TOPOLOGICAL FIELD THEORIES FROM QUANTUM 6J-SYMBOLS

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1. INTRODUCTION

We outline here a construction of a topological quantum field theory attached to a finite set of $\star$-representations of an associative $\star$-algebra satisfying the conditions indicated in Section 2 below. Details will appear elsewhere. It generalizes recent work of Turaev-Viro [6] and is related to the work of Reshetikhin-Turaev [4], [5] as well as Turaev-Wenzl [7]. Examples that fit into this framework are Chiral algebras, quantum groups at primitive roots of (-1), as well as finite groups.

2. THE BASIC STRUCTURE

$\mathfrak{A}$ is an associative $\star$–algebra (e.g. a quantum group or a chiral algebra) with a distinguished <u>finite</u> set of $\star$–representations

$$(1) \qquad I = \{\pi_i \mid i \in I\}$$

(not necessarily finite dimensional) acting on Hilbert spaces H_i and satisfying certain conditions to be specified below.

Let π and ρ be members of our family, then $V_{\pi,\rho}$ denotes the set of (bounded) intertwiners between π on H_π and ρ on H_ρ. Notice that

$$(2) \qquad u \in V_{\pi,\rho} \quad \Rightarrow \quad u^* \in V_{\rho,\pi} \, ,$$

$$(3) \qquad u \in V_{\pi,\rho}, v \in V_{\rho,\eta} \quad \Rightarrow \quad v \circ u \in V_{\pi,\eta} \, .$$

If π is irreducible then $V_{\pi,\rho}$ has a natural inner product $\langle \cdot, \cdot \rangle$ given by

$$(4) \qquad u, u' \in V_{\pi,\rho} : \ u'u^* = \langle u', u \rangle \cdot 1_\pi \, .$$

The main assumption on the algebraic structure of the representations is that there exists a "product", to be denoted $\otimes$, on the set of representations,

$$(5) \qquad (\pi \text{ on } H_\pi, \rho \text{ on } H_\rho) \rightsquigarrow \pi \otimes \rho \text{ on } H_{\pi \otimes \rho},$$

satisfying the following technical assumptions:

Talk given by Hans P. Jakobsen

- <u>Factorization.</u> Given $u \in V_{\pi,\pi'}, v \in V_{\rho,\rho'}$, there is an assignment, bilinear in (u, v), of a canonical intertwiner $u \otimes v \in V_{\pi \otimes \rho, \pi' \otimes \rho'}$ such that for $v' \in V_{\rho',\rho''}$, and $u' \in V_{\pi',\pi''}$,

(6) $$(u' \otimes v')(u \otimes v) = (u'u) \otimes (v'v) \,,$$

(7) $$1_\pi \otimes 1_\rho = 1_{\pi \otimes \rho} \,,$$

(8) $$(u \otimes v)^* = u^* \otimes v^* \,.$$

- <u>Associativity.</u> $(\pi_i \otimes \pi_j) \otimes \pi_k \equiv \pi_i \otimes (\pi_j \otimes \pi_k) \,.$
- <u>Reducibility.</u> For each pair $i, j \in I$ there is a (unique) finite set of indices $k_1, \ldots, k_{n(i,j)} \in I$ such that

(9) $$\pi_i \otimes \pi_j \equiv \bigoplus_{s=1}^{n(i,j)} \pi_{k_s} \,.$$

- <u>Neutral representation.</u> There is a distinguished index $0 \in I$ such that

(10) $$\forall i \in I : \quad \pi_0 \otimes \pi_i \equiv \pi_i \otimes \pi_0 \equiv \pi_i \,,$$

and for each $i \in I$ there is a unique $i^\vee$ such that

(11) $$\pi_0 \subseteq \pi_i \otimes \pi_{i^\vee}$$

and the multiplicity of π_0 in $\pi_i \otimes \pi_{i^\vee}$ is 1. Finally, it is assumed that

(12) $$(i^\vee)^\vee = i \,,$$

which means that

(13) $$\pi_0 \subseteq \pi_{i^\vee} \otimes \pi_i$$

with multiplicity 1.

The following quantities are of fundamental importance

Definition 2.1. $V_{ij}^k = V_{\pi_i \otimes \pi_j, \pi_k} \,.$

An element α of V_{ij}^k is thus an intertwiner

(14)
$$
\begin{array}{c}
H_i \otimes H_j \\
\downarrow \alpha \\
H_k
\end{array} \quad .
$$

We can now define the fundamental building blocks—the so-called $6j$-symbols $F_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix}$, or fusion operators, which are linear maps

(15) $$F_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} : V_{jp}^i \otimes V_{kl}^p \to V_{ql}^i \otimes V_{jk}^q$$

defined symbolically by

$$(16) \qquad \langle F_{pq} \begin{bmatrix} j & k \\ i & \ell \end{bmatrix} (\alpha \otimes \beta), (\delta \otimes \gamma) \rangle = j \otimes p \qquad q \otimes \ell$$

or simply by

$$(17) \qquad F_{pq} \begin{bmatrix} j & k \\ i & \ell \end{bmatrix} = j \otimes p \qquad q \otimes \ell \ .$$

We have written $i, i \otimes j$ etc. instead of $H_i, H_{i \otimes j}$, etc. Before starting to analyze the properties of the so constructed $6j$-symbols more closely, we state a few additional technical assumptions, listed as va)–vc) below, that turn out to be needed. For this purpose we fix once and for all for each $i \in I$ intertwiners

$$(18) \qquad \phi_i \in V^i_{0i} \ , \ _i\phi \in V^i_{i0} \ , \ \psi_i \in V^0_{ii^\vee}$$

which are *partial isometries*, i.e.

$$(19) \qquad \phi_i \phi^*_i = 1_i = {}_i\phi_i\phi^* \ ,$$

$$(20) \qquad \psi_i \psi^*_i = 1_0 \ .$$

According to iv), these intertwiners are unique up to phases, and ϕ_i and $_i\phi$ are unitary. We define

$$(21) \qquad F_i = \langle F_{00} \begin{bmatrix} i & i^\vee \\ i & i \end{bmatrix} {}_i\phi \otimes \psi_{i^\vee}, \phi_i \otimes \psi_i \rangle$$

or, diagramatically,

$$(22) \qquad F_i = i \otimes 0 \qquad 0 \otimes i \ .$$

We assume that $\phi_i, {}_i\phi$ and $\psi_i, i \in I$, can be chosen such that:

va) For each $i \in I$, $\quad F_i \neq 0$ and

$$(23) \qquad\qquad\qquad\qquad\qquad F_i = F_{i^\vee}.$$

vb) For all $i, j, k \in I$ and $\alpha \in V_{ij}^k$,

$$(24) \qquad\qquad\qquad\qquad 1_i \otimes \phi_j = {}_i\phi \otimes 1_j \, ,$$

$$(25) \qquad\qquad\qquad \phi_k \circ (1_0 \otimes \alpha) = \alpha \circ (\phi_i \otimes 1_j) \, ,$$

$$(26) \qquad\qquad\qquad {}_k\phi \circ (\alpha \otimes 1_0) = \alpha \circ (1_i \otimes {}_j\phi) \, ,$$

i.e. the following diagrams commute,

(27)

(28)

(29)

vc) For arbitrary $i, j, k \in I$ and $\alpha \in V_{ij}^{k}$, $\beta \in V_{j^{\vee}i^{\vee}}^{k^{\vee}}$ the following diagram is commutative:

$$
(30)
$$

3. Tetrahedral symmetry

A labeled triangle is a triangle with an index from I and an arrow attached to each of its edges. If we wish to change the direction of an arrow on a side having the label "i", we must at the same time switch to the label "$i^{\vee}$", and vice versa.

We wish to attach a space of intertwiners to each labeled oriented triangle. We do this as in the top of Figure 1 for the indicated orientation. For the same triangle but with the opposite orientation we associate $V_{j^{\vee}i^{\vee}}^{p^{\vee}}$. The following shows that this prescription is well-defined:

Theorem 3.1. *Let $i, j, k \in I$ be arbitrary.*

- *Given $\alpha \in V_{ij}^{k}$ there are unique intertwiners $\tilde{\alpha} \in V_{k^{\vee}i}^{j^{\vee}}$ and $\underset{\sim}{\alpha} \in V_{jk^{\vee}}^{i^{\vee}}$ rendering the following diagrams commutative*

$$
(31)
$$

- *The mappings $\alpha \to \tilde{\alpha}$ and $\alpha \to \underset{\sim}{\alpha}$ from V_{ij}^{k} to $V_{k^{\vee}i}^{j^{\vee}}$ and $V_{jk^{\vee}}^{i^{\vee}}$, respectively, are isomorphims, and the inverse of $\alpha \to \tilde{\alpha}$ on V_{ij}^{k} is the mapping $\beta \to \underset{\sim}{\beta}$ on $V_{k^{\vee}i}^{j^{\vee}}$,*
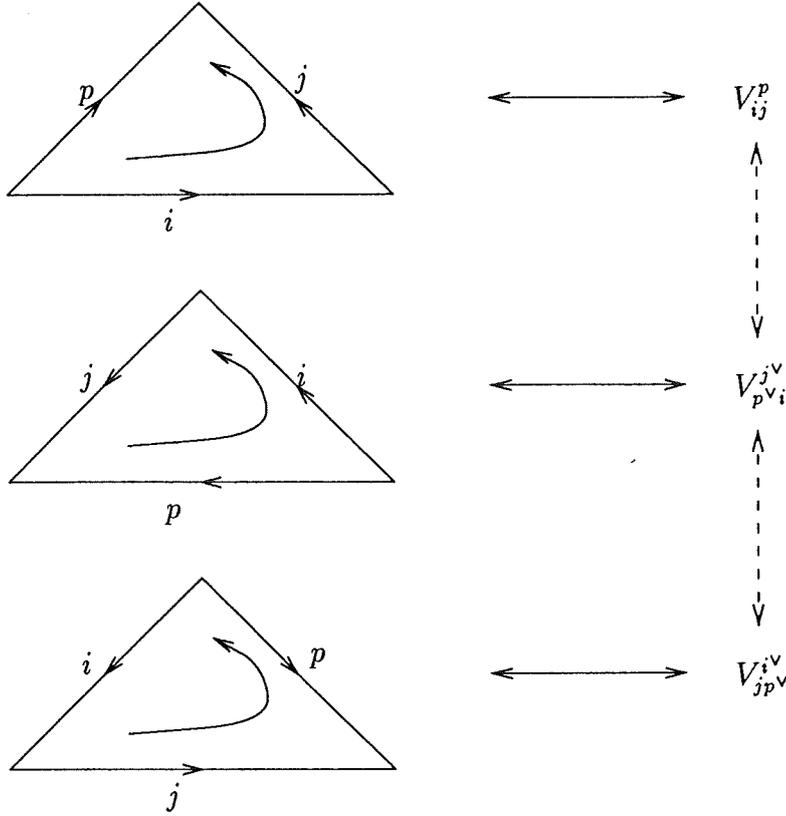
FIGURE 1

i.e.

$$(32) \qquad\qquad \underset{\sim}{(\tilde{\alpha})} = \widetilde{(\underset{\sim}{\alpha})} = \alpha , \quad \alpha \in V_{ij}^k .$$

- *The mappings $\alpha \to \tilde{\alpha}$ and $\alpha \to \underset{\sim}{\alpha}$ are unitary w.r.t. the natural inner products (4), i.e.*

$$(33) \qquad\qquad < \alpha, \beta > = < \tilde{\alpha}, \tilde{\beta} > = < \underset{\sim}{\alpha}, \underset{\sim}{\beta} > , \quad \alpha, \beta \in V_{ij}^k .$$

Having defined the spaces V_t associated with oriented labeled triangles, we are ready to define a vector space $U_{(\Sigma, \mathcal{S})}$ associated with any compact oriented two-dimensional surface Σ and any triangulation $\mathcal{S}$ of Σ, as follows. Let a labeling L of $(\Sigma, \mathcal{S})$ be given by attaching to each link ℓ in $\mathcal{S}$ an index $i_\ell \in I$ and an arrow, thus inducing a labeling of each triangle in $\mathcal{S}$. We then set

$$(34) \qquad\qquad U_{(\Sigma, \mathcal{S})}^L = \bigotimes_{t \in \mathcal{S}} V_t ,$$
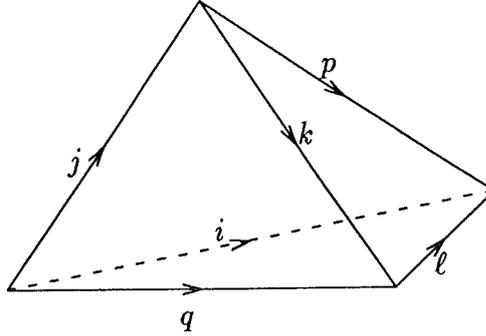
FIGURE 2

where the tensor product extends over all labeled triangles in $\mathcal{S}$ and some fixed ordering of the latter has been chosen. Furthermore, we set

$$(35) \qquad\qquad U_{(\Sigma,\mathcal{S})} = \bigoplus_{L} U^{L}_{(\Sigma,\mathcal{S})} \, ,$$

where the direct sum extends over all labelings L of $(\Sigma, \mathcal{S})$ *with a fixed configuration of arrows.* It is important to note that $U_{(\Sigma,\mathcal{S})}$ is independent of the chosen configuration of arrows on the links in $\mathcal{S}$.

We considerer more closely the particular case where $(\Sigma, \mathcal{S})$ is the triangulated boundary of an oriented tetrahedron T. In this case we set $U^{L}_{(\Sigma,\mathcal{S})} = U^{L}_{T}$ and $U_{(\Sigma,\mathcal{S})} = U_{T}$.

It follows that we may associate

$$(36) \qquad\qquad F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} \in V^{p}_{j\vee} \otimes V^{\ell}_{k\vee_p} \otimes V^{i}_{ql} \otimes V^{q}_{jk}$$

with the configuration in Figure 2. Our first major result is the following:

Theorem 3.2. *The vector*

$$(37) \qquad\qquad W(T) = F_{p} \cdot F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix}$$

has the symmetry of the Tetrahedral symmetry group

Remark 3.1. *We shall not give the details of the proof here. The most important method is that of tensoring a given diagram. eg. (17) from either the left or the right by a fixed representation.*

4. The bilinear form

The duality relations between intertwiner spaces are given by the following theorem:

Theorem 4.1. *The bilinear form* $(\cdot\,,\,\cdot) : V_{ij}^{k} \times V_{i^{\vee}k}^{j} \to \mathbb{C}$ *defined by*

(38)
$$(\alpha,\beta) = F_k \cdot i^{\vee} \otimes k$$

$$i^{\vee} \otimes i \otimes j$$

$$\alpha \nearrow \qquad \searrow$$

$$0 \otimes j$$

$$\beta \searrow \qquad \nearrow$$

$$j$$

(39)
$$= F_k \cdot \left\langle F_{k0} \begin{bmatrix} i^{\vee} & i \\ j & j \end{bmatrix} \beta \otimes \alpha\,,\; \varphi_j \otimes \psi_{i^{\vee}} \right\rangle$$

is non-degenerate and symmetric, i.e.,

(40)
$$(\alpha,\beta) = (\beta,\alpha)\,,$$

and fulfills

(41)
$$(\alpha,\beta) = (\tilde{\alpha},\beta)$$

for arbitrary $\alpha \in V_{ij}^{k}$, $\beta \in V_{i^{\vee}k}^{j}$.

It follows from this that there is a natural duality between the spaces V_t and V_{t^*} corresponding to the same labeled triangle, but with opposite orientations. Another corollary to the construction of this bilinear form is that $\tilde{\tilde{\alpha}} = \alpha$.

If $(\Sigma^*,\mathcal{S})$ denotes the same triangulated surface as $(\Sigma,\mathcal{S})$ but with opposite orientation, then, with the same ordering of triangles in $\mathcal{S}$, we get that

(42)
$$U_{(\Sigma^*,\mathcal{S})}^{L} = \bigotimes_{t\in\mathcal{S}} V_{t^*}\,,$$

and hence we may define the symmetric non-degenerate bilinear form

(43)
$$(\cdot\,,\,\cdot)_{(\Sigma,\mathcal{S})}^{L} : U_{(\Sigma,\mathcal{S})}^{L} \times U_{(\Sigma^*,\mathcal{S}^*)}^{L} \to \mathbb{C}$$

by

(44)
$$\left(\bigotimes_{t\in\mathcal{S}} \alpha_t, \bigotimes_{t\in\mathcal{S}} \beta_{t^*}\right)_{(\Sigma,\mathcal{S})}^{L} = \prod_{\ell\in\mathcal{S}} F_{i_\ell}^{-1} \prod_{t\in\mathcal{S}} (\alpha_t,\beta_{t^*})_t$$

for an arbitrary labeling L of $(\mathcal{S},\Sigma)$ and arbitrary $\alpha_t \in V_t$, $\beta_{t^*} \in V_{t^*}$, and where the product over $\ell \in \mathcal{S}$ extends over all links in $\mathcal{S}$ and i_ℓ is the index attached to the link ℓ. The factor $\Pi_{\ell\in\mathcal{S}} F_{i_\ell}^{-1}$ has been inserted for later convenience.

The direct sum over all labelings L with a fixed configuration of arrows yields a non-degenerate symmetric bilinear form

$$(45) \qquad (\cdot \, , \, \cdot)_{(\Sigma, \mathcal{S})} : U_{(\Sigma, \mathcal{S})} \times U_{(\Sigma^*, \mathcal{S}^*)} \to \mathbb{C}$$

implying the identification

$$(46) \qquad U_{(\Sigma^*, \mathcal{S}^*)} = U^*_{(\Sigma, \mathcal{S})} \, .$$

Of course, different choices of orderings of the triangles in $\mathcal{S}$ yield canonically isomorphic spaces, which we hence suppose are identified.

5. THE CONSTRUCTION

Suppose that T_1 and T_2 are labelled oriented tetrahedra such that the oriented labelled triangle t is a face of T_1 whereas t^* is a face of T_2. Then (ignoring the order in which things appear)

$$(47) \qquad U_{T_1} = V_t \otimes U_1, \quad U_{T_2} = V_{t^*} \otimes U_2 \, .$$

We may then associate the vector

$$(48) \qquad W(T_1)W(T_2) \in U_1 \otimes U_2,$$

obtained by **contracting** $W(T_1) \otimes W(T_2)$ along V_t, with the 3–manifold obtained by gluing T_1 and T_2 together along t.

More generally, let M be a 3–dimensional compact oriented manifold with boundary ∂M, let $\mathcal{T}$ be a triangulation of M whose tetrahedra we denote by $T_1, \ldots, T_N$, and let $\partial \mathcal{T}$ denote the corresponding induced triangulation of ∂M. Labelling each link l in $\mathcal{T}$ by an arrow and an element $i_l \in I$, the tetrahedra $T_1, \ldots, T_N$ may be viewed as oriented labelled tetrahedra, and we may associate with this labelled triangulation of M the vector

$$(49) \qquad \left(\prod_{n=1}^{N} W(T_n) \right)_{\mathrm{int} \, \mathcal{T}}$$

obtained by contracting the tensor $\otimes_{i=1}^{N} W(T_i)$ with respect to all the spaces V_{t_j} corresponding to the interior labelled triangles in $\mathcal{T}$. Thus,

$$(50) \qquad \left(\prod_{n=1}^{N} W(T_n) \right)_{\mathrm{int} \, \mathcal{T}} \in \bigotimes_{t \in \partial \mathcal{T}} V_t \subseteq U_{\partial M, \partial \mathcal{T}}.$$

Definition 5.1 (Fundamental).

$$(51) \qquad U_{M,\mathcal{T}} \;=\; U_{\partial M, \partial \mathcal{T}} = \bigoplus_{(i_t)_{t\in\partial\mathcal{T}}} \bigotimes_{t\in\partial\mathcal{T}} V_t \,,$$

$$(52) \quad Z(M,\mathcal{T}) \;=\; F^{-(|\mathcal{T}|-\frac{1}{2}|\partial\mathcal{T}|)} \sum_{(i_t)_{t\in\partial\mathcal{T}}} \prod_{l\in\mathcal{T}\setminus\partial\mathcal{T}} \frac{1}{F_{i_l}} \Big(\prod_{i=1}^{N} W(T_i) \Big)_{\text{int } \mathcal{T}} \in U_{\partial M, \partial \mathcal{T}} \,,$$

where the factor $F > 0$ will be defined below, and where

$(53) \qquad\qquad |\,\mathcal{T}\,| \;=\;$ # vertices in $\mathcal{T}$,

$(54) \qquad\qquad |\,\partial\mathcal{T}\,| \;=\;$ # vertices in $\partial\mathcal{T}$,

$$(55) \qquad\qquad F \;=\; \sum_{k,l\in I} \frac{F_p \cdot N_{kl}^p}{F_k \cdot F_l}, \quad \text{with } N_{kl}^p = \dim(V_{kl}^p) \,.$$

It is easy to see that F is independent of p.

6. Fundamental identities

The following identity is known as the Pentagon Identity:

$$\sum_{q\in I} F_{qm}^{(23)} \begin{bmatrix} n & j \\ r & k \end{bmatrix} F_{ir}^{(12)} \begin{bmatrix} n & q \\ u & l \end{bmatrix} F_{pq}^{(23)} \begin{bmatrix} j & k \\ i & l \end{bmatrix}$$

$$(56) \qquad = P^{23} F_{pq}^{(13)} \begin{bmatrix} m & k \\ u & l \end{bmatrix} F_{im}^{(12)} \begin{bmatrix} n & j \\ u & p \end{bmatrix}$$

as operators from $V_{ni}^u \otimes V_{jp}^i \otimes V_{kl}^p$ into $V_{rl}^u \otimes V_{mk}^r \otimes V_{nj}^m$ for all choices of indices.
The next two identities are related to the so-called Hexagon Identity:

$$(57) \qquad\qquad F_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \equiv F_{qp^\vee} \begin{bmatrix} i^\vee & j \\ l^\vee & k \end{bmatrix}^*$$

$$(58) \qquad\qquad \sum_{m\in I} F_{qm} \begin{bmatrix} j & k \\ i & l \end{bmatrix}^* F_{pm} \begin{bmatrix} j & k \\ i & l \end{bmatrix} = \delta pq \cdot 1_{V_{jp}^i \otimes V_{kl}^p}$$
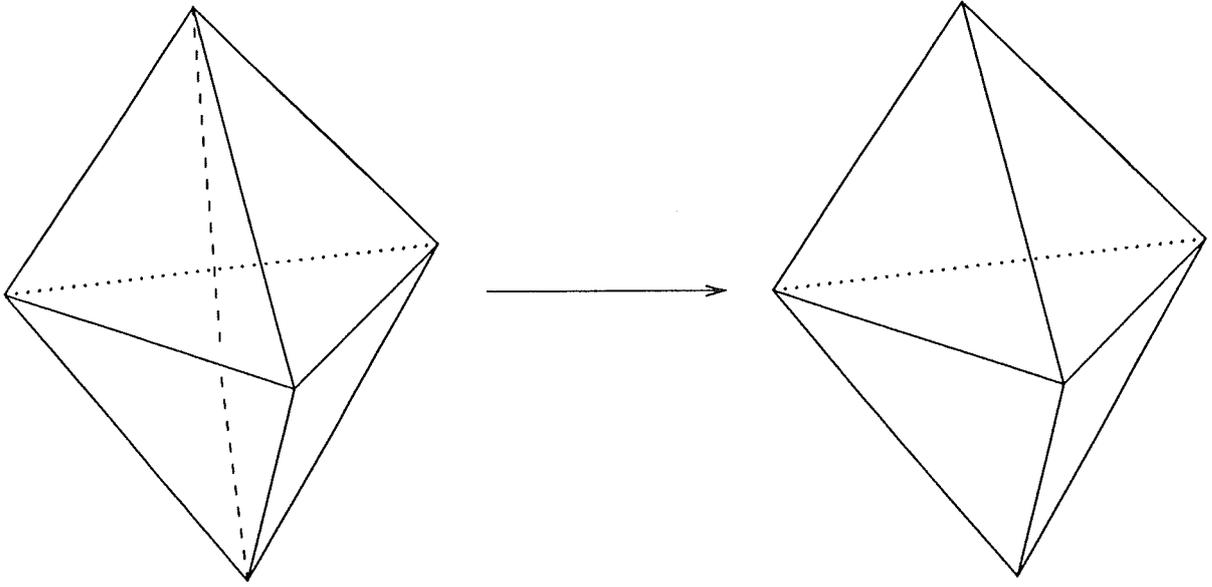
as operators on $V_{jp}^i \otimes V_{kl}^p$.

FIGURE 3

The first identity can be obtained by reexpressing

$$n \otimes j \otimes k \otimes \ell$$

(59)

$$n \otimes j \otimes p \qquad m \otimes k \otimes \ell$$

$$n \otimes i \qquad r \otimes \ell$$

$$u$$

in two different ways by inserting either $n \otimes q \otimes \ell$ or $m \otimes p$ in the center of the diagram and then decomposing by means of some natural (partial) isometries. In a similar, though easier, way, the other identities can be obtained by decomposing diagrams.

The pentagon identity may be written

$$(60) \qquad \sum_{q \in I} \frac{1}{F_q} \Big(W(T_1) W(T_2) W(T_3) \Big)_{\text{int } T_1} = \Big(W(T_1') W(T_2') \Big)_{\text{int } T_2},$$

or, diagrammatically, as in Figure. 3.

Likewise, we get two diagrammatic identities ("moves") from the other identities as indicated in Figure 4.
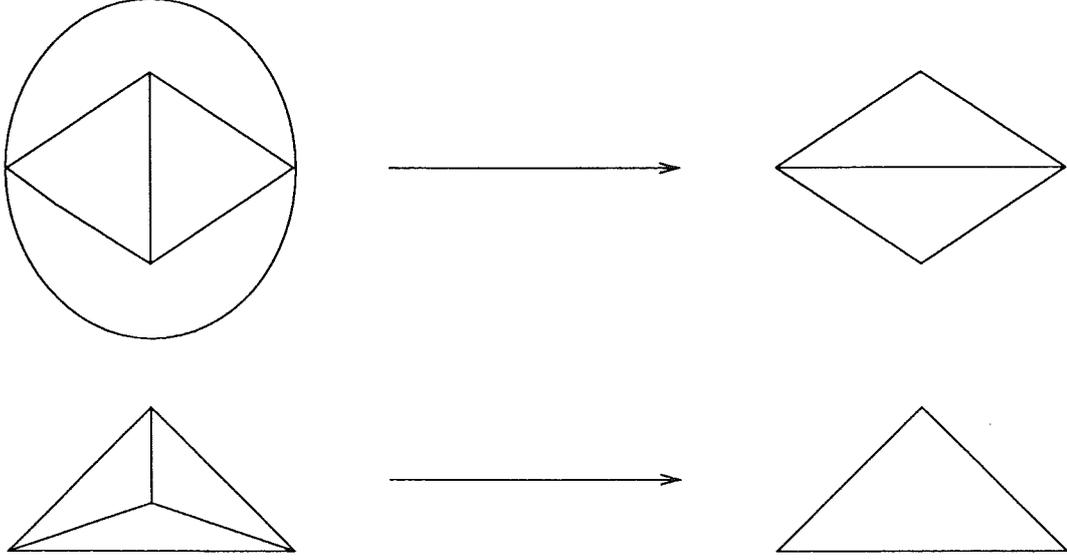
FIGURE 4. Top: Two tetrahedra sharing two triangles are collapsed into two triangles. Bottom: Two tetrahedra sharing three triangles are collapsed into one triangle.

Our main concern then is to prove that $Z(M, \mathcal{T})$ is actually independent of $\mathcal{T}$, i.e. it only depends on the homeomorphism class of M. The proof of this statement parallels that of [6] and consists of two steps. First we show that $Z(M, \mathcal{T})$ only depends on $\partial \mathcal{T}$, i.e. if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two triangulations of M that coincide on its boundary, then $Z(M, \mathcal{T}_1) = Z(M, \mathcal{T}_2)$. The second step consists in exhibiting certain subspaces $U'_{\partial M, \partial \mathcal{T}} \subseteq U_{\partial M, \partial \mathcal{T}}$ such that $Z(M, \mathcal{T}) \in U'_{\partial M, \partial \mathcal{T}}$, and canonical isomorphisms

$$(61) \qquad h_{\partial \mathcal{T}_2, \partial \mathcal{T}_1}(\partial M) : U'_{\partial M, \partial \mathcal{T}_1} \to U'_{\partial M, \partial \mathcal{T}_2}$$

for each pair of triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of M, such that

$$(62) \qquad h_{\partial \mathcal{T}_2, \partial \mathcal{T}_1}(\partial M) Z(M, \mathcal{T}_1) = Z(M, \mathcal{T}_2) .$$

This allows us to identify the spaces $U'_{\partial M, \partial \mathcal{T}}$, which we shall call $U_{\partial M}$, and the vectors $Z(M, \mathcal{T})$, which we shall call $Z(M)$.

The first step is accomplished as follows. Given any two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of a compact manifold M which coincide on ∂M, one can be deformed into the other by a sequence of steps of the types indicated in Figure 3 and in Figure 4, and such that at no step the triangulation is altered on the boundary ∂M. The easiest way to prove this result for closed manifolds is to use the classical results of J.W. Alexander [1] by showing that the barycentric subdivisions of that paper may be generated by moves as above, and their inverses. The generalization to manifolds with boundary may be found in [6].

Given any two triangulations $\mathcal{S}_1$ and $\mathcal{S}_2$ of an oriented compact closed two-dimensional surface Σ, let $M(\Sigma)$ denote the manifold $\Sigma \times [0,1]$ oriented such that Σ^* may be identified with $\Sigma \times \{0\}$ and Σ with $\Sigma \times \{1\}$. Moreover, let $\mathcal{T}(\mathcal{S}_1, \mathcal{S}_2)$ be a triangulation of $M(\Sigma)$ which coincides with $\mathcal{S}_1^*$ on $\Sigma \times \{0\}$ and with $\mathcal{S}_2$ on $\Sigma \times \{1\}$. The existence of such a $\mathcal{T}(\mathcal{S}_1, \mathcal{S}_2)$ is easily proved.

We set

$$(63) \qquad Z_{\mathcal{S}_1, \mathcal{S}_2}(\Sigma) = Z(M(\Sigma), \mathcal{T}(\mathcal{S}_1, \mathcal{S}_2))$$

which, by our previous results, is independent of the choice of $\mathcal{T}(\mathcal{S}_1, \mathcal{S}_2)$.

It follows that $Z_{\mathcal{S}_1, \mathcal{S}_2}(\Sigma) \in U_{\Sigma^*, \mathcal{S}_1} \otimes U_{\Sigma, \mathcal{S}_2}$ defines an operator

$$(64) \qquad h_{\mathcal{S}_2, \mathcal{S}_1}(\Sigma) : U_{\Sigma, \mathcal{S}_1} \to U_{\Sigma, \mathcal{S}_2},$$

and with this we achieve our objective.

7. MAIN RESULT

Let $f : \Sigma_1 \to \Sigma_2$ be a homeomorphism between oriented surfaces, let $\mathcal{S}$ be an arbitrary triangulation of Σ_1, and L a labeling of $(\Sigma_1, \mathcal{S})$. Clearly, f induces a triangulation $f(\mathcal{S})$ and a labeling $f(L)$ of Σ_2, and we may then define a linear mapping $U^L(f, \mathcal{S}) : U_{(\Sigma_1, \mathcal{S})}^L \to U_{(\Sigma_2, f(\mathcal{S}))}^{f(L)}$ by

$$(65) \qquad U^L(f, \mathcal{S})(\bigotimes_{t \in \mathcal{S}} \alpha_t) = \bigotimes_{t \in f(\mathcal{S})} \alpha_t$$

for arbitrary $\alpha_t \in V_t$, $t \in \mathcal{S}$. Taking direct sums, equation (65) defines a linear mapping

$$(66) \qquad U(f, \mathcal{S}) : U_{(\Sigma_1, \mathcal{S})} \to U_{(\Sigma_2, f(\mathcal{S}))}$$

which obviously is unitary. It is now easy to verify that $U(f, \mathcal{S})$ in turn induces a unitary operator

$$(67) \qquad U(f) : U_{\Sigma_1} \to U_{\Sigma_2} .$$

Our main result is the following:

Theorem 7.1. *Let $\mathfrak{A}$ be an algebra and $\{\pi_i \mid i \in I\}$ a finite set of irreducible representations of $\mathfrak{A}$ fulfilling the previous requirements. Then the finite dimensional Hilbert spaces $(U_\Sigma, \langle \cdot, \cdot, \cdot, \rangle_\Sigma)$, where Σ is any closed oriented compact surface, and the vectors $Z(M) \in U_{\partial M}$, where M is any compact oriented three-dimensional manifold, satisfy the following properties:*

1. For any homomorphism $f : \Sigma_1 \to \Sigma_2$ between oriented surfaces, there is an isomorphism $U(f) : U_{\Sigma_1} \to U_{\Sigma_2}$, such that

$$(68) \qquad U(f_2 \circ f_1) = U(f_2) \circ U(f_1)$$

for any pair of homomorphisms $f_1 : \Sigma_1 \rightarrow \Sigma_2$, $f_2 : \Sigma_2 \rightarrow \Sigma_3$ *between oriented surfaces.*

If $F : M_1 \rightarrow M_2$ *is a homomorphism between oriented three-manifolds and* $f : \partial M_1 \rightarrow \partial M_2$ *denotes the restriction of* F *to* ∂M_1, *then*

$$(69) \qquad U(f)Z(M_1) = Z(M_2) \, .$$

2. For any disjoint union $\Sigma_1 \cup \cdots \cup \Sigma_n$ *of oriented surfaces* $\Sigma_1, \ldots, \Sigma_n$,

$$(70) \qquad U_{\Sigma_1 \cup \cdots \cup \Sigma_n} = \bigotimes_{i=1}^{n} U_{\Sigma_i} \, .$$

3. The bilinear form $(\cdot, \cdot)_\Sigma$ *gives rise to the duality*

$$(71) \qquad U_{\Sigma^*} = U_\Sigma^* \, .$$

4. For any M,

$$(72) \qquad Z(M^*) = Z(M)^* \, ,$$

where $z \rightarrow z^*$ *denotes the natural antilinear isomorphism* $U_\Sigma \rightarrow U_\Sigma^* \equiv U_{\Sigma^*}$.

5. For any pair of oriented manifolds M_1, M_2 *that can be glued together to an oriented manifold* $M_1 \amalg_\Sigma M_2$ *along an oriented surface* Σ,

$$(73) \qquad Z(M_1 \amalg_\Sigma M_2) = (Z(M_1), Z(M_2))_\Sigma \, .$$

Remark 7.1. *According to the definitions in ref. [2], in establishing Theorem 7.1 we have constructed a topological quantum field theory.*

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