UNITARITY OF HIGHEST WEIGHT MODULES FOR QUANTUM GROUPS

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ABSTRACT. We determine the highest weights that give rise to unitarity when q is real. We further show that when q is on the unit circle and $q \neq \pm 1$, then unitary highest weight representations must be finite dimensional and q must be a root of unity. We analyze the special case of the "ladder" representations for $su(m, n)$. Finally we show how the quantized Ladder representations and their analogues for other Lie algebras play an important role.

1. INTRODUCTION

Unitary highest weight representations for simple complex Lie algebras have been important both in analysis as well as mathematical physics for a long time and for many different reasons, [14], [22], [19], [20], [16], [18], [5], [6], [7], [4], [3].

Whereas many of the early attempts were analytic, the recent ones have been more algebraic. Now, completely algebraic proofs of the classification of the full set of such representations have been given [11], [8].

With these results and methods in place it then becomes natural to extend the investigations into the realm of quantum groups, the latter being in the shape of quantized enveloping algebras. This is exactly the purpose of the present article.

On our way to this goal we have discovered some objects which may be called 'quantized hermitian symmetric spaces', or perhaps more correctly, quantum deformed hermitian symmetric spaces (see [9] and, for the case $\mathfrak{g} = su(n, m)$, [10]). These are quadratic algebras. We discuss these spaces briefly in Section 3. Besides this, the main ingredient is the quantized Shapovalov determinant due to De Concini and Kac.

We consider q's which are either real or on the unit circle. There is a big difference between these cases, the latter reduces more or less directly to the study of finite dimensional representations at roots of unity. This assertion is proved in Section 5. Sections 2 and 4 give

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definitions and background material. The case where q is real is treated in Section 6 and here the Shapovalov determinant is presented. Finally, having this determinant at our disposal, we return to the primitive roots of unity in Section 7 for a quick observation concerning unitarity in this domain.

2. Background. Quantum groups

Given an $n \times n$ Cartan matrix $A = (a_{ij})$ of finite type, choose $d_i \in \{1, 2, 3\}$ such that $(d_i a_{ij})$ is symmetric. The quantum group $\mathcal{U}_q$ is defined by the generators $E_i, F_i, K_i, K_i^{-1} (1 \le i \le n)$ and relations ('quantized Serre relations')

$$\begin{aligned} K_i K_j &= K_j K_i,\ K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad (1) \\ K_i E_j K_i^{-1} &= q^{d_i a_{ij}} E_j,\ K_i F_j K_i^{-1} = q^{-d_i a_{ij}} F_j, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, \\ \sum_{s=0}^{1-a_{ij}} (-1) \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s &= 0, \qquad i \neq j, \\ \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s &= 0, \qquad i \neq j. \end{aligned}$$

As is very well known, $\mathcal{U}_q$ is a Hopf algebra with co-multiplication $\triangle$, antipode S, and co-unit ϵ defined by

$$\begin{aligned} \triangle(E_i) &= E_i \otimes 1 + K_i \otimes E,\ \triangle(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad (2) \\ \triangle(K_i) &= K_i \otimes K_i,\ S(E_i) = -K_i^{-1} E_i,\ S(F_i) = -F_i K_i,\ S(K_i) = K_i^{-1}, \text{ and} \\ \epsilon(E_i) &= \epsilon(F_i) = 0,\ \epsilon(K_i) = 1. \end{aligned}$$

For our purposes there are two interesting families of anti-linear anti-involutions, ω_1 and ω_2, of this algebra

$$\omega_1(E_i) = \epsilon_i \cdot F_i,\ \omega_1(F_i) = \epsilon_i \cdot E_i,\ \omega_1(K_i) = K_i^{-1},\ \omega_1(q) = q^{-1}, \quad (3)$$

and

$$\omega_2(E_i) = \epsilon_i \cdot F_i,\ \omega_2(F_i) = \epsilon_i \cdot E_i,\ \omega_2(K_i) = K_i,\ \omega_2(q) = q, \quad (4)$$

where $\forall i : \epsilon_i^2 = 1$.

At the moment we consider $\mathcal{U}_q$ to be an algebra over $\mathbb{C}(q)$, the field of rational functions in q. Later on, when we 'localize', we consider subalgebras over $\mathbb{C}[q,q^{-1}]$ or even $\mathbb{Z}[q,q^{-1}]$. These will be subalgebras generated, in standard notation (see e.g. [1]), by the elements

$$\begin{aligned} E_i^{(r)}, r &\geq 0,\ i=1,\dots,n, && (5)\\ F_i^{(r)}, r &\geq 0,\ i=1,\dots,n, && (6)\\ K_i, K_i^{-1}, \left[\begin{array}{c} K_i;c \\ t \end{array}\right], i &= 1,\dots,n,\ c\in\mathbb{Z},\ t\in\mathbb{N}. && (7)\end{aligned}$$

(Actually, when we specialize q to a real value (different from -1 and 0) we can just as well consider the original generators $E_i, F_i, K_i^{\pm 1}$.)

If we replace E by $E' = EK^{-1/2}$ and F by $F' = FK^{1/2}$ then we get the physicists coproduct Δ_D (D for Drinfeld)

$$\triangle_D\, E_i' = E_i'\otimes K_i^{-1/2} + K_i^{1/2}\otimes E_i' \text{ and } \triangle_D\, F_i' = F_i'\otimes K_i^{-1/2} + K_i^{1/2}\otimes F_i'. \tag{8}$$

When q is real, this is much better for unitarity questions and hence is the one we will use there, whereas when q is on the unit circle, we will use the coproduct (2). Specifically, when q is real and if we use an involution of the form (4), then the tensor product of unitarizable modules is again unitary provided the tensor product is based on Δ_D.

When q is real there is no problem with the square roots $K_i^{\pm 1/2}$ in (8) since the weights of the K_i's either are of the form $e^{\alpha_1\cdot t}, e^{\alpha_2\cdot t},\dots$, or $-e^{\alpha_1\cdot t}, -e^{\alpha_2\cdot t},\dots$ for real numbers $\alpha_1, \alpha_2,\dots$ (depending on the definition of the highest weight representation).

3. Definitions of the modules.

Let $\mathcal{U}_q$ be a quantum group as above, defined in terms of an $n\times n$ Cartan matrix A. Assume furthermore that we are in the case of a hermitian symmetric space. Then the set of roots Δ can be written as a disjoint union

$$\Delta = \Delta_c \cup \Delta_n \tag{9}$$

where Δ_c are the compact roots and Δ_n are the non-compact roots. Specifically, if $\mathfrak{g}$ denotes the simple complex Lie algebra defined by A, then there is a decomposition, compatible with (9),

$$\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+, \tag{10}$$

where $\mathfrak{k}$ is a maximal compact subgroup with a 1 dimensional center such that Δ_c is the root system for $[\mathfrak{k},\mathfrak{k}]$, and $\mathfrak{p}^\pm$ are abelian Lie subalgebras consisting of root spaces corresponding to the roots $\Delta_n^\pm$, respectively. Moreover,

$$[\mathfrak{k},\mathfrak{p}^\pm] \subseteq \mathfrak{p}^\pm. \tag{11}$$

Finally, there is exactly one simple non-compact root $\beta \in \Delta_n^+$. We denote the corresponding elements in $\mathcal{U}_q$ by E_β, F_β, and $K_\beta^{\pm 1}$. Let $\Pi_c = \{\mu_1, \dots, \mu_{n-1}\}$ denote a choice of simple compact roots, and denote similarly the corresponding elements of $\mathcal{U}_q$ by E_{μ_i}, F_{μ_i}, and $K_{\mu_i}^{\pm 1}$, for $i = 1, \dots, n-1$. Then the elements $E_{\mu_i}, F_{\mu_i}, K_{\mu_i}^{\pm 1}$, and $K_\beta^{\pm 1}$ together generate the quantum analogue $\mathcal{U}_q(\mathfrak{k})$ of the enveloping algebra $\mathcal{U}(\mathfrak{k})$. This is a subalgebra of $\mathcal{U}_q(\mathfrak{g})$.

Let $\mathcal{U}_q^-(\mathfrak{g})$ denote the subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by the elements $F_{\mu_i}, i = 1, \dots, n-1$ and F_β. Let $\mathfrak{p}_q^-$ denote the subspace

$$\mathfrak{p}_q^- = \mathrm{Span}\left\{\mathrm{adF}_{\mu_{i_1}} \cdots \mathrm{adF}_{\mu_{i_r}}(\mathrm{F}_\beta \mathrm{K}_\beta) \mid \mathrm{r} = 0, 1, \dots \right\}, \tag{12}$$

of $\mathcal{U}_q^-(\mathfrak{g})$, where adF_{μ_i} denotes the quantum adjoint action. In an arbitrary Hopf algebra, the adjoint action is defined by the formula

$$(ad(a))(u) = \sum a^i u S(b^i) \text{ if} \tag{13}$$

$$\Delta a = \sum a^i \otimes b^i. \tag{14}$$

The K_β in (12) is inserted to give nicer formulas and is not important here. (C.f. Remark 3.2.) Let $\mathcal{A}^-$ denote the subalgebra generated by $\mathfrak{p}_q^-$. It is then clear that we have a decomposition

$$\mathcal{U}_q^-(\mathfrak{g}) = \mathcal{A}^- \mathcal{U}^-(\mathfrak{k}), \tag{15}$$

where $\mathcal{U}^-(\mathfrak{k})$ is defined in the obvious way. Indeed, we can define an analogous structure $\mathcal{A}^+$ for the positive part of the quantized enveloping algebra in such a way that we get a decomposition of $\mathcal{U}_q(\mathfrak{g})$. In fact, we have the following result (c.f. [9] and, for the case $\mathfrak{g} = su(n,m)$, [10]).

Theorem 3.1. *There exists a decomposition*

$$\mathcal{U}_q(\mathfrak{g}) = \mathcal{A}^- \cdot \mathcal{U}_q(\mathfrak{k}) \cdot \mathcal{A}^+, \tag{16}$$

where the algebras $\mathcal{A}^\pm$ are quadratic algebras. The decomposition is invariant under the adjoint action of $\mathcal{U}_q(\mathfrak{k})$.

Remark 3.2. *The action of $\mathcal{U}(\mathfrak{k})$ on an element $B \in \mathcal{A}^-$ may be obtained by demanding that the action on an element $B \otimes u$ with $u \in \mathcal{U}_q(\mathfrak{k})$ should satisfy that the left action on $Bu \in \mathcal{U}_{\text{II}}(\mathfrak{g})$ should turn into a tensor product action,*

$$F_\mu(B \otimes u) = (F^i * B) \otimes G^i u \quad \textit{if} \tag{17}$$

$$\Delta(F_\mu) = \sum F^i \otimes G^i, \tag{18}$$

with a similar formula for E_μ. Indeed, this leads just to the adjoint action. However, it leads in some sense to simpler formulas for $\mathcal{A}^-$ if we interchange E_i and F_i by e.g. the automorphism ϕ,

$$\phi(E_i) = F_i,\ \phi(F_i) = E_i,\ \phi(K_i) = K_i,\ \phi(q) = q^{-1}. \tag{19}$$

Equivalently, we may consider the coproduct

$$\Delta_\phi = (\phi \otimes \phi) \circ \Delta \circ \phi. \tag{20}$$

This leads to the formulas

$$F_\mu * B = F_\mu B - (K_\mu B K_\mu^{-1}) F_\mu, \tag{21}$$

$$E_\mu * B = (E_\mu B - B E_\mu) K_\mu. \tag{22}$$

Using this action one can define a quadratic algebra directly as generated by the $\mathcal{U}_{\text{II}}(\mathfrak{k})$ orbit through F_β. We shall, however, not pursue this point further here.

Lemma 3.3. *Let q be generic. The relations in $\mathcal{A}^-$ all come from the relations in degree 2, i.e. they are generated by a subspace V_a of $\mathcal{A}_1^- \otimes \mathcal{A}_1^-$. The subspace V_a is a $\mathfrak{k}$ module generated by one or two highest weight vectors of the quantized anti-symmetric kind*

$$Z_{-\beta} \otimes W_{-\beta-\mu} - q^p Z_{-\beta-\mu} \otimes W_{-\beta}, \tag{23}$$

where p is an appropriate power. It is only in the case where $\mathfrak{g} = su(n,m)^{\mathbb{C}}$ that there are two highest weight representations.

Proof. It follows from Lusztig's canonical bases that e.g. the Hilbert series of $\mathcal{A}^-$ at a generic q will be the same as for $q = 1$. The rest is just standard representation theory. □

We now introduce the modules to be considered.

Definition 3.4. *Let $r_1, \dots r_{n-1}$ be non-negative integers, let $\Lambda_0 = (r_1, \dots, r_{n-1})$, let $\lambda \in \mathbb{R}$, and let $\delta = (\delta_1, \dots, \delta_n)$, where each $\delta_i = \pm 1$. The Verma module $M_\Lambda^\delta(\mathfrak{g})$ of highest weight $\Lambda = (\Lambda_0, \lambda)$ and sign δ is the module generated by a non-zero vector v_Λ for which*

$$\mathcal{U}_q^+(\mathfrak{g}) v = 0 \tag{24}$$

$$K_{\mu_i} v = \delta_i q^{d_i r_i} v, \text{ and } K_\beta v = \delta_\beta q^{\lambda d_\beta}, \tag{25}$$

and where $\mathcal{U}_q^-(\mathfrak{g})$ acts freely (recall that $\mathcal{U}_q^-(\mathfrak{g})$ has no zero divisors). We denote by $M_\Lambda^\delta(\mathfrak{k})$ the $\mathcal{U}_q(\mathfrak{k})$ module defined by the same data. If $\delta = (1, \dots, 1)$ we just write M_Λ.

Actually, we will only consider the sign $\delta = (1, 1, \dots, 1)$. The reason is that in general, the signs δ_i can be counterbalanced by the signs ϵ_i on the anti-involutions.

4. Unitarity

The notion of unitarity is introduced in the following more or less standard way: Throughout we let ω be a fixed anti-linear anti-automorphism of $\mathcal{U}_q(\mathfrak{g})$. To each of the following notions we really ought to append the phrase 'with respect to ω', but we will omit this. Given $a = a(q) = \frac{\sum c_i q^i}{\sum d_j q^j} \in \mathbb{C}(q)$ let $\overline{a} = \frac{\sum \overline{c_i}\omega(q^i)}{\sum \overline{d_j}\omega(q^j)} \in \mathbb{C}(q)$.

Definition 4.1. *A sesquilinear hermitian form H on a vector space V over $\mathbb{C}(q)$ with values in $\mathbb{C}(q)$ is a map $V \times V \longrightarrow \mathbb{C}(q)$ that satisfies*

$$\begin{aligned} H(au, v) &= aH(u, v),\ H(u, av) = \overline{a}H(u, v),\ \text{and} \quad (26)\\ H(u, v) &= \overline{H(v, u)},\ \text{for } a \in \mathbb{C}(q),\ \text{and}\ u, v \in V. \quad (27)\end{aligned}$$

It is easy to see that the module M_Λ carries a unique sesquilinear form H_Λ with values in $\mathbb{C}(q)$ such that

$$H_\Lambda(v_\Lambda, v_\Lambda) = 1 \text{ and } H_\Lambda(gu, v) = H_\Lambda(u, \omega(g)v) \text{ for } g \in \mathcal{U}_q(\mathfrak{g}),\ u, v \in M_\Lambda. \quad (28)$$

More generally, so does M_Λ^δ.

Definition 4.2. *If H is a sesquilinear form on a vector space V which carries a representation π of $\mathcal{U}_q(\mathfrak{g})$ such that*

$$H(\pi(g)u, v) = H(u, \pi(\omega(g))v) \text{ for } g \in \mathcal{U}_q(\mathfrak{g}),\ u, v \in V, \quad (29)$$

we say that H is invariant with respect to *π.*

The following notion only makes sense for modules and algebras that have been evaluated/localized at a $q \in \mathbb{C}$.

Definition 4.3. *We say that π is unitarizable on V if there exists an invariant sesquilinear form H on V which is positive semi-definite.*

Returning to the module M_Λ, it is clear from the decomposition (9) (and more directly from the fact that the simple compact roots $\mu_1, \dots, \mu_{n-1}$ are the simple roots for the semi-simple part $[\mathfrak{k}, \mathfrak{k}]$ of $\mathfrak{k}$) that the kernel $N_\Lambda(\mathfrak{k})$ of the hermitian form on the $\mathcal{U}_q(\mathfrak{k})$ module also defined by Λ, will generate a subset $\mathcal{A}^- \cdot N_\Lambda(\mathfrak{k})$ of the kernel $N_\Lambda(\mathfrak{g})$ of H. Indeed, Λ defines a finite dimensional simple $\mathcal{U}_q(\mathfrak{k})$ module V_Λ and

$$M_\Lambda/(\mathcal{A}^- \cdot N_\Lambda(\mathfrak{k})) = \mathcal{A}^- \otimes V_\Lambda \quad (30)$$

as a vector space. In fact, as a $\mathcal{U}_q(\mathfrak{k})$ module. Also observe that we have the following result:

Lemma 4.4. *Let ω_2 be as in (4). Let H_1 and H_2 be invariant forms on vector spaces V_1, V_2 carrying representations π_1, π_2. Assume H_i is invariant w.r.t. π_i, $i = 1, 2$. Then $H_1 \otimes H_2$ is invariant w.r.t. $\pi_1 \otimes \pi_2$, provided that the latter is defined by means of Δ_D.*

5. The case $|q| = 1$.

Suppose that q is on the unit circle (and different from ± 1). Suppose we have a unitary highest weight representation with a highest weight vector v_Λ satisfying

$$H(v_\Lambda, v_\Lambda) = 1 \quad \text{and} \quad K_\beta v_\Lambda = q^{d_\beta \lambda} \tag{31}$$

for some real λ. Then, if $\omega(E_\beta) = -F_\beta$, the fact that $H(F_\beta^{(s)} v_\Lambda, F_\beta^{(s)} v_\Lambda) \geq 0$ implies, through (29) and the formula ([2])

$$E_\beta^{(s)} F_\beta^{(s)} = \sum_{0 \leq t \leq s} F_\beta^{(s-t)} \left[\begin{array}{c} K_\beta; 2t-2s \\ t \end{array} \right] E_\beta^{(s-t)}, \tag{32}$$

that

$$(-1)^s ([s]_{d_i}^!)^2 \left[\begin{array}{c} K_i; 0 \\ s \end{array} \right]_{d_i} \tag{33}$$

must be positive. Here,

$$\left[\begin{array}{c} K_i; c \\ t \end{array} \right]_{d_i} = \prod_{s=1}^{t} \frac{K_i q^{d_i(c-s+1)} - K_i^{-1} q^{-d_i(c-s+1)}}{q^{d_i s} - q^{-d_i s}}, \text{ and} \tag{34}$$

$$[s]_{d_i}^! = \prod_{j=1}^{s} \frac{q^{d_i j} - q^{-d_i j}}{q^{d_i} - q^{-d_i}}. \tag{35}$$

Now, if $q^{d_i} = \exp(i \cdot \theta)$ is not a root of unity,

$$(-1)^s \prod_{j=1}^{s} \sin(j\theta) \sin((\lambda + 1 - j)\theta) \tag{36}$$

must be non-negative for all s. Let $\theta_0 = -(\lambda+1)\theta$. Then in particular,

$$\forall s \in \mathbb{N}: \quad \sin(\sim\theta)\sin(\theta_{\nvdash} + \sim\theta) \geq \nvdash, \tag{37}$$

and this is impossible since, under the current assumptions, $\{e^{i \cdot s \cdot \theta}\}_{s=1}^{\infty}$ is dense in S^1.

When q is a (primitive) m^{th} root of unity one ends up with the study of the unitarity of finite dimensional representations. At the moment we shall be satisfied with giving two examples involving the so-called 'Ladder Representations' for $SU(p, q)$. Later, in Section 7, we shall prove a general result.

Let us for simplicity assume that $q = e^{i\frac{\pi}{p}}$, where p is a positive integer. First we consider a scalar representation. This is where we in the case where $q = 1$ find the 'wave equation' representation:

$$K_\beta v = e^{i\frac{\pi}{p}\lambda} \cdot v \text{ for the simple non-compact root } \beta, \tag{38}$$

$$K_\alpha v = 0 \text{ for all simple compact roots.} \tag{39}$$

Observe that the vectors $F_\beta^{(j)} \cdot v$ are highest weight vectors for $\mathcal{U}_{\rm II}(\mathfrak{k})$. Unitarity as above then implies that

$$\sin(s\theta)\sin((s-\lambda-1)\theta) > 0 \tag{40}$$

for $s = 1, \dots, s_0 - 1$ and that $\sin((s_0 - \lambda - 1)\theta) = 0$. ($s_0$ is then the number of $\mathfrak{k}$-types.) Now, in general there will of course be other '$\mathfrak{k}_q$-types', corresponding to various minors. But here we leave those out, by demanding that

$$(Z_1 Z_4 - q Z_2 Z_3)v \equiv 0. \tag{41}$$

More precisely, the element in $\mathcal{A}^-_{\in} = \mathcal{A}^-_{\infty} \cdot \mathcal{A}^-_{\infty}$ of highest weight -$\beta - \gamma_1$, where γ_1 is the smallest non-compact root which is orthogonal to β, should be in the kernel of the hermitian form. We have that $\gamma_1 = \beta + \mu + \nu$, where μ, ν are the two simple compact roots that have non-zero inner product with β. The condition is easily seen to be equivalent to

$$E_\beta(F_\beta F_\nu F_\mu F_\beta - \frac{1}{q+q^{-1}} F_\nu F_\beta F_\mu F_\beta)v = 0. \tag{42}$$

We then get that

$$\tan(\lambda\theta) = -\tan(\theta), \tag{43}$$

i.e. that $\lambda\theta = -\theta + r\pi$ for some integer r. Now observe that we have periodicity in the λ-solutions: If λ_0 is a solution, then so is $\lambda_0 + \frac{2a\pi}{\theta}$ for any integer a. We need then only consider the cases $r = 0, 1$ above. Of these, $r = 1$ leads to $\lambda = p - 1$ and hence $\sin(-\lambda\theta) < 0$. This is in conflict with the equation (43) above. The case $r = 0$ has the 'classical' solution $\lambda = -1$. With this value, it is easy to see that s_0 exists, indeed we have

$$s_0 = p \quad (= \frac{\pi}{\theta}). \tag{44}$$

Next we consider the usual 'vector case':

$$K_\beta v = e^{i\frac{\pi}{p}\lambda} v \text{ for the simple non-compact root } \beta, \tag{45}$$

$$K_\alpha v = 0 \text{ for all simple compact roots except one, } \nu, \text{ and} \tag{46}$$

$$K_\nu v = e^{i\frac{\pi}{p}n}, \text{ for one of the simple compact roots } \nu \text{ for which } a_{\nu\beta} \neq 0. \tag{47}$$

where $n \in \mathbb{N}, \lambda \in \mathbb{R}$. We now demand that a certain first order expression should vanish, namely that

$$E_\beta(F_v F_\beta + aF_\beta F_\nu)v = 0, \text{ where} \tag{48}$$

$$a = -\frac{\sin((n+1)\theta)}{\sin(n\theta)}. \tag{49}$$

The choice of a implies that this is a highest weight vector for $\mathcal{U}(\mathfrak{k})$. It follows then from (48) that

$$\sin(\lambda\theta)\sin(n\theta) = \sin((\lambda+1)\theta)\sin((n+1)\theta). \tag{50}$$

Again we have periodicity. The classical solution is $\lambda = -n-1$. Just as for the scalar case, the periodicity must in fact be in terms of 2π. So, let us assume that $\lambda = -n-1$. Let the integer u be determined such that

$$2\pi u \leq (n+1)\theta \leq 2\pi u + \pi \iff \tag{51}$$

$$2up \leq n+1 \leq (2u+1)p \tag{52}$$

(this may not always be possible). If it is, and if both inequalities in (52) are sharp then

$$s_0 = (2u+1)p - n. \tag{53}$$

The reason for this is that with the removal of the ideal generated by the $\mathfrak{k}_q$-type (41), the only ones left are those similar to the scalar case. Indeed, we can take over the conclusion from that case immediately. Furthermore observe that

$$1 < s_0 < p+1. \tag{54}$$

The case where either $2\pi u = (n+1)\theta$, or $2\pi u + \pi = (n+1)\theta$ correspond to the degenerate case where $s_0 = 1$.

Clearly, if there is no integer u such that (52) holds, then there is no unitarity. Summarizing,

Proposition 5.1. *There is unitarity at the vector representation (45) - (47) in the Ladder series if and only if there exists an integer u such that $2up \leq n+1 \leq (2u+1)p$. Corresponding to this, the possible values of λ are $-n-1+2\pi \cdot p \cdot a$, for $a \in \mathbb{Z}$.*

6. Unitarity for real t.

The Shapovalov determinant of the hermitian form in the quantum case was introduced and determined by De Concini and Kac ([2]) (c.f. Joseph [12]). Essentially, it is the determinant of the hermitian form H_Λ introduced above and computed in a fixed basis, namely the canonical basis of $\mathcal{U}_q^-(\mathfrak{g})$ introduced by Lusztig ([15]). Actually, De Concini and Kac work with bilinear forms and with a fixed involution ω, but it is clear that a change of involution by some signs ϵ_i will at most change the determinant by a sign, and switching between bilinear and sesquilinear has no significance what so ever. If $\Lambda \in X$, the determinant of the contravariant hermitian form on the weight space $M_\zeta(\Lambda)^{-\eta}$ of the Verma module $M_\zeta(\Lambda)$ of highest weight Λ (and sign $(1, 1, \dots, 1)$) is given by

$$\det_{\Lambda,\zeta}(\eta) = \prod_{\beta\in\triangle^+}\prod_{m\in\mathbb{N}}\left([m]_{d_\beta}\frac{\zeta^{d_\beta(\Lambda+\rho-\frac{m}{2}\beta,\beta^\vee)}-\zeta^{-d_\beta(\Lambda+\rho-\frac{m}{2}\beta,\beta^\vee)}}{\zeta^{d_\beta}-\zeta^{-d_\beta}}\right)^{\mathrm{Par}(\eta-m\beta)}. \tag{55}$$

Remark 6.1. *Later on it is a part of the proof of unitarity to analyze the behavior of the Shapovalov form as a function of the λ in $\Lambda = (\Lambda_0, \lambda)$ around a point λ_0 where the determinant vanishes. Specifically, we take an interest in the $N \in \mathbb{N}$ for which (locally) $\det_{\Lambda,\zeta}(\eta) = (\lambda - \lambda_0)^N \cdot F_\eta(\zeta, \lambda)$ with $F_\eta(\zeta, \lambda_0) \neq 0$. Notice that this makes sense when e.g. ζ is real and non-zero and that in this case, the degree N of vanishing does not depend on ζ. Below, we will be using a basis which is formally different since it is directly related to the decomposition (9). However, a change of basis will, naturally, not influence the degree of vanishing.*

Consider from now on the case of the anti-linear involution ω_2, characterized by

$$\omega_2(K_i) = K_i \text{ for } i = 1, \dots, n \text{ and} \tag{56}$$

$$\omega_2(F_i) = E_i \text{ for } i = 1, \dots, n-1 \text{ and } \omega_2(F_\beta) = -E_\beta. \tag{57}$$

Theorem 6.2. *Let $q = e^t$. The set of unitary representations is the same as for $q = 1$.*

Proof. First of all we consider the finite dimensional representation V_Λ of $\mathcal{U}_q(\mathfrak{k})$ defined by $\Lambda = (\Lambda_0, \lambda)$,

$$V_\Lambda = M_\Lambda(\mathfrak{k})/N_\Lambda(\mathfrak{k}). \tag{58}$$

The norm square of an element in the highest weight module defined by $q = e^t$ and (Λ_0, λ) then depends smoothly on t and λ. As far as $\mathcal{U}_q(\mathfrak{k})$ is concerned, λ is of no relevance. Furthermore, we have unitarity at $t = 0$. If the unitarity and/or the weight content of V_Λ were to change at some non-zero t, it would lead to a new zero in the Shapovalov form. But, clearly, the form does not depend on t. Hence this is impossible.

Thus, it follows that V_Λ is unitary for all t and has the same weight multiplicities (the last, of course, is very well known).

Turning now to the full $\mathcal{U}_q(\mathfrak{g})$ modules, the classical case with $t = 0$ is well known ([7]).

Let the space of λ's for which there is unitarity at $t = 0$ be given as

$$\mathcal{L}_{\text{unit}} = \{\lambda \mid \lambda \leq \lambda_{crit}\} \cup \{\lambda_r, \dots, \lambda_0\}, \tag{59}$$

where $\lambda_{crit} < \lambda_r < \ldots < \lambda_0$. We shall occasionally refer to the discrete set as the Wallach set. Observe that we do not count λ_{crit} as a member of this set. The Wallach set may thus be empty.

Consider the unitarity in terms of the two variables t and λ. If λ is below the critical value '*the first possible place of non unitarity*', λ_{crit}, then, by definition, the Shapovalov form cannot be zero. Observe that this does not depend on t. Hence, since there is unitarity at $t = 0$ there is unitarity throughout the set (which is closed by continuity)

$$\{(\lambda, t) \mid \lambda \leq \lambda_{crit}\}. \tag{60}$$

Consider now (c.f. (30) that

$$M_\Lambda = \left(\mathcal{A}^- \cdot N_\Lambda(\mathfrak{k})\right) \oplus \left(\mathcal{A}^- \otimes M_\Lambda(\mathfrak{k})/N_\Lambda(\mathfrak{k})\right). \tag{61}$$

This is a $\mathcal{U}_q(\mathfrak{k})$ invariant vector space decomposition, and the first summand is in the radical of the hermitian form. It descends, of course, to a decomposition of each weight space. When computing the determinant there will thus be factorization, with a contribution from the first summand which is independent of λ.

Let us then look at a point $\lambda > \lambda_{crit}$ which is not in the Wallach set, i.e. a point where there is no unitarity when $t = 0$. Keeping this λ fixed, if there is unitarity for some $t \neq 0$ it implies in particular that the (lowest order) polynomial (vector valued) which we know has a negative norm at $t = 0$ must vanish for some $t \neq 0$. This however leads to a zero in the Shapovalov form. Specifically, the lowest order vanishing term must define a primitive vector, hence there will be a zero in the hermitian form, hence in the Shapovalov form, which is not implied by the zeros due to $\mathfrak{k}_q$. But we know that the hermitian form on the second summand in (61) is non-degenerate at $q = 1$. Thus there are no more zeros in the form than what is implied by the first summand in (61).

Finally, when λ is in the Wallach set we have unitarity for $t = 0$. If there are some t's where there is no unitarity, it must be because some of the elements that vanish have become non-vanishing. Of course, some of the elements with positive norm might also change sign, but that can only happen if they move into the kernel of the hermitian form and take up the places of some which have left the kernel. Ultimately, it follows from the Shapovalov form that this cannot happen, but to get around this vanishing versus non-vanishing, we prove unitarity at λ_0, '*the last possible place of unitarity'*, where one easily can see what goes on. After that we use that the general point in the Wallach set can be obtained as the tensor product of such a representation with a certain number of copies of the so-called '*wave-equation representation'*.

The representation space of (Λ_0, λ_0) corresponding to the last point of unitarity is annihilated by a first order differential operator when $t = 0$. Equivalently, a first order $\mathfrak{k}$-type $V_\gamma \subset \mathcal{A}_1^- \otimes V_\Lambda$ has norm zero (is in the radical of the hermitian form). Now, if something that vanishes at 0 'stops vanishing', it must be the first order element itself since if it remains vanishing, so does the ideal it generates. But, for a fixed t, the first order element has a positive norm as λ goes to infinity and a negative norm as t goes to minus infinity. Hence, for each t there is a $\lambda(t)$ where the norm vanishes. This has got to be the one given by the Shapovalov form, i.e. the one which does not depend on t. Thus, the first order element vanishes at λ_0 for all t. As pointed out above, one has of course to make sure that the ideal $\mathcal{A}^- \cdot V_\gamma$ stays the same for all t, in particular, that it does not shrink. But that follows from Lemma 3.3. Hence, we have unitarity for all t at $\lambda = \lambda_0$.

The wave-equation representation is characterized by the fact that its $\mathfrak{k}$-types are of the form $(0, \lambda_1) - n\beta$, for $n = 0, 1, 2, 3, \dots$. $(\Lambda_0 = 0.)$ It is easy to see that it is unitary for all t (c.f. below, or the computations in Section 5 which carry over to the case where $q = e^t$). □

A more constructive proof, especially of the unitarity in the Wallach set, may be given based on the following observation:

Proposition 6.3. *In the classical case $q = 1$ there is a series of unitary representations for which the set of $\mathfrak{k}$-types either is of the form $\{n \cdot \xi \mid n = 1, 2, \dots\}$ or $\{r \cdot \xi_1 + s \cdot \xi_2 \mid (r, s) \in \mathbb{N}_{\not{0}} \times \mathbb{N}_{\not{0}} \setminus (\not{0}, \not{0})\}$ and such that any representation in the Wallach set is a tensor product of representations from this series.*

Remark 6.4. *For $=su(n, m)$ these are the so-called Ladder Representations. From now on we shall use this name also in the cases of $=sp(n, \mathbb{R})$ and $=so^*(2n)$. The possibility with two parameters only occur for the cases e_6 and e_7. In case $=so(n, 2)$ the Wallach set is empty except in the 'wave-equation' case. The usefulness of Proposition 6.3 here is that the unitarity of the series, due to the size of the $\mathfrak{k}$ spectrum, is easy to prove (c.f. below).*

Proof. [Of Proposition 6.3] For $\mathfrak{g} = su(m,n), sp(n,\mathbb{R})$ this is a consequence of the 'Kashiwara-Vergne Conjecture' ([13]), proved for $sp(n,\mathbb{R})$ in [6] and for $su(n,m)$ in [5]. In all cases the representations are characterized by a Λ_0 which is either 0 (the 'wave-equation') or is zero on all but one of the simple compact roots, and if $\Lambda_0(h_i) > 0$ for the h_i of a simple compact root α_i, then $(\beta,\alpha_i) \neq 0$. (For $sp(n,\mathbb{R})$ there is the further restriction that only $\Lambda_0(h_i) = 1$ is allowed for this root.) It is then easy to see, using the classification of unitaries in [7] (see also [8]) the claim for $so^*(2n), e_6, e_7$. As mentioned in Remark 6.4, for $so(n,2)$ it is only the representations $\Lambda = (0,\lambda)$ that have a Wallach set, and here the discrete set consists of one point, corresponding to the trivial one dimensional representation. □

Remark 6.5. *For $so(n,2)$ there are also Ladder representations. The case where n is even is richest in this respect. But we do not need them here.*

The constructive version of the proof of Theorem 6.2 then consists of establishing the unitarity in the quantum case ($q = e^t$) of the representations in Proposition 6.3. In the case of the wave-equation for an arbitrary $\mathfrak{g}$ as well as the Ladder representations for $su(m,n), sp(n,\mathbb{R})$, and $so^*(2n)$, this follows from the computation (32)-(33) since it suffices to investigate the norm of the highest weight vector in each $\mathfrak{k}$-type, and these vectors are of form $F_\beta^{(j)}v_\Lambda$, $j = 0,1,2,\dots$. In the cases of e_6, e_7 the unitarity follows because one is left with computations which are equivalent to ones in $so(8,2)$, and $so(10,2)$, respectively, and where the value of λ corresponds to positivity (a fact which is easily established using that $so(n,2)$ has essentially no Wallach set).

7. A general result for the case where q is on the unit circle

With the use of the Shapovalov determinant we can also establish the following result

Proposition 7.1. *Corresponding to the usual compact real form of a simple Lie algebra $\mathfrak{g}$, let $\Lambda = (r_1,\dots,r_n)$ be dominant and integral. The finite dimensional representation with highest weight v_Λ such that*

$$\begin{aligned} \mathcal{U}_q^+(\mathfrak{g})v_\Lambda &= 0 \qquad (62)\\ K_{\mu_i}v &= q^{d_ir_i}v_\Lambda \text{ for } i=1,\dots,n, \end{aligned}$$

is unitary for $q = e^{i\cdot\theta}$, $0 \leq \theta \leq \pi$, *provided that*

$$\forall \beta \in \Delta^+ : \frac{\pi}{\theta} \geq \langle \Lambda + \rho, \beta \rangle - 1. \tag{63}$$

Proof. Set $\frac{\pi}{\theta_0} = max_{\beta\in\Delta^+}\langle\Lambda+\rho,\beta\rangle - 1$. It is then clear from the Shapovalov determinant (this is also clear for other reasons) that the finite dimensional module defined by equation (62) has the same weights as for $q = 1$ as long as q is given as $q = e^{i\cdot\theta}$ with $0 \leq \theta \leq \theta_0$. Since we have unitarity at $q = 1$ and since a transition to non-unitarity has got to pass through a place where the kernel of the hermitian form increases, and hence the highest weight module decreases, the claim follows easily. □

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