Quantized Hermitian Symmetric Spaces

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Abstract

This talk is basically a progress report for a project whose aim is to study quantum deformations of hermitian symmetric spaces. Most of it, in particular relating to the algebraic aspects of certain quadratic algebras, is joint work with A. Jensen, S. Jøndrup, and H.C. Zhang.

1 Introduction

The issue we wish to address here is that of quantum deformations of hermitian symmetric spaces. This is a quantization which goes in another direction than the well established attempts of quantization by means of translating classical observables into self adjoint operators. Instead, in the spirit of "non-commutative geometry", one tries to "quantize" the underlying spaces.

What is a hermitian symmetric space $\mathcal{D}$ anyway? By definition it is a connected complex manifold with a Hermitian structure in which each point is an isolated fixed point of an involutive holomorphic isometry. Automatically, it becomes Kählerian. We will only be interested in irreducible spaces of the non-compact type. The classification of such spaces was begun by Cartan ([2]) and finished by Harish-Chandra ([5]).

Operationally, we may, see e.g. [6, Theorem 6.1], define it to be a space

$$\mathcal{D} = G/K, \tag{1}$$

where G is a connected noncompact simple Lie group with center $\{e\}$ and K has non-discrete center and is a maximal compact subgroup of G.

Equivalently, $\mathcal{D}$ is a bounded symmetric domain (i.e. a domain in which each point is an isolated fixed point of an involutive holomorphic diffeomorphism of $\mathcal{D}$ onto itself). G may then be taken to be the group of bi-holomorphic maps of $\mathcal{D}$ into itself, and K as the subgroup of isometries.

Example 1.1

$$\mathcal{D} = \{n \times n \textit{ complex matrices } z \mid z^* z < 1\}\,. \tag{2}$$

Here, $G = SU(n,n)$ *and* $K = S(U(n) \times U(n))$.

We have the following useful algebraic version of the situation:

$$\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+, \text{ and } \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{p}^-) \cdot \mathcal{U}(\mathfrak{k}) \cdot \mathcal{U}(\mathfrak{p}^+). \tag{3}$$

It is well known that we may choose

$$\mathcal{D} \subset \mathfrak{p}^+. \tag{4}$$

Furthermore, $\mathfrak{p}^-$ and $\mathfrak{p}^+$ are *abelian* $\mathfrak{k}$-modules and much of the geometry and analysis can be described in terms of spaces of holomorphic functions, e.g. polynomials, on these spaces. The $\mathfrak{k}$-module structure is given by 2-dimensional weight diagrams, [7], e.g. (for $SU(m,n)$)

$$\begin{array}{ccccccc}
\vdots & & \vdots & & & & \\
\uparrow & & \uparrow & & \nearrow & & \\
Z_{31} & \overset{\mu_1}{\rightarrow} & Z_{32} & \overset{\mu_2}{\rightarrow} & Z_{33} & \rightarrow & \cdots \\
\uparrow \nu_2 & & \uparrow \nu_2 & & \uparrow \nu_2 & & \\
Z_{21} & \overset{\mu_1}{\rightarrow} & Z_{22} & \overset{\mu_2}{\rightarrow} & Z_{23} & \rightarrow & \cdots \\
\uparrow \nu_1 & & \uparrow \nu_1 & & \uparrow \nu_1 & & \\
Z_{11} & \overset{\mu_1}{\rightarrow} & Z_{12} & \overset{\mu_2}{\rightarrow} & Z_{13} & \rightarrow & \cdots
\end{array} \tag{5}$$

or (for $Sp(n,\mathbb{R})$)

$$\begin{array}{ccccccc}
 & & & & & & \nearrow \\
 & & & & Z_{33} & \rightarrow & \cdots \\
 & & & & \uparrow \mu_2 & & \\
 & & Z_{22} & \overset{\mu_2}{\rightarrow} & Z_{23} & \rightarrow & \cdots \\
 & & \uparrow \mu_1 & & \uparrow \mu_1 & & \\
Z_{11} & \overset{\mu_1}{\rightarrow} & Z_{12} & \overset{\mu_2}{\rightarrow} & Z_{13} & \rightarrow & \cdots
\end{array} \tag{6}$$

There are more examples later in the article.

This is the structure we wish to quantize, basically by inseting a "q" in an appropriate place.

2 Quantum groups

Given an $n \times n$ Cartan matrix $A = (a_{ij})$ of finite type, choose $d_i \in \{1,2,3\}$ such that $(d_i a_{ij})$ is symmetric. The quantum group $\mathcal{U}_q$ is defined by the generators $E_i, F_i, K_i, K_i^{-1} (1 \le i \le n)$ and relations ("quantized Serre relations")

$$
\begin{gathered}
K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\
K_i E_j K_i^{-1} = q^{d_i a_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q^{-d_i a_{ij}} F_j, \\
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, \\
\sum_{s=0}^{1-a_{ij}} (-1)^s \left[\begin{array}{c} 1 - a_{ij} \\ s \end{array} \right]_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0, \qquad i \ne j, \\
\sum_{s=0}^{1-a_{ij}} (-1)^s \left[\begin{array}{c} 1 - a_{ij} \\ s \end{array} \right]_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s = 0, \qquad i \ne j.
\end{gathered} \tag{7}
$$

As is very well known, $\mathcal{U}_q$ is a Hopf algebra with comultiplication $\triangle$, antipode S, and counit ϵ defined by

$$
\begin{aligned}
\triangle(E_i) &= E_i \otimes 1 + K_i \otimes E,\ \triangle(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \\
\triangle(K_i) &= K_i \otimes K_i, S(E_i) = -K_i^{-1} E_i, S(F_i) = -F_i K_i, S(K_i) = K_i^{-1}, \text{ and} \\
\epsilon(E_i) &= \epsilon(F_i) = 0,\ \epsilon(K_i) = 1.
\end{aligned} \tag{8}
$$

Actually, there is another coproduct, Δ_D, which for some purposes is more useful. For instance, it behaves better with respect to tensor products when q is a real parameter. In fact, it is the coproduct introduced by Drinfeld (whence the subscript).

$$
\triangle_D (E_i) = E_i \otimes K_i^{-1/2} + K_i^{1/2} \otimes E_i,\ \triangle_D(F_i) = F_i \otimes K_i^{-1/2} + K^{1/2} \otimes F_i. \tag{9}
$$

There are two interesting families of anti-linear anti-involutions, ω_1 and ω_2, of this algebra,

$$
\omega_1(E_i) = \epsilon_i \cdot F_i, \omega_1(F_i) = \epsilon_i \cdot E_i, \omega_1(K_i) = K_i^{-1}, \omega_1(q) = q^{-1}, \tag{10}
$$

and

$$
\omega_2(E_i) = \epsilon_i \cdot F_i, \omega_2(F_i) = \epsilon_i \cdot E_i, \omega_2(K_i) = K_i, \omega_2(q) = q, \tag{11}
$$

where $\forall i : \epsilon_i^2 = 1$.

Above, we consider $\mathcal{U}_q$ to be an algebra over $\mathbb{C}(q)$, the field of rational functions in the "dummy" variable q. At times, e.g. when studying unitarity, we shall "localize" the algebra at a specific complex number ζ, i.e. $q \longrightarrow \zeta$. We will not go into the details here but just mention that some care is needed for this process when q is a root of unity.

3 Quadratic algebras

Let V be a finite dimensional vector space over $\mathbb{C}$, let R be a subspace of $V \otimes V$, and let I_2 be the two-sided ideal in $T(V)$ generated by R. The quadratic algebra A is defined as

$$A = T(V)/I_2. \tag{12}$$

It is a graded algebra which is generated by the elements of degree 1.

Let

$$\mathbb{P} = P(V^*). \tag{13}$$

An element

$$f = \sum_{i,j} \alpha_{ij} v_i \otimes v_j \in R \tag{14}$$

defines a bilinearform on $\mathbb{P} \times \mathbb{P}$ by

$$f(p,q) = \sum_{i,j} \alpha_{ij} v_i(p) \otimes v_j(q) \in \mathbb{C}. \tag{15}$$

The associated variety is defined as

$$\Gamma = \mathcal{V}(R) = \{(p,q) \in \mathbb{P} \times \mathbb{P} \mid \forall f \in R : f(p,q) = 0\}. \tag{16}$$

In many interesting cases,

$$R = R(\Gamma) = \{f \in V \otimes V \mid f(p,q) = 0 \text{ for all } (p,q) \in \Gamma\}. \tag{17}$$

Moreover, in the cases to follow as well as for e.g. the Sklyanin algebra,

$$\Gamma = \{(p, \sigma(p)) \mid p \in E\}, \tag{18}$$

where

$$E = \pi_1(\Gamma), \tag{19}$$

and σ is an automorphism of E. Typically, E consists of a continuous part which is an algebraic variety together with a few extra points, lines etc. “at infinity”.

The idea now is to relate the representation theory of A to the geometry of E. As such it becomes an instance of *non commutative algebraic geometry* as introduced by Artin ([1]).

Of interest are of course *simple modules* as well as certain “geometric” modules:

Definition 3.1 *A graded module M is a* point module *if*

- *M is cyclic.*
- $H_M(t) = (1-t)^{-1}$.

It is a line module *if*

- *M is cyclic.*
- $H_M(t) = (1-t)^{-2}$.

Here,

$$H_M(t) = \sum t^n \dim M_n \tag{20}$$

is the Hilbert series. Of course, one can go on to define plane modules, etc.

Finally, one is interested in finding all *fat points.*

Definition 3.2 *A graded module M is called a fat point if it is 1-critical of Gelfand-Kirillov dimension 1 and of multiplicity 1.*

Basically, this means that from a certain step on, the spaces M_n all have the same dimension $e > 1$. Furthermore, if N is any graded submodule, then M/N is finite dimensional.

Remark 3.3 *Actually, the above definitions ought to be formulated in terms of a certain category Proj(A) of graded modules modulo finite dimensional modules. Eg. the fat points are closely related to the irreducible points in this category.*

4 Quadratic algebras associated to hermitian symmetric spaces

Let $\mathfrak{g}$ now correspond to a hermitian symmetric space and consider the corresponding quantum group $\mathcal{U}_q$. Analogous to the decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{p}^-) \cdot \mathcal{U}(\mathfrak{k}) \cdot \mathcal{U}(\mathfrak{p}^+), \tag{21}$$

we would like to have a decomposition

$$\mathcal{U}_q = A^- \cdot \mathcal{U}_q(\mathfrak{k}) \cdot A^+, \tag{22}$$

and where A^- and A^+ are quadratic algebras that furthermore are (as linear spaces) $\mathcal{U}_q(\mathfrak{k})$-modules.

Theorem 4.1 **([8])** *This is indeed possible.*

4.1 Special case: $su(n,m)$

In the case of $su(n,m)$ the set of simple compact roots break up into two orthogonal sets: $\sum_c = \{\nu_1, \ldots, \nu_{n-1}\} \cup \{\mu_1, \ldots, \mu_{m-1}\}$. Thus,

$$\forall i,j : E_{\mu_i} E_{\nu_j} = E_{\nu_j} E_{\mu_i}. \tag{23}$$

Assume moreover that these roots have been labeled in such a way that

$$\langle \beta, \mu_1 \rangle = \langle \beta, \nu_1 \rangle = \langle \mu_i, \mu_{i+1} \rangle = \langle \nu_j, \nu_{j+1} \rangle = -1, \tag{24}$$

for all $i = 1, \ldots, a-2$ and for all $j = 1, \ldots, b-2$.

Theorem 4.2 ([9]) *For a simple compact root vector E_μ and E_α an arbitrary element of the quantum algebra of weight α set*

$$(adE_\mu)(E_\alpha) = E_\mu E_\alpha - q^{\langle \alpha, \mu \rangle} E_\alpha E_\mu, \tag{25}$$

where, as usual, $\langle \alpha, \mu \rangle = \frac{2(\alpha,\mu)}{(\mu,\mu)}$. Define $Z_{0,0} = E_\beta$ and

$$Z_{i,j} = (adE_{\mu_i}) \ldots (adE_{\mu_0})(adE_{\nu_j}) \ldots (adE_{\nu_0})(E_\beta), \tag{26}$$

for $i = 0, \ldots, m-1$ and $j = 0, \ldots, n-1$, where adE_{μ_0} and adE_{ν_0} are defined to be the identity operator. Let $Z_1 = Z_{i,j}$, $Z_2 = Z_{i,j+y}$, $Z_3 = Z_{i+x,j}$, and $Z_1 = Z_{i+x,j+y}$. Then

$$\begin{aligned} Z_1 Z_3 &= q Z_3 Z_1 \\ Z_1 Z_2 &= q Z_2 Z_1 \\ Z_2 Z_4 &= q Z_4 Z_2 \\ Z_3 Z_4 &= q Z_4 Z_3 \\ Z_2 Z_3 &= Z_3 Z_2 \\ Z_1 Z_4 - Z_4 Z_1 &= (q - q^{-1}) Z_2 Z_3. \end{aligned} \tag{27}$$

Proof: This follows from the Serre relations by using Lusztig's automorphisms ([13]) of the quantum group (in fact, a representation of the Hecke algebra), given in part by

$$\begin{aligned} T_\mu(E) &= -E_\mu E + q^{-1} E E_\mu, \text{ and} \\ T_\mu(F) &= -E E_\mu + q E_\mu. \end{aligned} \tag{28}$$

(This is for the simply laced case). We finally remark that there are no further relations. This follows since it is clear that the algebra generated by the Z_{ij}'s must have the same Hilbert series as the polynomial algebra ($q = 1$). □

So, what we find in this special case is the so-called quantum matrix algebra $M_q(n)$. It has been studied by many people, e.g. [4],[14],[15], but mostly from the point of view of generalizing $Gl(n,\mathbb{C})$. Thus, the *quantum determinant* $\det_q$;

$$\det_q = \sum_{\pi\in\mathcal{S}_n} (-q)^{l(\pi)} Z_{1,\pi(1)} \dots Z_{n,\pi(n)}, \tag{29}$$

is usually set to 1.

By construction we clearly have

Proposition 4.3 *Let $\tilde{A}^+$ be the algebra generated by the Z_{ij}'s and let $\tilde{A}^-$ be defined analogously. Then*

$$\mathcal{U}_q = \tilde{A}^- \, \mathcal{U}_q(\mathfrak{k}) \, \tilde{A}^+ \, . \tag{30}$$

We shall later introduce another space, A^+, which can compete with $\tilde{A}^+$ about being the most "natural".

5 Modules

In the special case of $su_q(2,2)$, we have found in [9] that

$$E = \{(z_1, z_2, z_3, z_4) \mid z_1 z_4 = z_2 z_3\}\,, \tag{31}$$

together with two planes at infinity. The automorphism σ is given by, on the continuous part,

$$\sigma(z_1, z_2, z_3, z_4) = (z_1, q\cdot z_2, q\cdot z_3, q^2\cdot z_4). \tag{32}$$

The simple modules are given as follows (this is partly well-known) : Unless q satisfies $q^m = 1$ for some positive integer m, they are 1-dimensional.

If $q^m = 1$ is a primitive root of unity, the picture is more complicated. All simple modules are still finite-dimensional and there are still some families of 1-dimensional modules. But in addition there are the following representations together with similar ones with some of the generators permuted ([9]):

Let

$$I_m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & q^2 & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & q^{m-1} \end{bmatrix}, \quad \sigma_m = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{33}$$

and

$$\tau_k = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \text{ a } k\times k \text{ matrix .} \tag{34}$$

Z_2, Z_1 invertible:

$$Z_1 \to x_1\sigma^{-1}, \qquad Z_2 \to x_2 I_m \tag{35}$$
$$Z_3 \to x_3 I_m, \qquad Z_4 \to x_4\sigma I_m + \sigma I_m^{-1} x_5 \tag{36}$$

where $x_1x_4 = x_2x_3$.

Z_2 invertible, Z_1, Z_4 nilpotent:

$$Z_4 \to x_4\tau_k, \qquad Z_2 \to x_2 I_k \tag{37}$$
$$Z_3 \to x_3 I_k, \qquad Z_1 \to \frac{x_1}{1-q^2}\tau_k^t(1-I_k^2) \tag{38}$$

where $x_1x_4 = x_2x_3$ and $k = m$ or $\frac{m}{2}$ (the last case is of course only possible if m is even). Moreover,

$$\{\text{point modules}\} \longleftrightarrow \mathcal{V}(z_1z_4 - z_2z_3) \cup \mathcal{V}(z_2z_3) \longleftrightarrow E. \tag{39}$$

$$\{\text{line modules}\} \longleftrightarrow \{\text{lines on } \mathcal{V}(z_1z_4 - z_2z_3)\} \cup (2 \text{ planes at } \infty) \cup \mathbb{P}^2. \tag{40}$$

The higher dimensional cases $su_q(n,m)$ are more complicated. The space

$$E = \pi_1(\Gamma) \tag{41}$$

is, more-or-less, a union of the spaces for $su_q(2,2)$. There are still many open problems, but some, for instance to find the center in the case of a root of unity, have recently been solved [10] (see also [11]).

For reasons of having a nice point variety E, the following alternative candidate to a quadratic algebra associated to $su_q(n,m)$ might be preferred:

Lemma 5.1 *It is possible to multiply the elements $\tilde{Z}_{i,j}$ from the right by certain elements from the algebra U_K in such a way that the resulting elements $Z_{i,j}$ satisfy: Let $Z_1 = Z_{i,j}$, $Z_2 = Z_{i,j+a}$, $Z_3 = Z_{i+b,j}$ and $Z_4 = Z_{i+b,j+a}$. Then*

$$Z_1Z_3 = q^b Z_3Z_1 \tag{42}$$
$$Z_1Z_2 = q^a Z_2Z_1 \tag{43}$$
$$Z_2Z_4 = q^b Z_4Z_2 \tag{44}$$
$$Z_3Z_4 = q^a Z_4Z_3 \tag{45}$$
$$Z_2Z_3 = q^{b-a} Z_3Z_2 \tag{46}$$
$$Z_1Z_4q^{1-a} - q^{b-1}Z_4Z_1 = (q-q^{-1})Z_2Z_3. \tag{47}$$

Lemma 5.2 *The continuous part of the point variety of the quadratic algebra corresponding to these relations is the set*

$$\widetilde{E} = \{ z \mid z \text{ is an } n \times m \text{ matrix of rank 1} \}. \tag{48}$$

Definition 5.3 *By the quadratic algebra of type AIII(n,m) we mean the algebra A^+ above, subject to the relations (42)-(47).*

Remark 5.4 *The "quantization" of the hermitian symmetric space is not just the algebra A^+ above but the whole structure involving both A^+ as well as its counter part A^-, their interrelation, their modules, and the realization of $\mathcal{U}_q(\mathfrak{g})$-modules on these algebras.*

6 Quadratic algebras related to other hermitian symmetric spaces.

We present here, without details, the quadratic algebras we have constructed for the other hermitian symmetric spaces. The construction proceeds in analogy with the $su(n,m)$ case and, basically, the tools are Lusztig's automorphisms together with lengthy computations.

6.1 Type B$_n$.

The resulting quadratic relations are:

$$\begin{aligned} W_{i,i}W_{j,j} - W_{j,j}W_{i,i} &= \frac{1-q^2}{q+q^{-1}}\, W_{i,j}^2 (i<j), \\ W_{i,i}W_{j,k} - W_{j,k}W_{i,i} &= (1-q^2)W_{i,j}W_{i,k}\ (i<j \text{ and } j<k), \\ qW_{i,j}W_{j,k} - W_{j,k}W_{i,j} &= (q^{-2}-q^2)W_{i,k}W_{j,j}\ (i<j<k), \\ W_{i,j}W_{k,l} - W_{k,l}W_{i,j} &= q^{-1}W_{i,l}W_{j,k} - qW_{j,k}W_{i,l}\ (\ i<j<k), \\ W_{i,j}W_{k,l} - W_{k,l}W_{i,j} &= (q^{-1}-q)W_{i,l}W_{k,j}\ (i,k<j<l), \\ W_{i,i}W_{i,j} &= q^{-2}W_{i,j}W_{i,i}\ (i<j), \\ W_{i,,j}W_{j,j} &= q^{-2}W_{j,j}W_{i,j}\ (i<j), \\ W_{i,j}W_{i,k} &= q^{-1}W_{i,k}W_{i,j}\ (i<j<k), \\ W_{i,j}W_{k,l} &= W_{k,l}W_{i,j}\ (i\leq j, k<i, \text{ and } j<l). \end{aligned} \tag{49}$$

6.2 The case of $O^*(2n)$.

After a simple modification, this simply becomes a sub-case of B_n.

6.3 Some results for $\mathbf{D}_n$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & Z_n & & & & & & \\
 & & & & & \overset{\nu_{n-1}}{\nearrow} & & \overset{\nu_n}{\searrow} & & & & & \\
Z_1 & \xrightarrow{\nu_1} & \cdots & \xrightarrow{\nu_{n-2}} & Z_{n-1} & & & & Z_{n+2} & \xrightarrow{\nu_{n-2}} & \cdots & \xrightarrow{\nu_1} & Z_{2n} \\
 & & & & & \overset{\nu_n}{\searrow} & & \overset{\nu_{n-1}}{\nearrow} & & & & & \\
 & & & & & & Z_{n+1} & & & & & &
\end{array}
\tag{50}
$$

The relations are:

$$
\begin{aligned}
Z_i Z_{i+r+1} &= q^{-1} Z_{i+r+1} Z_i \text{ if } r \geq 0 \text{ and } r \neq 2(n-i), \qquad (51)\\
Z_i Z_{2n+1-i} - Z_{2n+1-i} Z_i &= -q Z_{i+1} Z_{2n-i} + q^{-1} Z_{2n-i} Z_{i+1} \text{ for } 1 = 1, \ldots, n-1.
\end{aligned}
$$

In this case we have found the following invariant element is

$$
Z_1 Z_{2n} - q^{-1} Z_2 Z_{2n-1} + \cdots + (-q)^{-n+1} Z_n Z_{n+1}. \tag{52}
$$

6.4 The case of $O(2n-1, 2)$.

$$
Z_1 \xrightarrow{\nu_1} \cdots \xrightarrow{\nu_{n-2}} Z_{n-1} \xrightarrow{\nu_{n-1}} Z_n \xrightarrow{\nu_{n-1}} Z_{n+1} \xrightarrow{\nu_{n-2}} \cdots \xrightarrow{\nu_1} \tag{53}
$$

The analogous relations are:

$$
\begin{aligned}
Z_i Z_{i+r} &= q^{-2} Z_{i+r} Z_i \text{ if } r > 0 \text{ and } r \neq 2(n-i), \qquad (54)\\
Z_i Z_{2n-i} - Z_{2n-i} Z_i &= -q^2 Z_{i+1} Z_{2n-i-1} + q^{-2} Z_{2n-i-1} Z_{i+1} \text{ for } 1 = 1, \ldots, n-1.
\end{aligned}
$$

In this case the invariant element is

$$
Z_1 Z_{2n-1} + \cdots + (-1)^{i-1} (q)^{2(i-1)} Z_i Z_{2n-i} + \cdots + (-)^n \frac{q^{-2n+2}}{1+q^{-2}} Z_n^2. \tag{55}
$$

6.5 The exceptional cases E_6 and E_7.

In these cases there are basically two relations: Let us associate to each positive non-compact root γ an element W_γ as above. We order these (partially) according to height. Observe that if $\gamma_1 \neq \gamma_2$ then

$$
<\gamma_1, \gamma_2> = \frac{2(\gamma_1, \gamma_2)}{(\gamma_1, \gamma_1)} \in \{0, 1\}. \tag{56}
$$

If $(\gamma_1, \gamma_2) = 0$, define Case A to be one in which $ht(\gamma_1) < ht(\gamma_2)$ and where exists a (unique) shortest compact root μ for which $\tilde{\gamma_1} = \gamma_1 + \mu$ and $\tilde{\gamma_2} = \gamma_2 - \mu$ also are non-compact (and perpendicular) roots, and Case B to be the one in which either $ht(\gamma_1) = ht(\gamma_2)$ or, if $ht(\gamma_1) < ht(\gamma_2)$, Case A does not occur. We can then give the quadratic relations:

$$
\begin{aligned}
W_{\gamma_1}W_{\gamma_2} - W_{\gamma_2}W_{\gamma_1} &= q^{-1}W_{\gamma_2}W_{\gamma_1} - qW_{\gamma_1}W_{\gamma_2} \quad \text{(Case } A\text{)}, \\
W_{\gamma_1}W_{\gamma_2} - W_{\gamma_2}W_{\gamma_1} &= 0 \quad \text{(Case } B\text{)}. \\
W_{\gamma_1}W_{\gamma_2} &= q^{-1}W_{\gamma_2}W_{\gamma_1} \quad (\text{if } (\gamma_1,\gamma_2)\neq 0 \text{ and } ht(\gamma_1) < ht(\gamma_2)).
\end{aligned} \tag{57}
$$

7 Unitarity

The main tool when q is real is the Shapovalov determinant of the hermitian form in the quantum case as proved by de Concini and Kac ([3]) (c.f. [12]): If $\Lambda \in X$, the determinant of the contravariant hermitian form on the weight space $M_\zeta(\Lambda)^{-\eta}$ of the Verma module $M_\zeta(\Lambda)$ of highest weight Λ (and type $(+,+,\ldots,+)$) is given by

$$
det_{\Lambda,\zeta}(\eta) = \prod_{\beta\in\Delta^+}\prod_{m\in\mathbb{N}} \left([m]_{d_\beta} \frac{\zeta^{d_\beta(\Lambda+\rho-\frac{m}{2}\beta,\beta^\vee)} - \zeta^{-d_\beta(\Lambda+\rho-\frac{m}{2}\beta,\beta^\vee)}}{\zeta^{d_\beta}-\zeta^{-d_\beta}}\right)^{\mathrm{Par}(\eta-m\beta)}. \tag{58}
$$

In the case of the anti-linear involution ω_2, characterized by

$$
\omega_2(K_i) = K_i, \tag{59}
$$

it follows from the decomposition

$$
\mathcal{U}_q = A^- \cdot \mathcal{U}_q(\mathfrak{k}) \cdot A^+, \tag{60}
$$

together with the Shapovalov determinant and the Drinfeld coproduct Δ_D, where the latter guarantees unitarity of tensor products of unitary modules, that we have

Theorem 7.1 *Let q be a* real *(non-zero) parameter. The set of unitarizable highest weight modules for the usual choice of signs ϵ_i, extended to the quantum case as ω_2, is the same as for $q = 1$.*

Proposition 7.2 *If $|q| = 1$, there are only finite-dimensional unitarizable modules, and q has to be a root of unity (with respect to ω_1).*

References

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