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Singular Holomorphic Representations and Singular Modular Forms

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Introduction

In the papers [3, 10, 11] Resnikoff and Freitag introduced the notion of a singular modular form on a tube domain $\mathscr{D}$, with respect to an arithmetic group of transformations of $\mathscr{D}$. They proved that, for an appropriately restricted class of arithmetic groups Γ, a modular form of weight k for Γ is singular if and only if k is one of the set of "singular weights" for $\mathscr{D}$.

The condition that a modular form f be singular may be expressed as the condition that f be annihilated by a certain constant coefficient differential operator. The main object of this paper is to interpret such conditions representation-theoretically. Namely, by a well-known procedure, f gives rise to a representation $\mathscr{V}_f$ of the enveloping algebra of the Lie algebra of the group of holomorphic automorphisms of $\mathscr{D}$ (cf. 3.1). We prove (Theorem 4.8) that $\mathscr{V}_f$ is an *irreducible* representation if f is a form of singular weight with respect to *any* arithmetic group Γ acting on $\mathscr{D}$; it then follows (Theorem 3.6) that f is annihilated by a specific constant coefficient vector-valued differential operator, which depends on the weight of f. This provides a generalization of the results of Resnikoff and Freitag. In the course of proving these theorems, we generalize some results of Selberg, Maass, Freitag, and Resnikoff [3, 8, 11] on the homogeneity of constant coefficient differential operators on tube domains (Theorem 3.2); our proof is rather different from theirs.

Our main step is the explicit computation (Proposition 2.9) of the irreducible quotients of the anti-holomorphic type enveloping algebra representations corresponding to the singular weights. Here we rely heavily on the Vergne-Rossi construction of singular holomorphic representations [13], and its further elaboration in [6].

Our results as stated clearly allow for considerable generalization; some directions of generalization are suggested in Remark 4.9. In an appendix, we apply a mild generalization of the techniques of [6] to simplify Freitag's construction [3, 19] of holomorphic differential forms on the Siegel modular varieties.

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0. Notation

We use the following standard notation: for a C^∞-manifold X and a real vector space V, we write $C^\infty(X, V)$ for the space of C^∞ V-valued functions on X. If G is a Lie group, with Lie algebra $\mathfrak{g}$, we write exp for the exponential map from $\mathfrak{g}$ to G.

If V is a complex vector space, $\mathrm{Sym}^n V$ denotes the nth symmetric power of V for a positive integer n, $\mathrm{Sym}^0 V = \mathbb{C}$, and $S(V) = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n V$. We let V^* denote $\mathrm{Hom}(V, \mathbb{C})$, and $S^*(V) = \bigoplus_{n=0}^{\infty} (\mathrm{Sym}^n V)^*$.

In the remainder of this section, we present our notation for tube domains. Proofs of the results indicated here can be found in [1, 13, 14].

Let V be a real vector space with Euclidean inner product $(\,,)$, and let C be an open irreducible, homogeneous cone in V, self-adjoint with respect to $(\,,)$ (for their theory, cf. Chap. II of [1]). Let $H_0 \subset \mathrm{GL}(V)$ denote the connected identity component of the subgroup of $\mathrm{GL}(V)$ leaving C fixed. Then H_0 acts as a group of linear transformations on the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of V. Let $\mathcal{D} = \{x + iy \in V_{\mathbb{C}} | x \in V, y \in C\}$ be the tube domain over C. Let U_0 be the group of affine transformations of $\mathcal{D}$ generated by the translations t_v, $v \in V$, defined by the formula $t_v(z) = z + v$, $z \in \mathcal{D}$. The semi-direct product $\mathcal{P} = H_0 \times U_0$ acts on $\mathcal{D}$. Let $\mathrm{Aut}(\mathcal{D})$ denote the group of holomorphic automorphisms of $\mathcal{D}$, $\mathrm{Aut}(\mathcal{D})^0$ its connected component; then $\mathcal{P}$ is a (covering group of a) real parabolic subgroup of $\mathrm{Aut}(\mathcal{D})^0$. Let G_0 denote a covering group of $\mathrm{Aut}(\mathcal{D})^0$ containing $\mathcal{P}$; denote its Lie algebra $\mathfrak{g}_0$, and denote the complexification of $\mathfrak{g}_0$ by $\mathfrak{g}$. The enveloping algebra of $\mathfrak{g}$ is denoted $U(\mathfrak{g})$.

Pick a base point $s \in C$, and let $K_0 \subset G_0$ denote the stabilizer of $is \in \mathcal{D}$. Then K_0 is a maximal compact subgroup of G_0, with Lie algebra $\mathfrak{k}_0$. The elements of $\mathfrak{g}_0$, and therefore of $U(\mathfrak{g})$, act both on the right and on the left on $C^\infty(G_0, S)$, for any complex vector space S: For $X \in \mathfrak{g}_0$, we have the right (resp. left) action

$$(r(X)*f)(g) = \frac{d}{dt} f(g \exp(tX))|_{t=0}$$

$$\left(\text{resp. } (l(X)*f)(g) = \frac{d}{dt} f(\exp(-tX)g)|_{t=0}\right) \quad \forall\, f \in C^\infty(G_0, S);$$

the notation $r(X)*f$ and $l(X)*f$ will be used for any $X \in U(\mathfrak{g})$. We have a natural map $G_0 \to \mathcal{D}$:

$$g \mapsto g \cdot s$$

which identifies $\mathcal{D}$ with the homogeneous space G_0/K_0, and the action of $l(X)$ preserves $C^\infty(\mathcal{D}, \mathbb{C}) \approx C^\infty(G_0/K_0, \mathbb{C}) \subset C^\infty(G_0, \mathbb{C})$, for any $X \in U(\mathfrak{g})$. Let $\mathfrak{k} \subset \mathfrak{g}$ denote the complexification of $\mathfrak{k}_0$. As a $\mathfrak{k}$-module under the adjoint action, we have the decomposition

$$\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-,$$

where, $\forall X \in \mathfrak{p}^+$ (resp. $\mathfrak{p}^-$), $l(X)$ gives rise to a vector field on $\mathcal{D}$ which, at every point of $\mathcal{D}$, lies in the holomorphic (resp. anti-holomorphic) tangent space. Note that $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian *Lie subalgebras* of $\mathfrak{g}$.

We let G be a connected algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$ such that the connected subgroup of G with Lie algebra $\mathfrak{g}_0$ is isomorphic to G_0; we regard G_0 as a subgroup of G. Let K, P^+, and P^- denote the connected subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{p}^+$, and $\mathfrak{p}^-$ respectively. It is well known that the multiplication map $P^+ \times K \times P^- \to G$ is injective, that G_0 is contained in its image P^+KP^-, and that the map $G_0 \to \mathfrak{p}^+$ obtained by projection on the first factor P^+ of P^+KP^-, followed by the logarithm $\log^+: P^+ \to \mathfrak{p}^+$, identifies G_0/K_0 with a bounded domain $\mathscr{D}_0$ in $\mathfrak{p}^+$. We let $\zeta: P^+KP^- \to \mathfrak{p}^+$ denote projection on P^+ followed by $\log^+$.

The Lie algebra $\mathfrak{k}_0$ contains a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$; let $\mathfrak{h} \subset \mathfrak{g}$ denote its complexification. For every root γ of $\mathfrak{g}$ relative to $\mathfrak{h}$, let $\mathfrak{g}^\gamma$ denote the corresponding root subspace; let $H_\gamma \in i\mathfrak{h}_0 \cap [\mathfrak{g}^\gamma, \mathfrak{g}^{-\gamma}]$ be the unique element such that $\gamma(H_\gamma) = 2$. Choose an ordering on the roots so that $\mathfrak{p}^+$ is a direct sum of positive root spaces; the corresponding roots are called *positive non-compact*. For each positive non-compact root γ, choose $E_\gamma \in \mathfrak{g}^\gamma$ such that

$$[E_\gamma, \bar{E}_\gamma] = H_\gamma \quad (\bar{\ } = \text{complex conjugation with respect to } \mathfrak{g}_0).$$

Note that $\bar{E}_\gamma \in \mathfrak{g}^{-\gamma} \subset \mathfrak{p}^-$; we let $E_{-\gamma} = \bar{E}_\gamma$. Choose a maximal set $\{\gamma_1, \dots, \gamma_r\}$ of strongly orthogonal non-compact positive roots in the usual way (cf. [4, p. 315]); we use the known properties of these roots freely. Define the *Cayley transform*

$$c = \prod_{i=1}^{r} \exp\left(-\frac{\pi}{4}[E_{\gamma_i} - E_{-\gamma_i}]\right) \in G.$$

We assume we have chosen the E_{γ_i} so that $c^{-1}U_0 c \subset P^+$, as is always possible. Then we may identify $V \simeq U_0$ (via the map $v \mapsto t_v$) with a real form of $\mathfrak{p}^+$ (via the composition of $u \mapsto c^{-1}uc \in P^+$ with $\log^+: P^+ \to \mathfrak{p}^+$). Moreover, $c^{-1}G_0 \supseteq P^+KP^-$, and the map

$$\zeta': G_0 \to \mathfrak{p}^+; \qquad \zeta'(g) = \zeta(c^{-1}g)$$

identifies $G_0/K_0 \simeq \mathscr{D}$ with the tube domain (also denoted $\mathscr{D}$) $V + iC \subset \mathfrak{p}^+ \simeq V_{\mathbb{C}}$. The fixed point of K_0 is identified with s. We note finally that, under these hypotheses, $c^{-1}H_0 c \subset K$.

We next define the canonical automorphy factor. Let $c^{-1} = c_+ c_0 c_-$, with $c_\pm \in P^\pm$, $c_0 \in K$. If $g \in G_0$, let $j(g) = j(g, is) = c_0^{-1} g_0$, where $c^{-1}g = g_+ g_0 g_-$, $g_\pm \in P^\pm$, $g_0 \in K$. If $z = g_z(is) \in \mathscr{D}$, for some $g_z \in G_0$, let

$$j(g, z) = j(gg_z) j(g_z)^{-1}, \qquad \forall\, g \in G_0.$$

Then $j(g, z)$ is well-defined, and has the following obvious properties:

(0.1.1) $\quad j(g_1 g_2, z) = j(g_1, g_2(z)) j(g_2, z) \quad g_1, g_2 \in G_0 \quad z \in \mathscr{D}$,

(0.1.2) $\quad j(k, is) = k \quad k \in K_0, \quad s$ as above,

(0.1.3) $\quad j(u, z) = 1, \quad \forall\, u \in U_0, \quad z \in \mathscr{D}$,

(0.1.4) $\quad j(h, z) = c^{-1}hc \quad \forall\, h \in H_0, \quad z \in \mathscr{D}$.

A let ϱ be any holomorphic representation of K on a finite-dimensional vector space V_ϱ, and let $j_\varrho(g, z) = \varrho(j(g, z)) \in \mathrm{GL}(V_\varrho)$, $\forall\, g \in G_0$, $z \in \mathscr{D}$. We may think of $j_\varrho(g, is)$

as an element of $C^\infty(G_0, \mathrm{End}(V_\varrho))$. Then it follows easily from the definitions that

(0.1.5) $$r(X)*j_\varrho(g, is) \equiv 0 \quad \forall\, \varrho \text{ as above}, \quad X \in \mathfrak{p}^-.$$

We write the Cartan decomposition of $\mathfrak{g}_0$, with respect to $\mathfrak{k}_0$, in the form $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. Let $X_i = \mathrm{Ad}(c)H_{\gamma_i}$, $i = 1, \ldots, r$. Then $X_i \in \mathfrak{p}_0$, and $\mathfrak{a}_0 = \bigoplus_{i=1}^{r} \mathbb{R}X_i$ is a maximal abelian subalgebra of $\mathfrak{p}_0$, and is a maximal split diagonalizable subalgebra of $\mathfrak{g}_0$. Since G_0/K_0 is isomorphic to a tube domain, the set of roots of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$ is of the form

$$\{\pm\tfrac{1}{2}(\alpha_i \pm \alpha_j) \quad 1 \leqq i \leqq j \leqq r\} \quad \text{(cf. [13, Sect. 2])}.$$

Let p be the dimension of the root space corresponding to $\frac{1}{2}(\alpha_i + \alpha_j)$, $i \neq j$ (this doesn't depend on the choice of $i \neq j$). The *singular weights* (or the Wallach set [15]) for $\mathcal{D}$ are the half-integers

$$\lambda_e = \frac{(e-1)p}{2}, \quad 1 \leqq e \leqq r.$$

The *regular weights* are the real numbers $\lambda > \frac{1}{2}(r-1)p$.

1. Homogeneous Differential Operators

1.1. Let $\varrho: K \to \mathrm{GL}(V_\varrho)$ be a holomorphic representation. Any function $f \in C^\infty(\mathcal{D}, V_\varrho)$ induces a function $\tilde{f}: G_0 \to V_\varrho$, under the identification $G_0/K_0 \simeq \mathcal{D}$ of Sect. 0. For such an f, define $\phi_{f,\varrho} \in C^\infty(G_0, V_\varrho)$:

(1.1.1) $$\phi_{f,\varrho}(g) = j_\varrho(g, is)^{-1} \tilde{f}(g) \qquad g \in G_0.$$

When it does not lead to ambiguity, we denote this correspondence $f \leftrightarrow \phi$. We note that

(1.1.2) $$\phi_{f,\varrho} \in C^\infty(G_0, V_\varrho)_\varrho \overset{\text{def}}{=} \{\phi \in C^\infty(G_0, V_\varrho) \mid \phi(gk) = \varrho(k)^{-1}\phi(g),\ \forall\, g \in G_0,\ k \in K_0\}.$$

A function $\phi \in C^\infty(G_0, V_\varrho)$ is said to be of *holomorphic type* if $r(X)*\phi \equiv 0$, $\forall\, X \in \mathfrak{p}^-$. The following fact is standard (cf. [13, (2.4.2)]):

1.2 **Lemma.** *Under the above hypotheses, f is holomorphic if and only if $\phi_{f,\varrho}$ is of holomorphic type.*

1.3. Let $W \subset S(\mathfrak{p}^+)$ be a finite-dimensional subspace invariant with respect to the adjoint action of K; denote the representation of K on W by W likewise. Let $\phi \in C^\infty(G_0, V_\varrho)_\varrho$, and consider $D_W\phi \in \mathrm{Hom}(W, C^\infty(G_0, V_\varrho))$, defined by the formula

(1.3.1) $$D_W\phi(X) = r(X)*\phi \qquad X \in W.$$

We may regard $D_W\phi$ canonically as an element of $C^\infty(G_0, V_\varrho \otimes W^*)_{\varrho \otimes W^*}$; the action of K on W^* is contragradient to the action on W. Now suppose $f \in C^\infty(\mathcal{D}, V_\varrho)$; define

(1.3.2) $$(\delta_{W,\varrho} f)(Z) = j_{\varrho \otimes W^*}(g, is) \cdot D_W \phi_{f,\varrho}(g),$$

where $gK_0 = z \in \mathscr{D} \simeq G_0/K_0$ and $j_{\varrho \oplus W^*}(g, is)$ acts on $V_\varrho \otimes W^*$ in the natural way. Then $\delta_{W,\varrho} f$ is a well-defined element of $C^\infty(\mathscr{D}, V_\varrho \otimes W^*)$. Moreover, $\delta_{W,\varrho}$ satisfies a homogeneity property: if, for $f \in C^\infty(\mathscr{D}, V_\varrho)$, $\gamma \in G_0$, we define

$$(f|_\varrho \gamma)(Z) = j_\varrho(\gamma, Z)^{-1} f(\gamma Z),$$

then it follows easily from the left invariance of $r(X)$ and formula (0.1.1) that

$$\delta_{W,\varrho}(f|_\varrho \gamma) = (\delta_{W,\varrho} f)|_{\varrho \otimes W^*} \gamma. \qquad (1.3.3)$$

1.3.4. If $\Gamma \subset G_0$ is a discrete subgroup, then $f \in C^\infty(\mathscr{D}, V_\varrho)$ is (for the purposes of this article) called a *C^∞-automorphic form for Γ of type ϱ* if $f|_\varrho \gamma = f \quad \forall\, \gamma \in \Gamma$. It follows from (1.3.3) that $\delta_{W,\varrho}$ takes such forms to C^∞-automorphic forms of type $\varrho \otimes W^*$ for Γ (any Γ).

1.4. **Lemma.** 1. *The* $\mathrm{Hom}(V_\varrho, V_\varrho \otimes W^*)$*-valued differential operator* $\delta_{W,\varrho}$ *is a polynomial in holomorphic tangent vectors – i.e., a holomorphic differential operator – with coefficients in* $C^\infty(\mathscr{D}, \mathrm{Hom}(V_\varrho, V_\varrho \otimes W^*))$.

2. *These coefficients, as functions on* $\mathscr{D} = \{X + iY \in V_{\mathbb{C}} | X \in V, Y \in C\}$*, depend only on* Y.

Proof. 1. In view of the definitions, it is enough to prove, for any anti-holomorphic function f on D_0, that $r(X) * \tilde{f} \equiv 0$ for all $X \in \mathfrak{p}^+$. This is essentially the same as the proof of Lemma 1.2: For any $g \in G_0$, let $f_g(Z) = f(gZ)$, $Z \in \mathscr{D}$. Then f is anti-holomorphic $\Leftrightarrow f_g$ is anti-holomorphic, $\forall\, g \in G_0 \Leftrightarrow l(X) * f_g \equiv 0 \quad \forall\, g \in G_0$, $X \in \mathfrak{p}^+$, by definition of $\mathfrak{p}^+$. Now

$$(l(X) * f_g)(is) = (l(X) * \tilde{f}_g)(e) = (r(X) * \tilde{f}_g)(e) = (r(X) * \tilde{f})(g).$$

This proves 1.4.1.

2. From (0.1.3) and (1.3.3) it follows that

$$\delta_{W,\varrho}(f(Z+v)) = (\delta_{W,\varrho} f)(Z+v) \qquad v \in V \subset \mathfrak{p}^+,$$

from which the assertion is clear.

1.5. **Corollary.** *Assume that W and ϱ have the property that, for any holomorphic $f \in C^\infty(\mathscr{D}, V_\varrho)$, $\delta_{W,\varrho} f$ is a holomorphic element of $C^\infty(\mathscr{D}, V_\varrho \otimes W^*)$. Then $\delta_{W,\varrho}$ is a holomorphic differential operator with constant coefficients.*

Proof. The hypotheses imply that the coefficients of $\delta_{W,\varrho}$ are holomorphic elements of $C^\infty(G_0, \mathrm{Hom}(V_\varrho, V_\varrho \otimes W^*))$. The corollary now follows from 1.4.2.

2. Singular Representations

2.1. Let $\lambda \in \mathbb{R}$. There is a unique homomorphism of Lie algebras $\Lambda_\lambda : \mathfrak{k} \to \mathbb{C}$ such that $\Lambda_\lambda(H_{\gamma_i}) = -\lambda$, $i = 1, \ldots, r$; in this way we identify scalar representations of $\mathfrak{k}$ with real numbers. The representation Λ_λ integrates to a representation ϱ_λ of the universal covering group $\tilde{K}$ of K (cf. [13, p. 6]).

Letting $\tilde{G}_0$, $\tilde{K}_0$, denote the universal covering groups of G_0 and K_0, respectively, we observe that $\tilde{G}_0/\tilde{K}_0 \simeq G_0/K_0$, and that $j(g, z)$ lifts to a map [also denoted $j(g, z)$] from $\tilde{G}_0 \times \mathscr{D}$ to $\tilde{K}$ (cf. [13]). For $\lambda \in \mathbb{R}$, let $j_\lambda(g, z) = \varrho_\lambda(j(g, z))$.

Consider the following representation U_λ of $\tilde{G}_0$ on the space $\mathscr{H}(\mathscr{D})$ of holomorphic functions on $\mathscr{D}$:

(2.1.1) $$(U_\lambda(g)f)(z)=j_\lambda(g^{-1},z)^{-1}f(g^{-1}z).$$

The results of Vergne and Rossi in [13] imply that

(2.1.2) When λ is a regular weight, U_λ is unitarizable and irreducible on $\mathscr{H}(\mathscr{D})$;

(2.1.3) When λ is a singular weight, U_λ is unitarizable and irreducible on the unique irreducible subspace $H_\lambda \subset \mathscr{H}(\mathscr{D})$.

The subspace H_λ may be described explicitly. The group H_0 has precisely $r+1$ orbits on the closure $\bar{C}$ of C in V [13, 4.1]. We may denote the orbits $\mathcal{O}_e$, $e=1,\ldots,r+1$; this numbering is unique if we require that $\mathcal{O}_e$ be in the closure of $\mathcal{O}_{e+1}$.

On each orbit $\mathcal{O}_e$, there is a measure $d\mu_e$, unique up to scalar multiples, which is semi-invariant with respect to the action of H_0 on $\mathcal{O}_e$ [13, 4.2]. The space $H_{(e)} \overset{\text{def}}{=} H_{\lambda_e}$ consists of functions F of the form

(2.1.4) $$F(z)=\int_{\mathcal{O}_e} e^{2\pi i(z,\xi)}\phi(\xi)d\mu_e(\xi) \qquad \phi\in L^2(\mathcal{O}_e, d\mu_e),$$

where $(\,,)$ is the bilinear extension to $\mathfrak{p}^+$ of the given Euclidean inner product on V. There is a map from the space $\mathscr{L}(\mathfrak{p}^+)$ of constant coefficient holomorphic differential operators on $\mathfrak{p}^+$, to the space $S^*(\mathfrak{p}^+)$ of polynomials on $\mathfrak{p}^+$, denoted $d\mapsto\hat{d}$, and characterized by the property

(2.1.5) $$d(e^{2\pi i(z,\xi)})=\hat{d}(\xi)e^{2\pi i(z,\xi)}; \qquad d\in\mathscr{L}(\mathfrak{p}^+), \qquad z,\xi\in\mathfrak{p}^+;$$

here d differentiates with respect to the z-variable.

Let $J_e\subset S^*(\mathfrak{p}^+)$ be the ideal of polynomials vanishing on $\mathcal{O}_e$, or equivalently on the Zariski closure $\mathscr{R}_e$ of $\mathcal{O}_e$ in $\mathfrak{p}^+$. Then, since $d\mu_e$ is semi-invariant, we see that

2.2. **Lemma.** *$d\in\mathscr{L}(\mathfrak{p}^+)$ annihilates $H_{(e)}\Leftrightarrow\hat{d}\in J_e$.*

The ideals J_e are described in Sect. 4 below. We note that J_e is prime, because $\mathscr{R}_e$, being the Zariski-closure of an orbit of the connected real Lie group H_0, is an orbit of the connected algebraic group H.

2.3. At this point we make the following trivial observation. The group K acts on $\mathscr{L}(\mathfrak{p}^+)$, which is canonically isomorphic to $S(\mathfrak{p}^+)$, via the adjoint action on $\mathfrak{p}^+$. Let $W\subset\mathscr{L}(\mathfrak{p}^+)$ be a K-invariant subspace, so that we have a map $\varrho_W: K\to GL(W)$; let $\varrho_W^*: K\to GL(W^*)$ denote the contragradient representation.

The map

$$C^\infty(\mathscr{D},\mathbb{C})\otimes W\to C^\infty(\mathscr{D},\mathbb{C}) \qquad f\otimes d\to df$$

induces a differential operator $C^\infty(\mathscr{D},\mathbb{C})\to C^\infty(\mathscr{D},W^*)$ which is homogeneous under H_0 in the sense that the following formula is satisfied:

(2.3.1) $$d_W(f(hz))=\varrho_W^*(j(h,z))^{-1}(d_Wf)(hz) \qquad h\in H_0, \qquad z\in\mathscr{D}$$
$$f\in C^\infty(\mathscr{D},\mathbb{C}).$$

This formula is a direct consequence of the chain rule, once we remember that $j(h,z)=c^{-1}hc\in K$, and that $hz=\mathrm{Ad}(c^{-1}hc)\cdot z \quad \forall\, h\in H_0,\ z\in \mathfrak{p}^+$.

2.4. As usual, $c\in G$ denotes the Cayley transform. Recall that, for $z\in\mathscr{D}_0\subset\mathfrak{p}^+$, we have $c^{-1}\exp(z)\in P^+KP^-$, and the map $z\to c^{-1}(z)\overset{\mathrm{def}}{=}\zeta(c^{-1}\exp(z))$ takes the bounded domain $\mathscr{D}_0$ to the tube domain $\mathscr{D}\subset\mathfrak{p}^+$. Thus, for any $f\in\mathscr{H}(\mathscr{D})$,

$$(c_{\lambda_e}f)(z)\overset{\mathrm{def}}{=}j_{\lambda_e}(c^{-1},z)^{-1}f(c^{-1}(z)) \qquad z\in\mathscr{D}_0 \tag{2.4.1}$$

is a holomorphic function on $\mathscr{D}_0$. The map c_{λ_e} has an obvious inverse, and thus induces an isomorphism of $H_{(e)}$ with a space $\mathscr{M}_e$ of holomorphic functions on $\mathscr{D}_0$.

2.5. **Lemma.** *The operator $d\in\mathscr{L}(\mathfrak{p}^+)$ annihilates $H_{(e)}$ if and only if d annihilates $\mathscr{M}_e$.*

Proof. There is an element $X=\dfrac{-\pi}{4}\sum_{i=1}^{r}(E_{\gamma_i}-E_{-\gamma_i})\in\mathfrak{g}$ such that $c=\exp(X)$. The action U_{λ_e} of G_0 on $H_{(e)}$ induces an action dU_e of $U(\mathfrak{g})$ on $H_{(e)}$; we let $\exp(X)$ be the operator $\sum_{n=0}^{\infty}\dfrac{dU_e(X)^n}{n!}$ on $H_{(e)}$. For any $f\in H_{(e)}$, the series defining $\exp(X)f$ converges uniformly on compact subsets, so it defines a function $\tilde{c}f$, which is obviously annihilated by all $d\in\mathscr{L}(\mathfrak{p}^+)$ which annihilate $H_{(e)}$. But $\tilde{c}f$ coincides with $c_{\lambda_e}f$ on the open set $\mathscr{D}_0\cap\mathscr{D}$, because they are both given locally by the same power series. Thus any $d\in\mathscr{L}(\mathfrak{p}^+)$ which annihilates $H_{(e)}$ annihilates $c_\lambda f$ for any $f\in H_{(e)}$; the reverse implication is proved by considering the inverse Cayley transform.

2.6. We may construct $\mathscr{M}_e$ in another way. Consider the space $\mathbf{D}_\lambda=U(\mathfrak{g})\otimes_{U(\mathfrak{k}\oplus\mathfrak{p}^-)}V_\lambda^*$, where $\mathfrak{k}$ acts via $-\Lambda_\lambda$ on V_λ^* and $\mathfrak{p}^-$ acts trivially; here $\lambda\in\mathbb{R}$. Now $\mathbf{D}_\lambda$, as a vector space, is isomorphic to $S(\mathfrak{p}^+)$. This may be identified, via the Killing form, with the space of polynomials $S^*(\mathfrak{p}^-)$ on $\mathfrak{p}^-$, as follows (cf. [6, Sect. 2]): If $B(X,Y)$ is the Killing form on $\mathfrak{g}$, then $X_1\circ X_2\circ\ldots\circ X_n\in\mathrm{Sym}^n(\mathfrak{p}^+)$ may be identified with the polynomial $p(Y)=B(Y,X_1)\ldots B(Y,X_n)$, $Y\in\mathfrak{p}^-$. This is clearly a ring homomorphism; it is an isomorphism since B places $\mathfrak{p}^+$ and $\mathfrak{p}^-$ in perfect duality. By transport of structure from $\mathbf{D}_\lambda$, $S^*(\mathfrak{p}^-)$ becomes a $U(\mathfrak{g})$-module, which we denote $\mathbf{D}(\lambda)$.

Similarly, we identify $S(\mathfrak{p}^-)$ with $S^*(\mathfrak{p}^+)$. If $p(Y)=B(Y,X_1)\ldots B(Y,X_n)\in\mathbf{D}(\lambda)$, $q(X)=B(X,Y_1)\ldots B(X,Y_m)\in S^*(\mathfrak{p}^+)$, define the pairing

$$\begin{aligned}\langle p,q\rangle &= \sum_{\sigma\in S_n}\prod_i B(X_i,Y_{\sigma(i)}) \qquad n=m\\ &=0 \qquad n\neq m;\end{aligned} \tag{2.6.1}$$

here S_n is the symmetric group on n letters. This extends to a natural pairing between $S^*(\mathfrak{p}^+)$ and $\mathbf{D}(\lambda)$. In fact, if we define

$$(\delta(X)q)(X_0)=\frac{d}{dt}q(X_0+tX)|_{t=0} \quad X,X_0\in\mathfrak{p}^+ \tag{2.6.2}$$

then, when p and q are as above, one sees easily that

$$\langle p,q\rangle=(\delta(X_1)\ldots\delta(X_n)q)(0). \tag{2.6.3}$$

In particular, $\mathbf{D}(\lambda)$ may be identified with the space of constant coefficient differential operators on $\mathfrak{p}^+$, and $S^*(\mathfrak{p}^+)$ may be identified, via transport of structure, with the space $\mathbf{S}(\lambda)$ of $\mathfrak{k}$-finite vectors in the representation of $U(\mathfrak{g})$ contragradient to $\mathbf{D}_\lambda$. [That $S^*(\mathfrak{p}^+)$ contains all the $\mathfrak{k}$-finite vectors is clear from (2.6.3).] The action of $U(\mathfrak{g})$ on $\mathbf{S}(\lambda)$ thus obtained is described in (2.12) of [6], where these constructions are worked out in greater detail.

For $\lambda \in \mathbb{R}$, $\tilde{G}_0$ acts on the space $\mathscr{H}(\mathscr{D}_0)$ of holomorphic functions on $\mathscr{D}_0$ as follows:

$$(2.6.4) \qquad W_\lambda(g) f = (c_\lambda U_\lambda(g) c_\lambda^{-1}) f \qquad g \in \tilde{G}_0, \quad f \in \mathscr{H}(\mathscr{D}_0).$$

The $\tilde{K}_0$-finite elements are just the polynomials on $\mathscr{D}_0$. The description in [6, Sect. 2] makes it clear that the infinitesimal action of $U(\mathfrak{g})$ on the K_0-finite vectors for the representation W_λ is just the action of $U(\mathfrak{g})$ on $\mathbf{S}(\lambda)$ described above.

Now assume $\lambda = \lambda_e$. Then $\mathbf{S}(\lambda)$ contains an irreducible submodule $\mathscr{M}_e^f$ consisting of the K_0-finite vectors in $\mathscr{M}_e$. Let $\mathscr{M}_e^0 \subset \mathbf{D}(\lambda)$ denote the subspace vanishing on $\mathscr{M}_e^f$ under the pairing $\mathbf{D}(\lambda) \otimes \mathbf{S}(\lambda) \to \mathbb{C}$ constructed above; let $L_\lambda = \mathbf{D}(\lambda)/\mathscr{M}_e^0$.

2.7. **Lemma.** *The module $\mathscr{M}_e^f$ contains the constant functions, and any $U(\mathfrak{g})$-submodule of $\mathbf{S}(\lambda)$ contains $\mathscr{M}_e^f$.*

Proof. By [6, (2.12)], the image of $U(\mathfrak{g})$ in $\operatorname{End}(\mathbf{S}(\lambda))$ contains $\mathscr{L}(\mathfrak{p}^+)$. But we may obtain the constant functions by applying an appropriate constant coefficient differential operator to *any* polynomial. Thus any non-zero $U(\mathfrak{g})$-submodule of $\mathbf{S}(\lambda)$ contains the constants. Since $\mathscr{M}_e^f$ is a non-zero *irreducible* submodule of $\mathbf{S}(\lambda)$, both assertions follow from this.

2.8. Let $I'_e \subset \mathscr{L}(\mathfrak{p}^+)$ denote the set of d such that $\hat{d} \in J_e$. The elements of $S(\mathfrak{p}^+)$ may be regarded, via their action on $\mathbf{S}(\lambda)$, as constant coefficient differential operators as in (2.6.2); this is just the usual isomorphism of $S(\mathfrak{p}^+)$ with $\mathscr{L}(\mathfrak{p}^+)$. With this identification, let $I_e \subset S(\mathfrak{p}^+)$ denote the ideal corresponding to I'_e; then I_e is the annihilator in $S(\mathfrak{p}^+)$ of $\mathscr{M}_e$, by Lemma 2.5.

Assume $\lambda = \lambda_e$, $e = 1, \ldots, r$.

2.9. **Proposition.** *Let* $\mathbf{A}$ *be a proper $U(\mathfrak{g})$-submodule of* $\mathbf{D}(\lambda) \simeq \mathbf{D}_\lambda$*; let* $\mathbf{N} = \mathbf{D}(\lambda)/\mathbf{A}$. *The following are equivalent:*

1. $\mathbf{N}$ *is an irreducible $U(\mathfrak{g})$-module.*
2. $\mathbf{A} = I_e$ *(under the identification $S(\mathfrak{p}^+) \simeq S^*(\mathfrak{p}^-) = \mathbf{D}(\lambda)$).*
3. $\mathbf{N}$ *is unitarizable.*

Proof. 1)$\Rightarrow$2). By Lemma 2.7 and duality, any proper $U(\mathfrak{g})$-submodule of $\mathbf{D}_\lambda$ is contained in J_e.

2)$\Rightarrow$1) $\mathbf{N} = \mathbf{D}(\lambda)/I_e$ is the $\mathfrak{k}$-finite contragradient to $\mathscr{M}_f^e$. Since the latter is irreducible, so is the former.

2)$\Rightarrow$3) Because $H_{(e)}$, and therefore $\mathscr{M}_e$, is unitarizable.

3)$\Rightarrow$2) Let $A^0 = \{f \in \mathbf{S}(\lambda) \mid \langle f, a \rangle = 0 \quad \forall\, a \in \mathbf{A}\}$. Assume $\mathbf{A} \neq I_e$. Then $A^0 \neq \mathscr{M}_e^f$. Now $\mathbf{D}/\mathbf{A}$ is unitarizable $\Rightarrow A^0 = \{\mathfrak{k}\text{-finite vectors in } \operatorname{Hom}(\mathbf{D}/\mathbf{A}, \mathbb{C})\}$ is unitarizable. Take any $U(\mathfrak{g})$-invariant subspace $M' \subset A^0$ such that $M' \oplus \mathscr{M}_e^f = A^0$. Then $M' = 0$ by Lemma 2.7.

3. Cyclic Representations

3.1. Let $\lambda \in \mathbb{R}$, and let V_λ denote the space $\mathbb{C}$, viewed as a $\mathfrak{k}$-module under the representation Λ_λ. Let f be a holomorphic V_λ-valued function on $\mathscr{D}$; let $\phi = \phi_{f, \varrho\lambda}$. Denote by $\mathscr{V}_f$ the space of functions on G_0 of the form $r(X)*\phi, X \in U(\mathfrak{g})$. Then $U(\mathfrak{g})$ acts on $\mathscr{V}_f$, and by (1.1.2) and Lemma 1.2, $\mathscr{V}_f$ is naturally a $U(\mathfrak{g})$-quotient of $\mathbf{D}_\lambda$. Assume now $\lambda = \lambda_e$ for $1 \leqq e \leqq r$. Let $\mathscr{W}_f = I_e * \mathscr{V}_f$; then $\mathscr{W}_f$ is a $U(\mathfrak{g})$-submodule of $\mathscr{V}_f$, and $\mathscr{V}_f/\mathscr{W}_f$ is the unique non-trivial irreducible quotient of $\mathscr{V}_f$ (by Proposition 2.9 and transport of structure).

Let $A \subset \mathbf{D}_\lambda$ be a $\mathfrak{k}$-invariant subspace. We say A is a *highest* $\mathfrak{k}$*-type* if $r(X)*a = 0 \quad \forall\, a \in A, X \in \mathfrak{p}^-$. Our case-by-case analysis in 4.1–4.5 will show that

(3.1.1) *Every ideal I_e is generated, over $S(\mathfrak{p}^+)$, by a K-invariant subspace A_e of* $\mathrm{Sym}^e(\mathfrak{p}^+)$.

We claim that

(3.1.2) *The subspace $W_e = A_e \otimes V^*_{\lambda_e}$ of $\mathbf{D}_\lambda$ is a highest $\mathfrak{k}$-type.*

In fact, let W' be the image of $\mathfrak{p}^- \otimes W_e$ in $\mathbf{D}_\lambda$ under the map $X \otimes f \mapsto r(X)*f, X \in \mathfrak{p}^-$, $f \in W_e$. If we denote the roots of $\mathfrak{h}$ in $\mathfrak{p}^+$ by $\gamma_1, \ldots, \gamma_m$, then the weights of $\mathfrak{h}$ on W_e are all of the form $-\Lambda_\lambda + \sum_{i=1}^{m} \alpha_i \gamma_i$, with $\sum_{i=1}^{m} \alpha_i = e$, and the roots on $\mathfrak{p}^-$ are $-\gamma_1, \ldots, -\gamma_m$; thus the weights on W' are of the form $-\Lambda + \sum_{i=1}^{m} \alpha_i \gamma_i$, $\sum_{i=1}^{m} \alpha_i = e-1$. But the image of $S(\mathfrak{p}^+) \otimes W_e$ in $\mathbf{D}_\lambda$ is invariant under the whole of $U(\mathfrak{g})$ by (3.1.1), and in particular contains W'. A comparison of weight spaces now shows that $W' = 0$.

3.2. **Theorem.** *Let $\delta_e = \delta_{A_e, \lambda_e} : C^\infty(\mathscr{D}, V_{\lambda_e}) \to C^\infty(\mathscr{D}, V_{\lambda_e} \otimes A_e^*)$. Then δ_e is a holomorphic vector-valued differential operator with constant coefficients, and*

$$\delta_e(f|_{\lambda_e}\gamma) = (\delta_e f)|_{\lambda_e, \otimes A_e^*}\gamma$$

for all $\gamma \in G_0$.

Proof. It follows from (3.1.2) that the image of $\phi = \phi_{f, \varrho\lambda}$ in $C^\infty(G_0, V_{\lambda_e} \otimes A_e^*)$, under the map $\phi \mapsto (X \mapsto r(X)*\phi,\ X \in A_e)$, is of holomorphic type, for any holomorphic function f. The lemma now follows from Lemma 1.2, Corollary 1.5, and (1.3.3).

3.3. *Remark.* As a statement about the covariance properties of the constant coefficient differential operator δ_e, Theorem 3.2 generalizes previous theorems of Selberg, Maass, Freitag, and Resnikoff (cf. [8, 3, 10]); see also Kostant [7] and Jakobsen [5]. In general, the operator in question is obviously homogeneous with respect to the subgroup $\mathscr{P} \subset G_0$ [cf. (2.3.1)]; the hard part is proving the transformation property with respect to an element of G_0 corresponding to inversion in the associated Jordan algebra. Most proofs of such transformation rules invoke a Fourier-Laplace transform at some stage. Our proof is no different in this respect, although the Fourier analysis is mostly in the paper [13] of Vergne-Rossi. However, a purely algebraic proof may also be given: The homogeneity of δ_e is equivalent to the statement that the unique irreducible quotient of the module $U(\mathfrak{g}) \otimes_{U(\mathfrak{k} \oplus \mathfrak{p}^+)} V_{\lambda_e}$, identified with a space of V_{λ_e}-valued polynomials on $\mathfrak{p}^+$, is strictly smaller than the set of all V_{λ_e}-valued polynomials on $\mathfrak{p}^+$; i.e., that certain (specific) K-type are "missing". Observe in this connection that from this point of view the generalization of Theorem 3.2 to higher-dimensional holomorphic representations of K, along the lines of 1.3 and 3.1, is straightforward. However, the full set of holomorphic representations of K that give rise in this way to covariant matrix-valued constant coefficient differential operators is not known. But partial results are available, and in particular, for the λ_e's in question, the needed determination of the space of "missing" K-types also follows from the algebraic results of Wallach [16].

In Sect. 4 we give a more precise description of the operators δ_e.

3.4. We now consider a discrete subgroup $\Gamma \subset G_0$ such that

1) Γ is arithmetic for some $\mathbb{Q}$-structure on G;
2) $\Gamma \cap U_0$ is a lattice L in $U_0 \simeq V$.

Let f be a holomorphic automorphic form on $\mathcal{D}$ of weight λ for Γ [cf. (1.3.4)]. Then f has a Fourier expansion

$$f(z) = \sum_{l' \in L'} a_{l'} e^{2\pi i (l', z)}, \tag{3.4.1}$$

where $(,)$ is the inner product of 2.1 on $\mathfrak{p}^+ \supset V \supset L$, and $L' = \{l' \in V \mid (l', l) \in \mathbb{Z} \quad \forall\, l \in L\}$. We assume that $a_{l'} \neq 0$ implies that l' is in the closure $\bar{C}$ of C in V; this assumption is automatic ("Koecher's principle") unless $G_0 \simeq \mathrm{SL}(2, \mathbb{R})$.

Evidently, if $d \in \mathcal{L}(\mathfrak{p}^+)$, then

$$df(z) = \sum a_{l'} \hat{d}(l') e^{2\pi i (l', z)} \tag{3.4.2}$$

Now the homogeneity property (1.3.3) of δ_e, restricted to the case $\gamma = h \in H_0$, is identical with the formula (2.3.1) satisfied by the constant coefficient operator d_{A_e}.

3.5. **Lemma.** $\delta_e = d_{A_e}$.

Proof. It follows from work of Schmid [18] that, under the action of H_0 on $\mathcal{L}(\mathfrak{p}^+)$ induced by its linear action on $\mathcal{D} \subset \mathfrak{p}^+$, the representations of H_0 occur with multiplicity one. The lemma now follows from (2.3.1) and the above remarks, since δ_e is necessarily of the form d_W for some $W \subset \mathcal{L}(\mathfrak{p}^+)$.

We have essentially proved the following theorem:

3.6. **Theorem.** *Let $\lambda = \lambda_e$, and let f be a holomorphic V_λ-valued function. Let $\mathcal{V}_f$ be defined as in* 3.1, *with $\phi = \phi_{f, e_\lambda}$. Assume f has the Fourier expansion* (3.4.1). *The following are equivalent:*

(a) $\delta_e f = 0$.
(b) $a_{l'} \neq 0 \Rightarrow l' \in \bar{\mathcal{O}}_e$.
(c) *$\mathcal{V}_f$ is an irreducible $U(\mathfrak{g})$-module.*
(d) *$\mathcal{V}_f$ is a unitarizable $U(\mathfrak{g})$-module.*

Proof. The equivalence of (a) and (b) follows immediately from (3.4.2) and 3.5. Now by the definition of δ_e, (a) is equivalent to the statement $r(X) * \phi = 0 \quad \forall\, X \in A_e$. The equivalence of (a), (c), and (d) now follows from Proposition 2.9.

In Sect. 4 we identify the spaces A_e, and the operators δ_e, in terms of Jordan algebras, and apply the theorem to known cases.

4. A List of Orbits and Applications

We describe the orbits $\mathcal{O}_e$ in terms of the standard matrix realizations of the domains $\mathcal{D}$. This material is well known; we review it (without proofs) in order to identify our operators δ_e with the homogeneous differential operators discussed by Freitag [3] and Resnikoff [10, 11].

The classes of irreducible tube domains are listed in turn; for this material we refer to [2, 12, 9, 1].

4.1. $G_0=\mathrm{SU}(r,r)$, $K_0=S(U(r)\times U(r))=(U(r)\times U(r))\cap \mathrm{SU}(r,r)$, in the standard notation; $C=\{r\times r$ positive-definite Hermitian matrices$\}$; $\mathcal{D}=\{r\times r$ complex matrices $z|(z-{}^t\bar{z})/2i>0\}$. Then $\mathfrak{p}^+$ is identified with the space M_r of all $r\times r$ complex matrices, and $H_0=\{h\in R_{\mathbb{C}/\mathbb{R}}(\mathrm{GL}(r,\mathbb{C}))|\det h=\det\bar{h}\}$; here $R_{\mathbb{C}/\mathbb{R}}$ is Weil's restriction of scalars functor [17], and in the condition on determinants we are thinking of h as an element of $\mathrm{GL}(r,\mathbb{C})$. The Zariski closure H of H_0 in G is $\{(g_1,g_2)\in\mathrm{GL}(r,\mathbb{C})\times\mathrm{GL}(r,\mathbb{C})|\det g_1=\det g_2\}$; H acts on M_r by $(g_1,g_2)(X)=g_1X{}^tg_2$, $X\in M_r$, $(g_1,g_2)\in H$. The orbits in M_r under this action are the sets $\mathcal{O}_e$ of $r\times r$ matrices of rank $e-1$, $1\leqq e\leqq r+1$. The ideal J_e is generated by the subspace $\tilde{A}_e\subset\mathrm{Sym}^e(\mathfrak{p}^+)^*$, $1\leqq e\leqq r$, generated by the $e\times e$ minors of the matrix realization of $\mathfrak{p}^+$.

The number p of Sect. 0 in this case equals 2, and the singular weights are $0,1,\ldots,r-1$.

The remaining cases follow the same format, and the presentations are correspondingly abbreviated.

4.2. $G_0=\mathrm{Sp}(r,\mathbb{R})$, $K_0=U(r)$
$C=\{r\times r$ positive-definite real symmetric matrices$\}$
$\mathcal{D}=\{r\times r$ complex symmetric matrices with positive-definite imaginary part$\}$
$\mathfrak{p}^+\approx S_r=\{$all complex symmetric matrices$\}$
$H_0=\mathrm{GL}(r,\mathbb{R})$; $H=\mathrm{GL}(r,\mathbb{C})$
$g(X)={}^tg^{-1}Xg^{-1}$; $g\in H$, $X\in S_r$.

H-Orbits: $\mathcal{O}_e=\{r\times r$ complex symmetric matrices of rank $e-1\}$, $1\leqq e\leqq r+1$.
J_e generated (as ideal) by $\tilde{A}_e\subset\mathrm{Sym}^e(\mathfrak{p}^+)^*$, $1\leqq e\leqq r$
$\tilde{A}_e$ generated (as space) by $e\times e$ minors on S_r
$p=1$; singular weights $=0,\frac{1}{2},1,\frac{3}{2},\ldots,\frac{r-1}{2}$.

4.3. $G_0=\mathrm{SO}^*(4r)$, $K_0=U(2r)$.
$C=\{r\times r$ positive-definite quaternionic Hermitian matrices$\}$
$\mathcal{D}=$ tube domain in $\mathfrak{p}^+\approx\mathcal{A}_r=\{2r\times 2r$ alternating complex matrices$\}$
$H_0=U^*(2r)$; $H=\mathrm{GL}(2r,\mathbb{C})$
$g(X)=gX{}^tg$ $g\in H$, $X\in\mathcal{A}_r$.

H-Orbits: $\mathcal{O}_e=\{2r\times 2r$ alternating matrices of rank $2e\}$, $1\leqq e\leqq r+1$.
J_e generated (as ideal) by $\tilde{A}_e\subset\mathrm{Sym}^e(\mathfrak{p}^+)^*$ $1\leqq e\leqq r$
$\tilde{A}_e$ generated (as space) by Pfaffians of restrictions of a variable alternating matrix to $2e$-dimensional subspaces of $\mathbb{C}^{2r}$
$p=4$; singular weights $=0,2,4,\ldots,2r-2$.

4.4. $G_0 = \mathrm{SO}(2,q)^0$ (connected component), $K_0 = \mathrm{SO}(2) \times \mathrm{SO}(q)$, $r=2$
C = forward light cone: $\{X_1 > \sqrt{X_2^2 + \ldots + X_q^2} \text{ in } \mathbb{R}^q\}$
$\mathscr{D}$ = tube domain in $\mathfrak{p}^+ \approx \mathbb{C}^q$
$H_0 = \mathrm{SO}(1,q-1) \times \{$positive real homotheties of $\mathbb{R}^q\}$
$H = \mathrm{SO}(Q) \times \mathbb{C}^\times$, where $Q(x) = x_1^2 - x_2^2 - \ldots - x_q^2$, $x = (x_1, \ldots, x_q) \in \mathbb{C}^q$
$(\beta, t)(X) = t \cdot \beta(X)$, $(\beta, t) \in H$, $X \in \mathbb{C}^q$.

H-Orbits: $\mathcal{O}_1$ = origin; $\mathcal{O}_2 = \{X \in \mathbb{C}^q - \{0\} | Q(X) = 0\}$; $\mathcal{O}_3 = \mathbb{C}^q - (\mathcal{O}_2 \cup \mathcal{O}_1)$
J_1 generated (as ideal) by $\tilde{A}_1 = (\mathfrak{p}^+)^*$
J_2 generated (as ideal) by $Q \in \mathrm{Sym}^2(\mathfrak{p}^+)^*$
$p = q-2$; singular weights $= 0, \frac{q-2}{2}$.

4.5. G_0 = a real form of E_7; K_0 = compact real form of $E_6 \times \mathrm{SO}(2)$
$C = \{$positive-definite 3×3 octonion matrices$\}$
$\mathscr{D}$ = tube domain in 27-dimensional exceptional complex Jordan algebra $\mathscr{E}_3 \approx \mathfrak{p}^+$; $r=3$
H_0 = real form of $E_6 \times \mathrm{GL}(1)$; $H = E_6 \times \mathrm{GL}(1)$
H-orbits on $\mathscr{E}_3$ (following [9]):
$\mathcal{O}_1$ = origin
$\mathcal{O}_2 = \{$locus of quadratic map $a \to a \times a$ of [9]$\} - \mathcal{O}_1$
$\mathcal{O}_3 = \{$elements of $\mathscr{E}_3$ of reduced norm zero$\} - \bar{\mathcal{O}}_2$
$\mathcal{O}_4 = \mathscr{E}_3 - \bar{\mathcal{O}}_3$
J_e generated (as ideal) by elements of $\mathrm{Sym}^e(\mathfrak{p}^+)^*$, $e = 1, 2, 3$.
$p = 8$; singular weights $= 0, 4, 8, 12$.

4.6. In [10, 11], Resnikoff introduces, for each group G_0 and each $e = 1, \ldots, r$, a constant coefficient differential operator $\nabla^{[e]}$ and a polynomial function $a \to a^{[e]}$ on the complex simple Jordan algebra attached to G_0, of degree $e+1$, each with values in a certain vector space Φ_e, such that

$$\widehat{\nabla^{[e]}}(a) = a^{[e]}. \tag{4.6.1}$$

It follows easily from his definitions and 4.1–4.5 that, in an appropriate basis, we may identify Φ_e with A_e^* and the Jordan algebra with $\mathfrak{p}^+$ in such a way that $a^{[e]}$ is the natural A_e^*-valued polynomial function on $\mathfrak{p}^+$ canonically associated to $\tilde{A}_e \subset \mathrm{Sym}^e(\mathfrak{p}^+)^*$. Thus, we may conclude

4.6.2. **Lemma.** *With the above identifications, Resnikoff's operator* $\nabla^{[e]}$ *is the same as the operator* δ_e.

4.7. Henceforward, we assume $\mathscr{D} = \prod_{i=1} \mathscr{D}^i$, $G_0 = \prod_{i=1} G_0^i$, where $\mathscr{D}^i$ is an irreducible tube domain of rank r_i and G_0^i is a real group, constructed as before, acting on $\mathscr{D}^i$. We generalize other notation likewise to the reducible case. We assume G has a $\mathbb{Q}$-rational structure, and let $\Gamma \subset G_0$ be an arithmetic group with respect to that $\mathbb{Q}$-structure. Our singular weights are indexed by μ-tuples $\underline{\lambda} = (\lambda_{e_1}^1, \ldots, \lambda_{e_\mu}^\mu)$. Here $\lambda_{e_i} = (e_i - 1)p^i/2$, $1 \leq e_i \leq r_i$, where p^i is the dimension of the appropriate root space in the Lie algebra $\mathfrak{g}_0^i$.

Let f be a holomorphic automorphic from on $\mathscr{D}$, of weight $\underline{\lambda}$ with respect to Γ; let $\phi=\phi_{f,\varrho_{\underline{\lambda}}}$. That the following lemma should be true was suggested to the authors by Freitag:

4.7.1. **Lemma.** *If $\underline{\lambda}$ is a singular weight, then $\phi\in L^2(\Gamma\backslash G_0)$.*

Proof. We need the basic facts about reduction theory, for which we refer to [16]. Let $A_0\subset G_0$ be the subgroup with Lie algebra $\mathfrak{a}_0$, and let N_0 be a maximal unipotent subgroup of G_0, normalized by A_0, which contains U_0. The Iwasawa decomposition is then $G_0=N_0A_0K_0$. Let $S\subset A_0$ be a maximal $\mathbb{Q}$-split torus, let M be the maximal connected $\mathbb{Q}$-anisotropic subgroup of G which centralizes S, and let $U'\subset N_0$ be the unipotent radical of a minimal $\mathbb{Q}$-parabolic subgroup $P\subset G$ containing A_0N_0. Let Ω be a compact subset in the group $(MU')_0$ of $\mathbb{R}$-valued points of MU'. Let Δ be the set of simple roots of G relative to the minimal $\mathbb{Q}$-parabolic pair (P,S). If t is a positive real number, and if S_0 is the group of $\mathbb{R}$-valued points of S, let

$$S^t=\{s\in S_0|\alpha(s)\geqq t \quad \text{for all} \quad \alpha\in\Delta\}.$$

Let $\mathfrak{S}=\mathfrak{S}_{t,\Omega}=\Omega S^tK_0$ be the corresponding Siegel domain. To check that $\phi\in L^2(\Gamma\backslash G_0)$ it suffices, by reduction theory, to check that ϕ is square integrable on $\mathfrak{S}$.

Let $\mathfrak{S}'\supseteqq\mathscr{D}$ be the image of $\mathfrak{S}$ under the map $G_0\to\mathscr{D}$ of Sect. 0. It follows easily from our assumption (cf. 3.4) that f is "holomorphic at the boundary" that f is *bounded* on $\mathfrak{S}'$ (cf. [23, p. 183]). We thus have to check that

$$\int_{\mathfrak{S}} |j_{\underline{\lambda}}(g,\ is)^{-2}|\,dg<\infty, \tag{4.7.1.1}$$

where dg is Haar measure on G_0 and we write $j_{\underline{\lambda}}$ instead of $j_{\varrho_{\underline{\lambda}}}$. This may be rewritten (cf. [4, Chap. X] and [16, Lemma 1.9])

$$\int_{\Omega}\int_{S^t}\int_{K_0} |j_{\underline{\lambda}}(\omega\sigma k,\ is)^{-2}|\,\delta(\sigma)^{-1}\,d\omega\, d\sigma\, dk, \tag{4.7.1.2}$$

where δ is the determinant of the adjoint representation of S on the Lie algebra of U', or, what is the same thing, the determinant of the adjoint representation of S on the Lie algebra of N_0, and $d\omega$, $d\sigma$, and dk are Haar measures on $(MU')_0$, S_0, and K_0, respectively. To check that (4.7.1.2) is finite, it suffices to check

$$\int_{S^t} |j_{\underline{\lambda}}(\sigma,\ is)|^{-2}\,\delta(\sigma)^{-1}\,d\sigma<\infty. \tag{4.7.1.3}$$

Let $\chi(a)=|j_{\lambda}(a,\ is)|^{-2}\,\delta(a)^{-1}$, for all $a\in A_0$. Then χ is a real-valued character of A_0, to which we may associate a linear form $d\chi:\mathfrak{a}_0\to\mathbb{R}$. For each $i=1,\dots,\mu$, let $(\alpha_1^i,\dots,\alpha_{r_i}^i)$ be the set of strongly orthogonal roots in G_0^i of A_0^i, and write

$$d\chi=\sum_{i=1}^{\mu}\left(c_{i1}\alpha_1^i+\sum_{j=2}^{r_i}c_{ij}(\alpha_j^i-\alpha_{j-1}^i)\right).$$

From the definition of S^t we see (cf. [16, Lemma 1.9]) that (4.7.1.3) is a consequence of the following statement:

$$c_{ij}<0,\quad \forall i,j. \tag{4.7.1.4}$$

We may check (4.7.1.4) on each $\mathbb{R}$-factor separately. We thus assume $\mathfrak{g}_0$ is $\mathbb{R}$-simple, $\mathcal{D}$ is irreducible of rank r, and $\lambda=\lambda_e$ is a singular weight. Now the contribution to $d\chi$ of δ^{-1} is

$$-\sum_{i=1}^{r}\alpha_i - p\sum_{1\leq i<j\leq r}\left(\left(\frac{\alpha_j+\alpha_i}{2}\right)+\left(\frac{\alpha_j-\alpha_i}{2}\right)\right)$$
$$=-\sum_{i=1}^{r}(p(i-1)+1)\alpha_i,$$

while the contribution from $|j_\lambda(a,is)|^{-2}$ is (cf. 2.1)

$$\frac{p(e-1)}{2}\left(\sum_{i=1}^{r}\alpha_i\right)=\frac{p(e-1)}{2}\left(\sum_{i=2}^{r}(r-i+1)(\alpha_i-\alpha_{i-1})+r\alpha_1\right).$$

If (4.7.1.4) is true for $e=r$, it is *a fortiori* true for $e<r$; we thus assume $e=r$. Then we compute:

(4.7.1.5) $$d\chi=\sum_{i=1}^{r}(i-r-1)\left(\frac{p}{2}(i-1)+1\right)(\alpha_i-\alpha_{i-1}),$$

where we write $\alpha_0=0$, for simplicity. Then (4.7.1.4), and thus the Lemma, follows from (4.7.1.5).

4.7.2. **Corollary.** *If $\underline{\lambda}$ is a singular weight, and f is an automorphic form with respect to Γ of weight, $\underline{\lambda}$, then $\mathscr{V}_f\subset L^2(\Gamma\backslash G_0)$.*

Proof. This follows directly from the Lemma by a standard argument based on Theorem 1, Sect. 8 of [22]. (This reference was pointed out to us by Labesse.)

4.8. **Theorem.** *Suppose f is a holomorphic modular form on $\mathcal{D}$, of singular weight $\underline{\lambda}=(\lambda^1_{e_1},\ldots,\lambda^\mu_{e_\mu})$, for Γ. Let $\nabla_i^{[e_i]}$ be the differential operator $\nabla^{[e_i]}$, defined above, acting on functions on $\mathcal{D}$ via the factor $\mathcal{D}^i$, $i=1,\ldots,\mu$. Then*

(4.8.1) $$\nabla_i^{[e_i]}f\equiv 0,\qquad i=1,\ldots,\mu.$$

Moreover, the representation $\mathscr{V}_f$ of $U(\mathfrak{g})$ is irreducible. Finally, if Γ satisfies conditions 1) and 2) of 3.4, then the Fourier coefficient $a_{l'}$ of f is zero unless $l'\in\prod_{i=1}\overline{\mathcal{O}_{e_i}}$.

Proof. It follows from Corollary 4.7.2 that $\mathscr{V}_f$ is unitarizable. The irreducibility of $\mathscr{V}_f$ now follows from Proposition 2.9. Then (4.8.1) follows from Lemma 4.6.2, as in the proof of Theorem 3.6, and the final statement is a consequence of Lemma 4.6.2 and Theorem 3.6.

4.9. *Remarks.* Special cases of Theorem 4.8 were obtained by Resnikoff [10] and Freitag [3]; in both cases the methods differ from ours. It is not hard to generalize our argument to the case of domains not of tube type.

It is possible to define singular forms, in certain cases, when $\underline{\lambda}$ is replaced by a vector-valued representation. Are the conclusions of Theorem 4.8 still true in that case? The representations for which singular forms can possibly exist have been classified: cf. [24–26].

When $G_0 = \mathrm{Sp}(n, \mathbb{R})$ and $\Gamma \subset \mathrm{Sp}(n, \mathbb{Q})$, Howe has proved a generalization of Theorem 4.8 by showing that a singular f can be represented as a linear combination of theta functions: cf. his preprint [21] and also Freitag's article [20]. One expects such a theorem to be true more generally.

We note that Lemma 4.7.1 and Corollary 4.7.2 hold more generally if we merely assume that $\underline{\lambda}$ is less than or equal to the maximum singular weight attached to $\mathscr{D}$. But unless $\underline{\lambda}$ is itself a singular weight, the representation $\mathbf{D}_{\underline{\lambda}}$ has no unitarizable quotient [13, 15]. It follows that *there are no holomorphic modular forms of weight $\underline{\lambda}$ in the given range unless $\underline{\lambda}$ is in fact singular.*[1]

Appendix: Construction of Holomorphic Differential Forms (After Freitag)

This appendix sketches Freitag's construction of differential forms on the Siegel modular varieties, from the viewpoint of the present paper. Essentially all computations are left to the reader. The exposition is based upon a seminar talk given by one of the authors at Harvard, at the invitation of Professor David Mumford.

A.1. In this section, G_0 is the metaplectic two-fold cover of $\mathrm{Sp}(n, \mathbb{R})$, and the notation in effect is that of 4.2; in particular K is a covering group of $GL(n, \mathbb{C})$ and $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are matrix spaces. Let $N = \dfrac{n(n+1)}{2} = \dim \mathscr{D}$.

For any holomorphic representation $\varrho: K \to GL(V_\varrho)$, let $\mathbf{D}_\varrho = U(\mathfrak{g}) \otimes_{U(\mathfrak{k} \oplus \mathfrak{p}^-)} V_\varrho^*$, where $\mathfrak{p}^-$ operates trivially, and $\mathfrak{k}$ by the (differential of the) contragredient representation to ϱ, on V_ϱ^*. As in Sect. 2 and [6, Sect. 2.14], we may realize the unique irreducible quotient H_ϱ of $\mathbf{D}_\varrho$ as a $\mathfrak{k}$-submodule of the space of V^*-valued polynomials on $\mathfrak{p}^-$ (we are interchanging $\mathfrak{p}^+$ and $\mathfrak{p}^-$ relative to Sect. 2 and [6]). When $\varrho = \det^k$, we write $\mathbf{D}_k$ and H_k instead of $\mathbf{D}_\varrho$ and H_ϱ. Thus, arguments analogous to those in Sect. 2 show that $H_{\frac{n-1}{2}}$ may be identified with the $\mathfrak{k}$-submodule of $S^*(\mathfrak{p}^-) \otimes V^*_{\frac{n-1}{2}}$ annihilated by the constant coefficient differential operator $\nabla^{[n]}$ of degree n in the coordinates of $\mathfrak{p}^-$. This $\nabla^{[n]}$ is the same as the one introduced in Sect. 4, except that $\mathfrak{p}^+$ is replaced by $\mathfrak{p}^-$. In particular

$$H_{\frac{n-1}{2}} \supset S^i(\mathfrak{p}^-)^* \otimes V^*_{\frac{n-1}{2}} \quad \text{for} \quad 0 \leq i < n;$$

$$\dim \left(S^n(\mathfrak{p}^-)^* \otimes V^*_{\frac{n-1}{2}} \middle/ H_{\frac{n-1}{2}} \cap S^n(\mathfrak{p}^-)^* \otimes V^*_{\frac{n-1}{2}} \right) = 1.$$

Likewise, the representation $H^{\otimes 2}_{\frac{n-1}{2}}$ is identified with a subspace of $S^*(\mathfrak{p}^- \oplus \mathfrak{p}^-) \otimes \left(V^{\otimes 2}_{\frac{n-1}{2}} \right)^*$, and

$$H^{\otimes 2}_{\frac{n-1}{2}} \supset S^i(\mathfrak{p}^- \oplus \mathfrak{p}^-)^* \otimes \left(V^{\otimes 2}_{\frac{n-1}{2}} \right)^* \qquad 0 \leq i < n, \tag{A.1.1}$$

$$\dim \left(S^i(\mathfrak{p}^- \oplus \mathfrak{p}^-)^* \otimes \left(V^{\otimes 2}_{\frac{n-1}{2}} \right)^* \middle/ H^{\otimes 2}_{\frac{n-1}{2}} \cap S^i(\mathfrak{p}^- \oplus \mathfrak{p}^-)^* \otimes \left(V^{\otimes 2}_{\frac{n-1}{2}} \right)^* \right) = 2. \tag{A.1.2}$$

We can make (A.1.2) more precise: Under the diagonal action of $\mathfrak{k}$, $S^n(\mathfrak{p}^- \oplus \mathfrak{p}^-)^*$ contains an $(n+1)$-dimensional subspace B isotypic for the (differential of the)

1 See Note added in proof of p. 244

representation $\det^{-2}$, and every one-dimensional $\mathfrak{k}$-submodule of $S^n(\mathfrak{p}^- \oplus \mathfrak{p}^-)^*$ is contained in B. If we let (Z_1^α, Z_2^α), $\alpha = 1, \ldots, N$, be the coordinates on $\mathfrak{p}^- \oplus \mathfrak{p}^-$, then $B = \bigoplus_{i=0}^{n} B_i$, where B_i consists of polynomials on $\mathfrak{p}^- \oplus \mathfrak{p}^-$ of degree i (resp. $n-i$) in $\{Z_1^\alpha\}$ (resp. $\{Z_2^\alpha\}$). It is easy to see that

(A.1.3) $$H_{\frac{n-1}{2}}^{\otimes 2} \cap \left(B \otimes \left(V_{\frac{n-1}{2}}^{\otimes 2}\right)^*\right) = \bigoplus_{i=1}^{n-1} B_i \otimes \left(V_{\frac{n-1}{2}}^{\otimes 2}\right)^*.$$

Since $H_{\frac{n-1}{2}}$ is a unitarizable $U(\mathfrak{g})$-module, the tensor product $H_{\frac{n-1}{2}}^{\otimes 2}$ is completely reducible under the diagonal action of $U(\mathfrak{g})$. Any $\mathfrak{k}$-submodule $W^* \subset H_{\frac{n-1}{2}}^{\otimes 2}$ which is annihilated by the diagonal action of $\mathfrak{p}^-$ gives rise to a homomorphism $\psi_W : \mathbf{D}_W \to H_{\frac{n-1}{2}}^{\otimes 2}$ of $U(\mathfrak{g})$-modules. If $W^* \subset S^i(\mathfrak{p}^- \oplus \mathfrak{p}^-) \otimes \left(V_{\frac{n-1}{2}}^{\otimes 2}\right)^*$ with $0 \leqq i < n$, then it follows from (A.1.1) and a simple argument that ψ_W lifts to a homomorphism $\tilde{\psi}_W : \mathbf{D}_W \to \mathbf{D}_{\frac{n-1}{2}}^{\otimes 2}$; $\tilde{\psi}_W$ must be injective, since $D_{\frac{n-1}{2}}^{\otimes 2}$ is $U(\mathfrak{p}^+)$-torsion-free.

As in [6, 2.12], $\mathfrak{p}^-$ acts on $S^*(\mathfrak{p}^- \oplus \mathfrak{p}^-)$ by differentiation of polynomials. Let $E = \left(V_{\frac{n-1}{2}}^{\otimes 2}\right)^*$. Then any E-valued polynomial in $\{Z_1^\alpha - Z_2^\alpha, \alpha = 1, \ldots, N\}$ generates a $\mathfrak{k}$-invariant subspace W^* annihilated by $\mathfrak{p}^-$, as above, and the $\mathfrak{p}^-$-torsion in $S^*(\mathfrak{p}^- \oplus \mathfrak{p}^-)$ is spanned by the set of such W^*. Thinking of $\{Z_1^\alpha\}$ as coordinates of symmetric matrices, as in 4.2, we see that the $(n-1) \times (n-1)$ minors of the matrix $(Z_1 - Z_2)$, tensored by E, span such a subspace $W_{n-1}^* \subset S^{n-1}(\mathfrak{p}^- \oplus \mathfrak{p}^-) \otimes E$. By the preceding remarks, we obtain an imbedding $\tilde{\psi}_{W_{n-1}} : \mathbf{D}_{W_{n-1}} \to \mathbf{D}_{\frac{n-1}{2}}^{\otimes 2}$, since $n-1 < n$.

A.2. Now let f_1 and f_2 be holomorphic $\mathbb{C}$-valued functions on $\mathscr{D}$; lift them to functions ϕ_1 and ϕ_2 on G_0, as in (1.1.1), with $\varrho = \det^{\frac{n-1}{2}}$. Then $\mathscr{V}_{f_1}$ and $\mathscr{V}_{f_2}$, defined as in 3.1, are $U(\mathfrak{g})$-quotients of $\mathbf{D}_{\frac{n-1}{2}}$. Let $\pi : \mathbf{D}_{\frac{n-1}{2}}^{\otimes 2} \to \mathscr{V}_{f_1} \otimes \mathscr{V}_{f_2}$ be the natural map, and let

$$\mathscr{V}_{\{f_1, f_2\}} = \pi \circ \tilde{\psi}_{W_{n-1}}(\mathbf{D}_{W_{n-1}}).$$

Then $\mathscr{V}_{\{f_1, f_2\}}$ is generated by a $\mathfrak{p}^-$-torsion subspace, isomorphic (over $\mathfrak{k}$) to W_{n-1}^*, which by the procedure of 1.1, duality, and restriction to the diagonal in $\mathscr{D} \times \mathscr{D}$, defines a holomorphic W_{n-1}-valued function $\{f_1, f_2\}$ on $\mathscr{D}$. It is easy to check that the bundle Ω^{N-1} of $N-1$ forms on is canonically isomorphic to the homogeneous vector bundle on $\mathscr{D} \approx G_0/K_0$ arising from the representation of K_0 on W_{n-1}; thus we may regard $\{f_1, f_2\}$ as an $(N-1)$-form on $\mathscr{D}$. If f_1 and f_2 are modular forms of weight $\frac{n-1}{2}$ on $\mathscr{D}$ with respect to some arithmetic group Γ, then $\{f_1, f_2\}$ is a Γ-invariant $(N-1)$-form. Since $\{f_1, f_2\}$ is obtained from a polynomial function of degree $n-1$ in $\{Z_1 - Z_2\}$, we see that

(A.2.1) $$\{f_1, f_2\} = (-1)^{n-1}\{f_2, f_1\}.$$

If d denotes covariant differentiation, then

(A.2.2) $$d\{f_1, f_2\} = (-1)^{n-1} d\{f_2, f_1\}.$$

Now the same argument as in the proof of Corollary 1.5 shows that $f_1 \otimes f_2 \mapsto \{f_1, f_2\}$ is a constant coefficient differential operator of total degree $n-1$ in $f_1 \otimes f_2$. Moreover, the table in 4.2 easily implies that there are $\mathfrak{k}$-invariant pairings

$$\beta_e : \left(A_e^* \otimes V_{\frac{n-1}{2}}\right) \otimes \left(A_{n-e-1}^* \otimes V_{\frac{n-1}{2}}\right) \to W_{n-1} \quad \text{[notation as in (3.1.1)]},$$

unique up to scalar multiplication, and the argument of 3.5 implies that

$$\text{(A.2.3)} \qquad \{f_1, f_2\} = \sum_{e=0}^{n-1} a_e \beta_e (\delta_e f_1 \otimes \delta_{n-1-e} f_2)$$

for some set of $a_e \in \mathbb{C}$. Similarly, $d\{f_1, f_2\}$ corresponds, by the procedure of 1.1, to (the restriction to the diagonal of) an element $\Phi \in \mathscr{V}_{f_1} \otimes \mathscr{V}_{f_2}$, which is annihilated by $\mathfrak{p}^-$. It follows from (A.1.3), by a simple argument, that

A.2.4. **Lemma.** *Let $B' \subset \mathbf{D}_{\frac{n-1}{2}}^{\otimes 2}$ be the* $\det^{-(n+1)}$*-isotypic subspace for K. Then the $\mathfrak{p}^-$-torsion in B' is of dimension two, generated by*

$$\left(A_{n-1} \otimes V_{\frac{n-1}{2}}^*\right) \otimes V_{\frac{n-1}{2}}^* \quad \text{and} \quad V_{\frac{n-1}{2}}^* \otimes \left(A_{n-1} \otimes V_{\frac{n-1}{2}}^*\right).$$

By the argument of 3.5, it follows from A.2.4 and (A.2.2) that

$$\text{(A.2.5)} \qquad d\{f_1, f_2\} = c(\nabla^{[n]} f_1 \cdot f_2 + (-1)^{n-1} f_1 \cdot \nabla^{[n]} f_2)\omega,$$

where ω is the differential form $dZ^1 \wedge \ldots \wedge dZ^N$ and c is a nonzero constant; here Z^α are the matrix coordinates on $\mathscr{D}$.

By a uniqueness argument analogous to the proof of 3.5, we can show that $\{f_1, f_2\}$, as defined here, is a constant multiple of the differential operator introduced in [3] with the same notation. Then (A.2.5) corresponds to Lemma 3.1 of [3]. It is easy to see that (A.2.5) and the particular normalizations of the β_e determine the constants a_e uniquely. In particular one may obtain from (A.2.3) and (A.2.5), with a minimum of computation, sufficient information about $\{f_1, f_2\}$ to carry out the argument in [19] proving the non-rationality, in case $n = 8k+1$, $n > 9$, of the Siegel modular variety $\mathrm{Sp}(n, \mathbb{Z}) \backslash \mathscr{D}$.

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Note added in proof. Professor E. Freitag has kindly informed us that he has proved this result by classical methods.