A spin-off from highest weight representations;
Conformal covariants, in particular for $O(3,2)$.

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0. Introduction

Let $\mathcal{D} = G/K$ be a hermitian symmetric space of the non-compact type and let $E_{\tau_i} = G \times_K V_{\tau_i}$ be holomorphic vector bundles over $\mathcal{D}$; $i=1,2$, defined by finite-dimensional representations τ_i of a maximal compact subgroup K of the group G of holomorphic transformations of $\mathcal{D}$. Denote by $\Gamma_h(E_{\tau_i})$ the space of holomorphic sections of E_{τ_i} and let U_{τ_i} denote the representation of G on $\Gamma_h(E_{\tau_i})$ obtained from left translation of G on E_{τ_i} .

Consider a differential operator

(0.1) $$D: s \in \Gamma_h(E_{\tau_1}) \to Ds \in \Gamma_h(E_{\tau_2}) .$$

Definition. D is covariant if

(0.2) $$\forall g \in G \ \forall s \in \Gamma_h(E_{\tau_1}): DU_{\tau_1}(g)s = U_{\tau_2}(g)Ds .$$

The bundles E_{τ_i} may be parallellized; then $\Gamma_h(E_{\tau_i})$ becomes the space $\mathcal{O}(V_{\tau_i})$ of V_{τ_i}-valued holomorphic functions on $\mathcal{D}$ and D becomes a matrix-valued differential operator.

We may restrict (0.2) to the Shilov boundary of $\mathcal{D}$; for appropriate realizations of $\mathcal{D}$ and for suitable choices of G , among the spaces obtained as such are n-dimensional Minkowski space as well as $U(n)$, $n=1,2,\ldots$. Secondly, the representations involved are of positive (or one-sided) energy. For these as well as more abstract reasons, one is interested in

Problem I: Determine all such (D,τ_1,τ_2), and more generally,

Problem II: i) Determine all invariant subspaces of $\mathcal{O}(V_{\tau_1})$ and identify the subquotients. ii) In particular, determine which subspaces correspond to kernels of differential operators.

Let $\mathfrak{g}$ denote the Lie algebra of G. We note here that, due to analyticity, D is covariant if and only if

$$(0.3) \qquad \forall f \in \mathcal{O}(V_{\tau_1}) \forall x \in \mathfrak{g} : DdU_{\tau_1}(x)f \subset dU_{\tau_2}(x)Df ,$$

where dU denotes the differential of the representation U. In fact, as explained in [4] or [7], the problem is completely (modulo coverings of G) equivalent, by duality, to the algebraic problem of determining all homomorphisms between certain highest weight modules (≡ vacuum vector representations) of $\mathfrak{g}$. We remark that such a homeomorphism into a vacuum vector representation is completely determined by a <u>second</u> vacuum in the given space. We further remark that by this transformation, even the space $\mathcal{D}$ seems to disappear from the discussion. In some sense this is true, and this implies that the results hold for a number of different realizations, but, in fact, one realization is still there; as a subset of a subspace of the complexification of $\mathfrak{g}$.

In the following sections we explain in more detail about the highest weight modules involved. Then we turn to the special case $\mathfrak{g} = so(3,2) = sp(n,\mathbb{R})$ and describe a complete solution to the classification problem. It should be noted that this is an example of a group with two root lengths. We have also recently obtained the full classification for $su(2,2)$, and state the result without proof. The details for $su(2,2)$ will appear in [8] in which also a more detailed (but still far from complete) bibliography is given.

Finally, it should be remarked that our Problem II, though quite formidable in its full generality, still is only a special case of the programs of Dobrev ([3], and references therein) and of Angelopoulos ([1]). (See also these proceedings.)

<u>1. Simple Lie algebras</u>

A Lie algebra $\mathfrak{g}$ is called <u>simple</u> if it contains no ideals except 0 and itself, and such that, furthermore, $\mathfrak{g}$ is non-abelian. Then $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$. We will assume that $\mathfrak{g}$ is complex, but usually we will have in mind that $\mathfrak{g}$ is the complexification of some specific real Lie algebra $\mathfrak{g}_{\mathbb{R}}$; $\mathfrak{g} = (\mathfrak{g}_{\mathbb{R}})^{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then, under the adjoint action, $\mathfrak{h}$ can be diagonalized simultaneously. Let $\alpha \in \mathfrak{h}^*$ and set

(1.1) $$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid \forall h \in \mathfrak{h}: [h,x] = \alpha(h)x\} .$$

Further set

(1.2) $$\Delta = \{\alpha \in \mathfrak{h}^* \mid g^\alpha \neq \{0\}\} .$$

Δ is called the set of <u>roots</u>, and $\mathfrak{g}^\alpha$ is the space of <u>root-vectors</u> belonging to α. It is a fact that $\forall \alpha \in \Delta: \dim_{\mathbb{C}} \mathfrak{g}^\alpha = 1$, and $\alpha \in \Delta \Leftrightarrow -\alpha \in \Delta$.

On $\mathfrak{g}$ there is a symmetric bilinear form B; the killing form, and the restriction of B to $\mathfrak{h}$ induces, via duality, a non-degenerate form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ and hence on Δ. On the real span of Δ, this form is real and positive definite. Furthermore,

(1.3.) $$\alpha,\beta \in \Delta \Rightarrow \langle\alpha,\beta\rangle \underset{\text{def.}}{=} \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} .$$

For $\alpha \in \Delta$ let $h_\alpha \in [g^\alpha, g^{-\alpha}] \subseteq \mathfrak{h}$ be determined by $\alpha(h_\alpha) = 2$. Then

(1.4) $$\langle\alpha,\beta\rangle = \beta(h_\alpha) .$$

Set

(1.5) $$S_\alpha(\beta) = \beta - \beta(h_\alpha)\alpha .$$

Then $S_\alpha(\Delta) = \Delta$, and the reflexions S_α, $\alpha \in \Delta$, generate the so-called Weyl group of Δ.

Finally the elements in a basis Σ of Δ are called the <u>simple</u> roots. Δ decomposes according to Σ into

(1.6) $$\Delta = \Delta^+ \cup \Delta^-$$

where Δ^+ denotes the set of roots whose coordinates w.r.t. Σ all are non-negative integers, and $\Delta^- = -\Delta^+$. (A good reference to this section is [6].)

2. Highest weight modules

Fix a basis Σ of Δ. Set

$$(2.1)\qquad \mathfrak{g}^+ = \sum_{\alpha\in\Delta^+} \mathfrak{g}^\alpha , \quad \mathfrak{g}^- = \sum_{\alpha\in\Delta^+} \mathfrak{g}^{-\alpha}, \quad \text{and let} \quad \rho = \frac{1}{2} \sum_{\alpha\in\Delta^+} \alpha \ .$$

Then

$$(2.2)\qquad \mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+ \ .$$

We let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. It follows from (2.2) that

$$(2.3)\qquad \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}^-)\mathcal{U}(\mathfrak{h})\mathcal{U}(\mathfrak{g}^+) \ .$$

Let $\Lambda \in \mathfrak{h}^*$. The <u>Verma module</u> $M(\Lambda)$ of highest weight Λ is defined as follows:

1) $M(\Lambda)$ is a representation space of $\mathcal{U}(\mathfrak{g})$ and contains a vector v such that

$$\begin{aligned} &\text{i)} && M(\Lambda) = \mathcal{U}(\mathfrak{g})\cdot v_\Lambda \\ (2.4)\quad &\text{ii)} && \forall h \in \mathfrak{h}: h\cdot v_\Lambda = \Lambda(h)\cdot v_\Lambda \\ &\text{iii)} && \forall x \in \mathfrak{g}^+ : x\cdot v_\Lambda = 0 \ . \end{aligned}$$

2) $M(\Lambda)$ is maximal in this respect. ($M(\Lambda) = \mathcal{U}(\mathfrak{g}^-) \otimes v_\Lambda$ as a representation of $\mathfrak{h}$) .

More generally, a highest weight module of h.w. Λ is a module that satisfies 1) above. A special instance of this is a <u>generalized Verma module</u> $M_p(\Lambda)$ which is a quotinet of $M(\Lambda)$ and corresponds to induction from a (not necessarily minimal) parabolic p. We shall be interested in generalized Verma modules corresponding to holomorphic induction, and for that reason we only furnish the details for this case:

Assume from now on that $\mathfrak{g}$ corresponds to a hermitian symmetric space (a good reference for what follows is [5]). Let

$$(2.5)\qquad \mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$$

be a Cartan decomposition of the underlying real Lie algebra. Then (2.2) decomposes further into

$$(2.6)\qquad \mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k}^- \oplus \mathfrak{h} \oplus \mathfrak{k}^+ \oplus \mathfrak{p}^+ \quad \text{where}$$

$$(2.7)\qquad \mathfrak{k} = (\mathfrak{k}_{\mathbb{R}})^{\mathbb{C}} = \mathfrak{k}^- \oplus \mathfrak{h} \oplus \mathfrak{k}^+ , \quad \mathfrak{p} = \mathfrak{p}_{\mathbb{R}}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^- , \quad \text{and}$$

$$[\mathfrak{p}^+,\mathfrak{p}^+] = [\mathfrak{p}^-,\mathfrak{p}^-] = 0 \ .$$

In the present case there is a unique simple non-compact root β ($\mathfrak{g}^{\beta} \subset \mathfrak{p}^{+}$) , and $\Sigma \setminus \{\beta\}$ is a set of simple roots for $\mathfrak{k}_1 = [\mathfrak{k},\mathfrak{k}]$. (We have that $\mathfrak{k} = \mathfrak{k}_1 \oplus \eta(\mathfrak{k})$ where $\eta(\mathfrak{k})$ is the 1-dimensional center of $\mathfrak{k}$; $\eta(\mathfrak{k}) = \mathbb{C}\cdot h_0$.)

Let $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{k}_1$ and let γ_r denote the <u>highest root</u> ($\mathfrak{g}^{\gamma_r}$ satisfies: $\mathfrak{g}^{\gamma_r} \subset \mathfrak{p}^{+}$ and $\forall x \in \mathfrak{g}^{\gamma_r}$ $\forall y \in \mathfrak{k}^{+}$: $[x,y] = 0$). Then $\Lambda \in \mathfrak{h}^*$ can be written as

(2.8) $\Lambda = (\Lambda_0,\lambda)$ where $\Lambda_0 = \Lambda|_{\mathfrak{h}_1}$ and $\lambda = \Lambda(h_{\gamma_r})$.

From now on we assume:

(2.9) $\forall\mu \in \Sigma\setminus\{\beta\}$: $\Lambda_0(h_\mu) \in \mathbb{N} \cup \{0\}$ and $\lambda \in \mathbb{R}$.

It follows that Λ_0 determines a finite dimensional representation $V(\Lambda_0)$ of $\mathfrak{k}_1$;

(2.10) $$V(\Lambda_0) = \mathcal{U}(\mathfrak{k}_1)/I$$

where I is a left ideal in $\mathcal{U}(\mathfrak{k}_1)$. If $V(\Lambda_0)$ is given together with its highest weight vector v_{Λ_0} , I is determined as $I = \{u \in \mathcal{U}(\mathfrak{k}_1) \mid u\cdot v_{\Lambda_0} = 0\}$. However, usually it is the other way around; I determines $V(\Lambda_0)$.

The generalized Verma module $M_{\mathfrak{k}}(\Lambda)$ is defined by

(2.11) $$M_{\mathfrak{k}}(\Lambda) = \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{k}\oplus\mathfrak{p}^{+})}{\otimes} V(\Lambda)$$

where $V(\Lambda_0)$ is extended to a representation $V(\Lambda)$ of $\mathcal{U}(\mathfrak{k} \oplus \mathfrak{p}^{+})$ by letting h_0 act by $\Lambda(h_0)$ and letting $\mathfrak{p}^{+}$ act as zero. Clearly,

(2.12) $$M_{\mathfrak{k}}(\Lambda) = \mathcal{U}(\mathfrak{p}^{-}) \otimes V(\Lambda)$$

as a representation of $\mathfrak{k}$. Further, if $\tilde{A}(\Lambda) = \mathcal{U}(g)\cdot I\cdot v_{\Lambda}$ then

(2.13) $$M_{\mathfrak{k}}(\Lambda) = M(\Lambda)/\tilde{A}(\Lambda)$$

is a realization of $M_{\mathfrak{k}}(\Lambda)$ as a quotient of $M(\Lambda)$.

3. Homomorphisms between highest weight modules

Let R_{Λ} and R_{Λ_1} be highest weight modules of h.w.'s Λ and Λ_1 , respectively. A <u>homomorphism</u> φ of R_{Λ_1} into R_Λ is a map from R_{Λ_1} to R_Λ which commutes with the representations. In particular, $\varphi(v_{\Lambda_1}) = \tilde{v}_{\Lambda_1} \in R_\Lambda$ is a vector which satisfies 1) ii) and iii) in (2.4) for Λ_1. We assume that φ is non-trivial i.e. that $\tilde{v}_{\Lambda_1} \neq 0$. Conversely, if a non-zero vector in R satisfies 1) ii) and iii), then one can clearly define a map $\bar{\varphi}: M(\Lambda_1) \to R_\Lambda$, and out of $\bar{\varphi}$ one can construct a map of any given quotient of $M(\Lambda_1)$ into an appropriate subquotient of R_Λ. It is also clear that it may happen that the induced map between a quotient of R_{Λ_1} and a quotient of R_Λ may be zero.

In particular, <u>a map $\varphi: M(\Lambda_1) \to M(\Lambda)$ may induce the trivial map from $M_k(\Lambda_1)$ to $M_k(\Lambda)$</u>.

In this area, the most important theorem is the BGG (Bernstein-Gelfand-Gelfand) theorem. To formulate it we need:

<u>Definition 3.1</u>. Let $\chi,\psi \in \mathfrak{h}^*$. A sequence of roots $\gamma_1,\ldots,\gamma_k$ is said to satisfy condition (A) for the pair (χ,ψ) if

i) $\chi = S_{\gamma_k} \cdots\cdots S_{\gamma_1}\chi$

ii) Put $\chi_0 = \psi$, and $\chi_i = S_{\gamma_i} \cdots S_{\gamma_1}\psi$. Then

$\forall i=1,\ldots,k: \langle\chi_{i-1},\gamma_i\rangle \in \mathbb{N}$.

Under these circumstances, (χ,ψ) is said to satisfy $(\bar{A})$.

<u>Theorem 3.2</u> (BGG; [2]). i) There is a non-zero homomorphism from $M(\Lambda_1)$ if and only if $(\Lambda_1+\rho, \Lambda+\rho)$ satisfies $(\bar{A})$. ii) If there is a homomorphism from a (sub-)quotient of $M(\Lambda_1)$ to a (sub-)quotient of $M(\Lambda)$, then $(\Lambda_1+\rho, \Lambda+\rho)$ satisfies $(\bar{A})$.

Let $\Delta_n^+ = \{\alpha \in \Delta^+ \mid \mathfrak{g}^\alpha \subset \mathfrak{p}^+\}$ be the set of <u>non-compact</u> positive roots. The following is proved in [7].

<u>Corollary 3.3</u>. If there is a non-trivial homomorphism from $M_k(\Lambda_1)$ to a (sub-)quotient of $M_k(\Lambda)$, then $(\Lambda_1+\rho, \Lambda+\rho)$ satisfies $(\bar{A})$ with a sequence $\gamma_1,\ldots,\gamma_k$ of positive non-compact roots.

4. $O(3,2) \cong Sp(2,\mathbb{R})$

The following realization of $\mathfrak{g} = o(3,2) = sp(2,\mathbb{R})$ is convenient because it displays $\mathfrak{p}^+$ and $\mathfrak{p}^-$ directly as $\mathfrak{k}$-representation spaces. Let

$$(4.1)\qquad k^+ = \left[\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{0} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & 0 \\ & & -1 & 0 \end{array}\right], \quad k^- = \left[\begin{array}{cc|cc} 0 & 0 & \multicolumn{2}{c}{0} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & -1 \\ & & 0 & 0 \end{array}\right], \quad \text{and} \quad h_k = [k^+,k^-],$$

$$(4.2)\qquad \mathfrak{p}^- = \left\{ \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ \hline z_a & z_b & 0 \\ z_b & z_c & \end{array}\right] \;\middle|\; z_a, z_b, \text{ and } z_c \in \mathbb{C} \right\}, \quad \text{and}$$

$$(4.3)\qquad \mathfrak{p}^+ = \left\{ \left[\begin{array}{c|cc} 0 & z_a & z_b \\ & z_b & z_c \\ \hline 0 & \multicolumn{2}{c}{0} \end{array}\right] \;\middle|\; z_a, z_b, \text{ and } z_c \in \mathbb{C} \right\}.$$

Then

$$(4.4)\qquad \mathfrak{k} = \mathbb{C}\cdot k^+ \oplus \mathbb{C}\, k^- \oplus \mathbb{C}\cdot h_k \quad \text{and} \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^- .$$

We let z_a, z_b, and z_c denote the elements of $\mathfrak{p}^-$ corresponding to an entry 1 at the appropriate place in (4.2), and we let z_a^+, z_b^+, and z_c^+ denote the analogous elements of $\mathfrak{p}^+$ in (4.3). Also let

$$(4.5)\qquad h_a = [z_a^+, z_a], \quad h_b = [z_b^+, z_b], \quad \text{and} \quad h_c = [z_c^+, z_c].$$

The elements h_k, h_a, h_b, and h_c are all of the form $h_{\overline{\gamma}}$ for roots $\overline{\gamma}$. We denote these by μ, β, α, and γ, respectively. As before, β is the simple non-compact root, and $\gamma = \gamma_r$ is the highest root.

We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ with $\mathbb{C}^2$. Let e_1 and e_2 be the usual basis vectors. Then

$$(4.6)\qquad \beta = 2e_2, \quad \alpha = e_1+e_2, \quad \gamma = 2e_1, \quad \text{and} \quad \mu = e_1-e_2 .$$

Furthermore,

$$(4.7)\qquad h_a = e_2, \quad h_b = e_1+e_2, \quad h_c = e_1, \quad \text{and} \quad h_k = e_1-e_2 .$$

A generalized Verma module Λ is determined by

$$(4.8)\qquad \Lambda = (\lambda, \lambda-n); \quad n \in \mathbb{N} \cup \{0\}, \quad \text{and} \quad \rho = (2,1).$$

In the following, n will be held fixed and λ will be allowed to vary. The following is obtained, by trial and error, from Corollary 3.3.

Lemma 4.1. If there is a non-trivial homomorphism $M_{\mathfrak{k}}(\Lambda_1) \to M_{\mathfrak{k}}(\Lambda)$ then the sequence of reflexions corresponding to the sequence of roots in condition (A) is, for λ in the given intervals satisfying throughout that $2\lambda \in \mathbb{Z}$; and furthermore, whenever S_γ or S_β take part, satisfying that $\lambda \in \mathbb{Z}$:

$$
(4.9)\qquad
\begin{array}{rcll}
\lambda < -1 & : & \text{None} & \\
-1 \leq \lambda \leq \frac{n}{2}-2 & : & S_\gamma & (n > 1) \\
\frac{n}{2}-1 \leq \lambda \leq n-2 & : & S_\alpha \,,\quad S_\gamma S_\alpha & (n > 1) \\
n-2 < \lambda < n & : & S_\alpha & \\
n \leq \lambda & : & S_\beta \,,\quad S_\alpha S_\beta \,,\quad S_\alpha \; (= S_\gamma S_\alpha S_\beta) &
\end{array}
$$

We have from (2.12) that $M_{\mathfrak{k}}(\Lambda)$ is genrated by expressions of the form $z_1 \cdot \ldots \cdot z_r \otimes v$ with $z_1, \ldots, z_r \in \mathfrak{p}^-$ and $v \in V(\Lambda)$. It is obvious that $\mathfrak{k}$ preserves the degree r of such an expression. To describe the representations of $\mathfrak{k}$ which occur in a general $M_{\mathfrak{k}}(\Lambda)$ we must first describe the $\mathfrak{k}$-representation in $\mathcal{U}(\mathfrak{p}^-)$: Let

$$(4.10)\qquad \det z = z_a z_c - \frac{1}{4} z_b^2 \,.$$

The highest weight vectors of $\mathcal{U}(\mathfrak{p})$ as a $\mathfrak{k}$-representation are then

$$(4.11)\qquad z_c^{\,r} \det z^s \quad ; \qquad r,s \in \mathbb{N} \cup \{0\}.$$

This is obvious from the representation theory of $U(2)$. Observe that the $\mathfrak{k}_1$ weight of (4.11) is r and that the degree is $r+2s$. The $\otimes$-product $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda)$ is then easily computed either directly from the $U(2)$-theory or from [9]. Recall that $V(\Lambda)$ denotes the $\mathfrak{k}$-representation defined by (Λ_0, λ). Observe that if $\Lambda = (\lambda, \lambda - n)$ and if a $\mathfrak{k}$-irreducible subspace of $\mathfrak{k}_1$-weight $\tilde{n}$ occurs in the $\otimes$-product of degree d expressions in $\mathcal{U}(\mathfrak{p}^-)$ with $V(\Lambda)$, then if $2d > n+\tilde{n}$, the $\mathcal{U}(\mathfrak{p}^-)$ terms must all contain a factor of $\det z$.

We list here the relevant commutators

$$
(4.12)\qquad
\begin{array}{lll}
[k^+, z_a] = -z_b \,, & [k^+, z_b] = -2z_c \,, & [k^+, z_c] = 0 \\
[k^-, z_a] = 0 \,, & [k^-, z_b] = -2z_a \,, & [k^-, z_c] = -z_b \\
[k^+, z_a^+] = 0 \,, & [k^+, z_b^+] = 2z_a^+ \,, & [k^+, z_c^+] = 2_b^+ \\
[k^+, h_c] = k^+ \,, & [k^+, h_b] = 0 \,, & [k^+, h_a] = -k^+ \\
[z_c^+, z_a] = 0 \,, & [z_c^+, z_b] = k^- \,, & [h_k, z_c] = 2z_c \\
[z_b^+, z_a] = k^- \,, & [z_b^+, z_c] = k^+ \,, & [z_a^+, z_b] = k^+
\end{array}
$$

The following is then straightforward

Lemma 4.2. Inside $\mathcal{U}(\mathfrak{g})$,

$$
\begin{aligned}
z_a^+ \det z^s &= s \det z^{s-1} z_c (h_a + 3/2 - s) \quad \text{modulo} \quad \mathcal{U}(\mathfrak{g})\cdot k^+ \\
z_b^+ \det z^s &= -s \det z^{s-1} (z_b/2 (h_b + 3 - 2s) - z_c k^-) \quad \text{modulo} \quad \mathcal{U}(\mathfrak{g}) k^+ \qquad (4.13) \\
z_c^+ \det z^s &= s \det z^{s-1} (z_a (h_c + 3/2 - s) - z_b k^-/2) \quad \text{modulo} \quad \mathcal{U}(\mathfrak{g}) k^+ .
\end{aligned}
$$

Observe that the representation space in $\mathcal{U}(\mathfrak{p}^-)$ whose highest weight vector is given by (4.11), is spanned by the elements $((\operatorname{ad} k^-)^i z_c^{\,r})\cdot \det z^s$ for $i=0,\ldots,r$.

Since $\mathfrak{p}^- \otimes (\overset{r}{\underset{s}{\otimes}} \mathfrak{p}^-) = \det z \overset{r-1}{\underset{s}{\otimes}} \mathfrak{p}^- \oplus \overset{r+1}{\underset{s}{\otimes}} \mathfrak{p}^-$, one can easily establish the following (it suffices to prove the first)

Lemma 4.3. Let $\varepsilon = (\beta+1)^{-1}(4\beta+2)^{-1}$. Inside $\mathcal{U}(\mathfrak{p}^-)$,

$$
\begin{aligned}
z_c (\operatorname{ad} k^-)^\alpha z_c^\beta &= (2\beta-\alpha+2)(2\beta-\alpha+1)\varepsilon (\operatorname{ad} k^-)^\alpha z_c^{\beta+1} + \\
&\quad 2\beta\cdot\alpha(\alpha-1)(2\beta+1)^{-1} \det z (\operatorname{ad} k^-)^{\alpha-2} z_c^{\beta-1} \\
z_b (\operatorname{ad} k^-)^\alpha z_c^\beta &= -2(2\beta-\alpha+1)\cdot\varepsilon (\operatorname{ad} k^-)^{\alpha+1} z_c^{\beta+1} + \qquad (4.14) \\
&\quad 4\alpha\beta(2\beta+1)^{-1} \det z (\operatorname{ad} k^-)^{\alpha-1} z_c^{\beta-1} \\
z_a (\operatorname{ad} k^-)^\alpha z_c^\beta &= \varepsilon (\operatorname{ad} k^-)^{\alpha+2} z_c^{\beta+1} + 4\beta(2\beta+1)^{-1} \det z (\operatorname{ad} k^-)^\alpha z_c^{\beta-1}
\end{aligned}
$$

For later use we observe that if a vector $v \neq 0$ in a $\mathfrak{k}$-representation space satisfies that $k^+\cdot v = 0$ and $h_k\cdot v = \tilde{n}\cdot v$ for an integer $\tilde{n} > 1$, then

$$(z_a + \tilde{n}^{-1} z_b k^- + \tilde{n}^{-1}(\tilde{n}-1)^{-1} z_c (k^-)^2)\cdot v \qquad (4.15)$$

is a highest weight vector in $\mathfrak{p}^- \otimes V$ of $\mathfrak{k}_1$-weight $\tilde{n}-2$. Likewise,

$$(z_b + 2\tilde{n}^{-1} z_c k^-)\cdot v \qquad (4.16)$$

is a highest weight vector; its weight is $\tilde{n}$, and it suffices that $\tilde{n}$ be a positive integer.

Lemma 4.4. Let $v \in (\overset{d}{\otimes} \mathfrak{p}^-) \otimes V(\Lambda)$ be a highest weight vector of weight $\tilde{n} = y+n-x$ and let $2d = x+y$. Then, if $\tilde{n} > 0$ (4.16) defines a non-zero element of $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda)$; and if $\tilde{n} > 1$, so does (4.15).

Proof. The two cases are similar, so we only consider (4.16). With no loss of generality we can assume that v does not contain a factor of $\det z$. In particular, we may assume that $x \leq n$. It follows that

$$(4.17) \quad v = (\mathrm{ad}\, k^-)^x z_c^{\frac{x+y}{2}} \cdot v \;+ \text{terms from } \mathcal{U}(\mathfrak{p}^-) \otimes \mathrm{span}\left\{(k^-)^i v_\Lambda\right\}_{i=1}^n .$$

Thus, it suffices to consider the v_Λ-coefficient of (4.16), i.e.

$$(4.18) \qquad z_b^-(\mathrm{ad}\, k^-)^x z_c^{\frac{x+y}{2}} + 2(y+n-x)^{-1} z_c (\mathrm{ad}\, k^-)^{x+1} z_c^{\frac{x+y}{2}} .$$

This, however, is easily computed to be, with $\delta = (x+y+2)^{-1}(x+y+1)^{-1}$,

$$(4.19) \quad \begin{aligned} &2\delta(y+1)[-1+y(y+n-x)^{-1}](\mathrm{ad}\, k^-)^{x+1} z_c^{\frac{x+y}{2}+1} \\ &+ (x+y+1)^{-1}\cdot 2\cdot x\cdot(x+y)[1+(x+1)(y+n-x)^{-1}]\det z \cdot (\mathrm{ad}\, k^-)^{x-1} z_c^{\frac{x+y}{2}-1} , \end{aligned}$$

and this is clearly always non-zero. ▯

Let us now turn to the problem of determining when there can be a homomorphism into $M_k(\Lambda)$. First of all, we proved in ([7], Proposition 1.6) that anything of the form $S_{\overline{\gamma}}$ with $\overline{\gamma}$ long defines a homomorphism. In case $2\lambda \notin 2\mathbb{Z}$, the same argument implies that S_α defines a homomorphism, since it is the only possible non-compact root at such λ's . In fact, it is possible to find another sequence which satisfies condition (A) for the pair $(S_\alpha(\Lambda+\rho), \Lambda+\rho)$ if and only if λ is an integer, and $\lambda \geq n-1$. By a result due to Boe, it follows (cf. [7], Proposition 1.4) that for $\frac{n}{2} - 1 \leq \lambda < n-1$ and $2\lambda \in \mathbb{Z}$, S_α does define a homomorphism. In the remaining cases for S_α as well as for $S_\gamma S_\alpha$ and $S_\alpha S_\beta$, one is led to consider a highest weight vector q in $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda)$ which satisfies

$$(4.20) \quad \begin{aligned} &a) \quad \mathfrak{p}^+ q = k^+ q = 0 \\ &b) \quad q = \det z^s \,\overline{q} \quad \text{for some} \quad s \in \mathbb{N} . \end{aligned}$$

We will always assume that the s in b) is the biggest possible such. Let us further assume that the weight Λ_1 of q is $(\lambda_1, \lambda_1 - n_1)$. It then follows from (4.13) that

$$(4.21) \qquad s \det z^{s-1} z_c(\lambda_1 + 3/2 + s)\bar{q} + \det z^s z_a^+ \bar{q} = 0 \ .$$

Due to the fact that the ideal generated by $\det z$ is <u>prime</u>, and due to the assumption that s is biggest possible ($\bar{q}$ does not contain a factor $\det z$), it follows that

$$(4.22) \qquad \lambda_1 + 3/2 + s = 0 \ .$$

This equation has three interesting consequences: i) only one s is possible, ii) λ_1, and hence λ must satisfy: $2\lambda \notin 2\mathbb{Z}$, and iii) $-\lambda_1 \geqq 5/2$ $(s \geq 1)$.

Returning to $S_\gamma S_\alpha$ and $S_\alpha S_\beta$, it is easy to see (cf. the remarks following (4.11)) that if a homomorphism exists, it must be defined for a q of the form (4.20). Hence, since both exist only if $2\lambda \in 2\mathbb{Z}$, this is impossible. Finally, there can be no multiplicities for S_α due to consequences i) and iii).

This still leaves open the question whether some quotinets exist which are not defined by homomorphisms. However, Lemma 4.4 together with an easy count of multiplicities (cf. the proof for $su(2,2)$ in [8]) gives that there are no such quotients. Observe that at the situation in $sp(2,\mathbb{R})$ corresponding to the place in $su(2,2)$ where a non-homomorphic quotient exists, namely $S_\gamma S_\alpha$ with $\lambda = (n-2)/2$ (n even), the corresponding $\mathfrak{k}$-type does not belong to $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda)$. We can then state:

<u>Theorem 4.5</u>. For $\mathfrak{g} = sp(2,\mathbb{R})$, all quotients of $M_{\mathfrak{k}}(\Lambda)$ are defined by homomorphisms, and there are no multiplicities. For n fixed, in the language of (4.9) the full list of non-trivial homomorphisms into $M_{\mathfrak{k}}(\Lambda)$ is obtained for λ in the intervals below satisfying the requirement that $2\lambda \in \mathbb{Z}$:

$$(4.23) \qquad \begin{array}{lcll} -1 \leq \lambda \leq \frac{n}{2} - 2 & : & S_\gamma & (n \geq 2) \\ \frac{n}{2} - 1 \leq \lambda \leq n - 3/2 & : & S_\alpha & (n \geq 1) \\ n - 1/2 \leq \lambda & : & S_\beta \text{ when } \lambda \in \mathbb{Z}, & S_\alpha \text{ when } \lambda \notin \mathbb{Z} . \end{array}$$

<u>5. $\mathfrak{g} = su(2,2)$</u>

Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$. Then the positive non-compact roots are

(5.1) $\Delta_n^+ = \{\beta = e_1-e_2, \quad \alpha_1 = e_1-e_3, \quad \alpha_2 = e_1+e_3, \quad \text{and } \gamma=\gamma_r=e_1+e_2\}$.

The positive compact roots are

(5.2) $\Delta_c^+ = \{\mu = e_2+e_3, \quad \text{and} \quad \nu = e_2-e_3\}$.

We have that $\rho = (2,1,0)$ and the Λ's that define the $M_{\mathfrak{k}}(\Lambda)$'s are of the form

(5.3) $\Lambda = (r, \frac{n+m}{2}, \frac{n-m}{2})$,

with $n,m \in \mathbb{N} \cup \{0\}$. Below we assume that $n \geq m$ and write $\Lambda+\rho = (z,x,y)$. Thus, $y \geq 0$, and in the following, $z \in \mathbb{Z}+y$ throughout.

<u>Theorem 5.1.</u> [8] Let $M_{\mathfrak{k}}(\Lambda)$ be the generalized Verma module of highest weight $\Lambda = (z,x,y)-\rho$. Then the subspace structure is defined by homomorphisms except in the case maked by Q . There are no multiplicities:

i) $x > y+1$:

$z \leq -x$	:	None
$-x < z \leq -y-1$	:	S_γ
$-y < z \leq y$	:	S_{α_2}
$y+1 \leq z \leq x-1$	:	$S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_1}S_{\alpha_2}$
$z = x$	:	None
$x+1 \leq z$	:	S_β .

ii) $x = y+1$:

$z \leq 1-x$	:	None
$1-x < z \leq x-1$	:	S_{α_2}
$z = x$	:	$S_{\alpha_1}S_{\alpha_2}$
$x+1 \leq z$	:	$S_\beta, S_{\alpha_1}S_{\alpha_2}$.

iii) $y = 0$:

$z \leq -x$: None

$-x < z \leq -1$: S_γ

$z = 1$: S_{α_1}, S_{α_2}, $S_{\alpha_1}S_{\alpha_2}$, $S_\gamma S_{\alpha_1}S_{\alpha_2}$ (Q) $(x \geq 2)$

$1 < z \leq x-1$: S_{α_1}, S_{α_2}, $S_{\alpha_1}S_{\alpha_2}$

$z = x$: $S_{\alpha_1}S_{\alpha_2}$

$x+1 \leq z$: S_β, $S_{\alpha_1}S_{\alpha_2}$

References

1. A. Angelopoulos, The unitary irreducible representations of $\overline{SO}_0(4,2)$, Commun. Math. Phys. 89, 41-57 (1983).

2. I.N. Bernstein, I.M. Gelfand, and G.I. Gelfand, Differential operators on the base affine space and a study of $\mathfrak{g}$-modules; in Lie Groups and Their Representations (I.M. Gelfand, Ed.) Adam Hilger, London; 1975.

3. V.K. Dobrev, Elementary representations and intertwining operators for SU(2,2). I., J. Math. Phys. 26 235-251 (1985).

4. M. Harris and H.P. Jakobsen, Singular holomorphic representations and singular modular forms, Math. Ann. 259, 227-244 (1982).

5. S. Helgason, Differential geometry, Lie groups, and symmetric spaces. Academic Press, New York, San Francisco, London; 1978.

6. J. Humphreys, Introduction to Lie algebras and representation theory. Springer Verlag, New York, Heidelberg, Berlin; 1972.

7. H.P. Jakobsen, Basic covariant differential operators on hermitian symmetric spaces, to appear in Ann. Sci. Éc. Norm. Sup.

8. H.P. Jakobsen, Conformal covariants, to appear in Publ. RIMS. Kyoto Univ.

9. G.D. James, The representation theory of the symmetric groups. Lecture Notes in Math. # 682, Springer Verlag, Berlin-Heidelberg-New York; 1978.