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Wave and Dirac Operators, and Representations of the Conformal Group

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Let M be the flat Minkowski space. The solutions of the wave equation, the Dirac equations, the Maxwell equations, or more generally the mass 0, spin s equations are invariant under a multiplier representation U_s of the conformal group. We provide the space of distributions solutions of the mass 0, spin s equations with a Hilbert space structure H_s, such that the representation U_s will act unitarily on H_s. We prove that the mass 0 equations give intertwining operators between representations of principal series. We relate these representations to the Segal–Shale–Weil (or "ladder") representation of $U(2, 2)$.

INTRODUCTION

In recent years the conformal group has been proposed for use in experimental physics as well as in more theoretical considerations (see refs. in [20b]). Let $R^4 = (t, \mathbf{x}) = (t, x_1, x_2, x_3)$ be the flat Minkowski space; we identify R^4 with the space $H(2)$ of hermitian 2×2 matrices by

$$x = \begin{pmatrix} t + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & t - x_1 \end{pmatrix}.$$

Then $\det x = t^2 - x_1^2 - x_2^2 - x_3^2 = x \cdot x$. The group

$$G = U(2, 2) = \{g \in \mathrm{GL}(4, C), \qquad g = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right),$$

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where a, b, c, d are 2×2 complex matrices satisfying $ad^* - bc^* = 1$, $cd^* = dc^*$, $ab^* = ba^*\}$ acts on $H(2)$ (up to a set of measure zero) by

$$g \cdot x = (ax + b)(cx + d)^{-1}.$$

This realizes in particular $SU(2, 2)$ as a covering group of the conformal group.

The group G contains $P_0 = \{g \text{ such that } c = 0, \det a = 1\}$ which is isomorphic to the universal covering group of the Poincaré group.

Let us consider some classical differential operators on Minkowski space:

(a) The wave operator $\square = (\partial^2/\partial t^2) - \sum_{i=1}^{3}(\partial^2/\partial x_i^2)$.

(b) Let us denote by $x \to \tilde{x}$ the space-reversal

$$\tilde{x} = \begin{pmatrix} t - x_1 & -x_2 - ix_3 \\ -x_2 + ix_3 & t + x_1 \end{pmatrix}.$$

We let $\partial/\partial x$ be the differential operator acting on $\mathbb{C}^2$-valued C^∞ functions on $H(2)$ by

$$\frac{\partial}{\partial x} = \begin{pmatrix} \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial x_1}, & \dfrac{\partial}{\partial x_2} + i\dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_2} - i\dfrac{\partial}{\partial x_3}, & \dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial x_1} \end{pmatrix}.$$

The Dirac operator acting on C^4-valued C^∞ functions on $H(2)$ is then

$$D = \nabla = \begin{pmatrix} 0 & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial \tilde{x}} & 0 \end{pmatrix},$$

and

$$D^2 = \begin{pmatrix} \square & 0 \\ 0 & \square \end{pmatrix}.$$

(c) Maxwell's equations.

Let ω be a 2-form on $H(2)$ (i.e., ω has 6 components

$$\omega = e_1(x)\, dx_2 \wedge dx_3 + e_2(x)\, dx_3 \wedge dx_1 + e_3(x)\, dx_1 \wedge dx_2 + h_1(x)\, dt \wedge dx_1 + h_2(x)\, dt \wedge dx_2 + h_3(x)\, dt \wedge dx_3).$$

We consider the equations $d\omega = \delta\omega = 0$, where δ is the formal adjoint of d with respect to the Lorentz metric $x \cdot x$.

We refer to the equations $\Box\phi = 0$; $D\phi = 0$; $\delta\omega = d\omega = 0$, as examples of classical mass 0 equations.

It was first noted by Cunningham and Bateman [3, 4] that Maxwell's equations, $\delta\omega = d\omega = 0$, are covariant under the group G. More recently, Kosmann (in a more general geometric context [11]) showed covariance of the classical mass 0 equations. As a special case of algebraic results, Kostant [12] derived quasi-invariance properties of the wave operator. All these results account for the well-known fact (cf. [8]) that the solutions of mass 0 equations are conformally invariant under some multiplier representation T_s of G.

On the other hand, it is possible to give a Hilbert space structure to a space of distribution solutions of the mass 0 equations, in which the restriction of T_s to the Poincaré group is unitary and is its usual representation of mass 0, spin s: (a) corresponds to spin 0, (b) to spin $\frac{1}{2}$, and (c) to spin 1 ([2]; see also [19]).

In [8], Gross proved unitarity of T_s under the full conformal group for $s = 1$, and discussed the general question. For $s = 0$, the proof of unitarity is a special case of [16b, 23]. (See also [18], for related computations).

We present in this paper a simple proof of the unitarity of the representations T_s of the group G; furthermore, we show that the corresponding differential operators are intertwining operators for representations of principal degenerate series. Let us explain our method to prove unitarity in the case of (a):

Solutions ψ of $\Box\psi = 0$ are Fourier transforms of distributions on the boundary $b(C)$ of the dual light cone; $b(C) = \{k = (k_0, k_1, k_2, k_3)$ such that $k_0^2 = \sum_{i=1}^{3} k_i^2\}$. We consider the surface measure $dm(k)$ on $b(C)$ and define a Hilbert space of solutions by:

$$\mathscr{H} = \left\{ \check{\phi}(x) = \int_{b(C)} e^{i\operatorname{Tr} xk} \phi(k)\, dm(k) \text{ with } \|\phi\|^2 = \int_{b(C)} |\phi(k)|^2\, dm(k) < \infty \right\}.$$

Let us consider the representation U_α of $U(2, 2)$ (which is a representation of a principal degenerate series):

$$(U_\alpha(g)f)(x) = \det(cx + d)^{-\alpha} f((ax + b)(cx + d)^{-1}), \quad \text{if } g^{-1} = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right). \tag{0.1}$$

It is easily seen that the restriction T_0 of U_1 to the maximal parabolic subgroup $P = \{g \in G$ such that $c = 0\}$ of G is unitary in $\mathscr{H}$. It remains to consider the action of

$$\sigma = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right),$$

since P and σ generate G. σ acts by $(T_0(\sigma)f)(x) = (\det x)^{-1}f(-x^{-1})$ on $\mathscr{H}$.

We define $\phi = \phi_+ + \phi_-$, where ϕ_+ has support on the boundary $b(C^+)$ of the forward light cone ($k_0 \geqslant 0$, $k_0{}^2 = \sum_{i=1}^3 k_i{}^2$).

We extend $\check{\phi}_+(x)$ as a holomorphic function

$$\check{\phi}_+(z) = \int_{b(C^+)} e^{i\operatorname{Tr} zk}\phi_+(k)\, dm(k)$$

on the domain $D = \{z = x + iy,\ x, y \in H(2),\ y > 0\}$. It is easy to compute the reproducing kernel $K(z, w)$ of the Hilbert space $\mathscr{H}^+ = \{\check{\phi}_+; \| \check{\phi}_+ \|^2 = \| \phi \|^2 \text{ in } L^2(dm(k))\}$. It is

$$K(z, w) = \det((z - w^*)/2i)^{-1}.$$

The transformation properties of this function under the group G are immediate, and leads to the unitarity. (In elementary terms, to prove that $T_0(\sigma)$ is unitary, we select a particular dense set of functions f_z in $\mathscr{H}^+$, namely, the Fourier transforms of the functions $k \to e^{-i\operatorname{Tr} kz^*}$ restricted to $b(C^+)$, for every $z \in D$, and we compute explicitly the relation $\langle T_0(\sigma)f_z, T_0(\sigma)f_w\rangle = \langle f_z, f_w\rangle$.)

If $\alpha = 3$ in (0.1), we can prove, by the same methods (see [16b]) that U_3 acts unitarily in a Hilbert space $\mathscr{H}_3$ of Fourier transforms of functions in the dual solid light cone:

$$C = \{k; k_1{}^2 + k_2{}^2 + k_3{}^2 \leqslant k_0{}^2\}:$$

$$\mathscr{H}_3 = \left\{\check{\phi}(x) = \int_C e^{i\operatorname{Tr} kx}\phi(k)\, dk;\ \| \check{\phi} \|^2 = \int_C | \phi(k)|^2(\det k)\, dk < \infty\right\}.$$

We prove that $\Box \circ U_1(g) = U_3(g) \circ \Box$, and similar relations for $\Box^n$. These are stronger properties than the known relations of covariance of $\Box$. (This relation shows why the space of solution of $\Box$ is invariant under $U_1(g)$. Furthermore, it allows us to form some composition series of the full representation U_1, and to relate a subrepresentation of U_1 to $\mathscr{H}_3$, thus carrying U_1 into a space of real mass functions. That the real mass subspace of the scalar functions was conformally invariant was conjectured by I.E. Segal.

We extend this method to the case of $s > 0$, in particular, (b) and (c).

We also show that D of (b), powers of this as well as higher dimensional Dirac operators, are intertwining operators for representations of principal degenerate series.

Let us explain the organization of the article, and make some bibliographic comments:

Chapter I states the invariance properties of kernels similar to $\det((z - w^*)/2i)^{-1}$ (Formula I.5.6).

In Chapter II, (1.2) is taken from [13]; we include it for the sake of completeness.

Sections II.3, II.4, and II.5 are included to motivate our approach, and are not needed in the proofs.

The representations of mass 0, spin s, can be considered as exceptional points arising in the analytic continuation of the holomorphic discrete series. We consider in Chapter III the "nonexceptional" part (e.g., $\alpha > 1$ in (0, 1)). This part was first constructed by Gross and Kunze [7]. In other words, this chapter deals with the definition of conformally invariant unitary structures on spaces of functions which are Fourier transforms of functions supported in the dual solid light cone.

Chapter IV is concerned with one of the two main problems of this article: To give a unitary structure on solutions of (a), (b), and (c), and to show conformal invariance. This chapter can be read independently of Chapter III, but the formulas here appear as analytic continuations of those in Chapter III. This procedure for "guessing" the fundamental formulas (IV.1.1) was suggested to us by K. M. Gross.

Chapter V deals with reducibility properties of principal degenerate series induced by finite dimensional representations of the maximal parabolic subgroup P. The representation in the mass 0 solution space is naturally imbedded as a unitary subspace inside a degenerate principal series representation, which in general probably is non-unitarizable as for spin 0 (cf. [21]) and spin 0 and spin $\frac{1}{2}$ (cf. [10]). We study in more detail the mass 0 equations, and we prove that the classical differential operators implement intertwining relations. We include here, for the case of spin 0, a proof communicated recently to us by M. Kashiwara.

In Chapter VI, we relate the mass 0, spin s, representations to the Segal–Shale–Weil harmonic-metaplectic representation of $U(2, 2)$ inside $L^2(C^2)$. This relates to what physicists call the "ladder-representations," a subject studied in particular by Anderson and Raczka [1] and by Mack and Todorov [15].

Extensions of some of these results have been obtained for $U(p, q)$ and $\mathrm{Sp}(n, R)$ (see [10, 22]).

I. Geometric Preliminaries

1. *The Flat Minkowski Space*

1.1. Let us consider $\mathbb{R}^4 = \{x = (x_0, x_1, x_2, x_3)\}$ with the quadratic form $Q(x) = x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2$; we denote by $Q(x, y) = x \cdot y$ the associated scalar product, and $(\mathbb{R}^4, Q)$ is called the flat Minkowski space.

1.2. We identify $\mathbb{R}^4$ with the vector space $H(2)$ of 2×2 hermitian matrices by

$$x = \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{pmatrix};$$

we have $\operatorname{Tr} x = 2x_0$, $\det x = Q(x)$, $dx = dx_0\, dx_1\, dx_2\, dx_3$.

1.3. Let $C^+ \subset H(2)$ be the open convex cone of positive definite hermitian matrices, i.e., $C^+ = \{y; y_0 > 0,\ Q(y) > 0\}$. The group $\mathrm{GL}(2, \mathbb{C})$ acts on $H(2)$ by $g \cdot x = gxg^*$; we have $d(g \cdot x) = |\det g|^4\, dx$. The cone C^+ is the orbit of Id for this action and the measure $(\det y)^{-2}\, dy$ is the invariant measure on C^+, under the action of $\mathrm{GL}(2; \mathbb{C})$.

1.4. We denote by $b(C^+)$ the boundary of the cone C^+, i.e., $b(C^+)$ is the cone of semidefinite hermitian positive matrices k with $\det k = 0$. If

$$k = \begin{pmatrix} k_0 + k_1 & k_2 + ik_3 \\ k_2 - ik_3 & k_0 - k_1 \end{pmatrix},$$

then $b(C^+) = \{k, k_0 \geqslant 0, k_0{}^2 = k_1{}^2 + k_2{}^2 + k_3{}^2\}$ is the boundary of the forward light cone.

1.5. Similarly we define C^- to be the open convex cone of negative definite hermitian matrices, and $b(C^-)$ its boundary, i.e.: $b(C^-) = \{k, k_0 \leqslant 0, k_0{}^2 = k_1{}^2 + k_2{}^2 + k_3{}^2\}$.

1.6. The group $\mathrm{SL}(2, \mathbb{C})$ acts on $H(2)$ by $g \cdot x = gxg^*$, the form $Q(x) = \det x$ is preserved under this action, and this identifies $\mathrm{SL}(2; \mathbb{C})$ with the universal covering group of $SO(1, 3)$. The boundary of the forward light cone (without its vertex), and the boundary of the backward light cone are orbits of $\mathrm{SL}(2, \mathbb{C})$ under this action.

1.7. We will consider the isomorphism of $H(2)$ with $H(2)^*$ given by $(k, x) = \operatorname{Tr} kx = 2(k_0x_0 + k_1x_1 + k_2x_2 + k_3x_3)$. The dual cone of C^+, $\{x; \operatorname{Tr}(kx) > 0, \text{ for } x \in \overline{C^+}\}$, is then equal to C^+.

2. *The Group* $U(2, 2)$

2.1. Consider $\mathbb{C}^4$ with basis e_1, e_2, e_3, e_4; we write any 4×4 complex matrix t by blocks;

$$t = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right),$$

where a, b, c, d are complex 2×2 matrices; if $a = \lambda I_d$ with $\lambda \in \mathbb{C}$, we write λ instead of a. For example

$$\left(\begin{array}{c|c} 1 & i \\ \hline i & 1 \end{array}\right) \quad \text{means:} \quad \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}.$$

Let

$$p_u = \left(\begin{array}{c|c} 0 & -i \\ \hline i & 0 \end{array}\right)$$

represent the hermitian form on $\mathbb{C}^4$: $p_u(x, y) = \langle p_u x, y \rangle$. Then p_u has signature $(2, -2)$. If

$$c = (\tfrac{1}{2})^{1/2} \left(\begin{array}{c|c} 1 & i \\ \hline i & 1 \end{array}\right) \quad \text{and} \quad p_0 = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right)$$

then $cp_0c^* = p_u$.

2.2. We define $G = \{g \in \mathrm{GL}(4; \mathbb{C});\ gp_ug^* = p_u\}$. G is the group $U(2, 2)$. In order for an element

$$g = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$$

to belong to G it is necessary and sufficient to have

$$ab^* = ba^*, \qquad ad^* - bc^* = 1, \qquad cd^* = dc^*.$$

3. *The Hermitian Symmetric Space* D

3.1. Let $D = \{\lambda$; complex subspaces of dimension 2 of $\mathbb{C}^4$ such that the restriction of p_u to λ is negative definite$\}$. The group G acts naturally on D (transforming λ to $g(\lambda)$, which is a complex subspace of dimension 2 of $\mathbb{C}^4$, and as g preserves p_u, obviously $g(\lambda)$ belongs to D). Since $\mathbb{C}e_1 \oplus \mathbb{C}e_2$ is a totally isotropic subspace for the form p_u, if λ belongs to D, then $\lambda \cap \mathbb{C}e_1 \oplus \mathbb{C}e_2 = \{0\}$, and there exists a unique linear map $z\colon \mathbb{C}e_3 \oplus \mathbb{C}e_4 \to \mathbb{C}e_1 \oplus \mathbb{C}e_2$ such that $\lambda = \{v + zv;\ v \in \mathbb{C}e_3 \oplus \mathbb{C}e_4\}$. As $p_u(v + zv, v + zv) = -i\langle v, zv\rangle + i\langle zv, v\rangle$, λ belongs to D if and only if $z - z^* \in iC^+$. In other words, we can parametrize D by $H(2) + iC^+ = \{z = x + iy;\ x = x^*,\ y \in C^+\}$. D is then identified with the open subset of 2×2 complex matrices z such that $(z - z^*)/2i$ is positive definite, and so D inherits the complex structure of a domain in $\mathbb{C}^4$; for this complex structure the Shilov boundary of D is $H(2)$. It is easy to write in these coordinates the natural action of G on D. If

$$g = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$$

and if z represents λ, then $cz + d$ is an invertible 2×2 matrix, and $g(\lambda)$ is represented by $z' = (az + b)(cz + d)^{-1} = g \cdot z$.

3.2. The point $e_0 = i\,\mathrm{Id}$ is often written simply i. The stabilizer K of i is the maximal compact subgroup of G given by

$$K = \left\{ g = \left(\begin{array}{c|c} a & -b \\ \hline b & a \end{array}\right);\ (a + ib, a - ib) \in U(2) \times U(2) \right\},$$

i.e., $(a + ib)(a + ib)^* = 1$. The map

$$\left(\begin{array}{c|c} a & -b \\ \hline b & a \end{array}\right) \to (a + ib, a - ib)$$

is a group isomorphism of K with $U(2) \times U(2)$.

3.3. It follows from the relations (2.2) that $g \cdot z^* = (g \cdot z)^*$, and then, up to a set of measure 0, G preserves the Shilov boundary $H(2)$ of D, and the action is given by $g \cdot x = (ax + b)(cx + d)^{-1}$.

3.4. The group of affine transformations of the domain D is denoted by P; i.e.,

$$P = \left\{ \left(\begin{array}{c|c} a & b \\ \hline 0 & (a^*)^{-1} \end{array}\right);\ ab^* = ba^*,\ a \in \mathrm{GL}(2; \mathbb{C}) \right\}.$$

P is a maximal parabolic subgroup of G. If $a \in \mathrm{GL}(2, \mathbb{C})$, we consider

$$g(a) = \left(\begin{array}{c|c} a & 0 \\ \hline 0 & (a^*)^{-1} \end{array}\right) \in P.$$

The action of $g(a)$ on D is $g(a) \cdot (x + iy) = axa^* + iaya^*$. If $x_0 = x_0{}^* \in H(2)$ then

$$u(x_0) = \left(\begin{array}{c|c} 1 & x_0 \\ \hline 0 & 1 \end{array}\right) \in P$$

and acts on D by $u(x_0) \cdot (x + iy) = x + x_0 + iy$. The group P is isomorphic to the semidirect product of $\mathrm{GL}(2, \mathbb{C})$ and $H(2)$; we remark that the domain D is already homogeneous for the action of P.

3.5. We identify the G-homogeneous domain D with G/K by $g \to g \cdot i$.

3.6. If $y \in H(2)$, we write

$$v(y) = \left(\begin{array}{c|c} 1 & 0 \\ \hline y & 1 \end{array}\right).$$

We consider P_-, the parabolic group of G opposite to P; i.e.,

$$P_- = \left\{\left(\begin{array}{c|c} a & 0 \\ \hline b & (a^*)^{-1} \end{array}\right),\ a^*b = b^*a\right\}.$$

Every element of P_- is written uniquely as $v(y)\, g(a)$; and every element of G, except for a set of Haar measure 0, can be written $g = v(y)\, g(a)\, u(x)$. If we consider the G-homogeneous compact manifold G/P_-, the preceding decomposition of an element of G gives an isomorphism of an open dense subset of G/P_- with $H(2)$; and the action of G on G/P_- is then given on this open set by $g \cdot x = (ax + b)(cx + d)^{-1}$.

3.7. We will also consider the Poincaré subgroup

$$P_0 = \mathrm{SL}(2, \mathbb{C}) \times_s H(2)$$

of the group P. We will call the one parameter subgroup

$$g(t) = \left(\begin{array}{c|c} t & 0 \\ \hline 0 & t^{-1} \end{array}\right), \qquad t > 0,$$

acting on D by $z \to t^2 z$, the scale transformations.

4. *Representations of the Group* $U(2, 2)$ *inside Spaces of Holomorphic Functions on* D

We will consider, if τ is a representation of $K = U(2) \times U(2)$ in a finite-dimensional vector space V_τ, the space $\mathscr{C}(\tau) = \{\psi;\ C^\infty$ functions $G \to V_\tau$ such that $\psi(gk) = \tau(k)^{-1} \cdot \psi(g)\}$ which is the space of C^∞-sections of the vector bundle $(G \times V_\tau)/K \to G/K$ over D. The group G acts on $\mathscr{C}(\tau)$ by left translations: $(l(g_0)\psi)(g) = \psi(g_0^{-1}g)$.

In the next section we will give explicitly a map $J_\tau(g, z)$: $G \times D \to \mathrm{GL}(V_\tau)$ satisfying:

$$\begin{aligned} J_\tau(g_1 g_2, z) &= J_\tau(g_1, g_2 \cdot z) \circ J_\tau(g_2, z), \\ J_\tau(k, i) &= \tau(k), \\ J_\tau(1, z) &= 1. \end{aligned} \tag{4.1}$$

4.2. Then if $\psi \in \mathscr{C}(\tau)$, the function $(P_\tau\psi)(g) = J_\tau(g, i) \cdot \psi(g)$ is invariant by right translations of K, i.e., is a function on $D = G/K$, and we write $(P_\tau\psi)(g \cdot i) = J_\tau(g, i) \cdot \psi(g)$. We define the representation T_τ of G on C^∞ functions on D with values in V_τ by

$$(T_\tau(g) \cdot f)(z) = J_\tau(g^{-1}, z)^{-1} f(g^{-1} \cdot z),$$

and it follows from the relations (4.1) that

$$P_\tau \circ l(g_0) = T_\tau(g_0) \circ P_\tau .$$

In fact, the map $J_\tau(g, z)$ will be holomorphic in $z \in D$ and hence the space $\mathscr{O}(D; V_\tau)$ of holomorphic functions on D with values in V_τ will be stable under the representation T_τ.

4.3. *The universal automorphic factor.*

Let

$$g = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$$

be in G and $z \in D$. We define $J_1(g, z) = cz + d$; J_1 is a map from $G \times D$ into $\mathrm{GL}(2, \mathbb{C})$, holomorphic in z. It is immediate to verify:

$$J_1(g_1 g_2, z) = J_1(g_1, g_2 \cdot z) \circ J_1(g_2, z). \tag{4.4}$$

We define $J_2(g, z) = J_1(g, z^*)^{*-1} = (zc^* + d^*)^{-1}$ and it follows from (4.4) and (3.3) that also:

$$J_2(g_1 g_2, z) = J_2(g_1, g_2 \cdot z) \circ J_2(g_2, z);$$

and also J_2 is holomorphic in z.

4.5. Let $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ be the reflection element in $\mathrm{SL}(2; \mathbb{C})$. We have $w_0 (g^*)^{-1} w_0^{-1} = \bar{g}\,(\det \bar{g})^{-1}$ for $g \in \mathrm{GL}(2; \mathbb{C})$ where, if

$$g = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}, \qquad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}, \qquad \bar{g} = \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 \\ \bar{\lambda}_3 & \bar{\lambda}_4 \end{pmatrix},$$

and so we have $\overline{J_1(g, z^*)} = \overline{\det(cz^* + d)} \cdot w_0 \circ J_2(g, z) \circ w_0^{-1}$.

4.6. Let τ_1 and τ_2 be two irreducible holomorphic representations of $\mathrm{GL}(2; \mathbb{C})$, and let $\tau = \tau_1 \otimes \tau_2$ be the holomorphic representation of $\mathrm{GL}(2; \mathbb{C}) \times \mathrm{GL}(2; \mathbb{C})$ acting on the complex finite dimensional vector space $V_\tau = V_{\tau_1} \otimes V_{\tau_2}$. Choose a scalar product on V_τ, such that if $(g_1, g_2) \in \mathrm{GL}(2; \mathbb{C}) \times \mathrm{GL}(2; \mathbb{C})$, $\tau(g_1^*, g_2^*) = \tau(g_1, g_2)^*$. The restriction of this holomorphic representation of $\mathrm{GL}(2; \mathbb{C}) \times \mathrm{GL}(2; \mathbb{C})$ to the real form $U(2) \times U(2)$ is irreducible and unitary in this scalar product. We define $J_\tau \colon G \times D \to \mathrm{GL}(V_\tau)$ by

$$\begin{aligned} J_\tau(g, z) &= \tau_1(J_1(g, z)) \otimes \tau_2(J_2(g, z)) \\ &= \tau_1(cz + d) \otimes \tau_2((zc^* + d^*)^{-1}). \end{aligned}$$

If $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in K$ we have $(a^* + ib^*) = (a - ib)^{-1}$ and $J_\tau(k, i) = \tau_1(a + ib) \otimes \tau_2(a - ib)$. Identifying the irreducible representation τ of $U(2) \times U(2)$ with the representation that we also call τ of K, by $\tau\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \tau(a + ib, a - ib)$, we have $J_\tau(k, i) = \tau(k)$. J_τ satisfies the relations (4.1), and we define $(T_\tau(g) f)(z) = J_\tau(g^{-1}, z)^{-1} f(g^{-1} \cdot z)$ acting on the space $\mathcal{O}(D, V_\tau)$.

4.7. Let $\tilde{G}$ be the universal covering group of G. Then we can define the multiplier $j_1^\alpha(\tilde{g}, z) = (\text{“}\det(cz + d)\text{”})^\alpha$ by choosing a branch of the simply connected manifold $\tilde{G} \times D$ such that $j_1^\alpha(1, z) = 1$. Similarly we define $j_2^\beta(g; z) = (\text{“}\det(zc^* + d^*)\text{”})^{-\beta}$ by choosing a branch of the simply connected manifold $\tilde{G} \times D$ such that $j_2^\beta(1, z) = 1$. We can then define, if $\alpha, \beta \in R$, and τ is an irreducible representation of K, the representation $T_{\tau,\alpha,\beta}(\tilde{g})$ of $\tilde{G}$ on $\mathcal{O}(D; V_\tau)$ given by (if $\tilde{g}$ has image g in G)

$$(T_{\tau,\alpha,\beta}(\tilde{g})f)(z) = j_1^\alpha(\tilde{g}^{-1}, z)^{-1} j_2^\beta(\tilde{g}^{-1}, z)^{-1} J_\tau(g^{-1}, z)^{-1} f(g^{-1} \cdot z).$$

5. *Reproducing Kernels*

In the rest of the article, we will be concerned with constructing Hilbert subspaces $\mathscr{H}_\tau$ of $\mathcal{O}(D, V)$ where T_τ will act unitarily. Then the reproducing kernel $K_\tau(z, w)$ (see II.3) will be a function on $D \times D$ with values in $\mathrm{End}(V_\tau)$ holomorphic in z and antiholomorphic in w and satisfying in particular:

$$K_\tau(g \cdot z, g \cdot w) = J_\tau(g, w) \circ K_\tau(z, w) \circ J_\tau(g, w)^*.$$

5.1. The universal kernel. If z and w are in D, then $z - w^*$ is also in D, and hence is invertible. We consider R_1: $D \times D \to \mathrm{GL}(2; \mathbb{C})$ by $R_1(z, w) = ((z - w^*)/2i)^{-1}$; we have $R_1(i, i) = 1$, and $R_2(z, w) = ((z - w^*)/2i) = -(R_1(z^*, w^*)^*)^{-1}$; we have $R_2(i, i) = 1$.

R_1 and R_2 are holomorphic in z, antiholomorphic in w. It is easy to verify that:

$$R_1(g \cdot z, g \cdot w) = J_1(g, z) \circ R_1(z, w) \circ J_1(g, w)^*.$$

Applying the operation $*^{-1}$ on this relation for z^* and w^* we get

$$R_2(g \cdot z, g \cdot w) = J_2(g, z) \circ R_2(z, w) \circ J_2(g, w)^*.$$

We define, if $\tau = \tau_1 \otimes \tau_2$ is the irreducible holomorphic representation of $\mathrm{GL}(2; \mathbb{C}) \times \mathrm{GL}(2; \mathbb{C})$

$$R_\tau\colon D \times D \to \mathrm{GL}(V_\tau)$$

by

$$\begin{aligned} R_\tau(z, w) &= \tau_1(R_1(z, w)) \otimes \tau_2(R_2(z, w)) \\ &= \tau_1((z - w^*)/2i)^{-1} \otimes \tau_2((z - w^*)/2i). \end{aligned} \tag{5.2}$$

It follows from (5.1) that

$$R_\tau(g \cdot z, g \cdot w) = J_\tau(g, z) \circ R_\tau(z, w) \circ J_\tau(g, w)^*. \tag{5.3}$$

5.4. On the simply connected manifold $D \times D$ we can define

$d_\alpha(z, w) = \det((z - w^*)/2i)^{-\alpha}$ for every α in $\mathbb{R}$ following the branch where $d_\alpha(i, i) = 1$; we then have if

$$R_{\tau,\alpha,\beta}(z, w) = R_\tau(z, w) \det((z - w^*)/2i)^{\beta-\alpha},$$

$$R_{\tau,\alpha,\beta}(g \cdot z, g \cdot w) = j_\alpha(\tilde{g}, z) j_\beta(\tilde{g}, z)\, J_\tau(g, z)\, R_{\tau,\alpha,\beta}(z, w) \circ \overline{j_\alpha(\tilde{g}, w)}\, \overline{j_\beta(\tilde{g}, w)}\, J_\tau(g, w)^*.$$

5.5. Let us define, if $w \in D$ and $v \in V_\tau$, the function $f_\tau(w, v)(\cdot)$ by $f_\tau(w, v)(z) = R_\tau(z, w) \cdot v$ which is in the space $\mathcal{O}(D, V_\tau)$. The relation 5.3 can be rewritten

$$T_\tau(g) \cdot f_\tau(w, v) = f_\tau(g \cdot w, (J_\tau(g, w)^*)^{-1} \cdot v). \tag{5.6}$$

Consequently, if $\mathcal{L}_\tau$ is the subspace of $\mathcal{O}(D, V_\tau)$ defined by the linear span of the functions $f_\tau(w, v)$ for all w in D and $v \in V$, then $\mathcal{L}_\tau$ is a stable subspace of the representation T_τ. We will be able to define, for some special τ's, a unitary structure on $\mathcal{L}_\tau$.

5.7. We define $SU(2, 2) = \{g \in G; \det g = 1\}$. We remark that for

$$g \in SU(2, 2), \qquad g = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right), \qquad \text{and} \qquad x \in H(2),$$

$\det(cx + d)$ is real. This can be seen by observing that if $c = 0$, then $d = (a^*)^{-1}$, and $g \in SU(2, 2)$ implies $(\det a) = (\det a^*) = \overline{(\det a)}$. Since $J(g_1 g_2, x) = J(g_1, g_2 x)\, J(g_2, x)$, the set of g's for which $\det(cx + d)$ is real for all x in $H(2)$, is a group. This group contains $P \cap SU(2, 2)$ and also

$$\sigma = \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right),$$

since of course $\det x$ is real. The remark now follows by observing that these elements generate $SU(2, 2)$.

II. Representations of the Group $U(2, 2)$ inside Spaces of Holomorphic Functions, and Reproducing Kernels

1.1. We let $\mathcal{O}(D, V_\tau)$ be the space of holomorphic functions on D with values in the finite-dimensional vector space V_τ (I.4.3) and consider the representation $(T_\tau(g)f)(z) = J_\tau(g^{-1}, z)^{-1}f(g^{-1} \cdot z)$ of (I. 4.6).

1.2. Let $\mathscr{H}_\tau$ be a Hilbert space of holomorphic functions on D with values in V_τ. We assume that for every $z \in D$, the map $F \to F(z)$ is continuous from $\mathscr{H}_\tau$ to V_τ. Hence there exists a function $K_\tau: D \times D \to \mathrm{End}(V_\tau)$ called the reproducing kernel of H_τ, such that, for every $w \in D$ and $v \in V_\tau$, $K_\tau(\cdot, w) \cdot v \in \mathscr{H}_\tau$, and:

$$\langle F(w), v\rangle = \langle F, K_\tau(\cdot, w) \cdot v\rangle_{\mathscr{H}_\tau}. \tag{1.3}$$

We have by definition

$$\langle K_\tau(\cdot, z_1) \cdot v_1, K_\tau(\cdot, z_2) \cdot v_2\rangle = \langle K_\tau(z_2, z_1) \cdot v_1, v_2\rangle,$$

and hence

$$K_\tau(z, w) = K_\tau(w, z)^*. \tag{1.4}$$

1.5. Let $v_1, v_2, \ldots, v_N \in V$, and $z_1, \ldots, z_N \in D$ and consider the linear subspace of $\mathscr{H}_\tau$

$$\mathscr{L}_\tau = \left\{ \sum_{i=1}^{N} K_\tau(\cdot, z_i) \cdot v_i \text{, for all possible choices of } v\text{'s and } z\text{'s} \right\}.$$

Then clearly from (1.3) $\mathscr{L}_\tau$ is dense in $\mathscr{H}_\tau$.

Since for

$$\psi = \sum_{i=1}^{N} K_\tau(\cdot, z_i) \cdot v_i,$$

$$\|\psi\|^2 = \sum_{i,j} \langle K_\tau(z_j, z_i)v_i, v_j\rangle,$$

we get that

$$\sum_{i,j=1}^{N} \langle K_\tau(z_j, z_i)v_i, v_j\rangle \geqslant 0. \tag{1.6}$$

2.1. To prove unitarity and irreducibility of certain representations of $U(2, 2)$ in $\mathscr{H}_\tau$ we will use the following

2.2. Proposition [13]. *Let G be a group of holomorphic transformations of a complex domain D. Let H be a Hilbert space of holomorphic functions on D with values in a vector space V having a reproducing kernel $K(z, w)$. Let $J(g, z)$ be a continuous function of $G \times D$*

with values in GL(V) *and such that for each fixed g in G, $J(g, z)$ is holomorphic on D, satisfying*

$$J(g_1 \cdot g_2, z) = J(g_1, g_2 \cdot z) \circ J(g_2, z), \tag{2.3}$$

$J(1, z) = 1$. *Then if* $(T_J(g)f)(z) = J(g^{-1}, z)^{-1} f(g^{-1} \cdot z)$, T_J *is a unitary representation of G in H if and only if*

$$K(g \cdot z, g \cdot w) = J(g, z) \circ K(z, w) \circ J(g, w)^*. \tag{2.4}$$

Furthermore, if D is G-homogeneous and if for $e_0 \in D$ the representation $g \to J(g, e_0)$ of the stabilizer of e_0 in G is unitary and irreducible in V, then T is an irreducible unitary representation of G in H.

2.5. *Conversely, let $J(g, z)$ be a function on $G \times D$ with values in* GL(V), *holomorphic in z and satisfying* (2.3), *and let $K(z, w)$ be a function on $D \times D$ with values in* End V *holomorphic in z and anti-holomorphic in w satisfying* (2.4). *Suppose that K satisfies the condition* (1.6), *then it is easy to reconstruct from $\mathscr{L}_K$ (the linear space of functions $\sum_{i=1}^{N} K(z, w_i) \cdot v_i$) a Hilbert space H_K of holomorphic functions on D, in which the representation T_J will act unitarily, by completing $\mathscr{L}_K$ with respect to the norm*

$$\left\| \sum_{i=1}^{N} K(\cdot, w_i) \cdot v_i \right\|^2 = \sum_{i,j} \langle K(w_j, w_i) v_i, v_j \rangle.$$

3.1. Clearly, if the kernel R_τ (I.5.2) satisfies the condition (1.6) the formula (I.5.3) shows that the representation T_τ will act unitarily in some Hilbert space $\mathscr{H}_\tau \subset \mathcal{O}(D; V_\tau)$ completed from all the functions $\sum_{i=1}^{N} R_\tau(\cdot, z_i) \cdot v_i$ in the scalar product $\sum_{i,j} \langle R_\tau(z_j, z_i) v_i, v_j \rangle$. We will analyze in the next paragraphs, for some representation τ, the condition (1.6) by Fourier–Laplace transform as in [16(a)].

4. If τ is a unitary representation of $K = U(2) \times U(2)$ and if $\mathscr{C}(\tau) = \{f;\ C^\infty$ functions from G to V_τ such that $f(gk) = \tau(k)^{-1} f(g)\}$, then we can define on $\mathscr{C}(\tau)$ an obvious unitary structure, invariant by left translations, given by

$$\| f \|^2 = \int_G \| f(g) \|_{V_\tau}^2 \, dg.$$

We will make this norm explicit, after the isomorphism P_τ (I.4), as an L^2 norm on D. Let $S = \{(\begin{smallmatrix} a & 0 \\ u & b \end{smallmatrix});\ a, b > 0$ and $u = u_1 + iu_2 \in \mathbb{C}\}$, and let ds be the left invariant Haar measure on S. Let $B = \{g(s)\, u(x);\ s \in S,\ x \in H(2)\}$ be the Iwasawa subgroup of G; the left invariant Haar measure on B is $db = ds\,dx$. Since any element of G can be written uniquely as $g = bk$, $b \in B$, $k \in K$, we get that $dg = db\,dk$ and if $f \in \mathscr{C}(\tau)$, $\|f\|^2 = \int_B \|f(b)\|^2_{V_\tau}\,db$. The map $s \mapsto ss^*$ gives a diffeomorphism of S with the cone C^+; so $(\det y)^{-2}\,dy = ds$. We have

$$J_\tau(g(s)\,u(x)) = J_\tau((s^*)^{-1}) = \tau_1(s^*)^{-1} \otimes \tau_2(s),$$
$$g(s)\cdot u(x)\cdot i = sxs^* + iss^*,$$
$$\psi(g(s)\,u(x)) = \tau_1(s^*) \otimes \tau_2(s^{-1})\cdot P_\tau\psi(sxs^* + iss^*),$$

and finally

$$\|\psi\|^2 = \int_D \langle \tau_1(y) \otimes \tau_2(y^{-1})(P_\tau\psi)(x+iy), (P_\tau\psi)(x+iy)\rangle (\det y)^{-4}\,dx\,dy.$$

Hence if we define

$$H^o(\tau) = \Big\{F \in \mathcal{O}(D, V_\tau);\ \text{such that}$$
$$\|F\|^2 = \int_D \langle \tau_1(y) \otimes \tau_2(y^{-1})F(x+iy), F(x+iy)\rangle$$
$$\times (\det y)^{-4}\,dx\,dy < +\infty\Big\},$$

then if $H^o(\tau)$ is not zero, the representation T_τ acts unitarily and irreducibly on the Hilbert space $H^o(\tau)$ of holomorphic functions on D. In this article we will define, even in the case where $H^o(\tau)$ is reduced to zero, a Hilbert space H_τ of holomorphic functions on D, where T_τ will act unitarily, for special τ's.

5. In order to motivate our approach, let us recall some facts about $H^o(\tau)$ (but we will not need them in this article). We define [16a], if F is in $H^o(\tau)$,

$$\hat{F}(k) = (1/\pi^2)\int_{\operatorname{Im} z = y} e^{-i\operatorname{Tr} kz}F(z)\,dx = (1/\pi^2)\,e^{\operatorname{Tr} ky}\int e^{-i\operatorname{Tr} kx}F(x+iy)\,dx.$$

By the Plancherel theorem:

$$\|F\|^2_{H_0(\tau)} = (1/2\pi)^4 \int e^{-2\operatorname{Tr} ky}\langle \tau_1(y) \otimes \tau_2(y)^{-1}\cdot \hat{F}(k), \hat{F}(k)\rangle (\det y)^{-4}\,dy\,dk.$$

We define $\mu(g) = \tau_1(g) \otimes \tau_2(g^{*-1})\,(\det g)^{-2}$ which is an irreducible representation of $\mathrm{GL}(2, \mathbb{C})$;

$$\Gamma(\mu)(k) = \int_{C^+} e^{-\operatorname{Tr} ky}\mu(y)(\det y)^{-2}\,dy.$$

Then $\Gamma(\mu)(k)$ converges if and only if $k \in C^+$ and $H^o(\tau) \neq \{0\}$. Then:

$$\|F\|^2 = \int_{C^+} \langle \Gamma(\mu)(2k)\hat{F}(k), \hat{F}(k)\rangle\, dk < +\infty,$$

and we also have:

$$F(x+iy) = (1/\pi^2)\int_{C^+} e^{i\operatorname{Tr} kz} F(k)\, dk$$

as an absolutely convergent integral [16b]. Roughly speaking, we will construct unitary representations of G in spaces H_τ of holomorphic functions having integral representations

$$F(x+iy) = (1/\pi^2)\int_{C^+} e^{i\operatorname{Tr} kz}\Gamma(\mu)(2k)^{-1}\psi(k)\, dk,$$

where $\psi(k)$ will satisfy the relation

$$\int_{C^+} \langle \Gamma(\mu)(2k)^{-1}\psi(k), \psi(k)\rangle\, dk < +\infty,$$

and $\Gamma(\mu)^{-1}$ will be constructed by analytic continuation.

III. THE FUNCTION $\Gamma(\mu)$

1. Let μ be any irreducible representation of GL(2, $\mathbb{C}$). We define:

$$\Gamma(\mu)(k) = \int_{C^+} e^{-\operatorname{Tr} ky}\mu(y)(\det y)^{-2}\, dy$$

and $\Gamma(\mu) = \Gamma(\mu)(1)$, whenever this integral is convergent. In general, $\Gamma(\mu) = \Gamma(\mu)(1)$ is a nonscalar matrix. We will recall the calculation of $\Gamma(\mu)$ for the following special cases, where $\Gamma(\mu)$ will indeed be a scalar. As $\Gamma(\mu)$ satisfies the transformation property

$$\Gamma(\mu)(gkg^*) = \mu(g^*)^{-1}\,\Gamma(\mu)(k)\,\mu(g)^{-1},$$

it follows that $\Gamma(\mu)(1)$ commutes with the restriction of μ to the unitary group $U(2)$; so in the case that this restriction is irreducible (basically if μ is a holomorphic or antiholomorphic representation of GL(2; $\mathbb{C}$), then $\Gamma(\mu)(1)$ will be scalar.

2. We consider τ_1, the natural representation of GL(2; $\mathbb{C}$) in $\mathbb{C}^2$, basis (e_1, e_2); and we form $\tau_n = \bigotimes_s^n \tau$, the n-symmetric tensor

product of τ_1; τ_n is an irreducible holomorphic representation of GL(2; $\mathbb{C}$) and acts in the vector space V_n with basis $(e_1{}^n, e_1^{n-1}e_2, \ldots, e_2{}^n)$ of dimension $n+1$. We identify V_n with the space of homogeneous polynomials of degree n on $(\mathbb{C}^2)^*$; with respect to the natural inner product, $\tau_n(g^*) = \tau_n(g)^*$; and define $\tau_0 = id$, the trivial one-dimensional representation of GL(2; $\mathbb{C}$). We now form the integral

$$\Gamma_1(\tau_n, \alpha)(k) = \int_{C^+} e^{-\mathrm{Tr}\, ky} \tau_n(y)(\det y)^{\alpha-2}\, dy, \tag{2.2}$$

and $\Gamma_1(\tau_n, \alpha) = \Gamma_1(\tau_n, \alpha)(1)$.

2.3. PROPOSITION [6, 7b, 16a]. *$\Gamma_1(\tau_n, \alpha)$ converges in a point $k \in C^+$ if and only if $\Gamma_1(\tau_n, \alpha)(1)$ converges; and $\Gamma_1(\tau_n, \alpha)(1)$ converges if and only if $\alpha > 1$. In this case*

$$\Gamma_1(\tau_n, \alpha)(k) = (\pi/2)\, \Gamma(\alpha - 1)\, \Gamma(\alpha + n)(\det k)^{-\alpha}\, \tau_n(k^{-1}). \tag{2.4}$$

We recall the proof of this proposition: $\Gamma_1(\tau_n, \alpha)(k)$ has the following transformation property:

$$\Gamma_1(\tau_n, \alpha)(gkg^*) = \tau_n(g^*)^{-1}\, \Gamma_1(\tau_n, \alpha)(k)\, \tau_n(g)^{-1}\, |\det g|^{-2\alpha}, \tag{2.5}$$

as it is seen easily, changing y to $y' = (g^*)^{-1} y g^{-1}$ in the integral. We will use the parametrization of the cone C^+ by $y = ss^*$ with $s = \begin{pmatrix} a & 0 \\ u & b \end{pmatrix}$, $a > 0$, $b > 0$, $u = u_1 + iu_2 \in \mathbb{C}$, i.e.,

$$y = \begin{pmatrix} a^2 & a\bar{u} \\ au & |u|^2 + b^2 \end{pmatrix}, \qquad dy = 2a^3\, da\, b\, db\, du_1\, du_2, \qquad \det y = a^2 b^2.$$

We have $\tau(s^*)e_1{}^n = a^n e_1{}^n$, and so $\langle \tau(y)\, e_1{}^n, e_1{}^n \rangle = a^{2n}$. We compute

$$\langle \Gamma_1(\tau_n, \alpha)\, e_1{}^n, e_1{}^n \rangle = \int_{a,b,u_1,u_2} e^{-(a^2+b^2+|u|^2)} a^{2n} a^{2(\alpha-2)} b^{2(\alpha-2)} 2a^3\, da\, b\, db\, du_1\, du_2 .$$

This integral clearly converges if and only if $\alpha > 1$, and then equals $(\pi/2)\, \Gamma(\alpha - 1)\, \Gamma(\alpha + n)$; this proves the necessity of the condition. Using (2.5) (i.e., changing s to $s_0 s$) we have that if

$$s_0 = \begin{pmatrix} a_0 & 0 \\ u_0 & b_0 \end{pmatrix},$$

then $\langle \Gamma_1(\tau_n, \alpha)(s_0{}^* s_0)\, e_1{}^n, e_1{}^n \rangle$ also converges. This means that for every k in C^+ the integral giving the coefficient $\langle \Gamma_1(\tau_n, \alpha)(k)\, e_1{}^n, e_1{}^n \rangle$ is finite. But the set of v's such that $\langle \Gamma_1(\tau_n, \alpha)(k)v, v \rangle < +\infty$ for every k in C^+, is, by the Schwarz inequality, a vector space which contains e_1^n. (2.5) shows that this vector space is invariant by the irreducible representation τ_n, hence it is equal to V_n. Also (2.5) implies that $\Gamma_1(\tau_n, \alpha)(1)$ is the scalar matrix $(\pi/2)\,\Gamma(\alpha - 1)\,\Gamma(\alpha + n)$, since the restriction of τ_n to $U(2)$ is irreducible. (2.4) then follows from (2.5).

2.6. COROLLARY. *If* $\alpha > 1$, *then*

$$\int_{C^+} e^{-\operatorname{Tr} yk}(\Gamma_1(\tau_n, \alpha)(k))^{-1}\, dk = c(\tau_n, \alpha)\, \tau_n(y^{-1})(\det y)^{-\alpha}(\det y)^{-2}$$

with $c(\tau_n, \alpha) = (\alpha - 1)\,\alpha(\alpha + n + 1) = \Gamma_1(\tau_n, \alpha + 2)/\Gamma_1(\tau_n, \alpha)$.

Proof. This follows directly from (2.3) by interchanging the variables y and k, (and changing the value of α). We rewrite (2.4) changing the role of k and y.

$$\tau_n(y)^{-1}(\det y)^{-\alpha-2} = \frac{1}{(\pi/2)\,\Gamma(\alpha + 1)\,\Gamma(\alpha + n + 2)} \int_{C^+} e^{-\operatorname{Tr} ky}\tau_n(k)(\det k)^{\alpha}\, dk, \tag{2.7}$$

the integral on the right being absolutely convergent for $\alpha > -1$. This implies that $\tau_n(y^{-1})(\det y)^{-\alpha-2}$ for $\alpha > -1$ is the Laplace transform of the field $k \to \tau_n(k)$ $(\det k)^{\alpha}$ of positive operators on C^+.

3. *Applications to the Positivity of the Kernels*

3.1. PROPOSITION. *If* $\alpha > -1$, *the kernel* $K_1^+(\tau_n, \alpha)$ *on* $D \times D$ *given by*

$$K_1^+(\tau_n, \alpha)(z, w) = \tau_n((z - w^*)/2i)^{-1} \det((z - w^*)/2i)^{-\alpha-2}$$

satisfies the condition (II.1.6).

Proof. By analytic continuation of formula (2.7) it follows that if z, w are in D, then

$$\tau_n\left(\frac{z - w^*}{2i}\right)^{-1} \det\left(\frac{z - w^*}{2i}\right)^{-\alpha-2}$$
$$= \frac{2^{n+4+2\alpha}}{(\pi/2)\,\Gamma(\alpha + 1)\,\Gamma(\alpha + n + 2)} \int_{C^+} e^{i\operatorname{Tr}(k(z-w^*))}\tau_n(k)(\det k)^{\alpha}\, dk$$

as an absolutely convergent integral. Let $v_1, \ldots, v_N \in V_n$ and

$z_1, \ldots, z_N \in D$, and let $w(k) = \sum_{j=1}^{N} e^{i\operatorname{Tr} kz_j} v_j$. Then if K is the function on the left-hand side

$$\sum_{i,j} \langle K(z_i, z_j)v_j, v_i \rangle = \beta \int_{C^+} \langle \tau_n(k)\, w(k), w(k) \rangle (\det k)^\alpha\, dk \geqslant 0,$$

where β is a positive constant, hence it is positive. We will make this corollary precise in the following proposition.

3.2. PROPOSITION. *Let $\alpha > -1$, and let $\mathscr{L}_1^+(\tau_n, \alpha) = \{\psi$; measurable functions on C^+ with values in V_n, such that*

$$\|\phi\|^2_{\tau_n,\alpha} = \frac{2^{n+4+2\alpha}}{(\pi/2)\, \Gamma(\alpha+1)\, \Gamma(\alpha+n+2)} \int_{C^+} \langle \tau_n(k)\, \phi(k), \phi(k) \rangle (\det k)^\alpha\, dk < \infty\}.$$

Let us define, for $\phi \in \mathscr{L}_1^+(\tau_n, \alpha)$ and $z \in D$,

$$(F_1^+(\tau_n, \alpha)\phi)(z) = \frac{2^{n+4+2\alpha}}{(\pi/2)\, \Gamma(\alpha+1)\, \Gamma(\alpha+n+2)} \int_{C^+} e^{i\operatorname{Tr} kz} \tau_n(k) \cdot \phi(k) (\det k)^\alpha\, dk.$$

Then the right-hand side is an absolutely convergent integral and defines a holomorphic function $(F_1^+(\tau_n, \alpha)\phi)$ on D with values in V_n. If

$$H_1(\tau_n, \alpha) = \{F_1^+(\tau_n, \alpha)\phi;\ \text{for } \phi \in \mathscr{L}_1^+(\tau_n, \alpha) \text{ with } \|F_1^+(\tau_n, \alpha)\phi\|^2 = \|\phi\|^2_{\tau_n,\alpha}\}$$

then $H_1(\tau_n, \alpha)$ has reproducing kernel $K_1^+(\tau_n, \alpha)$.

Proof. If $z = x + iy$, $y \in C^+$ *and* $v \in V_n$,

$$|\langle (F_1^+(\tau_n, \alpha)\phi)(z), v \rangle| \leqslant \frac{2^{n+4+2\alpha}}{(\pi/2)\, \Gamma(\alpha+1)\, \Gamma(\alpha+n+2)} \int_{C^+} e^{-\operatorname{Tr} ky}\, |\langle \tau_n(k)\, \phi(k), v \rangle| (\det k)^\alpha\, dk.$$

Since $|\langle \tau_n(k)\, \phi(k), v \rangle| \leqslant \langle \tau_n(k)\, \phi(k), \phi(k) \rangle^{1/2} \langle \tau_n(k)v, v \rangle^{1/2}$, by applying the Schwarz inequality, we get:

$$|\langle (F_1^+(\tau_n, \alpha)\phi)(z), v \rangle|^2 \leqslant \frac{2^{n+4+2\alpha}}{(\pi/2)\, \Gamma(\alpha+1)\, \Gamma(\alpha+n+1)} \int_{C^+} e^{-2\operatorname{Tr} ky} \langle \tau_n(k)v, v \rangle (\det k)^\alpha\, dk \cdot \|\phi\|^2_{\tau_n,\alpha}$$

$$= \langle \tau_n(y^{-1})v, v \rangle (\det y)^{-\alpha-2} \|\phi\|^2_{\tau_n,\alpha}. \qquad (3.3)$$

So the first part of the proposition is proved. The fact that $F_1{}^+(\tau_n, \alpha)\phi$ is holomorphic is standard (apply the Morera criterion), and also the inequality (3.3) proves that $H_1{}^+(\tau_n, \alpha)$ has a reproducing kernel $K = F_1{}^+(\tau_n, \alpha)\kappa$. We must then have: for every ϕ in $\mathscr{L}_1(\tau_n, \alpha)$

$$\begin{aligned}\langle (F_1{}^+(\tau_n, \alpha)\phi)(w), v\rangle_{v_n} &= \langle F_1{}^+(\tau_n, \alpha)\phi, K(\cdot, w)\cdot v\rangle_{H_1{}^+} \\ &= \langle \phi, \kappa(\cdot, w)\cdot v\rangle_{\mathscr{L}_1{}^+}.\end{aligned}$$

The first term is

$$\frac{2^{n+4+2\alpha}}{(\pi/2)\,\Gamma(\alpha+1)\,\Gamma(\alpha+n+2)}\int_{C^+} e^{i\operatorname{Tr} kw}\langle \tau_n(k)\,\phi(k), v\rangle(\det k)^{\alpha}\,dk;$$

the third term is

$$\frac{2^{n+4+2\alpha}}{(\pi/2)\,\Gamma(\alpha+1)\,\Gamma(\alpha+n+2)}\int_{C^+} \langle \tau_n(k)\,\phi(k), \kappa(k, \omega)\cdot v\rangle(\det k)^{\alpha-2}\,dk.$$

This equality implies that $\kappa(k, \omega) = e^{-i\operatorname{Tr} k\omega^*}\cdot v$, and then $K = K_1{}^+(\tau_n, \alpha)$.

4. *Unitary Representations of G in $\mathcal{O}(D, V_n)$*

Let $\tilde{G}$ be the universal covering of G. We define, if $\tilde{g} \in \tilde{G}$ has image

$$g = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \text{ in } G, \qquad J_1{}^+(\tau_n, \alpha)(\tilde{g}, z) = \tau_n(cz+d)\,j_1^{\alpha+2}(\tilde{g}, z).$$

Applying the method of paragraph 2, we then conclude

4.1. THEOREM. *If $\alpha > -1$, the representation*

$$(T_1{}^+(\tau_n, \alpha)(\tilde{g})f)(z) = J_1{}^+(\tau_n, \alpha)(\tilde{g}^{-1}, z)^{-1}\cdot f(g^{-1}\cdot z)$$

is a unitary irreducible representation of G inside $H_1{}^+(\tau_n, \alpha)$. If α is an integer strictly greater than -1, this representation is in fact a representation of the group $U(2, 2)$ itself, given by, if

$$g^{-1} = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right),$$

$$(T_1{}^+(\tau_n, \alpha)(g)f)(z) = \tau_n(cz+d)^{-1}\det(cz+d)^{-(\alpha+2)}\,f((az+b)(cz+d)^{-1}).$$

For example, if $n = 0$, this leads to the family of representations of $U(2, 2)$ already studied in [16b, 24],

$$(T_\alpha(g)f)(z) = \det(cz + d)^{-(\alpha+2)} f((az + b)(cz + d)^{-1}).$$

We remark that for $\alpha = 0$ this representation is unitary inside the Hardy space

$$H_2 = \left\{F \in \mathcal{O}(D);\ \sup_{y \in C^+} \int |F(x + iy)|^2\, dx = \|F\|^2 < +\infty\right\},$$

since $L_1^+(\tau_0, 0)$ coincides with $L^2(C^+, dk)$, and taking the boundary value, which exist in the L^2 sense, gives a unitary imbedding of $T(\tau_0, 0)$ into the unitary principal series representations given by

$$(T(g)f)(x) = \det(cx + d)^{-2} f((ax + b)(cx + d)^{-1})$$

5. Let $\Gamma_2(\tau_n, \alpha)(k) = \int_{c^+} e^{-\operatorname{Tr} ky} \tau_n(y)^{-1} (\det y)^{-\alpha-2}\, dy$.

5.1. For each element $u \in M(2; \mathbb{C})$, we denote by $c(u)$ the cofactor of u, defined by $u \circ c(u) = (\det u)\mathrm{Id}$; the map $u \to c(u)$ satisfies $c(uv) = c(v) \circ c(u)$, and if

$$u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}, \qquad c(u) = \begin{pmatrix} u_4 & -u_2 \\ -u_3 & u_1 \end{pmatrix};$$

in particular, for $x = (x_0, x_1, x_2, x_3)$ in $H(2)$, $c(x) = (x_0, -x_1, -x_2, -x_3)$ and c is the space-reversal. We denote by $c(\tau_1)$ the representation $g \to c(g^*)$ of $\mathrm{GL}(2; \mathbb{C})$, i.e., $c(\tau_1)(g) = \tau_1(c(g^*))$, and define $c(\tau_n)$ by $c(\tau_n)(g) = \tau_n(c(g^*))$. $c(\tau_n)$ is then an irreducible antiholomorphic representation of $\mathrm{GL}(2; \mathbb{C})$ in V_n. We have

$$c(\tau_n)(g) = \tau_n((g^*)^{-1})(\det g^*)^n.$$

5.2. The preceding Γ_2 function can then be written as

$$\Gamma_2(\tau_n, \alpha)(k) = \int_{c^+} e^{-\operatorname{Tr} ky} c(\tau_n)(y)(\det y)^{-(\alpha+n+2)}\, dy.$$

We can treat Γ_2 by the same method as for Γ_1. We will remark here that as $\operatorname{Tr}(c(u)) = \operatorname{Tr} u$ for $u \in M(2; \mathbb{C})$, after changing y into $c(y)$,

$$\Gamma_2(\tau_n, \alpha)(k) = \int_{c^+} e^{-\operatorname{Tr} c(k) y} \tau_n(y)(\det y)^{-(\alpha+n)-2}\, dy$$
$$= \Gamma_1(\tau_n, -(\alpha + n))(c(k)).$$

So we have the following:

5.3. PROPOSITION. *$\Gamma_2(\tau_n, \alpha)$ converges if and only if $-(\alpha - n) > 1$ and in this case:*

$$\Gamma_2(\tau_n, \alpha)(k) = (\pi/2)\, \Gamma(-\alpha)\, \Gamma(-\alpha - n - 1)\, \tau_n(k)(\det k)^{\alpha}.$$

We change y into $c(y) = y^{-1} \det y$ in the formula (2.7) (and k into $c(k)$ in the integral) and get:

$$\tau_n(y)(\det y)^{-(\alpha+n+2)} = \frac{1}{(\pi/2)\, \Gamma(\alpha+1)\, \Gamma(\alpha+n+2)} \int_C e^{-\mathrm{Tr}\, ky} \tau_n(c(k))(\det k)^{\alpha}\, dk \qquad (5.4)$$

for $\alpha > -1$. This implies that $\tau_n(y)(\det y)^{-(\alpha+n+2)}$ is the Laplace transform of the field $k \to \tau_n(c(k))(\det k)^{\alpha}$ of positive Hermitian operators on C^+; by analytic continuation we then have

$$\tau_n\left(\frac{z - w^*}{2i}\right) \det\left(\frac{z - w^*}{2i}\right)^{-(\alpha+n+2)} = \frac{2^{n+4+2\alpha}}{(\pi/2)\, \Gamma(\alpha+1)\, \Gamma(\alpha+n+1)} \int_{C^+} e^{i\mathrm{Tr}\, k(z-w^*)} \tau_n(c(k))(\det k)^{\alpha}\, dk. \qquad (5.5)$$

5.6. THEOREM. *Let* $\alpha > -1$. Let $\mathscr{L}_2^+(\tau_n, \alpha) = \{\phi$; *measurable functions on C^+ with values in V_n, such that*

$$\|\phi\|^2 = \frac{2^{n+4+2\alpha}}{(\pi/2)\, \Gamma(\alpha+1)\, \Gamma(\alpha+n+2)} \int_{C^+} \langle \tau_n(c(k))\, \phi(k), \phi(k)\rangle (\det k)^{\alpha}\, dk < \infty\}.$$

Let us define, for $\phi \in \mathscr{L}_2^+(\tau_n, \alpha)$ and $z \in D$,

$$(F_2^+(\tau_n, \alpha)\phi)(z) = \frac{2^{n+4+2\alpha}}{(\pi/2)\, \Gamma(\alpha+1)\, \Gamma(\alpha+n+2)} \int_{C^+} e^{i\mathrm{Tr}\, kz} \tau_n(c(k)) \cdot \phi(k)(\det k)^{\alpha}\, dk.$$

Then the right hand side is an absolutely convergent integral and defines a holomorphic function $F_2^+(\tau_n, \alpha)$ on D with values in V_n.

If $H_2^+(\tau_n, \alpha) = \{F_2^+(\tau_n, \alpha)\phi; \phi \in \mathscr{L}_2^+(\tau_n, \alpha)\}$ then $H_2^+(\tau_n, \alpha)$ is a Hilbert space of holomorphic functions having reproducing kernel $K_2^+(\tau_n, \alpha)$, where

$$K_2^+(\tau_n, \alpha)(z, w) = \tau_n((z - w^*)/2i) \det((z - w^*)/2i)^{-(\alpha+n+2)}.$$

The representation

$$(T_2^+(\tau_n, \alpha)(\tilde{g})f)(z) = J_2^+(\tau_n, \alpha)(\tilde{g}^{-1}; z)^{-1} f(g^{-1} \cdot z)$$

where $J_2^+(\tau_n, \alpha)(\tilde{g}; z) = \tau_n((zc^* + d^*)^{-1})\, j_2^{-(\alpha+n+2)}(g; z)$ is an irreducible unitary representation of $\tilde{G}$ inside $H_2^+(\tau_n, \alpha)$. If α is an integer greater than -1, this representation is a representation of the group $U(2, 2)$ itself given, for $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, by

$$(T_2^+(\tau_n, \alpha)(g)f)(z) = \tau_n(zc^* + d^*)\det(zc^* + d^*)^{-(\alpha+n+2)} f(g^{-1} \cdot z).$$

IV. The Exceptional Case

We will now consider, τ_n being fixed, the value $\alpha = -1$. We will see it still is possible to give a meaning to the integrals (III.2.7), or the ones in (III.3.2), etc., by analytic continuation: Let $b(C^+)$ be the boundary of the cone C^+ and let

$$dm(k) = \tfrac{1}{2}dk_1\, dk_2\, dk_3/(k_1^2 + k_2^2 + k_3^2)^{1/2}$$

be the measure on $b(C^+)$ parametrized by k_1, k_2, k_3 $(k_0^2 = k_1^2 + k_2^2 + k_3^2)$. The function $(\det k)^\alpha$ is locally integrable on C^+ if and only if $\alpha > -1$; its meromorphic continuation as a distribution has a pole for $\alpha = -1$, and we have $\lim_{\alpha\to-1}(1/\Gamma(\alpha + 1))(\det k)^\alpha\, dk = dm(k)$. The integral (III.2.7), for example, can be rewritten as

$$\tau_n(y)^{-1}(\det y)^{-(\alpha+2)} = \frac{1}{(\pi/2)\,\Gamma(\alpha + n + 2)} \int_{C^+} e^{-\mathrm{Tr}\, ky}\tau_n(k)\, \frac{(\det k)^\alpha\, dk}{\Gamma(\alpha + 1)},$$

and as $\tau_n(k)$ is a polynomial representation of $\mathrm{GL}(2, \mathbb{C})$, $\tau_n(k)$ has still a meaning for k being any (2×2) complex matrix, in particular, for k in $b(C^+)$. The integral (III.2.7) will then be replaced, for $\alpha = -1$, by the integral with respect to $dm(k)$ of the function $e^{-\mathrm{Tr}\, ky}\tau_n(k)$ restricted to the boundary. This indicates why the formulas (1.1), etc., will hold. However, we will prove the formulas in this chapter directly (independently of Section 3).

1.1. Proposition. *Let $b(C^+)$ be the boundary of the cone C^+ and let $dm(k) = dk_1\, dk_2\, dk_3/(2k_0)$; then*

$$\tau_n(y)^{-1}(\det y)^{-1} = C_n \int_{b(C^+)} e^{-\mathrm{Tr}\, yk}\tau_n(k)\, dm(k)$$

is an absolutely convergent integral, where C_n is a positive constant.

Proof. We have

$$dk_0 \wedge dk_1 \wedge dk_2 \wedge dk_3 = ((dk_1 \wedge dk_2 \wedge dk_3)/2k_0) \wedge d(\det k).$$

From the semi-invariance property of dk and $\det k$ we then get that $dm(gkg^*) = |\det g|^2 \, dm(k)$. We can see that

$$\alpha(y) = \int_{b(C^+)} e^{-\mathrm{Tr}\, yk} \tau_n(k) \, dm(k)$$

formally satisfies the transformation property:

$$\alpha(gyg^*) = \tau_n(g^*)^{-1} \alpha(y) \tau_n(g)^{-1} \, |\det g|^{-2}.$$

So by the same reasoning as in Proposition (III.2.3) we just have to check the convergence of the integral $\int_{b(C^+)} e^{-\mathrm{Tr}\, k} \langle \tau_n(k) \, e_1{}^n, e_1{}^n \rangle \, dm(k)$, and we can conclude that $\alpha(1)$ is a scalar matrix and that $\alpha(y) = \alpha(1)\tau_n(y)^{-1}(\det y)^{-1}$. We write $\epsilon_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $s = \begin{pmatrix} a & 0 \\ u & 1 \end{pmatrix}$; then

$$s\epsilon_0 s^* = \begin{pmatrix} a^2 & a\bar{u} \\ au & |u|^2 \end{pmatrix}$$

and $s \mapsto s\epsilon_0 s^*$ is a parametrization of $b(C^+)$ except for a set E of measure 0, given by $E = \{k_2 = k_3 = 0, \, k_0 = -k_1\}$. So we have in these coordinates $dm(k) = a \, da \, du_1 \, du_2$. As $\tau_n(\epsilon_0)$ is the projection of V_n onto $e_1{}^n$, $\langle \tau_n(s\epsilon_0 s^*) e_1{}^n, e_1{}^n \rangle = a^{2n} e_1{}^n$. The integral to be calculated is thus $\int e^{-(a^2+|u|^2)} a^{2n} a \, da \, du_1 \, du_2$, which clearly converges, so we get the formula,

$$\tau_n(y)^{-1}(\det y)^{-1} = C_n \int_{b(C^+)} e^{-\mathrm{Tr}\, ky} \tau_n(k) \, dm(k).$$

Since for each $k = g\epsilon_0 g^*$ in $b(C^+)$, $\tau_n(k)$ is a matrix of rank 1, proportional to the projection onto $\tau_n(g) \cdot e_1{}^n$, the formula (1.1) is an integral representation of the hermitian positive operator $\tau_n(y)^{-1}(\det y)^{-1}$ by projections of rank 1 with respect to a positive measure based on the boundary $b(C^+)$ of the light cone. We can then conclude in the same way:

1.2. Proposition. *The kernel* $K_1{}^+(\tau_n, -1)$ *on* $D \times D$ *given by*

$$K_1{}^+(\tau_n, -1)(z, w) = \tau_n((z - w^*)/2i)^{-1} \det((z - w^*)/2i)^{-1}$$

satisfies the condition (II.1.6), *and we have*:

$$\tau_n\left(\frac{z - w^*}{2i}\right)^{-1} \det\left(\frac{z - w^*}{2i}\right)^{-1} = C_n \int_{b(C^+)} e^{i\mathrm{Tr}\, k(z - w^*)} \tau_n(k) \, dm(k).$$

1.3. PROPOSITION. *Let* $\mathscr{L}_1^+(\tau_n; -1) = \{\phi;$ *measurable functions on* $b(C^+)$ *with values in* V_n, *such that*

$$\|\phi\|^2 = \int_{b(C^+)} \langle \tau_n(k)\,\phi(k), \phi(k)\rangle\, dm(k) < \infty\},$$

and let us define, for $\phi \in \mathscr{L}_1^+(\tau_n, -1)$ *and* $z \in D$,

$$(F_1^+(\tau_n, -1)\phi)(z) = \int_{b(C^+)} e^{i\operatorname{Tr} kz}\tau_n(k)\cdot\phi(k)\, dm(k);$$

then the right-hand side is an absolutely convergent integral and defines a holomorphic function $F_1^+(\tau_n, -1)\phi$ *on* D *with values in* V_n.

If

$$\mathscr{H}_1^+(\tau_n, -1) = \{F_1^+(\tau_n, -1)\phi \text{ for } \phi \in \mathscr{L}_1^+(\tau_n, -1),$$
$$\text{with } \|F_1^+(\tau_n, -1)\phi\|^2 = \|\phi\|^2\},$$

then $\mathscr{H}_1^+(\tau_n, -1)$ has reproducing kernel

$$\tau_n((z - w^*)/2i)^{-1} \det((z - w^*)/2i)^{-1}$$

up to a positive constant. The proof is entirely similar to the one of Proposition (III.3.2).

2. *Description of the Space* $L_1^+(\tau_n, -1)$

As $\tau_n(k)$ is a noninvertible operator, there are functions ϕ in $\mathscr{L}_1^+(\tau_n, -1)$ with norm 0; we denote by $\overline{\mathscr{L}_1^+}(\tau_n, -1)$ the Hilbert space formed by the equivalence classes of functions in $\mathscr{L}_1^+(\tau_n, -1)$ modulo those of norm zero.

Let k be in $b(C^+)$

$$k = \begin{pmatrix} k_0 + k_1 & k_2 + ik_3 \\ k_2 - ik_3 & k_0 - k_1 \end{pmatrix}.$$

k is a hermitian operator of rank 1; so it is proportional to a projection e; we write $k = \operatorname{Tr} ke$, with $e^2 = e$; then $k^2 = (\operatorname{Tr} k)k = 2k_0k$, $\tau_n(k) = 2^{-n}k_0^{-n}\tau_n(k^2)$, and $\langle\tau_n(k)\,\phi(k), \phi(k)\rangle = 2^{-n}k_0^{-n}\langle\tau_n(k)\,\phi(k), \tau_n(k)\,\phi(k)\rangle$. So if $\phi(k)$ has value in V_n, $\tau_n(k)\,\phi(k) = \psi(k)$ is a function on $b(C^+)$ taking its values in the one-dimensional subspaces $\tau_n(k)V_n = V_n(k)$; so we will be able to identify, via the map $\psi(k) = \tau_n(k)\,\phi(k)$, the space $\overline{\mathscr{L}_1^+}(\tau_n, -1)$ with the space

$$L_1^+(\tau_n, -1) = \Big\{\psi \text{ measurable functions on } b(C^+) \text{ such that}$$

$$\text{(a)}\quad \psi(k) \in V_n(k) \text{ for almost every } k \tag{2.1}$$

$$\text{(b)}\quad \|\psi\|^2 = \int_{b(C^+)} \frac{\langle \psi(k), \psi(k)\rangle}{(2k_0)^n}\, dm(k) < +\infty\Big\}.$$

We can then rephrase the preceding theorem as follows:

2.2. THEOREM. *Let $\psi \in L_1^+(\tau_n, -1)$. Let us define*

$$\tilde{\psi}(z) = \int_{b(C^+)} e^{i \operatorname{Tr} kz} \psi(k)\, dm(k).$$

Then the right-hand side is an absolutely convergent integral and defines a holomorphic function $\tilde{\psi}$ on D, with values in V_n.

Let $H_1^+(\tau_n, -1) = \{\tilde{\psi}$ for $\psi \in L_1^+(\tau_n, -1)$ with $\|\tilde{\psi}\|^2 = \|\psi\|^2\}$; then $H_1^+(\tau_n, -1)$ has reproducing kernel

$$R_1(\tau_n, -1)(z, w) = \tau_n((z - w^*)/2i)^{-1} \det((z - w^*)/2i)^{-1}.$$

The representation $T_1^+(\tau_n, -1)$ of G defined by

$$\left(g^{-1} = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)\right)$$

$$(T_1^+(\tau_n, -1)(g)f)(z) = \tau_n(cz + d)^{-1} \det(cz + d)^{-1} f((az + b)(cz + d)^{-1})$$

is a unitary irreducible representation of G inside $H_1^+(\tau_n, -1)$.

2.3. Similarly, we can consider $\mathscr{L}_2^+(\tau_n, -1) = \{\phi$; measurable functions on $b(C^+)$, such that $\int \langle \tau_n(c(k))\, \phi(k), \phi(k)\rangle\, dm(k) < +\infty\}$ and $L_2^+(\tau_n, -1) = \{\psi$; measurable functions on $b(C^+)$, such that

(a) for almost every k, $\psi(k) \in V_n(c(k))$

(b) $\int(\langle \psi(k), \psi(k)\rangle/(2k_0)^n)\, dm(k) < +\infty\}$ and get

2.4. THEOREM. *If $\psi \in L_2^+(\tau_n, -1)$ then the integral*

$$\tilde{\psi}(z) = \int_{b(C^+)} e^{i \operatorname{Tr} kz} \psi(k)\, dm(k)$$

is absolutely convergent and defines a holomorphic function on D with values in V_n. If $H_2^+(\tau_n, -1) = \{\tilde{\psi}$ for $\psi \in L_2^+(\tau_n, -1)$, with $\|\tilde{\psi}\|^2 =$

$\| \psi \|^2\}$, *then* $H_2^+(\tau_n, -1)$ *has reproducing kernel* $R_2(\tau_n, -1)(z, w) = \tau_n((z - w^*)/2i) \det((z - w^*)/2i)^{-(n+1)}$, *and the representation*

$$(T_2(\tau_n, -1)(g)f)(z) = \tau_n(zc^* + d^*) \det(zc^* + d^*)^{-(n+1)} f(az + b)(cz + d)^{-1})$$

is a unitary irreducible representation of G inside $H_2^+(\tau_n, -1)$.

3. *Restriction of the Representation* $T_1(\tau_n, -1)$ *to the Poincaré Subgroup*

We consider the Poincaré subgroup $P_0 = \mathrm{SL}(2, \mathbb{C}) \times_s H(2)$ imbedded in G as described in I. For k in $H(2)$, we consider the character χ_k: $x \to e^{-i \operatorname{Tr} kx}$ of the normal abelian subgroup $H(2)$ of P_0; the natural action of SL(2; $\mathbb{C}$) on χ, by $(g \cdot \chi)(x) = \chi(g^{-1}x(g^*)^{-1})$, is given by $g \cdot \chi_k \mapsto \chi_{k'}$ with $k' = (g^*)^{-1}kg^{-1}$. Let $\epsilon_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$; the stabilizer of ϵ_0 under this action of SL(2; $\mathbb{C}$) is the subgroup

$$S = \begin{pmatrix} a & 0 \\ u & a^{-1} \end{pmatrix},$$

with $|a| = 1$, and the orbit of ϵ_0 is the cone $b(C^+)$ (apart from the origin). We denote by $d\dot{g}$ the invariant measure on $\mathrm{SL}(2, \mathbb{C})/S$ corresponding to $dm(k)$. We denote by χ_n the character on S given by

$$\chi_n \begin{pmatrix} a & 0 \\ u & a^{-1} \end{pmatrix} = a^n.$$

We form the representation T_n induced to P_0 by the unitary character $(s, x) \mapsto \chi_n(s)\, e^{-i \operatorname{Tr} \epsilon_0 x}$ of the subgroup $S \times {}_s H(2)$; the representation T_n is an irreducible representation of P_0 which is one part of the representation of mass 0, and spin $s = \frac{1}{2}n$.

3.1. Proposition. *The restriction of the representation* $T_1(\tau_n, -1)$ *of the group G to the Poincaré subgroup is the one part of the representation of mass* 0 *and spin* $\frac{1}{2}n$; *the restriction of the representation* $T_2(\tau_n, -1)$ *of the group G to the Poincaré subgroup is the other part of the representation of mass* 0 *and spin* $\frac{1}{2}n$.

Proof. Let $\mathscr{A}_n = \{\phi; \mathrm{SL}(2; \mathbb{C}) \to \mathbb{C}$, such that $\phi(gs) = \chi_n(s)^{-1}\phi(g)\}$. The norm $\phi \mapsto \int_{\mathrm{SL}(2,\mathbb{C})/S} |\phi|^2\, d\dot{g}$ is an invariant norm by left translation of SL(2; $\mathbb{C}$); we denote by A_n the Hilbert space of measurable functions in $\mathscr{A}_n$ with finite norm; the representation of mass 0 and spin $\frac{1}{2}n$ is realized in A_n by the following formula (see [22])

$$(T_n(g_0)\phi)(g) = \phi(g_0^{-1}g), \qquad g_0 \in \mathrm{SL}(2, \mathbb{C}),$$
$$(T_n(x_0)\phi)(g) = \exp(-i \operatorname{Tr} \epsilon_0 g^{-1} x_0 (g^*)^{-1})\, \phi(g), \qquad x_0 \in H(2).$$

At the point $k = g\epsilon_0 g^*$, the one-dimensional subspace $V_n(k)$ is spanned by the vector $\tau_n(g)\, e_1{}^n = v_g$. We have,

$$\text{if} \quad g = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right), \qquad \langle v_g , v_g\rangle = \langle \tau_n(g^*g)\, e_1{}^n, e_1{}^n\rangle$$
$$= (|\, a\,|^2 + |\, c\,|^2)^n;$$

as $\operatorname{Tr} k = |\, a\,|^2 + |\, c\,|^2$, $\langle v_g , v_g\rangle = (\operatorname{Tr} k)^n = (2k_0)^n$. We consider $b(C^+)$ isomorphic to $\mathrm{SL}(2; \mathbb{C})/S$ by the map $g \mapsto (g^*)^{-1}\epsilon_0 g^{-1}$; we define, if $\phi \in A_n$ and $n \geqslant 0$, $(A\phi)(g) = \phi(g)\, \tau_n((g^*)^{-1}) \cdot e_1{}^n$. Then $A\phi$ is invariant by right translation by S, and defines a function on $b(C^+)$, which at each point $k = (g^*)^{-1}\epsilon_0 g^{-1}$ has values in $V_n(k)$. We have

$$\|\, A\phi\,\|^2_{L_1^+} = \int_{b(C^+)} (\langle (A\phi)(k), (A\phi)(k)\rangle/(2k_0)^n)\, dm(k),$$

and since $\langle \tau_n(g^*)^{-1}\, e_1{}^n, \tau_n(g^*)^{-1}\, e_1{}^n\rangle = (\operatorname{Tr}((g^*)^{-1}\epsilon_0 g^{-1}))^n$,

$$\|\, A\phi\,\|^2_{L_1^+} = \int_{G/S} |\, \phi(g)|^2\, d\dot{g};$$

and we have an isomorphism A of A_n with $L_1{}^+(\tau_n , -1)$. We consider the representation $\hat{T}_1(\tau_n , -1)$ in $L_1{}^+(\tau_n , -1)$ given by for $\phi \in L_1{}^+(\tau_n , -1)$

$$(\hat{T}_1(\tau_n , -1)(g_0)\phi)^\vee = T_1(\tau_n , -1)(g_0)\check{\phi}.$$

It is easy to check that if

$$g(a) = \left(\begin{array}{c|c} a & 0 \\ \hline 0 & a^{*-1} \end{array}\right) \qquad \text{is in } G \text{ with } a \in \mathrm{GL}(2; \mathbb{C}),$$

and

$$u(x_0) = \left(\begin{array}{c|c} 1 & x_0 \\ \hline 0 & 1 \end{array}\right) \qquad \text{is in } G \text{ with } x_0 \in H(2),$$

then

$$\begin{aligned} (\hat{T}_1(\tau_n , -1)(g(a))\phi)(k) &= \tau_n(a^*)^{-1}(\det a)\, \phi(a^*ka), \\ (\hat{T}_1(\tau_n , -1)(u(x_0))\phi)(k) &= e^{-i\operatorname{Tr} kx_0}\phi(k), \end{aligned} \tag{3.2}$$

and it is immediate to verify that A intertwines the two representations of the Poincaré group $\mathrm{SL}(2; \mathbb{C}) \times {}_s H(2)$. Similarly, if $k = g\epsilon_0 g^*$, the one-dimensional subspace $V_n(c(k))$ is spanned by the vector $w_g =$

$\tau_n(c(g^*))\, e_2{}^n(\tau_n(c(\epsilon_0))V_n = e_2{}^n)$, and $\langle w_g, w_g\rangle = (\mathrm{Tr}\, k)^n$. We define, if $n > 0$ and if $\phi \in A_{-n}$, $(A\phi)(g) = \phi(g)\,\tau_n(g)\cdot e_2{}^n$. Then A is invariant by right translation by S, and defines a function on $b(C^+)$ which at each point $k = (g^*)^{-1}\epsilon_0 g^{-1}$ has value in $V_n(c(k))$, and we have

$$\| A\phi \|_{L_2^+}^2 = \int_{G/S} | \phi(g)|^2\, d\dot{g}.$$

We consider the representation $\hat{T}_2(\tau_n, -1)$ in $L_2{}^+(\tau_n, -1)$. It is easy to check:

$$\begin{aligned}(\hat{T}_2(\tau_n, -1)(g(a))\phi)(k) &= \tau_n(a)(\det a)^{-n}(\det a)^*\,\phi(a^*ka),\\ (\hat{T}_2(\tau_n, -1)(u(x_0))\phi)(k) &= e^{-i\mathrm{Tr}\, kx_0}\phi(k),\end{aligned} \tag{3.3}$$

and it is immediate to check that A intertwines the two representations.

V. The Wave and Dirac Equations

We consider the space $H(\tau_n, -1)$ of holomorphic functions on D; by taking boundary values, this Hilbert space can be considered as a space of distributions on Minkowski space. We shall see that these distributions are solutions of certain differential equations: for example, since they are Fourier transforms of distributions with support on $b(C^+)$ $(k_0{}^2 = k_1{}^2 + k_2{}^2 + k_3{}^2;\ k_0 \geqslant 0)$, they will be solutions to the wave equation $\square\phi = 0$, where

$$\square = (\partial^2/\partial x_0{}^2) - (\partial^2/\partial x_1{}^2) - (\partial^2/\partial x_2{}^2) - (\partial^2/\partial x_3{}^2).$$

We shall see that $H(\tau_n, -1)$ is contained in the null space of a certain Dirac operator; furthermore, we will prove some properties of covariance of these operators. Finally, we specialize to spin 0 and spin $\frac{1}{2}$ and show that even the most degenerate representations, by taking quotients by appropriate null spaces, can be carried into the interior of the (forward) lightcone.

1. *The Principal Series Representations*

Let

$$P_- = \left\{\begin{pmatrix} a & 0 \\ u & (a^*)^{-1}\end{pmatrix} \text{ with } a \in \mathrm{GL}(2, \mathbb{C}),\ u \in H(2)\right\} \subseteq U(2, 2).$$

Let μ be an irreducible representation of $\mathrm{GL}(2, \mathbb{C})$ in a vector space V_μ; we consider also μ as a representation of P_- by

$$\mu\left(\begin{array}{c|c} a & 0 \\ \hline u & (a^*)^{-1} \end{array}\right) = \mu(a).$$

We form $\mathscr{C}(\mu) = \{\phi;\ C^\infty$ functions on G with values in V_μ, such that $\phi(gp) = \mu(p^{-1})\phi(g)\}$.

G acts by left translation on $\mathscr{C}(\mu)$. The subspace of functions in $\mathscr{C}(\mu)$ which are K-finite is the algebraic Harish–Chandra–module of the principal degenerate series.

Using the map (I.3.4) u: $H(2) \to G$ to define an isomorphism of $H(2)$ with an open dense subset of the compact manifold $G/P_- = U(2)$, we can identify $\mathscr{C}(\mu)$ with a subspace of the C^∞-functions on $H(2)$, by $(I\phi)(x) = \phi(u(x))$. (This map is clearly not surjective, since not all C^∞-functions on $H(2)$ can be extended as C^∞-sections of the corresponding vector bundle to the conformal compactification G/P_- of $H(2)$.)

The action of G by left translation becomes

$$(U_\mu(g)f)(x) = \mu((cx + d)^*) f((ax + b)(cx + d)^{-1}) \quad \text{for } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.1}$$

We consider the map $g \to ((g^*)^{-1}, g)$ of $\mathrm{GL}(2, \mathbb{C})$ into $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})$. (On the Lie algebra level the map $A \to (-A^*, A)$ defines an isomorphism of the complexification of $\mathrm{gl}(2, \mathbb{C})$ (considered as a real algebra) with $\mathrm{gl}(2, \mathbb{C}) \times \mathrm{gl}(2, \mathbb{C})$. If τ_1 and τ_2 are two holomorphic representations of $\mathrm{GL}(2, \mathbb{C})$, then $\mu(g) = \tau_1(g^{*-1}) \otimes \tau_2(g)$ is an irreducible representation of $\mathrm{GL}(2, \mathbb{C})$. (But not all irreducible representations of $\mathrm{GL}(2, \mathbb{C})$ are obtained; for example, $|\det g|$ is not, whereas $|\det g|^2 = \det g\ \det g^*$ is). We suppose that

$$\mu(g) = \tau_1((g^*)^{-1}) \otimes \tau_2(g).$$

Then the action U_μ of G can be written $(U\mu(g)f)(x) = \tau_1(cx + d)^{-1} \otimes \tau_2(cx + d)^* f((ax + b)(cx + d)^{-1})$, for $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We define

$$J_\mu(g, x) = \tau_1(cx + d) \otimes \tau_2((cx + d)^*)^{-1}.$$

We will consider the space $\mathscr{A}(\mu)$ obtained by taking boundary values of the functions $z \to R_\tau(z, w) \cdot v$ for $\tau = \tau_1 \otimes \tau_2$ (I.5.2), i.e.,

$$x \to \tau_1((x - w^*)/2i)^{-1} \otimes \tau_2((x - w^*)/2i) \cdot v = R_\tau(x, w) \cdot v.$$

Since if $x \in H(2)$, $w \in D$, $x - w^*$ is in D, hence in particular is invertible, $\mathscr{A}(\mu) = \{\sum_{i,j=1}^{n} R_\tau(x, w_i) \cdot v_i\}$ is a subspace of the C^∞-functions on $H(2)$. The formula (I.5.6) shows that $U\mu$ leaves $\mathscr{A}(\mu)$ invariant and that the map $v \to R_\tau(x, i)v$ defines an injection of V_τ into $\mathscr{A}(\mu)$ intertwining the representation τ of K in V_τ(I.4.6) with the restriction of $U\mu$ to K.

1.2. LEMMA. *$\mathscr{A}(\mu)$ is contained in the image of $\mathscr{C}(\mu)$ by I.*

Proof. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, and $w \in D$, then $b - w^*d$ is an invertible matrix; the function

$$\phi(w, v)(g) = (\tau_1((b - w^*d)/2i)^{-1}) \otimes \tau_2((b^* - d^*w^*)/2i) \cdot v$$

is easily seen to belong to $\mathscr{C}(\mu)$, and clearly $I(\phi(w, v))(x) = R_\tau(x, w) \cdot v$.

This lemma indicates that whenever an irreducible representation μ of $\mathrm{GL}(2, \mathbb{C})$ is extendable to a holomorphic representation of $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})$, the principal series will be reducible; the subspace $A(\mu)$ (which consists of functions on $H(2)$ having a holomorphic extension to D) will be invariant and proper.

2. *Imbedding of the Representation $T(\tau_n, -1)$ into the Principal Series*

We consider the dense invariant subspace $\mathscr{D}_1(\tau_n, -1)$ of $H(\tau_n, -1)$ consisting of the functions $z \to \sum_{i=1}^{N} R_1(\tau_n, -1)(z, w_i) \cdot v_i$. The preceding paragraph shows that there exists, by taking boundary values, a natural imbedding of $\mathscr{D}_1(\tau_n, -1)$ into an invariant subspace $\mathscr{A}_1(\tau_n, -1)$ of the principal series representations, defined by $\mathscr{A}_1(\tau_n, -1) = \{\sum_{i=1}^{N} R_1(\tau_n, -1)(x, w_i) \cdot v_i\}$, and the corresponding principal series representation is given by

$$(U_1(\tau_n, -1)(g)f)(x) = \tau_n(cx + d)^{-1} \det(cx + d)^{-1} f((ax + b)(cx + d)^{-1}),$$
$$g^{-1} = \begin{pmatrix} a & b \\ c & b \end{pmatrix}. \tag{2.1}$$

Similarly, we let $\mathscr{D}_2(\tau_n, -1)$ be the dense invariant subspace of $H_2(\tau_n, -1)$ consisting of the functions $\sum_{i=1}^{N} R_2(\tau_n, -1)(z, w_i)v_i$, which is imbedded into $\mathscr{A}_2(\tau_n, -1) = \{\sum_{i=1}^{N} R_2(\tau_n, -1)(x, w_i)v_i\}$, and the corresponding representation is given by

$$(U_2(\tau_n, -1)(g)f)(x) = \tau_n(cx + d)^*(\det(cx + d)^*)^{-(n+1)} f((ax + b)(cx + d)^{-1}). \tag{2.2}$$

Moreover, since for $f \in H_1(\tau_n, -1)$

$$\begin{aligned} |\langle f(z), v\rangle|^2 &= |\langle f, K_1(\tau_n, -1)(\cdot, z) \cdot v\rangle|^2 \\ &\leqslant \|f\|^2 \| K_1(\tau_n, -1)(\cdot, z) \cdot v \|^2 \\ &= \|f\|^2 \langle \tau_n(y)^{-1}(\det y)^{-1} \cdot v, v\rangle, \end{aligned}$$

any function in $H_1(\tau_n, -1)$ has boundary value as a distribution, because it has at most polynomial growth near the boundary ($y = 0$). We call $D_1(\tau_n, -1)$ and $D_2(\tau_n, -1)$ the Hilbert spaces of distributions obtained by boundary values of $H_1(\tau_n, -1)$ and $H_2(\tau_n, -1)$. We remark that formula (I.4.5) shows that if

$$(U(\tau_n, -1)f)(x) = \overline{\tau_n(cx+d)}^{-1} \overline{\det(cx+d)}^{-1} f((ax+b)(cx+d)^{-1})$$

then the map $f \to \tau_n(w_0)f$ with w_0 from (I.4.5) defines an isomorphism of $U_1(\tau_n, -1)$ with $U_2(\tau_n, -1)$.

We will construct the Dirac operator D_n, acting on $C^\infty(H(2), V_n) =$ the space of C^∞-functions on $H(2)$ with values in V_n, such that the space $D_1(\tau_n, -1)$ will be annihilated by D_n. Furthermore, we will prove that D_n intertwines the representation $U_1(\tau_n, -1)$ with a representation U_μ of the principal series.

3. *Commutation Relations of Differential Operators*

Let μ and μ' be two irreducible representations of $GL(2, \mathbb{C})$; let D be a differential operator (matrix-valued) going from $C^\infty(H(2), V_\mu)$ into $C^\infty(H(2), V_{\mu'})$. We want to study the properties D should have in order to be intertwining between U_μ and $U_{\mu'}$, i.e.,

$$\forall g \in G\colon D \circ U_\mu(g) = U_{\mu'}(g) \circ D.$$

An immediate observation is that D must commute with translations, hence must be constant coefficient.

Let $\check{\phi}(x) = \int e^{-i\operatorname{Tr} kx} \phi(k)\, dk$ define the Fourier isomorphism of the Schwartz-space on $H(2)$. We define the polynomial function $d(k)$ on $H(2)$ with values in End $(V_\mu, V_{\mu'})$ by

$$(d\phi)^\vee = D(\check{\phi}), \qquad \text{where} \quad (d\phi)(k) = d(k)\cdot\phi(k).$$

Then the commutation relation

$$D \circ U_\mu(a) = U_{\mu'}(a) \circ D \qquad \text{for every } a \in GL(2, \mathbb{C})$$

is equivalent to the condition

$$d(a^*ka) = \mu'(a)^{-1}\, d(k)\, \mu(a). \tag{3.1}$$

4. *The Dirac Operators*

We consider $S^+ = \mathbb{C}e_1^+ \oplus \mathbb{C}e_2^+$, $V_1 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, and denote by $v \to v^+$ the corresponding identification $xe_1 + ye_2 \to xe_1^+ + ye_2^+$. Likewise, we consider $S^- = \mathbb{C}e_1^- \oplus \mathbb{C}e_2^-$, and $v \to v^-$.

Let k be any (2×2) complex matrix. We define

$$d_1(k) = \left(\begin{array}{c|c} 0 & k \\ \hline c(k) & 0 \end{array}\right)$$

acting on $S = S^+ \oplus S^-$. We call S the space of spinors. (We have that $d_1(k)^2 = \det kI_d = \underset{\sim}{Q}(k)I_d$, so $k \to d_1(k)$ defines the irreducible representation of the Clifford algebra constructed from the quadratic form $\underset{\sim}{Q}(k)$.) Observe that $d_1(k)S^+ \subseteq S^-$; $d_1(k)S^- \subseteq S^+$. Also, we recall that

$$\square = (\partial^2/\partial x_0^2) - (\partial^2/\partial x_1^2) - (\partial^2/\partial x_2^2) - (\partial^2/\partial x_3^2).$$

We consider the symmetrized tensor product $\otimes_s^n S$, naturally imbedded into $\otimes^n S$.

The identification $v \to v^+$ of V_1 with S^+; $v \to v^-$ of V_1 with S^-, give imbeddings $\beta^+ \colon V_n \to \otimes^n S^+ \subset \otimes^n S$; $\beta^- \colon V_n \to \otimes^n S^- \subset \otimes^n S$ by

$$\beta_n^+(v \otimes \cdots \otimes v) = v^+ \otimes \cdots \otimes v^+,$$

$$\beta_n^-(v \otimes \cdots \otimes v) = v^- \otimes \cdots \otimes v^-.$$

By β we denote the symmetrization on $\otimes^n S$; $\beta(v_1 \otimes \cdots \otimes v_n) = \sum_{\tau \in S_n} (1/n!)\, v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)}$ which then is a projection onto $\otimes_s^n S$. We define injections $i^+ \colon V_{n-1} \otimes V_1$ into $\otimes^n S$, $i^- \colon V_{n-1} \otimes V_1$ into $\otimes^n S$ by

$$i^+(f \otimes w) = \beta(\beta_{n-1}^+(f) \otimes w^-),$$

$$i^-(f \otimes w) = \beta(\beta_{n-1}^-(f) \otimes w^+).$$

(Observe that if we let

$$v^{n-1} = \underbrace{v \otimes \cdots \otimes v}_{n-1},$$

then for instance $i^+(v^{n-1} \otimes v) = (1/n) \sum v^+ \otimes \cdots \otimes v^- \otimes \cdots \otimes v^+$.) By $d_n(k)$ we denote the operator on $\otimes^n S$ defined by

$$d_n(k)(v_1 \otimes \cdots \otimes v_n) = \sum_i v_1 \otimes \cdots \otimes d_1(k)v_i \otimes \cdots \otimes v_n .$$

Then $k \to d_n(k)$ is a linear map from $H(2)$ into $\mathrm{End}(\otimes^n S, \otimes^n S)$. Finally, we define $d_n^+(k) \colon V_n \to \otimes^n S$ by $d_n^+(k)f = d_n(k)(\beta_n^+ f)$, and similarly $d_n^-(k)f = d_n(k)(\beta_n^-(f))$.

4.1. LEMMA.

$$d_n^+(V_n) \subseteq i^+(V_{n-1} \otimes V_1)$$
$$d_n^-(V_n) \subseteq i^-(V_{n-1} \otimes V_1)$$

Proof. It is sufficient to prove it for elements of the form $v^n = v \otimes \cdots \otimes v$, $v \in V_1$. Then

$$\begin{aligned} d_n^+(k)(v^n) &= \sum_{i=1}^{n} v^+ \otimes \cdots \otimes \underbrace{d_1(k)v^+}_{i} \otimes \cdots \otimes v^+ \\ &= ni^+(v^{n-1} \otimes c(k)v), \\ d_n^-(k)(v^n) &= \sum_{i=1}^{n} v^- \otimes \cdots \otimes \underbrace{d_1(k)v^-}_{i} \otimes \cdots \otimes v^- \\ &= ni^-(v^{n-1} \otimes kv). \end{aligned}$$

In particular, we can consider $d_n^+(k)$ and $d_n^-(k)$ as linear maps from V_n to $V_{n-1} \otimes V_1$.

We consider the representation

$$\alpha_1(a) = \begin{pmatrix} (a^*)^{-1} & 0 \\ 0 & a \det a^{-1} \end{pmatrix} \qquad \text{of} \qquad \mathrm{GL}(2, \mathbb{C}) \text{ in } S.$$

We form $\alpha_n(a) = \otimes^n \alpha_1(a)$ acting in $\otimes^n S$ and $\mu_n(a) = (\det a)^{-1}\alpha_n(a)$. On the subgroup $SU(2, 2)$ of $U(2, 2)$, defined by $\det g = 1$, the function $\det(cx + d)$ is real, so if we consider $C^\infty(H(2), V_n)$ imbedded in $C^\infty(H(2), \otimes^n S)$ via β^+ or β^-, the representation $U_1(\tau_n, -1) \oplus U_2(\tau_n, -1)$ of $SU(2, 2)$ can be considered as a subrepresentation of the representation U_{μ_n}. We define μ_n' by

$$\begin{aligned} \mu_n'(a) &= \det a^{-2}\alpha_n(a) \\ &= (\det a)^{-1} \mu_n(a). \end{aligned}$$

4.2. LEMMA. *Let* $\mathrm{GL}_r(2, \mathbb{C})$ *be the subgroup of* $\mathrm{GL}(2, \mathbb{C})$ *given by*

$$\mathrm{GL}_r(2, \mathbb{C}) = \{a \in \mathrm{GL}(2, \mathbb{C});\ \det a = \det a^*\}.$$

Then $d_n(a^*ka) = \mu_n'(a)^{-1} d_n(k)\, \mu_n(a)$ *for all* $a \in \mathrm{GL}_r(2, \mathbb{C})$.

Proof. The relation is easily verified for $n = 1$, and having it in that case,

$$\mu_n'(a)\, d_n(a^*ka)(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n)$$
$$= \sum_i \alpha_1(a)v_1 \otimes \cdots \otimes \mu_1'(a)\, d_1(a^*ka)v_i \otimes \cdots \otimes \alpha_1(a)v_n$$
$$= \sum_i \alpha_1(a)v_1 \otimes \cdots \otimes d_1(k)\, \mu_1(a)v_i \otimes \cdots \otimes \alpha_1(a)v_n$$
$$= (\det a)^{-1} \sum_i \alpha_1(a)v_1 \otimes \cdots \otimes d_1(k)\, \alpha_1(a)v_i \otimes \cdots \otimes \alpha_1(a)v_n$$
$$= d_n(k)\, \mu_n(a)(v_1 \otimes \cdots \otimes v_n).$$

4.3. Lemma. *If $k \in b(C^+)$, the kernel of the operator $d_n^+(k)$ is equal to the subspace $V_n(k) = \tau_n(k)V_n$. Likewise, the kernel of $d_n^-(k)$ is equal to the subspace $V_n(c(k)) = \tau_n(c(k))V_n$.*

Proof. Let, for $g \in \mathrm{GL}_r(2, \mathbb{C})$, $k' = gkg^*$. Then $V_n(k') = \tau_n(g) \cdot V_n(k)$, and since $\mu_n(a)$ on $\beta^+(V_n)$ coincides with the representation $\tau_n(a^*)^{-1}(\det a)^{-1}$, Lemma 4.2 proves that

$$\operatorname{Ker} d_n^+(gkg^*) = \tau_n(g)(\operatorname{Ker} d_n^+(k)).$$

So it is sufficient to prove that both subspaces coincide for $\epsilon_0 = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, but in that case, it is easily seen that they both are $\mathbb{C}e_1^n$. Similarly we see that the kernel of $d_n^-(k)$ at ϵ_0 is e_2^n, which is the space $V_n(c(k))$.

We now define the Dirac operator D_n, acting on $C^\infty(H(2), \otimes^n S)$ as being the Fourier transform of the operator $d_n(k)$. In particular, D_n sends $C^\infty(H(2), V_n)$, identified with $C^\infty(H(2), \otimes^n S^+)$ into

$$C^\infty(H(2), V_{n-1} \otimes V_1) \qquad \Big(d_n^+(k) = \sum (1 \otimes \cdots \otimes c(k) \otimes \cdots \otimes 1)\Big).$$

Similarly, D_n sends $C^\infty(H(2), V_n)$ identified with $C^\infty(H(2), \otimes^n S^-)$ into $C^\infty(H(2), V_{n-1} \otimes V_1)(d_n^-(k) = \sum (1 \otimes \cdots \otimes k \otimes \cdots \otimes 1))$. We refer to these operators as D_n^+ and D_n^-.

4.4. Proposition. *The space $D_1(\tau_n, -1)$ consists of solutions to the equation $D_n^+u = 0$ $(n > 0)$, and the space $D_2(\tau_n, -1)$ consists of solutions to the equation $D_n^-u = 0$. Finally, the space $D_1(\tau_0, -1)$ consists of solutions to $\Box f = 0$.*

Proof. $D_1(\tau_n, -1)$ consists of distributions which are Fourier transforms of distributions whose support is in $b(C^+)$ and has values in $V_n(k) = \operatorname{Ker} d_n^+(k)$. By similar reasoning the whole proposition follows.

5. *Intertwining Relations for the Dirac Operators* D_n^+ *and* D_n^-

We have

$$D_n^+ : C^\infty(H(2), V_n) \to C^\infty(H(2), V_{n-1} \otimes V_1).$$

Let $U_1(\tau_n, -1)$ be the representation of $SU(2, 2)$ which contains $D_1(\tau_n, -1)$, the space of solutions to $D_n^+\phi = 0$, as invariant subspace. Let U_1' be the representation of $SU(2, 2)$ given by restricting the representation $U_{\mu'_n}$ to the corresponding subspace of functions with values in $i^+(V_{n-1} \otimes V_1)$.

5.1. Proposition. $D_n^+(U_1(\tau_n, -1)(g)) = U_1'(g) \circ D_n^+$ *for* $g \in SU(2, 2)$.

Proof. By Lemma 4.2, this relation is true for every

$$g \in P_0 = \begin{pmatrix} a & v \\ 0 & (a^*)^{-1} \end{pmatrix}, \qquad a \in \mathrm{GL}_r(2, \mathbb{C}),$$

which is a maximal parabolic of $SU(2, 2)$. Let x be an element of the Lie algebra $su(2, 2)$ of $SU(2, 2)$. We define

$$(dU_1(x)\phi)(h) = (d/dt)(U_1(\exp tx)\phi)(h)\,|_{t=0}\,;$$

similarly for $dU_1'(x)$. Then $dU_1(x)$ a first order differential operator, and the relation in (5.1) is equivalent to (b) $D_n^+\, dU_1(x) = dU_1'(x) D_n^+$ for every $x \in su(2, 2)$. We remark that since D_n^+ has constant coefficients, (b) is equivalent to (c) $(D_n^+\, dU_1(x)\phi)(0) = (dU_1'(x) D_n^+\phi)(0)$.

We know that (c) holds for every element of the Lie algebra of P_0, thus we need only check (c) for the elements $y = \left(\begin{smallmatrix} 0 & 0 \\ y & 0 \end{smallmatrix}\right)$ with $y = y^*$ of $su(2, 2)$. We have

$$(dU_1(y)\phi)(x) = (d/dt)\, \tau_n(-tyx + 1)^{-1} \det(-tyx + 1)^{-1} \phi(x(-tyx + 1)^{-1})\,|_{t=0},$$

and a similar formula for $dU_1'(y)$; in particular, $(dU_1'(y)\phi)(0) = 0$, and the relation to be checked is $(D_n^+\, dU_1(y)\phi)(0) = 0$. Let $d\tau_n$ be the differential of the representation τ_n and $g(x) = d\tau_n(yx) + (\operatorname{Tr} yx) I_d$ as an operator on V_n. Then $g(0) = 0$. If $u \in H(2)$, we let $(\xi(u) f)(x) = (d/dt) f(x + tu)|_{t=0}$. Then $(dU_1(y)\phi)(x) = g(x) \cdot \phi(x) + (\xi(xyx) \cdot \phi)(x)$. Let e_0, e_1, e_2, e_3 be a basis of $H(2)$; then $\xi(xyx) = \sum e_i(x)\, \xi(e_i)$ where the $e_i(x)$'s are homogeneous polynomials of degree 2.

Since D_n is first order, and $g(0) = 0$,

$$(D_n^- \, dU_1(y)\phi)(0) = D_n^+(x \to g(x)\,\phi(0))(0),$$

hence it only depends on the value of ϕ at 0. Let us choose $\phi_1 \in \mathscr{S}_1(\tau_n, -1)$ such that $\phi_1(0) = \phi(0)$, which is always possible. Since ϕ_1 as well as $U_1(g)\phi_1$ for all g in G are solutions to $D_n^+u = 0$, we obviously get

$$D_n^+(dU_1(y)\phi_1)(0) = 0 = D_n^+(x \to g(x) \cdot \phi_1(0))(0).$$

Evidently Proposition 5.1 implies that the space of solutions to $D_n^+ = 0$ is invariant under the multiplier representation $U_1(\tau_n, -1)$. We remark that this proof at the contrary uses the weaker assumption that there exists a sufficiently large ($\phi(0)$ arbitrary in the case of a first order differential operator) invariant subspace of solutions, to conclude the covariance relation of the operator D_n^+.

In an entirely similar fashion we can of course prove:

5.2. Proposition. *Let $U_2(g)$ be the representation of $SU(2, 2)$ given by restricting $U_{\mu'_n}$ to the subspace of functions with values in $i^-(V_{n-1} \otimes V_1)$. Then $D_n^-(U_2(\tau_n, -1)(g)) = U_2'(g) \circ D_n^-$ for all g in $SU(2, 2)$.*

6. *The Wave Operator*

We consider here the case of a representation induced from a character $\chi(a) \cdot |\det a|^{-2} = \chi(a)(\det a)^{-1}(\det a^*)^{-1}$ of the parabolic

$$P_- = \left(\begin{array}{c|c} a & 0 \\ \hline u & (a^*)^{-1} \end{array}\right);$$

$a \in \mathrm{GL}(2, \mathbb{C})$, where χ is an arbitrary character of $\mathrm{GL}(2, \mathbb{C})$. If $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we have

$$(U_\chi(g)f)(x) = \chi((cx + d)^*)\,|\det(cx + d)|^{-2} f((ax + b)(cx + d)^{-1}).$$

We also form the representation induced by $\chi(a^*)^{-1}\,|\det a|^{-2}$, and get

$$(U_\chi'(g)f)(x) = \chi(cx + d)^{-1}\,|\det(cx + d)|^{-2} f((ax + b)(cx + d)^{-1}).$$

By the general theory of intertwining operators, there exists at least one intertwining operator between U_χ and U_χ', as indicated in the works of Gårding [5], and Kunze and Stein [14]. We are here interested in the case where the intertwining relation is provided by a differential operator.

6.1. PROPOSITION (M. Kashiwara). *Let D be a constant coefficient differential operator D: $C^\infty(H(2)) \to C^\infty(H(2))$. If $D \circ U_x(g) = U_{x'}(g) \circ D$ for $g = g(a)$, $a \in GL_r(2, \mathbb{C})$, then $D \circ U_x(g) = U_{x'}(g) \circ D$ for all $g \in G$.*

Proof. As in Proposition 5.1, we need only prove

$$(D\, dU_x(v)\phi)(0) = 0 \qquad \text{for} \quad v = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}, \qquad v = v^*,$$

in $su(2, 2)$. We are given that for

$$g(A) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & -A^* \end{array}\right), \qquad A \in \mathrm{gl}(2, \mathbb{C})$$

$$D\, dU_x(g(A)) = dU_{x'}(g(A))D,$$

from which we get

$$D(\xi(Ax + xA^*)f)(0) = d\chi(A + A^*)(Df)(0). \tag{6.2}$$

Now, since $xvx = \frac{1}{2}(xv)x + \frac{1}{2}x(vx)$, we can express the vector field $\xi(xvx)$ as a linear combination of vector fields $\xi(Ax + xA^*)$, for $A \in \mathrm{gl}(2, \mathbb{C})$. Specifically, let e_0, e_1, e_2, e_3 be a basis of $H(2)$. Then $xvx = \frac{1}{2}\sum_{i=0}^{3} x_i(e_ivx + xve_i)$ and $\xi(xvx) = \frac{1}{2}\sum_{i=0}^{3} x_i\xi(e_ivx + xve_i)$. We assume proved

$$\xi(xvx) = \left(\frac{1}{2}\sum_{i=0}^{3} \xi(e_ivx + xve_i)x_i\right) - 2\,\mathrm{Tr}\, xv \tag{6.3}$$

which implies

$$\begin{aligned}(D\, dU_x(v)\phi)(0) &= D((-d\chi(xv) + 2\,\mathrm{Tr}(xv) + \xi(xvx))\phi)(0) \\ &= D\left(\sum_{i=0}^{3} (-d\chi(e_iv) + \tfrac{1}{2}\xi(e_ivx + xve_i))\, x_if\right)(0).\end{aligned}$$

Applying (6.2), this becomes

$$\sum_{i=0}^{3} -d\chi(e_iv)\, D(x_if)(0) + \tfrac{1}{2}\, d\chi(e_iv + ve_i)\, D(x_if)(0) = 0.$$

To prove (6.3), we introduce the formal adjoint ξ^* of a vector field ξ, i.e., $\int (\xi f)(x)\, g(x)\, dx = \int f(x)(\xi^* g)(x)\, dx$ for $f, g \in C^\infty$, f compactly supported. From the relation

$$\int f(axa^*)\, g(x)\, dx = |\det a|^{-4} \int f(x)\, g(a^{-1}x(a^*)^{-1})\, dx, \qquad \text{for} \quad a \in \mathrm{GL}(2, \mathbb{C}),$$

we get that

$$\xi^*(Ax + xA^*) = -2(\mathrm{Tr}\, A + \mathrm{Tr}\, A^*) - \xi(Ax + xA^*),$$

and from

$$\int f(x(-tvx + 1)^{-1})\, g(x)\, dx = \int f(x) \det(+tvx + 1)^{-4} g(x(tvx + 1)^{-1})\, dx,$$

$v \in H(2)$, we get that $\xi^*(xvx) = -4\, \mathrm{Tr}\, xv - \xi(xvx)$. But $\xi^*(xvx) = \frac{1}{2} \sum \xi^*(e_i vx + xve_i)x_i$, and (6.3) follows.

From (6.1) and (3.1) we now get

6.4. Proposition. *Let* $\mu(a) = (\det a)^n(\det a/|\det a|)^q$ *where* q *is any integer. Then* $\square^n$ *intertwines* U_μ *and* $U_{\mu'}$.

Remark. In the case $n = 1$, q even, the subspace $\square f = 0$ is a proper subspace of the principal series representation. In the case $n = 1$, q odd, $\square$ is an invertible operator between the principal series (in particular, $\square f = 0$ has no K-finite solutions).

7. *More about Spin* 0

We specialize to the representations

$$(U_m(g)f)(x) = \det(cx + d)^{-(m+2)} f((ax + b)(cx + d)^{-1})$$

of $SU(2, 2)$, and have, by Proposition 6.2 that $\square^n U_{-n}(g) = U_n(g)\square^n$, for every g in $SU(2, 2)$. It may be of some interest that this relation can be proved directly on the group level by observing that since the transformation properties are obvious on P_0, we need only consider

$$(C_n f)(x) = (\det x)^{-n} f(-x^{-1}).$$

We write $C_n = (1/d^n)C_0$, $d = \det x$, and we let

$$K = x_0(\partial/\partial x_0) + x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2) + x_3(\partial/\partial x_3).$$

Then elementary calculations show that

$$\begin{aligned}
\square C_0 &= (1/d^2)\, C_0\square - 4C_0 dK \\
KC_0 &= -C_0 K \\
C_0 d^n &= d^{-n} C_0 \\
[\square^n, d] &= 4(n+1)\, n\square^{n-1} + 4nK\square^{n-1} \qquad (n \in \mathbb{N}) \\
\left[\square, \frac{1}{d^n}\right] &= -\frac{4n}{d^{n+1}} K + \frac{4(n-1)n}{d^{n+1}} \qquad (n \in \mathbb{Z}) \\
[\square, K] &= 2\square \\
\left[K, \frac{1}{d^n}\right] &= -\frac{2n}{d^n}.
\end{aligned} \tag{7.1}$$

From this it is easy to get an induction going, to conclude that

$$\square^n d^{n-2} C_0 = (1/d^{n+2})\, C_0 \square^n,$$

from which the relation follows. A priori, there is a discontinuity at $d = 0$. We shall see that in the applications we are going to make, this does not appear.

Now recall that

$$\int_{C^+} \exp(i \operatorname{Tr}(z - w^*)k) \det k^\alpha \, dk = K_\alpha \det((z - w^*)/2i)^{-\alpha-2}$$

for $\alpha > -1$ where K_α is a generic constant.

7.2. Lemma. $\square_x \det(x - w^*)^{-r} = 4(r-1)r \det(x - w^*)^{-r-1}$ *for all* r *in* $\mathbb{R}$.

Proof. This is easily proved directly. However, it also follows from the preceding formula, by analytic continuation. From this we conclude that there are constants c_n such that

$$\begin{aligned}
&\square_x^n \det((x - w^*)/2i)^{-2} \\
&\quad = c_n \det((x - w^*)/2i)^{-2-n}, \qquad \text{for } n \text{ in } \mathbb{N}.
\end{aligned} \tag{7.3}$$

To justify the use of the relation (7.1) which we are going to make, we observe that we have

7.4. LEMMA. *For $n \geqslant 0$, the function $\phi_w: G \to \mathbb{C}$ given by (for $w \in D$)*

$$\phi_w\left(\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)\right) = (\det d)^n \det(b - w^* d)^{-2}$$

is C^∞ on G, and belongs to the space

$$\mathscr{C}(-n) = \left\{\phi;\ C^\infty \text{ functions } G \to C, \text{ such that } \phi\left(g\left(\begin{array}{c|c} a & 0 \\ \hline v & a^* \end{array}\right)\right) = \det a^{*2-n}\phi(g)\right\}.$$

Moreover, $\phi_w(u(x)) = \det(x - w^*)^{-2}$.

For n in $\mathbb{Z}$, we consider the space

$$\mathscr{A}^+(n) = \left\{\sum_{i=1}^{N} c_i(\det(x - w_i^*)/2i)^{-(n+2)}\right\}$$

which, by Lemma 1.2, is an invariant subspace of the principal series representation U_n, and consists of boundary values of holomorphic functions on D.

For $n \geqslant 0$, $\mathscr{A}^+(n)$ consists of functions whose Fourier transforms are functions supported by the forward light cone; and we know (III) that we can complete $\mathscr{A}^+(n)$ to a Hilbert space D_n^+ of distributions on Minkowski space in which U_n extends to a unitary representation.

For any n, we consider

$$\mathscr{C}(n) = \left\{\phi;\ C^\infty \text{ functions } G \to \mathbb{C}, \text{ such that } \phi(gp) = (\det a^*)^{2+n}\,\phi(g), \text{ for } p = \left(\begin{array}{c|c} a & 0 \\ \hline v & a^{*-1} \end{array}\right)\right\},$$

and the map $I: \mathscr{C}(n) \to C^\infty(H(2))$ given by

$$(I\phi)(x) = \phi(u(x)).$$

Lemma 7.2 and (7.3) then prove

7.5. PROPOSITION. *Let $n \geqslant 0$. Then $\mathscr{A}^+(n)$ is contained in the image $\square^n(I\mathscr{C}(-n))$.*

Similarly we consider

$$\mathscr{A}^-(n) = \left\{ \sum_{i=1}^{N} c_i \det((x - w_i)/2i)^{-(n+2)} \right\} = \overline{\mathscr{A}^+(n)}.$$

$\mathscr{A}^-(n)$ is an invariant subspace of the principal series representation U_n, and consists of boundary values of antiholomorphic functions on D.

For $n \geqslant 0$, $\mathscr{A}^-(n)$ consists of functions whose Fourier transforms are supported by the backward light cone. Similarly, we can complete $\mathscr{A}^-(n)$ to a Hilbert space D_n^- of distributions on Minkowski space, and U_n extends to a unitary representation in this space, and $\mathscr{A}^-(n)$ is also contained in the image of $\square^n(I\mathscr{C}(-n))$.

We denote by $\mathscr{B}^+(-n)$ the subset of functions ϕ in $I(\mathscr{C}(-n))$ for which $\square^n\phi \in \mathscr{A}^+(n)$. $\square^n$ is then a surjective map from $\mathscr{B}^+(-n)$ to $\mathscr{A}^+(n)$. Similarly we let $\mathscr{B}^-(-n)$ be the subset of functions ϕ in $I(\mathscr{C}(-n))$ for which $\square^n\phi \in \mathscr{A}^-(n)$. We let $\mathscr{S}(-n) = \{\phi \in I(\mathscr{C}(-n)),$ such that $\square^n\phi = 0\}$. The relation $\square^n U_{-n}(g) = U_n(g)\square^n$ proves that $\mathscr{S}(-n)$, $\mathscr{B}^+(-n)$, and $\mathscr{B}^-(-n)$ are invariant subspaces for the representation U_{-n}.

We also consider $\mathscr{A}^+(-n)$ and $\mathscr{A}^-(-n)$ for $n \geqslant 0$, and observe that for $n \leqslant -2$, $\mathscr{A}^+(n) = \mathscr{A}^-(n)$, and in this case the spaces consists of polynomial functions on $H(2)$ of bounded degree ($\mathscr{A}^+(-2) = \mathscr{A}^-(-2) = \mathbb{C}$), hence they are finite dimensional subspaces of the

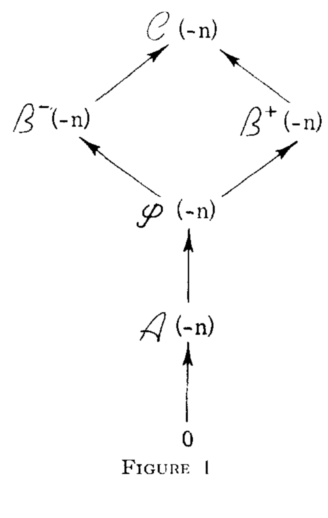

FIGURE 1

representation U_{-n}. We put $\mathscr{A}(-n) = \mathscr{A}^+(-n) \oplus \mathscr{A}^-(-n)$. For $n \geqslant 1$, we have the following diagram of inclusions of invariant subspaces under the representation U_{-n} (see Fig. 1). (For more detailed information on the composition series of this representation see [21].)

If $\phi \in C^\infty(H(2))$ we write $[\phi]_n$ for the equivalence class containing ϕ; equivalence being modulo solutions to $\square^n f = 0$, i.e.,

$$[\phi]_n = 0 \qquad \text{if and only if} \quad \square^n \phi = 0.$$

The map $\phi \to [\phi]_n$ is a surjection of $\mathscr{B}^+(-n)$ onto $\mathscr{B}^+(-n)/\mathscr{S}(-n)$; we maintain the notation U_{-n} for the representation obtained on this quotient.

We can then conclude:

7.6. Proposition. *If $\phi \in \mathscr{B}^+(-n)$, then*

$$\square^n U_{-n}(g)[\phi]_n = U_n(g)\, \square^n([\phi]_n).$$

Since, for $n \geqslant 0$, $\square^n$ gives an isomorphism between $\mathscr{B}^+(-n)/\mathscr{S}(-n)$ and $\mathscr{A}^+(n)$, we can turn the space $[\mathscr{B}^+(-n)]$ of equivalence classes of functions modulo $\square^n \phi = 0$, into a pre-Hilbert space, in which U_n will act unitarily, by setting

$$\langle [\phi], [\phi'] \rangle_n = \langle \square^n \phi, \square^n \phi' \rangle_{\mathscr{A}(n)}\,.$$

We observe that Lemma 7.3 shows that we can take a section of the map $\square^n\colon \mathscr{B}^+(-n) \to \mathscr{A}(n)$ by

$$\sum_{i=1}^{N} c_i \det((x - w_i^*)/2i)^{-(2+n)} \to \sum_{i=1}^{N} c_i \det((x - w_i^*)/2i)^{-2}.$$

So if $\mathscr{L}$ is the linear span of the functions $\det((x - w_i^*)/2i)^{-2}$, which are the Fourier transforms of the functions $f_{w_i}(k) = e^{-i\operatorname{Tr} k w_i^*}$ in $L^2(C^+)$, then $\mathscr{L}$ is dense in $[\mathscr{B}^+(-n)]$, and we have for $F_{w_i}(x) = \det((x - w_i^*)/2i)^{-2}$:

$$\begin{aligned} \langle F_{w_i}, F_{w_j} \rangle_n &= \det((w_j - w_i^*)/2i)^{-(n+2)} \\ &= \int_{C^+} f_{w_i}(k)\, \overline{f_{w_j}(k)} (\det k)^n \, dk \\ &= \langle (-\square_x)^n F_{w_i}, F_{w_j} \rangle_{L^2(H(2))}\,. \end{aligned}$$

So, for all functions ϕ, ϕ' in $\mathscr{L}$, we have

$$\langle[\phi], [\phi']\rangle = \int_{H(2)} ((-\Box)^n \phi(x)) \overline{\phi'(x)}\, dx.$$

We finally remark that the different powers of the multiplier in the U_n's are in direct correspondence with the different weightings of the spin 0, mass m representations of the Poincaré group in the given direct integrals.

8. *More about Spin* $\frac{1}{2}$

We consider here the representations

$$(U_1(\tau, \alpha)(g)f)(x) = \tau(cx + d)^{-1} \det(cx + d)^{-(\alpha+2)} f((ax + b)(cx + d)^{-1}),$$

$$(U_2(\tau, \alpha)(g)f)(x) = \tau(cx + d)^{*} \det(cx + d)^{-(\alpha+3)} f((ax + b)(cx + d)^{-1}).$$

We let $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ be the basis of $H(2)$ given by the Pauli matrices, i.e., $\sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\sigma_1 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\sigma_2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma_3 = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$ Then, $D_1^- = \sum_{i=0}^3 \sigma_i(\partial/\partial x_i)$; $D_1^+ = c(D_1^-) = \sigma_0(\partial/\partial x_0) - \sum_{i=0}^3 \sigma_i(\partial/\partial x_i)$, and by Proposition (5.11) and (5.2),

$$D_1^+ U_1(\tau, -1) = U_2(\tau, 0) D_1^+,$$
$$D_1^- U_2(\tau, -1) = U_1(\tau, 0) D_1^-.$$

We will now show that we get the same picture as for spin 0, but in order to do so, we shall have to take the following simple relations for granted. We keep the notation of paragraph 7, extending it in an obvious way (e.g.,

$$K \to \begin{pmatrix}K & 0\\ 0 & K\end{pmatrix}; \qquad \Box \to \begin{pmatrix}\Box & 0\\ 0 & \Box\end{pmatrix}).$$

Then, for $H \in H(2)$

$$\begin{aligned}
D_1^- H C_0 &= 4C_0 + d^{-1} c(H)\, C_0 D_1^-,\\
D_1^+ c(H) C_0 &= 4C_0 + d^{-1} H C_0 D_1^+,\\
[D_1^-, d^n] &= 2nc(H)\, d^{n-1},\\
[D_1^+, d^n] &= 2nH d^{n-1},\\
[\Box, c(H)] &= 2D_1^-,\\
[\Box, H] &= 2D_1^+,\\
[K, c(H)] &= c(H),\\
[K, H] &= H,\\
D_1^+ C_0 &= -d^{-1} C_0 D_1^- + 2d^{-1} HKC_0,\\
D_1^- C_0 &= -d^{-1} C_0 D_1^+ + 2d^{-1} c(H)\, KC_0.
\end{aligned}$$

8.1. LEMMA. *For $n \geqslant 0$,*

$$\square^n D_1^- d^{n-2} H C_0^- = d^{-n-3} c(H)\, C_0 \square^n D_1^-$$
$$\square^n D_1^+ d^{n-2} c(H) C_0 = d^{-n-3} H C_0 \square^n D_1^+.$$

Proof. We will only prove the latter, since the two cases are entirely similar. We take $n = 0$:

$$\begin{aligned} D_1^+ d^{-2} c(H) C_0 &= d^{-2} D_1^+ c(H) C_0 + [D_1^+, d^{-2}]\, c(H) C_0 \\ &= d^{-2}(4C_0 + d^{-1} H C_0 D_1^+) - 4Hc(H)\, d^{-3} C_0 \\ &= d^{-3} H C_0 D_1^+. \end{aligned}$$

We will now assume that the relation holds true up to n and compute

$$\begin{aligned} &\square^{n+1} D_1^+ d^{n+1-2} c(H) C_0 \\ &= \square^{n+1}(d D_1^+ d^{n-2} c(H) C_0 + 2H d^{n-2} c(H) C_0) \\ &= d\square(d^{-n-3} H C_0 \square^n D_1^+) + [4(n+2)(n+1)\, \square^n + 4(n+1)\, K\square^n] \\ &\quad \times D_1^+ d^{n-2} c(H) C_0 + 2\square^{n+1} d^{n-1} C_0 \\ &= d^{-n-2} \square H C_0 \square^n D_1^+ \\ &\quad + d(-4(n+3)\, d^{-n-4} K + 4(n+3)(n+2)\, d^{-n-4})\, H C_0 \square^n D_1^+ \\ &\quad + [4(n+2)(n+1) + 4(n+1)K][d^{-n-3} H C_0 \square^n D_1^+] \\ &\quad + 2d^{-n-3} C_0 \square^{n+1} \\ &= d^{-n-2} H (d^{-2} C_0 \square - 4C_0 dK)\, \square^n D_1^+ \\ &\quad + 2d^{-n-2}(-d^{-1} C_0 D_1^- + 2d^{-1} HKC_0)\, \square^n D_1^+ \\ &\quad - 4(n+3)\, d^{-n-3} KHC_0 \square^n D_1^+ + 4(n+2)(n+3)\, d^{-n-3} H C_0 \square^n D_1^+ \\ &\quad + 4(n+2)(n+1)\, d^{-n-3} H C_0 \square^n D_1^+ \\ &\quad + 4(n+1)\, K d^{-n-3} H C_0 \square^n D_1^+ + 2d^{-n-3} C_0 \square^{n+1} \\ &= d^{-n-4} H C_0 \square^{n+1} D_1^+ + 4d^{-n-3} HKC_0 \square^n D_1^+ + 4d^{-n-3} HKC_0 \square^n D_1^+ \\ &\quad - 4(n+3)\, d^{-n-3} H C_0 \square^n D_1^+ - 4(n+3)\, d^{-n-3} HKC_0 \square^n D_1^+ \\ &\quad + 4(n+2)(n+3)\, d^{-n-3} H C_0 \square^n D_1^+ \\ &\quad + 4(n+2)(n+1)\, d^{-n-3} H C_0 \square^n D_1^+ \\ &\quad - 8(n+1)(n+3)\, d^{-n-3} H C_0 \square^n D_1^+ + 4(n+1)\, d^{-n-3} H C_0 \square^n D_1^+ \\ &\quad + 4(n+1)\, d^{-n-3} HKC_0 \square^n D_1^+ \\ &= d^{-n-4} H C_0 \square^{n+1} D_1^+. \end{aligned}$$

As in the case of spin 0, once we know a relation is true for $g \in P_0$ (and in the following case we do, by Lemma 4.5), then it needs only be checked at the element

$$\left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right),$$

which is the element corresponding to conformal inversion, since it, together with P_0 generate $SU(2, 2)$. But, if we let

$$\nabla = D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix},$$

we have

$$D^2 = \begin{pmatrix} \square & 0 \\ 0 & \square \end{pmatrix},$$

and Lemma 8.1 is easily seen exactly to be this remaining verification in order to have proved the

8.2. PROPOSITION. *Let*

$$U(\tau, \alpha) = \begin{pmatrix} U_1(\tau, \alpha) \\ U_2(\tau, \alpha) \end{pmatrix} = U_1(\tau, \alpha) \oplus U_2(\tau, \alpha).$$

Then, for $n \geqslant 0$,

$$(\nabla)^{2n+1} U(\tau, -1 - n) = U(\tau, n)(\nabla)^{2n+1}.$$

Now we recall the formulas: (III.3.1, III.5.5). For $\alpha > -1$

$$\left(\frac{z - w^*}{2i}\right)^{-1} \det\left(\frac{z - w^*}{2i}\right)^{-(\alpha+2)}$$

$$= \frac{2^{5+2\alpha}}{(\pi/2)\,\Gamma(\alpha+1)\,\Gamma(\alpha+3)} \int_{C^+} e^{i \operatorname{Tr} k(z - w^*)} k (\det k)^\alpha \, dk$$

$$\det\left(\frac{z - w^*}{2i}\right)^{-(\alpha+2)}$$

$$= \frac{2^{4+2\alpha}}{(\pi/2)\,\Gamma(\alpha+1)\,\Gamma(\alpha+2)} \int_{C^+} e^{i \operatorname{Tr} k(z - w^*)} (\det k)^\alpha \, dk$$

$$\left(\frac{z - w^*}{2i}\right) \det\left(\frac{z - w^*}{2i}\right)^{-(\alpha+3)}$$

$$= \frac{2^{5+2\alpha}}{(\pi/2)\,\Gamma(\alpha+1)\,\Gamma(\alpha+3)} \int_{C^+} e^{i \operatorname{Tr} k(z - w^*)} c(k) (\det k)^\alpha \, dk.$$

In order to study quotients of representations, we need the following simple facts:

8.3. LEMMA. *For $v \in \mathbb{C}^2$, $w \in D$,*

$$D_1^- \det(x - w^*)^{-n} v = -2n(x - w^*)^{-1} \det(x - w^*)^{-n} v$$

$$D_1^+ \det(x - w^*)^{-n} v = -2n(x - w^*) \det(x - w^*)^{-n-1} v$$

$$D_1^-(x - w^*) \det(x - w^*)^{-n} v = (4 - 2n) \det(x - w^*)^{-n} v$$

$$D_1^+(x - w^*)^{-1} \det(x - w^*)^{-n+1} v = (4 - 2n) \det(x - w^*)^{-n} v,$$

relations which follow, for example, (by analytic continuation) from the preceding relations.

We will describe similar to the case of spin 0 some subspaces of the C^∞-functions on $H(2)$ with values in $S = S^+ \oplus S^-$. We recall that $V_1 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ is identified via $v \to v^+$ to $S^+(v \to v^-$ to $S^-)$. We introduce, for $v_1, \ldots, v_n \in V_1$, $w_1, \ldots, w_n \in D$,

$$\mathscr{A}(\tau^+, n) = \left\{ \sum_{i=1}^{N} \det((x - w_i^*)/2i)^{-(n+2)}((x - w_i^*)/2i)^{-1} \cdot v_i^+ \right\}$$
$$\subset C^\infty(H(2), S^+),$$

$$\mathscr{A}(\tau^-, n) = \left\{ \sum_{i=1}^{N} \det((x - w_i^*)/2i)^{-(n+3)}((x - w_i^*)/2i) \cdot v_i^- \right\}$$
$$\subset C^\infty(H(2), S^-),$$

and $\mathscr{A}(S, n) = \mathscr{A}(\tau^+, n) \oplus \mathscr{A}(\tau^-, n)$. For $n \geqslant 0$, $\mathscr{A}(S, n)$ consists of functions whose Fourier transforms are functions supported in the forward lightcone C^+.

We introduce

$$\mathscr{L}(\tau^+) = \left\{ \sum_{i=1}^{N} \det((x - w_i^*)/2i)^{-2} v_i^+ \right\} \subset C^\infty(H(2), S^+)$$

$$\mathscr{L}(\tau^-) = \left\{ \sum_{i=1}^{N} \det((x - w_i^*)/2i)^{-2} v_i^- \right\} \subset C^\infty(H(2), S^-)$$

and $\mathscr{L}(S) = \mathscr{L}(\tau^+) \oplus \mathscr{L}(\tau^-)$. Using the relation $\nabla\!\!\!/^{2n+1} = \nabla\!\!\!/ \square^n$ and Lemma 8.3, we see that $\nabla\!\!\!/^{2n+1}$ is a surjective map of $\mathscr{L}(S)$ onto $\mathscr{A}(S, n)$.

We consider the principal series representations $U(\tau, -1 - n)$ (or $U(\tau, n)$), acting on the subspaces $\mathscr{C}(S, -1 - n)$, (or $\mathscr{C}(S, n)$) of

C^∞-functions on $H(2)$ with values in S that can be extended as C^∞ sections of the corresponding line bundle over G/P.

We know that $\mathscr{A}(S, n) \subset \mathscr{C}(S, n)$ and is an invariant subspace under the representation $U(\tau, n)$. Furthermore, for $n \geqslant 0$, $\mathscr{A}(S, n)$ can be provided with the structure of a pre-Hilbert space in which $U(\tau, n)$ acts unitarily.

We introduce $\mathscr{B}(S, -1-n) \subset \mathscr{C}(S, -1-n)$ to be the subspace of functions $\phi \in \mathscr{C}(S, -1-n)$ such that $\nabla^{2n+1}\phi \in \mathscr{A}(S, n)$; it is clearly an invariant subspace of $\mathscr{C}(S, -1-n)$ under the representation $U(\tau, -1-n)$. Similar to the case of spin 0, the space $\mathscr{L}(S)$ is contained, for $n \geqslant 0$, in $\mathscr{B}(S, -1-n)$, providing a section of the surjective map

$$\nabla^{2n+1}\colon \mathscr{B}(S, -1-n) \to \mathscr{A}(S, n).$$

The space $\mathscr{L}(S)$ is composed of functions whose Fourier transforms are in $L^2(C^+)$.

Also we introduce $\mathscr{S}(S, -1-n) = \{\phi \in \mathscr{C}(S, -1-n)$, such that $\nabla^{2n+1}\phi = 0\}$ and $\mathscr{S}(S, -1-n)$ is an invariant subspace of $\mathscr{C}(S, -1-n)$ under the representation $U(\tau, -1-n)$.

Similarly we have

$$\mathscr{A}(S, -1-n) \subset \mathscr{S}(S, -1-n) \subset \mathscr{B}(S, -1-n) \subset \mathscr{C}(S, -1-n)$$

as a sequence of inclusion of invariant subspaces under the representation $U(\tau, -1-n)$. Similarly, if $n \geqslant 2$, we see that $\mathscr{A}(S, -1-n)$ is composed of polynomial functions on $H(2)$ of bounded degree; hence is a finite dimensional subspace of $U(\tau, -1-n)$.

We denote by $[\phi]_{2n+1}$ the class of ϕ modulo $\nabla^{2n+1}f = 0$; the map $\phi \to [\phi]_{2n+1}$ is a surjective map from $\mathscr{B}(S, -1-n)$ onto

$$\mathscr{B}(S, -1-n)/\mathscr{S}(S, -1-n).$$

We still denote by $U(\tau, -1-n)$ the corresponding representation of the group $U(2, 2)$ on this composition factor.

Clearly ∇^{2n+1} gives an isomorphism of $[\mathscr{B}(S, -1-n)]_{2n+1}$ onto $\mathscr{A}(S, n)$, so $[\mathscr{B}(S, -1-n)]_{2n+1}$ inherits the structure of a pre-Hilbert space, in which the representation $U(\tau, -1-n)$ will act unitarily.

VI. The Harmonic Representation

We consider the group $G \subset \mathrm{GL}(\mathbb{C}^4) \subset \mathrm{GL}(\mathbb{R}^8)$. Let p_u be the hermitian form on $\mathbb{C}^4 = V$ as in (I.2.1) and define a symmetric bilinear (real) form B on $V = \mathbb{R}^8$ by $B = \operatorname{Im} p_u$. Then, since an element of G leaves p_u stable, this determines an imbedding of G into $\mathrm{Sp}(B)$, where $\mathrm{Sp}(B)$ is the symplectic group of transformations of $\mathbb{R}^8$ leaving the form B stable. We can then define the metaplectic representation of $\mathrm{Sp}(B)$ in $L^2(\mathbb{R}^4)$ and restrict this representation (projective) to G. We will study this restriction R.

We consider the Heisenberg group N with Lie algebra $\eta = V \oplus \mathbb{R}E$, where we have defined the Lie bracket by $[v, v'] = -2B(v, v')E$ (E is a basis of the one-dimensional center $\mathbb{R}E$ of η). We consider $V_1 = \mathbb{C}e_1 + \mathbb{C}e_2$, $V_2 = \mathbb{C}e_3 + \mathbb{C}e_4$. Both V_1 and V_2 are totally isotropic for the form B. If $u_1 = \alpha_1 e_1 + \alpha_2 e_2 \in V_1$ and $u_2 = \beta_1 e_3 + \beta_2 e_4 \in V_2$ $\operatorname{Im} p_u(u_1, u_2) = \operatorname{Re}(\alpha_1\bar{\beta}_1 + \alpha_2\bar{\beta}_2)$. We will identify V_1 and V_2 with $\mathbb{C}^2$ by $u_1 = (\alpha_1, \alpha_2)$, $u_2 = (\beta_1, \beta_2)$, and we write $\langle u_1, u_2\rangle = \alpha_1\bar{\beta}_1 + \alpha_2\bar{\beta}_2$. There exist a unique irreducible unitary representation T of N having restriction $T(\exp tE) = e^{it}\,\mathrm{Id}$ to the center $\exp \mathbb{R}E$ of N. We can realize T as acting in $L^2(\mathbb{C}^2)$ as follows:

$$(T(\exp u_1)f)(u) = e^{-2i\operatorname{Re}\langle u, u_1\rangle} f(u), \qquad u_1 \in V_1,$$
$$(T(u_2)f)(u) = f(u - u_2), \qquad u_2 \in V_2.$$

The group G acts as a group of automorphisms of N by

$$g \cdot (\exp(u + tE) = \exp(g \cdot u + tE), \qquad u \in V, \quad t \in \mathbb{R}.$$

Since G leaves the center of N stable, the representation $T_g(n) = T(g^{-1} \cdot n)$ is unitarily equivalent to T (theorem of Stone, Von Neumann). Hence there exists, for each $g \in G$ a unitary operator $R(g)$ defined up to a scalar of modulus one such that $R(g)\, T(n)\, R(g)^{-1} = T(g \cdot n)$, and this defines a projective representation of G. Actually the following choices of $R(g)$ determines a representation of $U(2, 2)$ [7(a)]: If $a \in \mathrm{GL}(2, \mathbb{C})$,

$$(R(g(a))f)(u) = \det a\, f(a^* u).$$

If $x_0 \in H(2)$,

$$(R(u(x_0))f)(u) = e^{-i\langle x_0 u, u\rangle} f(u).$$

If

$$\sigma = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right),$$

$$(R(\sigma)f)(u) = (-1/\pi^2) \int \exp(-2i \operatorname{Re}\langle u, v\rangle)\, f(v)\, dv.$$

We consider

$$\mathscr{L}_n = \{\phi \in L^2(\mathbb{C}^2) \text{ such that } \phi(e^{i\theta}u) = e^{-in\theta}\phi(u)\}.$$

Then $\mathscr{L}_n$ is stable under the representation R, and we denote by R_n the restriction of R to $\mathscr{L}_n$. We have $L_2(\mathbb{C}) = \bigoplus_n \mathscr{L}_n$.

The character $g \mapsto \det g$ is a unitary character of $U(2, 2)$ which is trivial on $SU(2, 2)$. We consider $\tilde{R}(g) = (\det g)^{-1}R(g)$. $\tilde{R}$ and R coincides on $SU(2, 2)$; and we define $\tilde{R}_n$ to be the restriction of R to $\mathscr{L}_n$.

1.1. Proposition. *If $n \geqslant 0$ the representations R_n and $T_1(\tau_n, -1)$ are equivalent. If $n \leqslant 0$, the representations $\tilde{R}_n$ and $T_2(\tau_n, -1)$ are equivalent.*

Proof. Let $u = u_1e_1 + u_2e_2 \in \mathbb{C}^2$; we form the 2×2 complex matrix

$$M(u) = \begin{pmatrix} u_1 & 0 \\ u_2 & 0 \end{pmatrix}$$

and have $\operatorname{Tr}(M(u)\, M(v)^*) = \langle u, v\rangle$. Also, if $g \in \mathrm{GL}(2, \mathbb{C})$, $M(gu) = gM(u)$. We define

$$b(u) = M(u)\, M(u)^* = \begin{pmatrix} |u_1|^2 & u_1\bar{u}_2 \\ u_2\bar{u}_1 & |u_2|^2 \end{pmatrix}.$$

We remark that $b(u) = b(u')$ if and only if $u' = e^{i\theta}u$ for some $\theta \in \mathbb{R}$; we have $b(g \cdot u) = gb(u)g^*$. As $dm(k)$ is the unique semi-invariant measure on $b(C^+)$ under the transformations $k \mapsto gkg^*$ of $\mathrm{GL}(2, \mathbb{C})$, it follows that up to a positive scalar $\int_{\mathbb{C}^2} \phi(b(u))\, du = \int_{b(C^+)} \phi(k)\, dm(k)$ for any continuous compactly supported function ϕ on $b(C^+)$. At the point $b(u) = M(u)\, M(u)^*$ be one-dimensional subspace $V_n(b(u))$ is spanned by the vector

$$\tau_n(M(u)) \cdot e_1{}^n = (u_1e_1 + u_2e_2)^n$$

(as $\tau_n(M(u)^*) \cdot V_n = e_1{}^n$).

If $\phi \in \mathscr{L}_n$, $n \geqslant 0$, we define

$$(I\phi)(u) = \phi(u)\, \tau_n(M(u)) \cdot e_1{}^n = \phi(u)(u_1e_1 + u_2e_2)^n.$$

Then $(I\phi)(ue^{i\theta}) = (I\phi)(u)$, so we can define $(I\phi)(M(u)\, M(u^*)) =$

$I\phi(b(u)) = \phi(u)\,\tau_n(M(u)) \cdot e_1{}^n$ as a function on $b(C^+)$ with values in the one-dimensional subspaces $\tau_n(b(u)) \cdot V_n$. We have

$$\langle \tau_n(M(u))\, e_1{}^n, \tau_n(M(u))\, e_1{}^n \rangle = \langle \tau_n(M(u)^*\, M(u))\, e_1{}^n, e_1{}^n \rangle$$
$$= (|\, u_1\,|^2 + |\, u_2\,|^2)^n = \mathrm{Tr}(b(u))^n;$$

also

$$\|\, I\phi\,\|_{L_1^+}^2 = \int_{b(C^+)} \frac{\langle I\phi(k), I\phi(k)\rangle}{(\mathrm{Tr}\, k)^n}\, dm(k)$$
$$= \int_{\mathbb{C}^2} |\, \phi(u)|^2\, \frac{\langle \tau_n(M(u))\, e_1{}^n, \tau_n(M(u))\, e_1{}^n\rangle}{\mathrm{Tr}(b(u))^n}\, du$$
$$= \|\, \phi\,\|_{L^2(\mathbb{C})}^2 .$$

So I defines a unitary isomorphism between $\mathscr{L}_n$ and $L_1{}^+(\tau_n\,, -1)$. We will prove that this map intertwines the representation $\hat{T}(\tau_n\,, -1)$ with the representation R_n. It follows from the formulas (IV.3.2) that I intertwines the operators coming from the maximal parabolic group P. We need then only check the action of

$$\sigma = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right).$$

We define, if $w \in D$, $f \in V_n$, the functions

$$\Phi(w, f)(k) = e^{-i\mathrm{Tr}\, kw^*}\tau_n(k) \cdot f.$$

The collection of functions $\Phi(w, f)$ is a dense set of functions in $L_1{}^+(\tau_n\,, -1)$ (IV.1.3), and by (IV.1.2) we have

$$\check{\Phi}(w, f)(z) = \det((z - w^*)/2i)^{-1}\, \tau_n((z - w^*)/2i) \cdot f$$
$$= R_1(\tau_n\,, -1)(z, w) \cdot f.$$

A particular case of the relation (I.5.6) becomes

$$T_1(\tau_n\,, -1)(g_0)\, \check{\phi}(w, f) = \check{\phi}(g_0 \cdot w, (J_1(\tau_n\,, -1)(g_0\,, w))^{*-1} \cdot f), \quad (1.2)$$

where $J_1(\tau_n\,, -1)(g_0\,, w) = \det(c_0 w + d_0)\, \tau_n(c_0 w + d_0)$.

We will then use the following idea: although the operator $\hat{T}_1(\tau_n\,, -1)(\sigma)$ acting on $L_1{}^+(\tau_n\,, -1)$ is not easily expressed, nevertheless its action on a dense set of special functions is given by the explicit formula (1.2). We have

$$\Phi(w, f)(M(u)\, M(u)^*)$$
$$= \exp(-i\, \mathrm{Tr}(M(u)\, M(u)^*\, w^*))\, \tau_n(M(u)) \cdot \tau_n(M(u)^*) \cdot f.$$

We define $\psi(w,f)(u) \cdot e_1{}^n = e^{-i\langle u, w\cdot u\rangle} \tau_n(M(u)^*) \cdot f$. So $I(\psi(w,f)) = \Phi(w,f)$.

To prove our proposition, it remains only to check that if

$$\sigma = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right), \qquad R(\sigma)\,\phi(w,f) = (-1)^n(\det w^*)^{-1}\,\phi(-w^{-1}, \tau_n(w^*)^{-1}\cdot f),$$

i.e.,

$$(-1/\pi^2)\int_{\mathbb{C}^2} \exp(-2i\,\mathrm{Re}\langle u, v\rangle)\exp(-i\langle w^*v, v\rangle)\,\tau_n(M(v)^*\cdot f)\,dv$$
$$= (-1)^n \det(w^*)^{-1}\exp(i\langle (w^*)^{-1}u, u\rangle)\,\tau_n(M(u)^*)\cdot\tau_n(w^*)^{-1}\cdot f.$$

As both sides are antiholomorphic functions of w in D it is enough to check this relation for $w = iy$, i.e.,

$$\int_{\mathbb{C}^2} \exp(-2i\,\mathrm{Re}\langle u, v\rangle)\, e^{-\langle yv, v\rangle}\tau_n(M(v))^*\cdot f\,dv$$
$$= \pi^2(-i)^n(\det y)^{-1}\, e^{-\langle y^{-1}u, u\rangle}\tau_n(M(u)^*)\cdot\tau_n(y)^{-1}\cdot f. \tag{1.3}$$

We have for $n = 0$, $y = id$, that this relation is true since

$$\int_{\mathbb{C}^2} \exp(-2i\,\mathrm{Re}\langle u, v\rangle)\, e^{-\langle v,v\rangle}\,dv = \pi^2 e^{-\langle u,u\rangle}$$

Applying the operator $(\partial/\partial u_1)^i(\partial/\partial u_2)^j$ to both sides, we get the relation (1.3) for $f = e_1{}^i e_2{}^j$ in V_{i+j} (since $\tau_n(M(u)^*)\cdot e_1{}^i e_2{}^j = (\bar{u}_1)^i(\bar{u}_2)^j\, e_1{}^n$) and $y = id$),

$$\int_{\mathbb{C}^2} \exp(-2i\,\mathrm{Re}\langle u, v\rangle)\, e^{-\langle v,v\rangle}\tau_n(M(v))^* f\,dv$$
$$= \pi^2(-i)^n\, e^{-\langle u,u\rangle}\tau_n(M(u)^*)\cdot f. \tag{1.4}$$

Changing v to $g\cdot v$ in the integral (1.4), $g \in \mathrm{GL}(2, \mathbb{C})$, we get the relation (1.3) for $y = g^*g$.

Similarly, the one-dimensional subspace $V_n(c(b(u)))$ is spanned by $c(M(u)^*)\cdot e_2{}^n = (-\bar{u}_2 e_1 + \bar{u}_1 e_2)^n$. We define for $\phi \in \mathscr{L}_{-n}$ $(n > 0)$

$$(I\phi)(u) = \phi(u)\,\tau_n(c(M(u))^*)\cdot e_2{}^n = \phi(u)(-\bar{u}_2 e_1 + \bar{u}_1 e_2)^n.$$

This gives an isomorphism of $\mathscr{L}_{-n}$ onto $L_2{}^+$ which intertwines $\tilde{R}_n$ and $\hat{T}_2(\tau_n, -1)$.

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