Matrix Chain Models and their q-deformations

Hans Plesner Jakobsen

Abstract. The Lie Algebras for the open, respectively closed, Matrix Chain Models offer an interesting laboratory for the study of various non-trivial mixtures of Heisenberg, Kac-Moody, and Virasoro algebras as well as $sl(\infty)$. The quantum deformed analogue of these algebras offer a similar opportunity[1].

1. Introduction

In a number of papers, [**19**], [**20**], [**14**], [**15**], and [**16**], Rajeev and Lee, and subsequently Lee together with the present author, have studied various aspects of the open string model, also known as the Open Matrix Chain model, both from the point of view of physics and mathematics. We refer to these for additional motivation, applications to physics, and background results.

Here, we first give some additional results relating to orderings, to approximately finite, and to the non-split nature of various algebras. Moreover, a remarkable scaling property is revealed. After that we discuss various possible quantizations, and in special cases investigate relations to quantized Heisenberg, Virasoro, and Kac-Moody algebras. In this special case, for the Virasoro algebra, there is a close relation to a recent preprint by J.T. Hartwig, D. Larson, and S. Silvestrov ([**11**]), the details of which still remain to be investigated.

Much work has been done on the cohomology of conformal algebras ([**1**], [**3**], [**4**], [**5**], [**7**], [**8**], [**9**]), see in particular the remarks on p. 78–79 in "Vertex Algebras for Beginners" by V. Kac ([**17**]) where non-split extensions of the Virasoro Algebra by conformal modules are classified. Our work must evidently be related to this although our structures a priori are simpler. The analogies for the deformed algebras remain to be worked out.

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2. The open string algebra.

Consider the set $\{0, 1, \ldots, (\Lambda - 1)\}$, and let $\mathcal{W}_\Lambda$ denote the set of all finite words (of arbitrary length) made up of the integers in this set, together with the empty word $\emptyset$. For $K \in \mathcal{W}_\Lambda$, let s^K be the function which is 1 on the symbol K and 0 on all different symbols.

Set $|K| = r$ for a word $K = k_1 k_2 \ldots k_r \in \mathcal{W}$, $|\emptyset| = 0$, and define the *momentum* p of K by

$$\sum_{i=1}^{|K|=r} k_i.$$

If $K = k_1, \ldots, k_r$ then in a natural way $s^K \equiv e_{k_1} \otimes \cdots \otimes e_{k_r}$. We define:

$$\sigma^I_J s^K = \left(\sum_{K_1 K_2 K_3 = K} \delta^{K_2}_J s^{K_1 I K_3} \right). \tag{1}$$

Definition 2.1. *The* **open string algebra** Ξ_o *is the Lie algebra generated by these operators under the usual operator bracket. The above representation is* **the defining representation**, *denoted by* $\mathcal{T}_o$.

2.1. Scaling. That the Open String Algebra is very big may also be ascertained from the following fact which shows that it contains proper injective images of itself: Choose Λ *incommensurable* words $W_0, W_1, \ldots W_{\Lambda-1} \in \mathcal{W}$. Consider the map

$$\mathcal{W} \ni K = k_1 k_2 \ldots k_n \overset{\mathcal{I}}{\mapsto} W_1 W_2 \ldots W_n \in \mathcal{W}.$$

This map is clearly injective, and also proper unless the words W_0, $W_1, \ldots W_{\Lambda-1}$ are just a permutation of the letters in the alphabet. We suppress the words from $\mathcal{I}$. One simple example would be $\forall I = 0 \ldots \Lambda - 1 : W_i = ii$ (doubling). It then follows from the defining commutators of the algebra (see also (8)) that

Proposition 2.2. *The map, also denoted $\mathcal{I}$, from Ξ_o to itself given by*

$$\mathcal{I} : \sigma^I_J \mapsto \sigma^{\mathcal{I}(I)}_{\mathcal{I}(J)}$$

is a Lie algebra monomorphism. It is proper unless $\mathcal{I}$ is a permutation of the letters in the alphabet.

2.2. Orderings, Weyl decomposition, involutions, Verma-like modules, unitarity. We say about two sequences A, B that $A < B$ if either $|A| < |B|$ or, when $|A| = |B|$, if, at the first index from the left where they are different, A has the biggest index.

$X^A_B > X^C_D \Leftrightarrow AD < BC$.

$$X^A_B \text{ is } \textbf{positive} \text{ if and only if } A < B.$$
$$s^A > s^B \Leftrightarrow A < B.$$

Then X^A_B is positive iff $\forall K : X^A_B(s^K) > s^K$. We refer to this as the *standard ordering*.

More generally, one may consider a function $\mathcal{O}$,

$$\mathcal{S}_\Lambda \ni K \mapsto \mathcal{O}(K) \in \mathbb{N}_0$$

(or, $\mathcal{O}$ may even take values in some $\mathbb{N}^r$ for some r, where the target space is ordered by some total ordering.)

Assume that $\mathcal{O}$ satisfies

$$\mathcal{O}(A) > \mathcal{O}(B) \Rightarrow \tag{2}$$
$$\mathcal{O}(XA) > \mathcal{O}(XB) \text{ and } \mathcal{O}(AY) > \mathcal{O}(BY) \quad \forall X, Y$$

as well as

$$\mathcal{O}(A) = \mathcal{O}(B) \Rightarrow \tag{3}$$
$$\mathcal{O}(XA) = \mathcal{O}(XB) \text{ and } \mathcal{O}(AY) = \mathcal{O}(BY) \quad \forall X, Y.$$

We may then define an ordering by $\mathcal{O}$:

$$X^A_B \text{ is } \mathcal{O}\textbf{-positive} \text{ if and only if } \mathcal{O}(A) < \mathcal{O}(B).$$

Notice that ordinary lexicographical ordering does not fit into this scheme. For a thorough discussion of length lexicographic as well as other orderings, see ([**12**]).

We now define

$$\begin{aligned} \Xi^+_{\mathcal{O}} &= \text{Span}\{X^A_B \mid X^A_B \text{ is positive}\}, \\ \Xi^0_{\mathcal{O}} &= \text{Span}\{X^A_B \mid \mathcal{O}(A) = \mathcal{O}(B)\}, \\ \Xi^-_{\mathcal{O}} &= \text{Span}\{X^A_B \mid X^A_B \text{ is negative}\}. \end{aligned}$$

It follows from the assumptions on the ordering that Each of these space is a Lie subalgebra and $[\Xi^0_{\mathcal{O}}, \Xi^\pm_{\mathcal{O}}] \subseteq \Xi^\pm_{\mathcal{O}}$. Moreover, $[\Xi^0_{\mathcal{O}}, \Xi^0_{\mathcal{O}}] \subseteq \Xi^0_{\mathcal{O}}$. Moreover, we have a Weyl decomposition:

$$\Xi_o = \Xi^-_{\mathcal{O}} \oplus \Xi^0_{\mathcal{O}} \oplus \Xi^+_{\mathcal{O}}. \tag{4}$$

We have the following converse:

PROPOSITION 2.3. *Suppose a Weyl decomposition into three subalgebras as in (4) and with is induced from an ordering $\mathcal{O}$ of $\mathcal{W}$ as above and such that $[\Xi^0_{\mathcal{O}}, \Xi^\pm_{\mathcal{O}}] \subseteq \Xi^\pm_{\mathcal{O}}$. Then $\mathcal{O}$ satisfies (2) and (3).*

Proof. This follows from e.g. (8) below. Indeed, given K, L and one of the situations in (2) or (3). Then one may choose $A = I = J$ prudently in (8) below to cover this situation.

Remark 2.4. *For the standard ordering, we actually get* $[\Xi^0_{\mathcal{O}}, \Xi^0_{\mathcal{O}}] = 0$ *since here,* $\Xi^0_{\mathcal{O}}$ *acts diagonally in the defining representation. On the other hand, e.g.* $\Xi^0_{\mathcal{O}}$ *is not diagonalizable in the adjoint representation as is easy to see by specific computations. The general decomposition (4) includes cases of parabolic subalgebras rather than just Borel subalgebras.*

3. The limits

In the cases above with (W_p some Lie algebra)

$$0 \to sl(\infty) \to W_p \to \big(\mathbb{C}[t, t^{-1}] \otimes gl(\infty) + \mathbb{C} \cdot c\big) \to 0$$

Furthermore, to any element $w \in W_p$ it is possible to find an increasing sequence of matrices $X_n(w) \in gl(n); n = 1, 2, \dots, r, \dots$ with only finitely many non-zero entries in each column and row and such that $w - \lim_{n\to\infty} X_n(w) = \overline{X(w)}$ commutes with $sl(\infty)$ and such that

$$(\left[\overline{X(w_p)}, \overline{X(w'_p)}\right] - \overline{X([w_p, w'_p])}) \in sl(\infty).$$

This means that some completed algebra $\overline{W_p}$ is actually split over $\overline{sl(\infty)}$. Specifically, we assume that each $\overline{X(w)}$ stays within a finite distance from the diagonal. Any such algebra can be handled by the methods of ([**14**]). Recall that we call a unitary highest weight representation of first $sl(\infty)$ and then W_p, approximately finite if it originates in a unitary highest weight representation of some $gl(N)$. Then,

Proposition 3.1. *Any (irreducible) unitary highest weight representation of an algebra* W_p *in this class is the tensor product of an approximately finite unitary representation with a unitary representation arising as a representation of* $\widehat{\mathrm{L}}'(gl(\infty)) = (\mathbb{C}[t, t^{-1}] \otimes gl(\infty) + \mathbb{C} \cdot c)$.

We have the following commutation relations in $\mathfrak{g}_{0,\infty}$:

(5)
$$\begin{aligned}
[\sigma_a, \sigma_b] &= (a-b)\sigma_{a+b} \text{ if } (\operatorname{sgn} ab) = 1\\
[\sigma_a, T_b] &= -b \cdot T_{a+b} \text{ if } a+b \neq 0\\
[T_a, T_b] &= 0 \text{ if } a+b \neq 0\\
[\sigma_a, \sigma_b] &= (a-b)\sigma_{a+b} - (\operatorname{sgn} a)\min\{a^2, b^2\}T_{a+b}\\
&\quad \text{if } (\operatorname{sgn} ab) = -1 \text{ and } a+b \neq 0,\\
[\sigma_{-a}, \sigma_a] &= -2a\sigma_0 + a^2T_0 - \frac{(a-1)a(2a-1)}{6}F_0\\
&\quad - \left(\frac{1}{12}(a+1)a(a-1)Z_0\right) \text{ for } a \geq 0\\
[T_{-a}, \sigma_a] &= [\sigma_{-a}, T_a] = -aT_0 + \frac{a(a-1)}{2}F_0 \text{ for } a \geq 0
\end{aligned}$$

(6)
$$\begin{aligned}
[T_{-a}, T_a] &= -aF_0 \text{ for } a \geq 0\\
[\sigma_a, F_0] &= [T_a, F_0] = 0.
\end{aligned}$$

Observe that if for $a < 0 : \tilde{\sigma}_a := \sigma_a - aT_a$ and for $a \geq 0 : \tilde{\sigma}_a := \sigma_a$ then

$$\begin{aligned}
[\tilde{\sigma}_a, \tilde{\sigma}_b] &= (a-b)\tilde{\sigma}_{a+b} \text{ for } a+b \neq 0, \text{ and}\\
[\tilde{\sigma}_{-a}, T_a] &= -aT_0 - \frac{a(a+1)}{2}F_0 \text{ for } a \geq 0,\\
[\tilde{\sigma}_{-a}, \tilde{\sigma}_a] &= -2a\tilde{\sigma}_0 + \left(\frac{1}{6}(a+1)a(a-1)F_0\right)\\
&\quad - \left(\frac{1}{12}(a+1)a(a-1)Z_0\right) \text{ for } a \geq 0.
\end{aligned}$$

Due to the term with F_0 in the last formula above, the Virasoro is indeed sitting in a non-split position. Also observe that the term with T_0 plays the role of a central extension. We study the extensions further below where, indeed, the most general extension is classified for the q analogue. The classical result follows easily from this.

4. Quantization of the open string algebra

Our point of departure is the defining representation and three observations

$$\sigma^I_J = \sum_{E,F\in\mathcal{W}} f^{EIF}_{EJF}, \tag{7}$$

$$(\sigma^I_J)_{B,A} = \sum_{E,F\in\mathcal{W}} \delta^B_{EIF}\delta^A_{EJF} = \sum_{B=B_1B_2B_3\in\mathcal{W}} \delta^{B_2}_I \delta^A_{B_1JB_3},$$

$$\begin{aligned}\sigma^I_J\sigma^K_L &= \sum_{\substack{X,Y\in\mathcal{W}\\ X,Y\neq\emptyset}} \delta^K_{XJY}\sigma^{XIY}_L + \sum_{\substack{X,Y\in\mathcal{W}\\ Y\neq\emptyset}} \delta^{XK}_{JY}\sigma^{IY}_{XL}\\ &+ \sum_{\substack{X,Y\in\mathcal{W}\\ X\neq\emptyset}} \delta^{KY}_{XJ}\sigma^{XI}_{LY} + \sum_{X,Y\in\mathcal{W}} \delta^{XKY}_J\sigma^I_{XLY}.\end{aligned} \tag{8}$$

Observe that

$$\begin{aligned}\sum_{\substack{X,Y\in\mathcal{W}\\ X,Y\gg\emptyset}} \delta^{XK}_{JY}\sigma^{IY}_{XL} &= \sum_{\substack{W\in\mathcal{W}\\ W\gg\emptyset}} \sigma^{IWK}_{JWL}, \text{ and}\\ \sum_{\substack{X,Y\in\mathcal{W}\\ X,Y\gg\emptyset}} \delta^{KY}_{XJ}\sigma^{XI}_{LY} &= \sum_{\substack{W\in\mathcal{W}\\ W\gg\emptyset}} \sigma^{KWI}_{LWJ}.\end{aligned}$$

These formulas explain why the commutator $\left[\sigma^I_J\sigma^K_L, \sigma^K_L\sigma^I_J\right]$ is a finite sum.

We discuss briefly three possible quantizations of this algebra. All take their point of departure at (7). In the first case, which is the simplest, we simply set

$$\overrightarrow{\sigma}^I_J(q^a) = \sum_{E,F\in\mathcal{W}} q^{a\phi(EJF)} f^{EIF}_{EJF}. \tag{9}$$

Here, and throughout, ϕ denotes a function from $\mathcal{W}$ to $\mathbb{Z}$ that has the additivity property: $\forall A,B\in\mathcal{W}: \phi(AB)=\phi(A)+\phi(B)$. Simple computations then yield that the analogue of (8) becomes

$$\begin{aligned} q^{-a(\phi(K)-\phi(L))}\overrightarrow{\sigma}^I_J(q^a)\overrightarrow{\sigma}^K_L(q^b) &=\\ \Big(\sum_{\substack{X,Y\in\mathcal{W}\\ X,Y\neq\emptyset}} \delta^K_{XJY}\overrightarrow{\sigma}^{XIY}_L(q^{a+b}) &+ \sum_{\substack{X,Y\in\mathcal{W}\\ Y\neq\emptyset}} \delta^{XK}_{JY}\overrightarrow{\sigma}^{IY}_{XL}(q^{a+b})\\ +\sum_{\substack{X,Y\in\mathcal{W}\\ X\neq\emptyset}} \delta^{KY}_{XJ}\overrightarrow{\sigma}^{XI}_{LY}(q^{a+b}) &+ \sum_{X,Y\in\mathcal{W}} \delta^{XKY}_J\overrightarrow{\sigma}^I_{XLY})(q^{a+b})\Big).\end{aligned} \tag{10}$$

Thus,

$$\begin{aligned}\left[\overrightarrow{\sigma}^I_J(q^a), \overrightarrow{\sigma}^K_L(q^b)\right]_{q,1} &=\\ q^{-a(\phi(K)-\phi(L))}\overrightarrow{\sigma}^I_J(q^a)\overrightarrow{\sigma}^K_L(q^b) &- q^{-b(\phi(I)-\phi(J))}\overrightarrow{\sigma}^K_L(q^b)\overrightarrow{\sigma}^I_J(q^a)\end{aligned} \tag{11}$$

gives a finite sum of operators $\overrightarrow{\sigma}^X_Y(q^a)$. Also notice that the set of operators $\overrightarrow{\sigma}^X_Y(q^a)$ for which $a=\phi(X)-\phi(Y)$ form a subalgebra.

The second quantization is on the face of it a little more dubious. Here, we namely set

$$\overleftarrow{\sigma}^{I}_{J}(q^a) = \sum_{E,F\in\mathcal{W}} q^{a\phi(E)} f^{EIF}_{EJF}. \tag{12}$$

and get the following analogue of (8):

$$\begin{aligned}\overleftarrow{\sigma}^{I}_{J}(q^a)\overleftarrow{\sigma}^{K}_{L}(q^b) &= \\ \sum_{\substack{X,Y\in\mathcal{W}\\X,Y\neq\emptyset}} q^{a\phi(X)}\delta^{K}_{XJY}\overleftarrow{\sigma}^{XIY}_{L}(q^{a+b}) &+ \sum_{\substack{X,Y\in\mathcal{W}\\Y\neq\emptyset}} q^{b\phi(X)}\delta^{XK}_{JY}\overleftarrow{\sigma}^{IY}_{XL}(q^{a+b}) \\ + \sum_{\substack{X,Y\in\mathcal{W}\\X\neq\emptyset}} q^{a\phi(X)}\delta^{KY}_{XJ}\overleftarrow{\sigma}^{XI}_{LY}(q^{a+b}) &+ \sum_{X,Y\in\mathcal{W}} \delta^{XKY}_{J} q^{b\phi(X)}\overleftarrow{\sigma}^{I}_{XLY})(q^{a+b}).\end{aligned}$$

Now we have:

$$\begin{aligned}\sum_{\substack{X,Y\in\mathcal{W}\\X,Y\gg\emptyset}} q^{b(\phi(X))}\delta^{XK}_{JY}\overleftarrow{\sigma}^{IY}_{XL}(q^{a+b}) &= \sum_{\substack{W\in\mathcal{W}\\W\gg\emptyset}} q^{b(\phi(J))}q^{b(\phi(W))}\overleftarrow{\sigma}^{IWK}_{JWL}, \\ &\qquad\text{and} \\ \sum_{\substack{X,Y\in\mathcal{W}\\X,Y\gg\emptyset}} q^{a(\phi(X)})\delta^{KY}_{XJ}\overleftarrow{\sigma}^{XI}_{LY}(q^{a+b}) &= \sum_{\substack{W\in\mathcal{W}\\W\gg\emptyset}} q^{a(\phi(K)})q^{a(\phi(W)})\overleftarrow{\sigma}^{KWI}_{LWJ}.\end{aligned}$$

This means that if we want to define a "quantized Lie bracket" $[\cdot,\cdot]_{2,q}$ in analogy to (10), we have got to multiply the second term in (13) by $q^{-b\phi(J))}$ and the third term in (13) by $q^{-a\phi(K))}$. After that we subtract the analogous expression with the interchanges $a \leftrightarrow b, I \leftrightarrow K$, and $J \leftrightarrow L$.

Finally, there is a third quantization, where we set $\{z\}_q = \frac{1-q^z}{1-q}$ and define

$$\sigma^{I}_{J}(q^a) = \sum_{E,F\in\mathcal{W}} \{\phi(EJF)\}_{q^a} f^{EIF}_{EJF}. \tag{13}$$

In some sense, this is the one which seems most in spirit with quantum combinatorics, but which also is farthest from the usual quantizations. Still, it is well behaved in many ways. We study it below in a simple case.

Here we get the following analogue of (8):

$$\begin{aligned}
&\overleftrightarrow{\sigma}^{I}_{J}(q^a)\overleftrightarrow{\sigma}^{K}_{L}(q^b)\\
&\sum_{\substack{X,Y\in\mathcal{W}\\ X,Y\neq\emptyset}}\delta^{K}_{XJY}(q^{a\phi(X+J+Y)}\overleftrightarrow{\sigma}^{XIY}_{L}(q^a)+q^{b\phi(L)}\overleftrightarrow{\sigma}^{XIY}_{L}(q^b))\\
&+\sum_{\substack{X,Y\in\mathcal{W}\\ X,Y\neq\emptyset}}\delta^{K}_{XJY}(1-q^{a\phi(X+J+Y)}-q^{b\phi(L)}\overleftrightarrow{\sigma}^{XIY}_{L}(q^0))+\\
&+\sum_{\substack{X,Y\in\mathcal{W}\\ Y\neq\emptyset}}q^{b\phi(X)}\delta^{XK}_{JY}\overleftrightarrow{\sigma}^{IY}_{XL}(q^{a+b})+\sum_{\substack{X,Y\in\mathcal{W}\\ X\neq\emptyset}}q^{a\phi(X)}\delta^{KY}_{XJ}\overleftrightarrow{\sigma}^{XI}_{LY}(q^{a+b})\\
&+\sum_{X,Y\in\mathcal{W}}\delta^{XKY}_{J}q^{b\phi(X)}\overleftrightarrow{\sigma}^{I}_{XLY})(q^{a+b})
\end{aligned}$$

+ terms with equal the corresponding terms for $\overleftrightarrow{\sigma}^{K}_{L}(q^b)\overleftrightarrow{\sigma}^{I}_{J}(q^a)$.

4.1. Left and right operators. Recall that in the classical model we have the right and left operators in the space spanned by the operators σ^X_Y. To save time, we give their quantum analogues, where we use the quantization (9). The classical limit is obvious.

$$\begin{aligned}
\ell^I_J(q^a) &= \sum_{F\in\mathcal{W}} q^{a\phi(JF)} f^{IF}_{JF}\\
r^I_J(q^a) &= \sum_{E\in\mathcal{W}} q^{a\phi(EJ)} f^{EI}_{EJ}\\
f^I_J(q^a) &= q^{a\phi(J)} f^I_J.
\end{aligned}\tag{14}$$

These operators, as do the f^I_J's actually span ideals. We also have the equations

$$\begin{aligned}
\sum_{i=0}^{\Lambda-1}\sigma^{Ii}_{Ji}(q^a) &= \sigma^I_J(q^a)-r^I_J(q^a),\\
\sum_{i=0}^{\Lambda-1}\sigma^{iI}_{iJ}(q^a) &= \sigma^I_J(q^a)-\ell^I_J(q^a),\\
\sum_{i=0}^{\Lambda-1}\ell^{Ii}_{Ji}(q^a) &= \ell^I_J(q^a)-f^I_J(q^a),\ \text{and}\\
\sum_{i=0}^{\Lambda-1}r^{iI}_{iJ}(q^a) &= r^I_J(q^a)-f^I_J(q^a)
\end{aligned}\tag{15}$$

If one uses the second mentioned quantization, the operators ℓ^I_J in the second equation of (15) become replaced by their classical limits and, subsequently, so does the third equation. The rest remain formally unchanged.

Returning to (14) we have

$$\begin{aligned}\ell^I_J(q^a)\ell^K_L(q^b) &= \sum_{W\in\mathcal{W}} \delta^{JW}_K \ell^{IW}_L(q^{a+b})q^{a(\phi(K)-\phi(L))}\\ &+ \sum_{\emptyset\neq W\in\mathcal{W}} \delta^J_{KW}\ell^I_{LW}(q^{a+b})q^{a(\phi(K)-\phi(L))}.\end{aligned}$$

These sums are clearly finite. Again the operators $\ell^I_J(q^a)$ where $\phi(I)-\phi(J)\equiv a$ form a subalgebra.

To cut down on the complexity we may, as in the classical case, restrict ourselves to the algebras of finite momentum. Also, we will just look at the left operators. Roughly speaking, *The momentum-P algebra* $\mathcal{B}_{\leq P}$ is the algebra spanned by all f^I_J, l^I_J, r^I_J, and σ^I_J such that

$$p(I)=p(J)\leq P \text{ or } p(I)=p(J)\leq P.$$

It contains the algebra $\mathcal{A}_P$ defined to be the operators of exact momentum P.

The general structure is as follows:

$$0\to\mathcal{A}_p\to\mathcal{B}_{\leq p}\to\mathcal{B}_{\leq p-1}\to 0.$$

Let us finish by a look at the case of the operators ℓ^I_J at momentum $P=1$: Specifically, set $I=0^i10^t$ and $J=0^j1$. Assume that $i=j$. We have

$$\ell^I_J(q^a)\ell^J_I(q^b)-\ell^J_I(q^b)\ell^I_J(q^a)=\ell^I_I(q^{a+b})q^{-r}-\ell^J_J(q^{a+b})q^r.$$

If we assume that $f^{0^i10^r}_{0^i10^r}\equiv q^{0^i10^r}f^1_1$ then it follows from (15) that *modulo momentum* ≥ 2,

$$\begin{aligned}\ell^{0^i10^r}_{0^i10^r} &= \ell^{0^i10^{r-1}}_{0^i10^{r-1}}-f^1_1q^{i+r}\\ &= \ell^{0^i1}_{0^i1}-f^1_1q^{i+1}\frac{1-q^r}{1-q}.\end{aligned}$$

This equation should be compared to the "classical" ($q=1$) situation, where one sees a Kac-Moody algebra at this place, the r corresponding to the r in $\otimes t^r$ - in standard notation.

5. q-Virasoro

We have

$$[a]_q = (1-q^{2a}) \tag{16}$$
$$[V_a,V_b]_q = ([b]_q-[a]_q)V_{a+b} \tag{17}$$
$$[V_a,T_b]_q = [b]_qT_{a+b} \tag{18}$$
$$[T_a,T_b]_q = 0 \tag{19}$$
$$\left(1+q^{2c}\right)\cdot\left[[V_a,V_b]_q,V_c\right]_q + \text{cyclic}=0 \tag{20}$$
$$\left(1+q^{2c}\right)\left[[V_a,T_b]_q,V_c\right]_q + \text{cyclic}=0 \tag{21}$$
$$\left(1+q^{2c}\right)\left[[T_a,T_b]_q,V_c\right]_q + \text{cyclic}=0. \tag{22}$$

We now introduce a new bracket $\tilde{[\cdot,\cdot]}_q$ corresponding to a central extension. We assume that the new bracket satisfies the same q-Jacobi identities as does the old. We assume that

$$\begin{aligned}\tilde{[V_a,V_b]}_q &= [V_a,V_b]_q+g_b\delta_{a+b,0}\\ \tilde{[V_a,T_b]}_q &= [V_a,T_b]_q+h_b\delta_{a+b,0}\\ \tilde{[T_a,T_b]}_q &= [T_a,T_b]_q+k_b\delta_{a+b,0}.\end{aligned}$$

Working from the bottom, it follows from the analogue of (22) that

$$(1+q^{2b})[a]_qk_b-(1+q^{2a})[b]_qk_a=0, \text{ hence}$$

PROPOSITION 5.1.

$$k_a=\frac{[a]_q}{(1+q^{2a})}F_0$$

for some constant (central element) F_0.

REMARK 5.2. *The computation of k_a rests on equation (22). But the equation actually just says $0+$ cyclic $=0$. Thus, the factor $(1+q^{-2c})$ might be replaced by any nonzero function of c, and e.g. the following extension given by $\tilde{k}_a$ might be considered:*

$$\tilde{k}_a=(q^q-q^{-a})\tilde{F}_0.$$

Now consider the analogue of (21) for the new bracket:

$$\begin{aligned}(1+q^{2c})[b]_q(-h_{a+b}) &+ (1+q^{2b})([a]_q-[c]_q)(h_b)\\ +(1+q^{2a})[b]_q(h_{b+c}) &= 0.\end{aligned}$$

It follows that $h_0=0$. By making sign changes and observing that $(1+q^{2a+2c})([-a]_q-[-c]_q)=(1+q^{-2a-2c})([c]_q-[a]_q)$ it follows that

$$\begin{aligned}(1+q^{-2c})[a+c]_q(-h_c) &+ (1+q^{-2a-2c})([c]_q-[a]_q)(h_{a+c})\\ +(1+q^{-2a})[a+c]_q(h_a) &= 0.\end{aligned}$$

We set $h_a = \frac{[a]_q}{(1+q^{-2a})} H_a$ and obtain, for $a+c \neq 0$,

$$([a]_q - [c]_q)H_{a+c} - [a]_q H_a = -[c]_q H_c.$$

It follows that

$$\begin{aligned} H_{a+1} &= \frac{1}{(q^2 - q^{2a})}([a]_q H_a - [1]_q H_1) \qquad (a \neq 1) \\ &= \frac{q^{-2(r+1)}[a]_q}{[a-(r+1)]_q} H_{a-r} + \frac{[-(r+1)]_q}{[a-(r+1)]_q} H_1 \qquad (a \neq r+1). \end{aligned}$$

It follows that

$$H_a = \alpha + \beta[-a-1]_q$$

for arbitrary constants α, β, and hence

Proposition 5.3.

$$h_a = (F_1 + [-a-1]_q F_2)\frac{[a]_q}{(1+q^{-2a})}$$

for central elements F_1, F_2.

Finally, the analogue of (20) becomes

$$\begin{aligned} (1+q^{2c})([b]_q - [a]_q)g_c &+ (1+q^{2a})([c]_q - [b]_q)g_a \\ +(1+q^{2b})([a]_q - [c]_q)g_b &= 0. \end{aligned}$$

It is easy to see that $g_c = \alpha$ is a solution for any constant α, but we assume that $g_{-c} = -g_c$ so $\alpha = 0$. Equally easy it may be seen that $\frac{[c]_q}{1+q^{2c}}$ is a solution. In our quest for the most general solution we will from now on assume that $g_1 = 0$. We set $G_c = (1+q^{2c})g_c$. Then

(23) $$([b]_q - [a]_q)G_c + ([c]_q - [b]_q)G_a + ([a]_q - [c]_q)G_b = 0.$$

It follows easily that

$$G_{a+1} = q^{-2}\frac{[a+2]_q}{[a-1]_q} G_a$$

and then, that

$$\forall b : G_b = q^{-2b}[b+1]_q[b]_q[b-1]_q\gamma$$

for some constant γ. Thus, (see also ([**11**])

Proposition 5.4.

$$g_b = \frac{q^{-2b}[b+1]_q[b]_q[b-1]_q}{1+q^{2b}} F_4.$$

6. General setup

Let $\mathcal{P}$ be a (finite-dimensional) algebra with basis p_i and suppose given a representation by automorphisms $O_a, a \in \mathbb{Z}$ of $\mathbb{Z}$. This could e.g. be the space of (tails of) functions $p : \mathbb{N} \mapsto \mathbb{Z}[q]$ and with the automorphism $(O_a p)(i) = q^{ra} p(i + a)$ if p is homogeneous of degree r. Define

$$[p_i \otimes e_a, p_j \otimes e_b] = (p_i \cdot O_a(p_j) - p_j \cdot O_b(p_i)) \otimes e_{a+b}. \tag{24}$$

Proposition 6.1. *The formula (24) defines the structure of a Lie algebra on* $\mathrm{Span}\{p \otimes e_a \mid p \in \mathcal{P}, a \in \mathbb{Z}\}$.

The above formulas can be obtained by first letting considering the span of the (tails of the) sequences $(1)_{n\in\mathbb{N}}$, (n),$n \in \mathbb{N}, (q^{rn})_{n\in\mathbb{N}}$. Specifically, $T^r_a \equiv q^{ri} E_{i,i+a}, V_b \equiv (1 - q^{-2j}) E_{j,j+b}$. For later reference, we also introduce $W_c = i \cdot E_{k,k+c}$. Throughout, $E_{i,j}$ denotes the usual matrix units. The equations then results from insisting that $O_a V_b(j) = q^{2a} V_b(j + a)$ and by considering the subalgebra generated by the functions T^r_a, W_c and V_b. The claim is, indeed, that they are closed under the Lie bracket. The remaining Lie brackets are:

$$\begin{aligned}
[T^r_a, T^s_b] &= 0,\\
[T^r_a, W_b] &= aT^r_{a+b},\\
[W_a, W_b] &= (b - a)W_{a+b},\\
[T^r_a, V_b] &= (1 - q^{2a})T^r_{a+b}, \text{ and}\\
[V_a, W_b] &= (1 - q^{2a})W_{a+b} + aV_{a+b}.
\end{aligned}$$

Set $T_a = T^0_a$. The commutators may then also be written:

$$\begin{aligned}
q^{2a} V_a V_b - q^{2b} V_b V_a &= ([b]_q - [a]_q)V_{a+b},\\
V_a T_b - q^{2b} W_b V_a &= [b]_q W_{a+b} + aV_{a+b},\\
V_a T_b - q^{2b} T_b V_a &= [b]_q T_{a+b},
\end{aligned}$$

while the remaining relations involving only the elements T_a and W_b are the usual, “classical” ones. Observe that when we cut down the algebra of functions to the one generated by the above mentioned elements, since the Lie bracket between homogeneous elements may be of lower degree - and inhomogeneous, it is not true that $O_a [X_b, Y_c] = [O_a(X_b), O_a(Y_c)]$. But there will still be relations such as the quantized Jacobi relations (20 -22) above. In some sense we are of course promoting the defining representation to a higher level - in line with $W_b W_{a+c} - W_c W_{a+b} = (c - b)W_{a+b}$ in the defining representation of the Witt (Virasoro) algebra. We get the following:

$$\Big[V_a, [W_b, W_c]_q\Big]_q + \text{ cyclic} = (c - b)[b]_q[c]_q W_{a+b+c}.$$

Likewise, there is a simple relation for the commutators with two W's and one V:

$$\begin{aligned} q^{2a}\Big[W_a,[V_b,V_c]_q\Big]_q &+ \Big[V_b,[V_c,W_a]_q\Big]_q \\ +\Big[V_c,[W_a,B_B]_q\Big]_q &= (b[a]_q[c]_q - c[a]_q[b]_q)V_{a+b+c}. \end{aligned}$$

7. q-Sugawara

It is well known that in the $q=1$ situation, the T_x operators may be used to define the Virasoro algebra. See e.g. [**18**, Exercise 14.8]:

$$\begin{aligned} \underline{a>0:}\quad \sigma_a &= \sum_{n=0}^{\infty} T_{-n}T_{n+a} + \frac{1}{2}\sum_{n=1}^{a-1} T_n T_{a-n} \\ \underline{a<0:}\quad \sigma_a &= \sum_{n=0}^{\infty} T_{-n+a}T_n + \frac{1}{2}\sum_{n=1}^{|a|-1} T_{n+a}T_{-n} \\ \sigma_0 &= \sum_{n=0}^{\infty} T_{-n}T_n - \frac{1}{2}T_0^2 \end{aligned}$$

Moreover,

$$[\sigma_a,\sigma_b] = (a-b)\sigma_{a+b} + \delta_{a+b,0}\left(\frac{1}{12}(a-1)a(a+1)Z_0\right).$$

Moving to the q generic situation, one is lad to new considerations. First of all, we set

$$q^{-a}T_aT_b - q^{-b}T_bT_a = \delta_{a+b,0}\,[a]_q'\,F_0,$$

where $[a]_q'$ is some q integer which is odd as a function of a. Immediately one is led to the consistency requirement

$$\forall a\in\mathbb{Z} : F_0T_a = q^{-2a}T_aF_0.$$

Thus we see that the q analogue of a central extension may be an extension by an element satisfying covariance rather than invariance.

The q analogue of the Sugawara operators then becomes

$$\begin{aligned} \underline{a>0:}\quad \sigma_a(q) &= \sum_{n=0}^{\infty} T_{-n}q^{n+a}T_{n+a} + \frac{1}{2}\sum_{n=1}^{a-1} T_n q^{a-n} T_{a-n}, \\ \underline{a<0:}\quad \sigma_a(q) &= \sum_{n=0}^{\infty} T_{-n+a}q^nT_n + \frac{1}{2}\sum_{n=1}^{a-1} T_{n+a}q^{-n}T_{-n}, \\ \sigma_0(q) &= \sum_{n=0}^{\infty} T_{-n}q^nT_n - \frac{1}{2}T_0^2. \end{aligned}$$

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Mathematics Institute, Universitetsparken 5, DK–2100 Copenhagen Ø, Denmark, jakobsen@math.ku.dk