HIGHER ORDER TENSOR PRODUCTS OF WAVE EQUATIONS

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Introduction. Let T be a unitary (irreducible) representation of a group G in a space H of holomorphic functions on a homogeneous domain $D \subseteq \mathbb{C}^n$. The problem considered here is that of describing the decomposition of $T^n = \underset{n}{\otimes} T$. Of special interest is the situation in which the elements of H are solutions to "wave-equations". This is of relevance to theoretical physics ([9],[2]). For the same reason, attention is given to the action of the symmetric group; an action which, in particular, defines the subspaces of symmetric and anti-symmetric tensors.

The method we use was developed by Martens [5], and extended in [4]. The basic idea is to consider a filtration in $H^n = \underset{n}{\otimes} H$ according to the vanishing on the diagonal of $D \times \ldots \times D$. This gives the decomposition of T^n when T is strongly supported by the forward light cone ([4], Proposition 2.5). We paraphrase this result

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in §1. In §2 we show how to further extend this idea to cover the case of a wave-equation. A detailed knowledge of the corresponding representations is needed, and for this reason we restrict our attention to $G = SU(m,m)$. For the sake of simplicity we only treat the case of a scalar ("spin zero") wave-equation.

1. Holomorphic discrete series. To fix the notation and to introduce the method, we first consider the case in which the representation is strongly supported by the solid forward light cone (see [4] and below). The holomorphic discrete series are examples of such, but not all such are discrete series.

Let $G = SU(m,m)$ and let τ be an irreducible unitary representation of the maximal compact subgroup K of G in a finite dimensional vector space V_τ . The elements of K are of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$; $(a+ib),(a-ib) \in U(m) \times U(m)$, and $\det(a+ib)(a-ib) = 1$. Let $u_1 = (a+ib)$ and $u_2 = (a-ib)$. To begin with we shall work with the unbounded realization $\mathcal{D}$ of G/K . Specifically $\mathcal{D} = \{z \in gl(m,\mathbb{C}) | \frac{z-z^*}{2i} \in C_+\}$, where C_+ is the space of strictly positive definite $m \times m$ matrices. Corresponding to τ there is a representation U_τ of G , holomorphically induced from τ , which acts on the space $\mathcal{O}(\mathcal{D}, V_\tau)$ of holomorphic functions $f: \mathcal{D} \to V_\tau$ by

1.1 $$(U_\tau(g)f)(z) = J_\tau(g^{-1},z)^{-1}f(g^{-1}z) .$$

If $g = \binom{a\ b}{c\ d} \in G$, $gz = \frac{az+b}{cz+d}$, and if $k \in K$, $J_\tau(k,i) = J_\tau(k) = \tau(k)$. U_τ is said to be supported by the solid forward light cone C^+ if it is unitary in a Hilbert space $H(\tau) \subset \mathcal{O}(\mathcal{D},V_\tau)$ and if the elements of $H(\tau)$ are Fourier-Laplace transforms of functions from C^+ to V_τ . If $H(\tau)$ is a reproducing kernel Hilbert space, it is easy to see that there exists a continuous function $F_\tau: C^+ \to \mathrm{Hom}(V_\tau)$ such that the kernel K_τ is given by

1.2 $$K_\tau(z,w) = \int_{C^+} F_\tau(k)e^{i\,\mathrm{tr}\,(z-w^*)k}dk .$$

If, for all $k \in C^+$, $F_\tau(k)$ is nonsingular, we say that U_τ is strongly supported by C^+ .

Assume that U_τ is strongly supported by C^+ . Consider

1.3 $$U_\tau^n = \bigotimes_n U_\tau = U_\tau \otimes \dots \otimes U_\tau .$$

This representation acts on the space of holomorphic functions $f(z_1,\dots,z_n)$ from $\mathcal{D} \times \dots \times \mathcal{D}$ to $\bigotimes_n V_\tau$ in the obvious way. It is unitary on the subspace $H^n(\tau) = \bigotimes_n H(\tau)$. We consider its restriction to this space. Introduce new variables

1.4 $y_1 = z_1 + \ldots + z_n$, and $y_i = z_1 - z_i$ for $i = 2,\ldots,n$.

Recall that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, then ([4])

1.5 $$(gz_1 - gz_2) = (z_2c^* + d^*)^{-1}(z_1 - z_2)(cz_1 + d)^{-1} .$$

The representation U_τ^n is the restriction of a representation of $G \times \ldots \times G$ to the diagonal subgroup G_1 $(\simeq G)$. Its decomposition is given by [4], Proposition 2.5. We rephrase this result:

Let, for $j = 0,1,2,\ldots$,

1.6 $$H_j^n(\tau) = \{f \in H^n(\tau) \mid (\partial/\partial y_2)^{\alpha_2} \ldots (\partial/\partial y_n)^{\alpha_n} f(z,\ldots,z) = 0 \text{ for all } |\alpha| = \alpha_2 + \ldots + \alpha_n \leq j\} .$$

Here, for each y_i and α_i , $(\partial/\partial y_i)^{\alpha_i}$ runs through the set of all constant coefficient differential operators of degree α_i in the m^2 entries of y_i . The spaces $H_j^n(\tau)$ form a decreasing sequence of closed, invariant subspaces of $H^n(\tau)$. To be consistent we let $H^n(\tau) = H_{-1}^n(\tau)$. The functions in $H_j^n(\tau)$ are, to the lowest order in $y_2,\ldots,y_n$, homogeneous polynomials of degree $j+1$.

<u>Definition 1.1</u>. $\mathbb{C}_\tau^{n-1}[y_2,\ldots,y_n] = \mathbb{C}_\tau^{n-1}$ denotes the set of polynomial functions $p\colon (y_2,\ldots,y_n) \to \otimes_n V_\tau$. $\mathbb{C}_\tau^{n-1}(j)$ denotes the subspace of homogeneous polynomials of degree j .

K acts on $\mathbb{C}_\tau^{n-1}$ by $J_{\tau,n}$:

1.7 $(J_{\tau,n}(k)p)(y_2,\ldots,y_n) = (\tau^n(k)p)(u_2^{-1}y_2u_1,\ldots,u_2^{-1}y_nu_1)$.

Here $\tau^n = \underset{n}{\otimes}\tau$, and $k = (u_1,u_2)$. Let $\underset{i\in I}{\oplus}\rho_i(\tau,n)$ be the decomposition of $J_{\tau,n}$ into irreducible representations (there may be multiplicities) and let $U_{\rho_i(\tau,n)}$ denote the representation of G holomorphically induced from $\rho_i(\tau,n)$.

<u>Proposition 1.2</u> ([4]). If U_τ is strongly supported by C^+ ,

1.8 $$U_\tau^n = \underset{i\in I}{\oplus} U_{\rho_i(\tau,n)} .$$

Analogously to Definition 1.1 we can define $\mathbb{C}_\tau^n(j)$ for the variables $z_1,\ldots,z_n$. K is represented on this space by an action similar to (1.7).

<u>Definition 1.3</u>. T_j^n denotes the subspace of $\mathbb{C}_\tau^n(j)$ that is annihilated by $(\partial/\partial y_1) = (\partial/\partial z_1 + \ldots + \partial/\partial z_n)$.

T_j^n is clearly a submodule which, in a natural way, is isomorphic to $\mathbb{C}_\tau^{n-1}(j)$.

If $I[y_1]$ denotes the ideal in $\mathbb{C}_\tau^n$ generated by the entries of y_1 , T_j^n is a complement in $\mathbb{C}_\tau^n(j)$ of $\mathbb{C}_\tau^n(j) \cap I[y_1]$. Thus, if $\mathbb{C}_0^1[y_1](i) = \mathbb{C}_0^1(i)$ denotes the space of $\mathbb{C}$-valued homogeneous polynomials of degree i , we have

Lemma 1.4.

1.9 $$\mathbb{C}^n_\tau(j) = \bigoplus_{i=0}^{j} \mathbb{C}^1_0(i)\cdot T^n_{j-i} \quad .$$

K acts on $\mathbb{C}^1_0$ by, for $q \in \mathbb{C}^1_0$ and $k = (u_1, u_2)$,

1.10 $$(k\cdot q)(y_1) = q(u_2^{-1} y_1 u_1) \quad .$$

It is clear that K leaves $\mathbb{C}^1_0(i)\cdot T^n_{j-i}$ invariant, and that its action on this space is given as the tensor product of (1.10) with its action on T^n_{j-i} . Equation (1.9) is then an equality between K-modules.

Let us return to the space $H^n(\tau)$. On this space the symmetric group $\mathcal{S}(n)$ acts by

1.11 $$\sigma\,(f_1 \otimes \ldots \otimes f_n) = f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)} \quad .$$

This action is the tensor product of $\mathcal{S}(n)$'s natural action on $\underset{n}{\otimes} V_\tau$ with its action on scalar valued functions on $\mathcal{B} \times \ldots \times \mathcal{B}$. The spaces $H^n_j(\tau)$ are invariant, and thus $\mathcal{S}(n)$ acts on $\mathbb{C}^{n-1}_\tau(j)$, for each j , and hence, by isomorphism, on T^n_j . $\mathcal{S}(n)$ also acts on $\mathbb{C}^n_\tau$ by (1.11), and the above action on T^n_j is in fact, as follows from these remarks, the restriction of this latter action to that subspace. Observe that the direct sum decomposition (1.9) is invariant under the action of $\mathcal{S}(n)$.

Let $\mathbb{C}^n_\tau(j)_s$ denote the subspace of $\mathbb{C}^n_\tau(j)$ that transforms under $\mathcal{S}(n)$ according to a given symmetry s, and let $(T^n_j)_s$, $(T^n)_s$, and $H^n(\tau)_s$ be defined analogously, where $T^n = \bigoplus_{j=0}^{\infty} T^n_j$. Then, if $\bigoplus_{i\in I_s} \rho_i(\tau,n)$ denotes the decomposition of K's action on $(T^n)_s$ into irreducible representations, and if $(U^n_\tau)_s$ denotes the restriction of U^n_τ to $H^n(\tau)_s$,

<u>Theorem 1.5</u>. $(U^n_\tau)_s = \bigoplus_{i\in I_s} U_{\rho_i(\tau,n)}$. The K-types of $(T^n)_s$ can be obtained from the K-types of $(\mathbb{C}^n_\tau)_s$ by the recursion formula

$$1.12 \quad (T^n_j)_s = \mathbb{C}^n_\tau(j)_s - \mathbb{C}^1_0(1)\cdot(T^n_{j-1})_s - \dots - \mathbb{C}^1_0(n)\cdot(T^n_0)_s \ .$$

2. <u>Wave-equations</u>. Let us for simplicity consider the constant coefficient differential operator $\det(\mathbb{D}) = \det(\partial/\partial z)$ which is the (formal) Fourier-Laplace transform of the function $\det k$ on $\overline{C^+}$ (cf. [3], §2). There are other "Dirac-type" matrix valued constant coefficient differential operators which can be treated analogously. Take $m \geq 2$.

Let W denote the representation of G on the space of holomorphic functions on $\mathcal{D}$ given by, for

$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$,

2.1 $$(W(g)f)(z) = \det(cz+d)^{1-m} f(g^{-1}z) \quad .$$

This representation corresponds to a discrete point in the Wallach set ([10]) beyond the analytic continuation. It is known from [7] (see also [3]) that there exists a reproducing kernel Hilbert space H of holomorphic solutions to $\det(\mathbb{D})\varphi = 0$ on $\mathcal{D}$, in which W is unitary. In fact ([7], §4.5-4.6), there exists a semi-invariant measure μ on the set $b(C^+) = \{k \in gl(m,\mathbb{C}) \mid k \geq 0 \text{ and } \det k = 0\}$ such that $f \in H$ if and only if

2.2 $$f(z) = \int_{b(C^+)} e^{i\, tr\,(zk)} \varphi(k) d\mu(k)$$

for some $\varphi \in L^2(b(C^+), d\mu)$. As before we let $\mathbb{C}_0^1 = \mathbb{C}_0^1[z]$ denote the space of polynomials in the m^2 entries of z .

Definition 2.1. $I(\det z)$ denotes the ideal in $\mathbb{C}_0^1$ generated by $\det z$.

Remark: $I(\det z)$ is clearly a prime ideal.

Let $\mathbb{E}_0^1 = \mathbb{C}_0^1[\partial/\partial z]$ denote the space of constant coefficient differential operators in the entries of z , and let $I(\det \mathbb{D})$ be defined analogously. There is a natural pairing between $\mathbb{E}_0^1$ and $\mathbb{C}_0^1$ defined as

$$2.3 \qquad \langle E_{j_1}, p_{j_2} \rangle = E_{j_1}(p_{j_2})(o)$$

if $E_{j_1} \in \mathbb{E}_0^1(j_1)$ and $p_{j_2} \in \mathbb{C}_0^1(j_2)$. When $j_1 = j_2$, (2.3) of course gives the canonical duality between $\mathbb{E}_0^1(j_1)$ and $\mathbb{C}_0^1(j_1)$.

Lemma 2.2. Let $p \in \mathbb{C}_0^1$. Then $(\det \mathbb{D})p = 0$ if and only if $\langle I(\det \mathbb{D}), p\rangle = 0$.

Proof: If $\langle I(\det \mathbb{D}), p\rangle = 0$, all derivatives of $(\det \mathbb{D})p$ vanish at 0 , and hence $(\det \mathbb{D})p = 0$. The converse is obvious.

We can expand any function in H in a power series at the point $i \in \mathcal{B}$. The degree of the leading term is then K-invariant. Suppose $p_j(z-i)f(z) \in H$, that $f(i) \neq 0$, and that p_j is a homogeneous polynomial of degree j . Since $\det \mathbb{D}(p_j(z-i)f(z)) = 0$, it follows, by comparing degrees, that $\det \mathbb{D}(p_j(z-i)) = 0$, and hence $(\det \mathbb{D})p_j = 0$. Under $W(k)$'s action on H , for $k = (u_1, u_2) \in K$, $p_j(z-i)$ is mapped into $\det u_1^{m-1} p_j(u_2^{-1}(z-i)u_1)$.

Lemma 2.3. If $p_j \in \mathbb{C}_0^1(j)$ and if $(\det \mathbb{D})p_j = 0$, there exists a holomorphic function f such that $f(i) \neq 0$, and $p_j(z-i)f(z) \in H$.

Proof: Consider a function F_φ in H of the form

2.4 $$F_\varphi(z) = \int_{b(C^+)} e^{i\,\mathrm{tr}\,(zk)} \varphi(k)d\mu(k) \quad .$$

If F_φ is to be of the form $F_\varphi(z) = p_j(z-i)f(z)$, $f(i) \neq 0$, clearly

2.5 (i) $$\int_{b(C^+)} e^{-\mathrm{tr}\,k} q_\alpha(k)\varphi(k)d\mu(k) = 0$$

for all polynomials q_α of degree $\alpha < j$,

(ii) $$\int_{b(C^+)} e^{-\mathrm{tr}\,k} q_j(k)\varphi(k)d\mu(k) \neq 0$$

for some polynomial q_j of degree j.

It is easy to see that any function of the form $\varphi(k) = e^{-\mathrm{tr}\,k}q(k)$, where $q(k)$ is a polynomial, is in $L^2(b(C^+),d\mu)$. Moreover, $\| e^{-\mathrm{tr}\,k}q(k)\|_2 = 0$ if and only if q is zero on $b(C^+)$. A straightforward orbit argument now implies that q is zero on the set of points where $\det k = 0$. But this set is the locus for the ideal $I(\det k)$, hence

2.6 $$\| e^{-\mathrm{tr}\,k}q(k)\|_2 = 0 \iff q \in I(\det k) \quad .$$

Assume that $\varphi(k) = e^{-\mathrm{tr}\,k}q(k)$, $q(k) \notin I(\det k)$. Then

2.7 $$F_\varphi(z) = \int_{b(C^+)} e^{i\,\mathrm{tr}\,(z+i)k} q(k)\,d\mu(k)$$
$$= E \int_{b(C^+)} e^{i\,\mathrm{tr}\,(z+i)k} d\mu(k)$$
$$= E \det (z+i)^{1-m} ,$$

where E , except for a transposition, is $q(\partial/\partial z)$. In particular, $E \notin I(\det \mathbb{D})$. Conversely, if $E \notin I(\det \mathbb{D})$, then $q(k) \notin I(\det k)$. The proof is completed by an easy Gram-Schmidt argument.

It is now convenient to introduce the bounded version $\mathcal{B}$ of G/K ; $\mathcal{B} = \{z \in gl(m,\mathbb{C}) |\ zz^* < 1\}$. The differential operator $\det \mathbb{D}$ can, in an obvious way, be extended from $\mathcal{B}$ to $gl(m,\mathbb{C})$, and then restricted to act on holomorphic functions on $\mathcal{B}$. The map C :

2.8 $$(Cf)(z) = (\det(i(z+1))^{1-m} f(\frac{z-1}{i(z+1)})$$

maps H unitarily onto a reproducing kernel Hilbert space H_b of holomorphic functions on $\mathcal{B}$, and it is easy to see that H_b consists of solutions to $(\det \mathbb{D})\varphi = 0$. We let $W_b = CWC^{-1}$ be the corresponding representation of G and observe that if $k = (u_1, u_2) \in K$,

2.9 $$(W_b(k)f)(z) = \det u_1{}^{m-1} f(u_2^{-1} z u_1) .$$

Corollary 2.4. f is a K-finite vector in H_b if and only if it is a polynomial and $(\det \mathbb{D})f = 0$.

Let λ' denote the representation of K in $\mathbb{E}_0^1$ which is the contragredient of the representation (2.9) of K in $\mathbb{C}_0^1$. If $k = (u_1,u_2) \in K$ define $\tilde{k} = (u_2,u_1) \in K$. λ' is then equivalent to the representation $k \to W_b(\tilde{k})$. This has the following straightforward consequence:

Corollary 2.5. The space $Q^1(\det \mathbb{D})$ of K-finite solutions to $(\det \mathbb{D})\varphi = 0$ is a complement to $I(\det z)$ in $\mathbb{C}_0^1$.

Since, for all i and j in $\mathbb{N}$, $\det z^i \cdot \mathbb{C}_0^1(j) \subset \mathbb{C}_0^1$, we may state

Corollary 2.6.

2.10 $$Q^1(\det \mathbb{D}) = \mathbb{C}_0^1 - \bigoplus_{j=0}^{\infty} \bigoplus_{i=1}^{\infty} (\det z)^i \cdot \mathbb{C}_0^1(j) .$$

For $n \geq 2$ we let $W^n = \otimes_n W_b$ and $H^n = \otimes_n H_b$. Even though the domain has changed, we can keep the notation from §1. A function in H_j^n is then of the form

2.11 $$p_{j+1}(y_2,\dots,y_n)f(z_1,\dots,z_n) ,$$

where p_{j+1} is a homogeneous polynomial of degree $j+1$.

Let us assume that f is not identically zero on the diagonal $z_1 = \ldots = z_n$. It follows that

2.12 $$(\det \mathbb{D}_{z_i})p_{j+1} = 0 \quad \text{for } i = 1,\ldots,n \ .$$

Let $Q^n(\det \mathbb{D})$ denote the subset of $\mathbb{C}^n_0$,

2.13 $$Q^n(\det \mathbb{D}) = \{p \in \mathbb{C}^n_0 \mid (\det \mathbb{D}_{z_i})p = 0 \quad \text{for } i = 1,\ldots,n\} \ ,$$

and let

2.14 $$P^n = Q^n(\det \mathbb{D}) \cap T^n \ .$$

P^n is invariant under K (2.9). Let $\bigoplus_{j \in J} \mu^n_j$ denote the decomposition of K's action on P^n into irreducible representations.

Theorem 2.7. $$W^n = \bigoplus_{j \in J} U_{\mu^n_j} \ .$$

Proof: It follows from (2.12) and §1 that W^n is contained in the right hand side. To prove the equality, we need only observe that P^n is contained in H^n (c.f. the proof of Proposition 2.5 in [4]), and this follows from Corollary 2.4.

Let $I^n(\det z)$ denote the ideal in $\mathbb{C}^n_0$ generated by the functions $\det z_1, \ldots, \det z_n$ and let, as before, $I[y_1]$ denote the ideal generated by the entries of $y_1 = z_1 + \ldots + z_n$. As a consequence of Corollary 2.5 and the above we have

Proposition 2.8. The decomposition into irreducible representations of the action of K on the complement to $I^n(\det z) + I[y_1]$ in $\mathbb{C}_0^n$ is given by $\bigoplus_{j \in J} \mu_j^n$ (as above).

The following is elementary (cf. [1], p. 22)

Lemma 2.9. $I^n(\det z)$ is prime.

Corollary 2.10. Let $q \in \mathbb{C}_0^n$, $n \geq 2$. If $q \cdot \det(z_1 + \ldots + z_n)$ is in $I^n(\det z)$, then q is in $I^n(\det z)$.

Proof: Since $I^n(\det z)$ is prime, either q or $\det(z_1 + \ldots + z_n)$ is in the ideal. If $n \geq 2$, clearly $\det(z_1 + \ldots + z_n)$ is not $(m \geq 2)$.

Lemma 2.11. For $n \geq 3$,

$$I^n(\det z) \cap I[y_1] = I^n(\det z) \cdot I[y_1] . \tag{2.15}$$

Proof: Let $f_1, \ldots, f_n \in \mathbb{C}_0^n$ and assume that $f_1 \det z_1 + \ldots + f_n \det z_n \in I[y_1]$. Introduce variables

$z_1,\ldots,z_{n-1},y_1$. Each f_i $(i = 1,\ldots,n)$ can be written as $f_i = \alpha_i + \beta_i$, where $\beta_i \in I[y_1]$ and α_i is a polynomial in $z_1,\ldots,z_{n-1}$. It follows that

2.16 $$f_1 \det z_1 + \ldots + f_{n-1} \det z_{n-1} + f_n \det z_n =$$

$$\alpha_1 \det z_1 + \ldots + \alpha_{n-1} \det z_{n-1} + \alpha_n \det (z_1+\ldots+z_{n-1}) + \alpha_n q_1 + q_2 ,$$

where $q_1 \in I[y_1]$ and $q_2 \in I^n(\det z)\cdot I[y_1]$. Thus,

$$\alpha_1 \det z_1 + \ldots + \alpha_{n-1} \det z_{n-1} + \alpha_n \det (z_1 + \ldots + z_{n-1})$$

belongs to $I[y_1]$, and hence is zero. By Corollary 2.10 α_n is in the ideal generated by $\det z_1,\ldots,\det z_{n-1}$, and hence $\alpha_n \cdot q_1 \in I^n(\det z)\cdot I[y_1]$.

$Q^n = Q^n(\det \mathbb{D})$ is a complement to $I^n(\det z)$ in $\mathbb{C}_0^n$ by Corollary 2.5. We let Q_j^n denote the subspace of Q^n of homogeneous polynomials of degree j , and let

2.17 $$P_j^n = Q^n \cap T_j^n .$$

From Lemma 2.11 we conclude

<u>Proposition 2.12</u>. For $n \geq 3$,

2.18 $$Q_j^n = \bigoplus_{i=0}^{j} \mathbb{C}_0^1(i)\cdot P_{j-1}^n .$$

This formula is based on the decomposition

$$2.19 \qquad \mathbb{C}_0^n - I^n(\det z) - I[y_1] + I^n(\det z)\cdot I[y_1] =$$

$$\mathbb{C}_0^n - I^n(\det z) - I[y_1](\mathbb{C}_0^n - I^n(\det z)) .$$

An equivalent formula can be obtained by using the decomposition

$$2.20 \qquad \mathbb{C}_0^n - I[y_1] - I^n(\det z)(\mathbb{C}_0^n - I[y_1]) .$$

The decomposition (2.18) of Q_j^n is, just as (1.9), invariant under the action of $\mathfrak{S}(n)$. With it, P_j^n can be expressed in terms of tensor products of the spaces Q_j^n and $\mathbb{C}_0^1(i)$. Moreover, for each j , Q_j^n can be obtained recursively from the spaces $\mathbb{C}_0^1(i)$, in a manner analogous to the derivations of (1.9) and (2.10).

As for the case $n = 2$, observe that

$$2.21 \qquad I^2(\det z) + I[y_1] = I(\det(z_1 - z_2)) + I[y_1]$$

and that

$$2.22 \qquad I(\det(z_1-z_2)) \cap I[y_1] = I(\det(z_1-z_2))\cdot I[y_1] .$$

Since $z_1 - z_2$ is equal to the variable y_2 we conclude:

Proposition 2.13. The space P^2 is, as a K-module, equivalent to the K-module Q^1 (Corollaries 2.5 and 2.6).

Let Q^1_e and Q^1_o denote the set of homogeneous polynomials of even and odd degrees, respectively, in Q^1, and let $\bigoplus_{i\in I_+} \gamma_i$ and $\bigoplus_{i\in I_-} \gamma_i$ denote the decomposition of K on Q^1_e and Q^1_o, respectively. Denote by H^2_+ and H^2_- the symmetric and antisymmetric subspaces of H^2_1 and define $W^2_\pm$ analogously.

Proposition 2.14. $$W^2_\pm = \bigoplus_{i\in I_\pm} U_{\gamma_i} .$$

3. Concluding remarks. The problem of decomposing tensor products of infinite-dimensional representations of $G = SU(m,m)$ has been reduced to the problem of decomposing tensor products of finite-dimensional representations of compact groups, e.g. decomposing $\mathbb{C}^n_\tau(j)$ under K. In the case where τ is a character, the decomposition of $\mathbb{C}^1_\tau$ was found by Schmid [8], for more general groups. A related study is that of Procesi [6]. In principle, then, the decomposition of $\mathbb{C}^n_\tau$, and more generally of $(\mathbb{C}^n_\tau)_s$, can be found from classical invariant theory. In practice, of course, this is quite hard, and we shall not get into this problem here.

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