M(q, \mathbb{C})$, etc. As in the proof of Propositions 4.2 and 4.3 we can find a space $V_\rho \neq \{0\}$ of first order polynomials on $M(p, \mathbb{C})$ such that V_ρ is invariant under the representation

$$(\rho(a,b)q)(z) = \lambda({}^tb^{-1})\,q(b^{-1}za) (= \tau(b)\,q(b^{-1}za)),$$

$$\forall v \in V_{\tau_1}, \ \forall q \in V_\rho\colon\ (\lambda({}^tz)\,q(z), v) \in I_{j+1}, \tag{5.8}$$

and any first order polynomial q on $M(p, \mathbb{C})$ which satisfies (5.8) belongs to V_ρ. If we can find a $q \in V_\rho$ such that $q\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = q\begin{pmatrix} z_1 & 0 \\ z_3 & 0 \end{pmatrix}$ we will be done. Using the invariance of V_ρ under $\rho(a,1)$, $a \in Gl(p, \mathbb{C})$, it follows readily that for any $q \in V_\rho$ the polynomial $q_r, q_r \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} = q\begin{bmatrix} z_1 & 0 \\ z_3 & 0 \end{bmatrix}$, satisfies (5.8). Thus it suffices to prove

Lemma 5.2. *$\exists q \in V_\rho$ such that $q\begin{bmatrix} z_1 & 0 \\ z_3 & 0 \end{bmatrix} \neq 0$.*

Proof. Assume the converse and write

$$q\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = q_1(z_1) + q_2(z_2) + q_3(z_3) + q_4(z_4).$$

Then for all $q \in V_\rho$ q_1 and q_3 are zero. If

$$a_u = \begin{pmatrix} 1 & u^* \\ 0 & 1 \end{pmatrix} \in M(p, \mathbb{C}) \qquad \rho(a_u, 1)q \in V_\rho$$

and hence, for all $u \in M(r, q, \mathbb{C})$ and all z_1, z_3, $q_2(z_1u^*) + q_4(z_3u^*) = 0$. Thus $q_2 = q_4 = 0$, and hence $V_\rho = \{0\}$. Contradiction.

We can then state:

Proposition 5.3. *The Hilbert space $H_{\tau_1,1,j}$ is annihilated by first order constant coefficient differential operators.*

Finally, by forming tensor products as in §4, we arrive at

Theorem 5.4. *The representations $U_{\tau_1,\tau_2,k}$ of $U(p,q)$, holomorphically induced from*

$$\tau_1(0, \ldots, 0, -m_1, \ldots, -m_j) \otimes \tau_2(n_1, \ldots, n_i, 0, \ldots, 0) \otimes \delta_k$$

are unitary if and only if $i + j \leqq k$.

§ 6. A Concluding Remark

As a corollary we get that any irreducible unitary holomorphic representation of $U(p,q)$ appears in the k-th fold tensor product of the harmonic representation for some k, as was conjectured in [8]. Since the harmonic representation itself is the sum of all representations living on the orbit(s) $\mathcal{O}_1$, this conjecture has a natural extension to tube domains and conceivably to arbitrary Hermitian symmetric spaces of the non-compact type.

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